Hermitian Curvature Flow and Curvature Positivity Conditions

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Abstract

In the present thesis, we study metric flows on, not necessarily Kähler, complex Hermitian manifolds. Using the framework of the Hermitian curvature flows, due to Streets and Tian [ST11], we find a distinguished metric flow (further referred to as the HCF), which shares many features of the Ricci flow. For a large family of convex sets of Chern curvature tensors, we prove its invariance under the HCF. Varying these convex sets, we demonstrate that the HCF preserves many natural curvature (semi)positivity conditions in complex geometry: Griffiths/dual-Nakano/m-dual positivity, positivity of the holomorphic orthogonal bisectional curvature, lower bounds on the second scalar curvature. The key ingredient in the proof of these results is a very special form of the evolution equation for the Chern curvature tensor, which we were able to obtain by introducing a *torsion-twisted connection*. Motivated by these results, we formulate a differential-geometric version of Campana-Peternell conjecture, which characterizes the rational homogeneous manifolds by certain curvature semipositivity properties. We propose a metric flow approach based on the HCF and make an initial progress towards the conjecture. Specifically, we characterize complex manifolds admitting a metric of quasipositive Griffiths curvature, and find obstructions on the torsion-twisted holonomy group of an Hermitian manifold with a semipositive dual-Nakano curvature. We illustrate the behavior of the HCF by explicitly computing it on all complex homogeneous manifold, equipped with submersion metrics.

Presentations

The results of this dissertation were presented at the following conferences and seminars

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Contents

Chapter 1

Introduction

1.1 Overview

The subject of this thesis lies at the intersection of algebraic, differential, and complex geometry. A general principle bridging these fields suggests that a certain algebraic or differential-geometric positivity of the tangent bundle implies strong topological and geometric restrictions on the underlying manifold, and in many cases makes it possible to classify all such manifolds. A prototypical illustration of the stated principle is the uniformization theorem for Riemann surfaces, which, in particular, states that any closed oriented surface admitting a metric of semipositive Gauss curvature is either conformally equivalent to the round sphere $(S^2, g_{\mathrm{round}})$, or is flat and isomorphic to the torus T^2. The relation between curvature positivity and geometric uniformization becomes much more interesting and rich in higher dimensions. Let us review some of the related results and problems.

A classical theorem of Bochner [Boc46] states that if a compact manifold M admits a metric g with a semipositive Ricci curvature, then $b_1(M) \leqslant n$. If, moreover, $b_1(M) = n$, then (M, g) is isometric to a flat torus. The proof of this theorem is based on the application of Bochner's identity, which relates the covariant derivatives of harmonic 1-forms to the Ricci curvature of the underlying manifold. This is one of the simplest results, demonstrating that curvature positivity implies "boundedness" of the topology of the underlying manifold. Bochner's theorem also illustrates another important idea: often curvature semipositivity together with very mild topological assumptions have strong geometric consequences.

In his seminal papers [Ham82; Ham86], Hamilton introduced the Ricci flow and used it to prove the classification of three/four-dimensional manifolds admitting metrics with positive Ricci curvature/positive curvature operator. The main idea in Hamilton's approach is to control the positivity of the curvature tensor under the Ricci flow using a form of parabolic maximum principle for tensors. Following this route, one can prove the pinching of the curvature tensor toward a constant curvature tensor. This program was used by Böhm and Wilking [BW08] in their description of manifolds admitting positive curvature operator in all dimensions. The assumption of the positivity for the curvature operator was substantially weakened in a paper of Brendle and Schoen [BS09], where the authors proved that manifolds with 1/4-pinched sectional curvature are diffeomorphic to the space forms. In [BS08] they extended these results by classifying manifolds with *weakly* 1/4-pinched sectional curvature. An important step in this proof is a version of strong maximum for the curvature tensor evolved under the Ricci flow.

In the world of projective manifolds (Kähler with $[\omega] \in H^2(M, \mathbb{Z})$), besides the differential-geometric notions of curvature positivity, there exist also purely algebraic positivity notions, e.g., concepts of ample, globally generated, nef, big vector bundles. This fact makes the relation between the curvature positivity and the uniformization problems even more interesting and rich. Frankel's conjecture states that, the only manifold admitting a Kähler metric of positive Griffiths curvature is the projective space $\mathbb{P}^n$. It was proved by Siu and Yau [SY80] by studying harmonic maps $S^2 \to M$. An algebraic counterpart of Frankel's conjecture was proposed by Hartshorne. It states that $\mathbb{P}^n$ is the only projective manifold with an ample tangent bundle. The conjecture was resolved by Mori [Mor79] by purely algebraic methods.

The Ricci flow on Kähler manifolds satisfies particularly strong existence, convergence and singularity formation properties. Using these results, Chen, Sun, and Tian [CST09] gave an alternative proof of Frankel's conjecture based on the Kähler-Ricci flow. In [Mok88], Mok combined the approach based on the Kähler-Ricci flow with delicate algebraic methods to prove a generalization of Frankel's conjecture. A simplification of Mok's argument, which uses solely the Ricci flow, was found by Gu [Gu09]. Ties between Kähler and algebraic geometry over $\mathbb{C}$ open vast opportunities for relating various questions about the Kähler-Ricci flow to purely algebraic problems. An illustration of this principle is the *analytical minimal model program*, proposed by Song and Tian [ST17].

Unlike the Kähler situation, there are very little efficient tools to study non-Kähler complex manifolds. Thus, given all the success of the Ricci flow, it is reasonable to try to extend it onto arbitrary, not necessarily Kähler, Hermitian manifolds. However, on a general Hermitian manifold (M, g, J), its Ricci curvature $\mathrm{Ric}(g)$ is not invariant under the operator of almost complex structure, therefore the evolved metric is not necessarily Hermitian. This issue raises the following question motivating our research.

Question 1.1. *Are there geometric flows useful for studying Hermitian manifolds? Do these flows, similarly to the Ricci and Kähler-Ricci flows, satisfy strong existence, convergence, and regularization properties?*

The main difficulty in approaching this question is that there exists a multitude of, seemingly natural, modifications of the Ricci flow, while the geometric significance of these modified Ricci curvatures and the corresponding 'Einstein' metrics (i.e, scale-static solutions to the flow equation) is not always apparent. This makes the study of the long-time existence and convergence for these flows a difficult task. To find a distinguished evolution equation for an Hermitian metric, we further refine Question 1.1.

Question 1.2. *Does there exist a modification of the Ricci flow for the Hermitian setting, which preserves natural complex-analytic curvature positivity conditions?*

Preservation of curvature positivity, imposed in Question 1.2 on an evolution equation for an Hermitian metric, is a reasonable and important restriction, since: (a) in the Riemannian setting it is satisfied by the Ricci flow; (b) its Riemannian analogue lies in the core of most of the classification and uniformization problems resolved with the use of the Ricci flow.

In this thesis, we answer Question 1.2 affirmatively by finding a distinguished member of Streets-Tian's family of Hermitian curvature flows [ST11]. Namely, we consider an evolution equation

$$\frac{dg_{i\bar{j}}}{dt} = -g^{m\bar{n}}\Omega_{m\bar{n}i\bar{j}} - \frac{1}{2}g^{m\bar{n}}g^{p\bar{s}}T_{mp\bar{j}}T_{\bar{n}\bar{s}i}, \tag{HCF}$$

where Ω and T are the curvature and the torsion of the Chern connection. The main feature of the above flow is that the evolution equation for the Chern curvature takes a very special form.

Proposition 3.22. *Under the HCF, $\Omega \in \mathrm{Sym}^{1,1}(\mathrm{End}(T^{1,0}M)) \subset \mathrm{End}(T^{1,0}M) \otimes \mathrm{End}(T^{0,1}M)$ evolves by equation*

$$\frac{d}{dt}\Omega = \Delta_{\nabla^T}\Omega + \Omega^{\#} + \mathrm{tr}_{\Lambda^2(M)}(\Omega_{\nabla^T} \otimes \overline{\Omega}_{\nabla^T}) + \mathrm{ad}_u\Omega$$

where $u \in \operatorname{End}(T^{1,0}M)$ *is given by* $u_i^j = -\frac{1}{2}\Omega_{i\overline{n}}{}^{\overline{n}j}$.

Here Δ_{∇^T} and Ω_{∇^T} are the Laplacian and the curvature of the *torsion-twisted* connection canonically attached to any Hermitian manifold, and $\Omega^{\#}, \operatorname{ad}_u\Omega$ are certain algebraic operators on the space of curvature tensors. We will spend a large part of the thesis setting up and motivating these algebraic and geometric notions.

We prove that the evolution equation for Ω satisfies a version of the maximum principle for tensors. Given a subset $S \subset \operatorname{End}(V)$, $V = \mathbb{C}^{\dim M}$ and a function $F\colon \operatorname{End}(V) \to \mathbb{R}$, we define a convex set of algebraic curvature tensors $C(S,F) \subset \operatorname{Sym}^{1,1}(\operatorname{End}(V))$. Using the maximum principle for tensors, we prove that many curvature positivity conditions are preserved along the flow (HCF).

Theorem 4.10. *Consider an* $\operatorname{Ad} G$*-invariant subset* $S \subset \operatorname{End}(V)$ *and a nice function* $F\colon \operatorname{End}(V) \to \mathbb{R}$. *Let* $g = g(t)$ *be a solution to the HCF on an Hermitian manifold* (M,g,J) *for* $t \in [0, t_{\max})$. *Assume that* $\Omega^{g(0)}$ *satisfies* $C(S,F)$, *i.e.,*

$$\Omega^{g(0)} \in C(S,F) \times_G P.$$

Then the same holds for all $t \in [0, t_{\max})$.

Here $C(S,F) \times_G P$ is a subbundle of $\operatorname{Sym}^{1,1}(\operatorname{End}(T^{1,0}M))$ associated with a $GL(V)$-space $C(S,F) \subset \operatorname{Sym}^{1,1}(\operatorname{End}(V))$. Convex sets $C(S,F)$ define many natural curvature (semi)positivity conditions, including Griffiths positivity and dual-Nakano positivity. This theorem extends analogous statements proved in [Ham82; Ban84; Ham86; Mok88; BW08; BS08; BS09; Wil13] in the context of the Ricci and Kähler-Ricci flows. We also prove a version of the strong maximum principle for Ω.

Theorem 4.12. *Consider an* $\operatorname{Ad} G$*-invariant subset* $S \subset \operatorname{End}(V)$ *and a nice function* $F\colon \operatorname{End}(V) \to \mathbb{R}$. *Let* $g = g(t)$ *be a solution to the HCF on an Hermitian manifold* (M,g,J) *for* $t \in [0, t_{\max})$. *Assume that* $\Omega^{g(0)}$ *satisfies* $C(S,F)$. *Then for any* $t \in (0, t_{\max})$ *the set*

$$N(t) := \{ s \in S \times_G P \mid \langle \Omega^{g(t)}, s \otimes \overline{s} \rangle_{\mathrm{tr}} = F(s) \}$$

is preserved by the ∇^T*-parallel transport. Moreover, if* $s \in N(t)$, *then the 2-form* $\operatorname{tr}(s \circ (\Omega_{\nabla^T})(\,\cdot\,,\,\cdot\,)) \in \Lambda^2(M,\mathbb{C})$ *vanishes.*

The parabolic maximum principles for the Chern curvature opens vast opportunities for geometric applications of the HCF. In the present thesis, we use the HCF to make an initial progress approaching the following conjecture.

Conjecture 6.1. *Any complex Fano manifold which admits an Hermitian metric of Griffiths/dual-Nakano semipositive curvature must be isomorphic to a rational homogeneous space.*

Griffiths semipositivity is known to be a very restrictive curvature positivity assumption (see, e.g., [Yan17]). It implies numerical effectiveness of the tangent bundle. Therefore, the above conjecture can be thought of as an Hermitian version of the algebro-geometric Campana-Peternell conjecture [CP91] (see also [DPS94; DPS95]). Applying the HCF to manifolds equipped with an Hermitian metric of Griffiths/dual-Nakano semipositive curvature, we obtain strong evidence supporting the conjecture. Using the results of Mori [Mor79] and the regularization properties of the HCF, we prove the following result.

Theorem 6.4. *Let (M, g, J) be a compact complex n-dimensional Hermitian manifold such that its Griffith curvature is* quasipositive*, i.e.,*

1. *Chern curvature Ω^g is Griffiths semipositive;*
2. *Ω^g_m is Griffiths positive at some point $m \in M$.*

Then M is biholomorphic to the projective space $\mathbb{P}^n$.

We also prove that on Hermitian manifolds with dual-Nakano semipositive curvature, the torsion-twisted holonomy Lie algebra is closely related to the zero set of the Chern curvature.

Theorem 6.6. *Let $g(t)$ be the solution to the HCF on (M, g, J). Assume that $g(0)$ is dual-Nakano semipositive. Then for $t > 0$, at any point $m \in M$, subspace $K = \{v \in \mathrm{End}(T^{1,0}M) \mid \langle \Omega, v \otimes \bar{\cdot} \rangle_{\mathrm{tr}} = 0\}$ is the* tr*-orthogonal complement of the torsion-twisted holonomy subalgebra:*

$$K = \{v \in \mathrm{End}(T^{1,0}M) \mid \mathrm{tr}(v \circ w) = 0 \ \forall\ w \in \mathfrak{hol}_{\nabla^T}\}.$$

In order to better understand the relation between the Hermitian curvature flow of metrics with semipositive curvature and Conjecture 6.1, we explicitly compute the HCF on complex homogeneous manifolds equipped with submersion metrics $\mathcal{M}^{\mathrm{sub}}$ (see Definition 5.6). The space of submersion metrics is essentially the only known source of Griffiths/dual-Nakano semipositive Hermitian metrics on general homogeneous manifolds. In this case, the HCF on a G-homogeneous manifold $M = G/H$ reduces to an ODE for $B \in \mathrm{Sym}^{1,1}(\mathfrak{g})$, $\mathfrak{g} := \mathrm{Lie}(G)$.

Theorem 5.18. *Let $M = G/H$ be a complex homogeneous manifold equipped with a submersion Hermitian metric $g_0 = p_*(h_0) \in \mathcal{M}^{\mathrm{sub}}(M)$, where $h_0 \in \mathrm{Sym}^{1,1}(\mathfrak{g}^*)$. Let $B(t)$ be the solution to the ODE*

$$\begin{cases} \dfrac{dB}{dt} = B^{\#}, \\ B(0) = h_0^{-1}. \end{cases}$$

Then $g(t) = p_*(B(t)^{-1})$ solves the HCF on (M, g_0, J). In particular $g(t) \in \mathcal{M}^{\mathrm{sub}}(M)$.

The expected behavior of the HCF on rational homogeneous manifolds motivates us to make a purely algebraic conjectures about the pinching of the ODE $\dfrac{dB}{dt} = B^{\#}$. Moreover, blow-up behavior of this ODE turn out to be closely related to algebraic structure of the Lie algebra $\mathfrak{g}$.

1.2 Organization of the Thesis

The rest of this thesis is organized as follows.

In Chapter 2, we provide the background for Hermitian geometry. We define various positivity concepts for Hermitian holomorphic vector bundles and setup the abstract space of algebraic curvature tensors. Next, we review the related work on the uniformization problems in Kähler and algebraic geometry. Starting with the classical Hartshorne's and Frankel's conjectures, we discuss their various generalizations, describe known approaches to their solutions, and motivate Conjecture 6.1.

We open Chapter 3 with a discussion of various adaptations of the Ricci flow to the context of Hermitian geometry. We briefly review the Chern-Ricci flow, the general family of Hermitian curvature flows, and its specialization, the pluriclosed flow. Next, we define a member of the HCF family, which will be in the focus of the present thesis. We compute the evolution equation for the curvature tensor and, motivated by its form, introduce the torsion-twisted connection. The main outcome of the chapter, is a very clean reinterpretation of the evolution equation for the curvature tensor, obtained through the lens of the torsion-twisted connection, its Laplacian, and its curvature.

In Chapter 4, we adopt Hamilton's maximum principle for tensors to the evolution equation for the Chern curvature under the HCF. Following the framework of Wilking in the context of the Ricci flow, we define a set of Lie-algebraic curvature (semi)positivity conditions and prove that these are preserved by the HCF. Using the approach of Brendle and Schoen, we deduce a version of the strong maximum principle for the HCF.

In Chapter 5, we turn our attention to the complex homogeneous manifolds $M = G/H$. We introduce the space of submersion metrics on homogeneous manifolds and prove that these metrics have dual-Nakano semipositive curvature. By explicitly computing the HCF on such (M, g, J), we demonstrate that the evolution equation for g is induced by an ODE on $\mathrm{Sym}^{1,1}(\mathrm{Lie}(G))$. Motivated by the expected pinching of the curvature tensor under the HCF, we make purely algebraic conjecture concerning the above ODE on $\mathrm{Sym}^{1,1}(\mathrm{Lie}(G))$.

In Chapter 6, we discuss possible applications of the curvature positivity preservation results, which were proved in Chapter 4. We formulate a version of Conjecture 6.1 and approach it by the means of the HCF. We prove that a complex manifold with Griffiths quasipositive Chern curvature is biholomorphic to a projective space. Next, we study manifolds with the dual-Nakano semipositive curvature. We prove that, after running the HCF for an arbitrary small time, the kernel of the Chern curvature becomes invariant under the torsion-twisted parallel transport. Moreover the trace-orthogonal complement of the kernel coincides with the torsion-twisted holonomy Lie-algebra. This provides an approach to the weak Campana-Peternell conjecture through the study of the torsion-twisted holonomy group. Finally, on a general Hermitian manifold, we construct scalar quantities, which are monotone under the HCF, and use them to study the HCF-periodic solutions.

We conclude the thesis with Chapter 7. We summarize our main results and discuss further questions, open problems, and research directions.

Chapter 2

Preliminaries

In this chapter, we discuss the preliminaries for the thesis. We provide a background for the basics of Hermitian geometry, including the Chern connection, its curvature, and torsion tensors, Bianchi identities, positivity concepts for Hermitian vector bundles. We define the space of algebraic curvature tensors and describe its basic algebraic structure. Next, we review in detail the uniformization conjectures due to Hartshorne [Har70] and Frankel [Fra61], and discuss their further possible generalizations in complex and algebraic geometry.

2.1 Hermitian Geometry Background

In this section, we provide definitions and set up basic notations for 'doing geometry' on an Hermitian manifold (M, g, J). Most of the material in this section is rather standard and is covered in many references on complex geometry [GH94; Voi02; Huy05; Dem12].

2.1.1 Hermitian Manifolds

All manifolds in this thesis are assumed closed (compact without a boundary). Let M be a smooth manifold, equipped with an integrable almost complex structure

$$J \colon TM \to TM, \quad J^2 = -\mathrm{Id}.$$

The operator J defines the decomposition of the complexified tangent space $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ into the $\pm\sqrt{-1}$ eigenspaces of J:

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M.$$

which induces the decomposition of all associated complexified tensor bundles by type. In particular, the space of complex-valued differential k-forms $\Lambda^k(M,\mathbb{C}) := \Lambda^k(T^*_{\mathbb{C}}M)$ splits as follows

$$\Lambda^k(M,\mathbb{C}) = \bigoplus_{p+q=k} \Lambda^{p,q}(M).$$

Sections of the bundle $\Lambda^{p,q}(M)$ are differential forms of type (p,q). The de Rham differential splits correspondingly as $d^{1,0} + d^{0,1} = \partial + \overline{\partial}$. From now on, the Greek letters (ξ, η, ζ, etc.) denote *complex* vectors and vector fields of type (1,0), while the capital Latin letters (X, Y, Z, etc.) denote *real* vectors and vector fields.

Remark 2.1. Integrability of the almost complex structure J means that (M,J) is a genuine complex manifold, i.e., it is modeled on open subsets of $\mathbb{C}^n$ with holomorphic transition functions. On a complex manifold, the Lie bracket of any two vector fields of type (1,0) is again of type (1,0):

$$[\xi,\eta] \in \mathcal{C}^\infty(M, T^{1,0}M), \quad \text{for } \xi,\eta \in \mathcal{C}^\infty(M, T^{1,0}M).$$

Celebrated Newlander-Nirenberg theorem states that the converse is also true, i.e., if on an almost complex manifold (M,J) the space $\mathcal{C}^\infty(M, T^{1,0}M)$ is invariant under the Lie bracket, then J is an integrable complex structure, see, e.g., [Voi02, Thm. 2.1].

Any complex manifold (M,J) admits a J-invariant Riemannian metric

$$g\colon TM \otimes TM \to \mathbb{R}.$$

Metric g extends to a $\mathbb{C}$-bilinear form on the complexified tangent bundle $T_{\mathbb{C}}M$, which, in turn, yields an Hermitian metric on $T^{1,0}M$ via $\xi,\eta \mapsto g(\xi,\overline{\eta})$. In what follows, we identify metric g, its complexification, and the corresponding Hermitian metric on $T^{1,0}M$, since either of them determines the rest.

Definition 2.2. A triple (M,g,J) consisting of a compact smooth manifold M, an integrable almost complex structure J on it, and a J-invariant Riemannian metric g is called an *Hermitian manifold.*

A real (1,1) form

$$\omega(X,Y) := g(JX,Y)$$

is called the *fundamental form* of (M,g,J). Hermitian manifold (M,g,J) is Kähler, if its fundamental form is closed: $d\omega = 0$.

A circle of questions related to the construction of metrics with prescribed properties, obstructions for the existence of certain metrics, interplay between the topology of (M,J) and the geometry of (M,g,J) is a subject of Hermitian geometry.

2.1.2 Chern Connection

In order to study Riemannian geometry, one uses the Levi-Civita connection ∇^{LC} on TM, which is canonically attached to any Riemannian manifold (M,g). The situation in Hermitian geometry is more subtle, since ∇^{LC} does not necessary preserve the operator of almost complex structure, so it does not distinguish different choices of J. A natural set up would be to consider a connection D on TM, which preserves both g and J:

$$Dg = 0, \quad DJ = 0. \tag{2.1}$$

These conditions define an affine subspace in the space of all connections on TM. Unfortunately, in general, equation (2.1) does not specify a unique connection. More precisely, the following result due to Gauduchon holds.

Theorem 2.3 ([Gau97])**.** *For an Hermitian manifold (M,g,J) let $\mathcal{A}_{g,J}$ be the affine space of connections, satisfying* (2.1)*. Then*

1. $\mathcal{A}_{g,J} = \{\nabla^{LC}\}$*, if (M,g,J) is Kähler;*

2. $\dim_{\mathbb{R}} \mathcal{A}_{g,J} = 1$ *otherwise.*

In the case $\dim_{\mathbb{R}} \mathcal{A}_{g,J} = 1$, all connections $D \in \mathcal{A}_{g,J}$ have non-trivial torsion $T^D \in \Lambda^2(M, TM)$:

$$T^D(X,Y) := D_X Y - D_Y X - [X,Y].$$

In order to find distinguished members of $\mathcal{A}_{g,J}$, it is natural to impose additional restrictions on T^D.

Theorem 2.4 ([Gau97])**.** *On any Hermitian manifold (M,g,J) there exist:*

- *a unique connection* $\nabla^C \in \mathcal{A}_{g,J}$ *such that its torsion satisfies*

$$T^{\nabla^C}(X, JY) = T^{\nabla^C}(JX, Y). \tag{2.2}$$

This connection is called the Chern connection*;*

- *a unique connection* $\nabla^B \in \mathcal{A}_{g,J}$ *such that the tensor*

$$B(X, Y, Z) := g(T^{\nabla^B}(X, Y), Z)$$

is totally skew-symmetric. This connection is called the Bismut connection*.*

The Chern and Bismut connections coincide if and only if (M, g, J) is Kähler. By Theorem 2.3 on a non-Kähler Hermitian manifold

$$\mathcal{A}_{g,J} = \{t\nabla^C + (1-t)\nabla^B \mid t \in \mathbb{R}\}.$$

In what follows, we will be working primarily with the Chern connection. To simplify notations, we will write ∇ for the Chern connection ∇^C and T for its torsion. We automatically extend ∇ and all other connections on TM to $\mathbb{C}$-linear connections on $T_{\mathbb{C}}M$ and, by Leibniz rule, to connections on all associated vector bundles, e.g., $T^*M, \Lambda^k(M)$, $\mathrm{End}(TM)$, etc. With the $\mathbb{C}$-linear extension of ∇, condition (2.2) is equivalent to the fact that the type (1,1) part of the torsion tensor $T \in \Lambda^2(M, T_{\mathbb{C}}M)$ vanishes, i.e.,

$$T(\xi, \overline{\eta}) = 0, \quad \xi, \eta \in T^{1,0}M.$$

Remark 2.5. Alternatively, one can define the Chern connection as the unique Hermitian connection on the holomorphic Hermitian vector bundle $(T^{1,0}M, g)$, which is compatible with the holomorphic structure (see, e.g., [KN69, Prop. 10.2]), i.e., $\nabla\colon \mathcal{C}^\infty(M, T^{1,0}M) \to \mathcal{C}^\infty(M, T^{1,0}M \otimes \Lambda^1(M, \mathbb{C}))$ such that:

1. ∇ preserves Hermitian metric, $\nabla g = 0$;
2. the (0,1)-part of ∇ coincides with operator of holomorphic structure $\overline{\partial}$, i.e., $\nabla_{\overline{\xi}}\eta = i_{\overline{\xi}}(\overline{\partial}\eta)$ for any type (1,0) vector field $\eta \in \mathcal{C}^\infty(M, T^{1,0}M)$ and any (0,1)-vector $\overline{\xi} \in T^{0,1}M$.

The advantage of this definition is that it makes sense for any *holomorphic* vector bundle $(\mathcal{E}, \overline{\partial_{\mathcal{E}}}) \to M$, equipped with an Hermitian metric $h \in \mathcal{C}^\infty(M, \mathcal{E} \otimes \overline{\mathcal{E}})$.

Definition 2.6 (Laplacian of a connection)**.** Consider a Riemannian manifold (M, g) equipped with a metric connection D^{TM}. Let $(\mathcal{E}, D^{\mathcal{E}})$ be a vector bundle $\mathcal{E} \to M$ with a connection $D^{\mathcal{E}}$. The *Laplacian* $\Delta_{D^{\mathcal{E}}}$ is the second order differential operator on $\mathcal{C}^\infty(M, \mathcal{E})$ given by the composition

$$\mathcal{C}^\infty(M,\mathcal{E}) \xrightarrow{D^{\mathcal{E}}} \mathcal{C}^\infty(M,\Lambda^1(M)\otimes\mathcal{E}) \xrightarrow{D^{TM}\otimes\mathrm{id}+\mathrm{id}\otimes D^{\mathcal{E}}} \mathcal{C}^\infty(M,\Lambda^1(M)\otimes\Lambda^1(M)\otimes\mathcal{E}) \xrightarrow{\mathrm{tr}_g\otimes\mathrm{id}} \mathcal{C}^\infty(M,\mathcal{E}),$$

where $\mathrm{tr}_g \colon \Lambda^1(M) \otimes \Lambda^1(M) \to \mathbb{R}$ is the metric contraction.

We will be using this definition in a situation, when (M, g) is an Hermitian manifold, $\mathcal{E}$ is a complex vector bundle, D^{TM} is the Chern connection, and $D^{\mathcal{E}}$ is some connection on $\mathcal{E}$. If $\mathcal{E}$ is a bundle associated with TM and $D^{\mathcal{E}}$ is the Chern connection, we will refer to this operator as the Chern Laplacian and denote it by Δ. In holomorphic coordinates the Chern Laplacian is given by

$$\Delta = \frac{1}{2}\sum_{m,n} g^{m\bar{n}}\big(\nabla_{\partial/\partial z^m}\nabla_{\partial/\partial\overline{z^n}} + \nabla_{\partial/\partial\overline{z^n}}\nabla_{\partial/\partial z^m}\big). \tag{2.3}$$

Note that Δ differs from the standard Riemannian Laplacian on (M, g).

We will need the following lemma relating the Laplacians of two different connections on $\mathcal{E}$.

Lemma 2.7. *Let $D_1^{\mathcal{E}} = D$ and $D_2^{\mathcal{E}} = D + A$ be two connections on a vector bundle $\mathcal{E}$, where $A \in \Lambda^1(M, \mathrm{End}(\mathcal{E}))$. We extend D and $D + A$ to connections on the associated tensor bundle $\Lambda^1(M) \otimes \mathrm{End}(\mathcal{E})$ using a connection D^{TM} on the tangent bundle. Then*

$$\Delta_{D+A} = \Delta_D + \mathrm{tr}_g(A \circ A + 2A \circ D) + \mathrm{div}^D A,$$

where $\mathrm{div}^D A = \mathrm{tr}_g(D_\bullet A)_\bullet$.

Proof. Let $\{e_i\}_{i=1}^{\dim M}$ be a local orthonormal frame. Then

$$\begin{aligned}
\Delta_{D+A} = &\sum_{i=1}^{\dim M}\big((D+A)_{e_i}(D+A)_{e_i} - (D+A)_{D^{TM}_{e_i}e_i}\big) = \sum_{i=1}^{\dim M}\big(D_{e_i}D_{e_i} - D_{D^{TM}_{e_i}e_i}\big)\\
&+ \sum_{i=1}^{\dim M}\big(D_{e_i}\circ A_{e_i} + A_{e_i}\circ D_{e_i} + A_{e_1}\circ A_{e_i} - A_{D^{TM}_{e_i}e_i}\big)\\
=&\Delta_D + \sum_{i=1}^{\dim M}\big(D_{e_i}(A)_{e_i} + A_{D^{TM}_{e_i}e_i} + 2A_{e_i}\circ D_{e_i} + A_{e_1}\circ A_{e_i} - A_{D^{TM}_{e_i}e_i}\big)\\
=&\Delta_D + \mathrm{div}^D A + \mathrm{tr}_g(2A\circ D + A\circ A).
\end{aligned}$$

□

Observe that the difference $\Delta_{D+A} - \Delta_D$ is always a first-order differential operator on $\mathcal{C}^\infty(M, \mathcal{E})$.

Moreover, by choosing $A \in \Lambda^1(M, \mathrm{End}(\mathcal{E}))$ we can make the difference to have an arbitrary principle symbol of order one, and, given a principle symbol, A is uniquely defined.

2.1.3 Chern Curvature

Definition 2.8. The *curvature* of a connection D on a vector bundle $\mathcal{E}$ is the tensor $\Theta^{\mathcal{E}} \in \Lambda^2(M, \mathrm{End}(\mathcal{E}))$:

$$\Theta^{\mathcal{E}}(X,Y)e := (D_X D_Y - D_Y D_X - D_{[X,Y]})e,$$

where $X, Y \in TM$, $e \in \mathcal{E}$. It is straightforward to check that the value of $\Theta^{\mathcal{E}}$ at a given point $m \in M$ does not depend on the extension of X, Y, and e to tensor fields in the neighbourhood of m.

Remark 2.9. In this thesis, we will be dealing mostly with the curvatures of the Chern connections on holomorphic Hermitian bundles $(\mathcal{E}, h)$. In this situation the complexified Chern curvature has additional symmetries, in particular, $\Theta^{\mathcal{E}} \in \Lambda^{1,1}(M, \mathrm{End}(\mathcal{E}))$.

Definition 2.10. The *Chern curvature* of an Hermitian manifold (M, g, J) is the curvature of the connection ∇ on TM.

$$\Omega := \Theta^{TM}.$$

We also introduce a tensor with 4 vector arguments by lowering one index via g.

$$\Omega(X,Y,Z,W) := g\big((\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z, W\big).$$

If there might be an ambiguity of which metric is used to define curvature/torsion tensors, we use the corresponding superscript Ω^g, T^g.

The defining properties of the Chern connection imply that the Chern curvature tensor Ω satisfies a number of symmetries.

Proposition 2.11. *The (complexified) Chern curvature $\Omega(X,Y,Z,W)$ lies in*

$$\Lambda^{1,1}(M) \otimes \Lambda^{1,1}(M) \simeq \Lambda^{1,1}(M, \Lambda^{1,1}(M)) \subset \Lambda^2(M, T^*_{\mathbb{C}}M \otimes T^*_{\mathbb{C}}M).$$

Explicitly, for any real vectors $X, Y, Z, W \in TM$ one has

- $\Omega(X,Y,Z,W) = -\Omega(Y,X,Z,W)$, $\Omega(X,Y,Z,W) = -\Omega(X,Y,W,Z)$;
- $\Omega(JX,JY,Z,W) = \Omega(X,Y,Z,W)$, $\Omega(X,Y,JZ,JW) = \Omega(X,Y,Z,W)$,

or, equivalently, for $\xi, \eta, \zeta, \nu \in T^{1,0}M$

$$\Omega(\xi, \eta, \cdot, \cdot) = \Omega(\cdot, \cdot, \zeta, \nu) = 0, \quad \Omega(\xi, \overline{\eta}, \zeta, \overline{\nu}) = \overline{\Omega(\eta, \overline{\xi}, \nu, \overline{\zeta})}.$$

Symmetries of Ω imply that $\Omega(\xi, \overline{\xi}, \eta, \overline{\eta}) \in \mathbb{R}$. It is easy to check that the values $\Omega(\xi, \overline{\xi}, \eta, \overline{\eta})$ for $\xi, \eta \in T^{1,0}M$ completely determine tensor Ω.

Often, in computations we will use coordinate notation for different tensors and assume Einstein summation for repeated upper/lower indices, e.g.,

$$\begin{aligned}
&\nabla_i := \nabla_{\partial/\partial z^i},\\
&\Omega_{i\bar{j}k\bar{l}} := \Omega(\partial/\partial z^i, \partial/\partial \overline{z}^j, \partial/\partial z^k, \partial/\partial \overline{z}^l),\\
&T_{ij}^k\, \partial/\partial z^k := T(\partial/\partial z^i, \partial/\partial z^j),\\
&T_{ij\bar{l}} := g(T(\partial/\partial z^i, \partial/\partial z^j), \partial/\partial \overline{z}^l).
\end{aligned}$$

Unlike the Riemannian case, in Hermitian geometry, the Chern curvature does not satisfy the classical Bianchi identities, since the Chern connection has torsion. However, in this case slightly modified identities, involving torsion still hold [KN63, Ch. III, Thm. 5.3].

Proposition 2.12 (Bianchi identities for the Chern curvature)**.** *For any vectors* $X, Y, Z \in TM$ *one has respectively the first (algebraic) and the second (differential) Bianchi identities*

$$\begin{aligned}
&\sum_{\mathfrak{S}_3} \Omega(X, Y)Z = \sum_{\mathfrak{S}_3} \big(T(T(X, Y), Z) + \nabla_X T(Y, Z)\big),\\
&\sum_{\mathfrak{S}_3} \big(\nabla_X \Omega(Y, Z) + \Omega(T(X, Y), Z)\big) = 0,
\end{aligned} \tag{2.4}$$

where the sum is taken over all cyclic permutations. Splitting these identities by complex type and using the vanishing of the $(1,1)$*-part of* T *and symmetries of* Ω*, we get that for any vectors* $\xi, \eta, \zeta \in T^{1,0}M$

$$\begin{aligned}
&\Omega(\xi, \overline{\eta})\zeta - \Omega(\zeta, \overline{\eta})\xi = \nabla_{\overline{\eta}} T(\zeta, \xi),\\
&\sum_{\mathfrak{S}_3} \big(T(T(\xi, \eta), \zeta) + \nabla_\xi T(\eta, \zeta)\big) = 0,\\
&\nabla_\zeta \Omega(\xi, \overline{\eta}) - \nabla_\xi \Omega(\zeta, \overline{\eta}) = \Omega(T(\xi, \zeta), \overline{\eta}).
\end{aligned}$$

Equivalently, in the coordinates

$$\begin{gathered}\Omega_{i\bar{j}k\bar{l}}=\Omega_{k\bar{j}i\bar{l}}+\nabla_{\bar{j}}T_{ki\bar{l}}, \quad \Omega_{i\bar{j}k\bar{l}}=\Omega_{i\bar{l}k\bar{j}}+\nabla_{i}T_{\bar{l}\bar{j}k}, \\ T_{ij}^{s}T_{sk}^{l}+T_{jk}^{s}T_{si}^{l}+T_{ki}^{s}T_{sj}^{l}+\nabla_{i}T_{jk}^{l}+\nabla_{j}T_{ki}^{l}+\nabla_{k}T_{ij}^{l}=0, \\ \nabla_{m}\Omega_{i\bar{j}k\bar{l}}=\nabla_{i}\Omega_{m\bar{j}k\bar{l}}+T_{im}^{p}\Omega_{p\bar{j}k\bar{l}}, \quad \nabla_{\bar{n}}\Omega_{i\bar{j}k\bar{l}}=\nabla_{\bar{j}}\Omega_{i\bar{n}k\bar{l}}+T_{\bar{j}\bar{n}}^{\bar{s}}\Omega_{i\bar{s}k\bar{l}}.\end{gathered} \tag{2.5}$$

In the computations below we will be extensively using the first and the second Bianchi identities, involving the Chern curvature and its derivative respectively. Presence of the torsion terms in the Bianchi identities for Ω indicates that the four Ricci contractions of the Chern curvature tensor differ from each other (see [LY17] for the explicit description of the differences between these contractions):

$$\begin{aligned} S_{i\bar{j}}^{(1)} &:= \Omega_{i\bar{j}m\bar{n}}g^{m\bar{n}}, \quad S_{i\bar{j}}^{(2)} := \Omega_{m\bar{n}i\bar{j}}g^{m\bar{n}}, \\ S_{i\bar{j}}^{(3)} &:= \Omega_{n\bar{j}i\bar{m}}g^{m\bar{n}}, \quad S_{i\bar{j}}^{(4)} := \Omega_{i\bar{n}m\bar{j}}g^{m\bar{n}}. \end{aligned} \tag{2.6}$$

We call these contractions the Chern-Ricci tensors. Symmetries of Ω imply that the first and the second Chern-Ricci tensors define Hermitian products on $T^{1,0}M$. In general this is not the case for $S^{(3)}$ and $S^{(4)}$, since $S_{i\bar{j}}^{(3)} \neq \overline{S_{j\bar{i}}^{(3)}} = S_{i\bar{j}}^{(4)}$. By the second Bianchi identity (2.4), the real (1,1)-form

$$\rho := \sqrt{-1}S_{i\bar{j}}^{(1)}dz^i \wedge d\overline{z^j}$$

is closed, and according to the Chern-Weil theory it represents the class $2\pi c_1(M) \in H^2(M,\mathbb{R})$. All four Chern-Ricci contractions (2.6) will play certain role below.

There are also two scalar contractions of Ω: the standard scalar curvature $\mathrm{sc} = g^{i\bar{j}}g^{k\bar{l}}\Omega_{i\bar{j}k\bar{l}} = \mathrm{tr}_g S^{(1)} = \mathrm{tr}_g S^{(2)}$ and a quantity, which will be referred to as the *second scalar curvature* and will play important role below:

$$\widehat{\mathrm{sc}} = g^{i\bar{l}}g^{k\bar{j}}\Omega_{i\bar{j}k\bar{l}} = \mathrm{tr}_g S^{(3)} = \mathrm{tr}_g S^{(4)}.$$

By rising the last two indices of the Chern curvature tensor, we can interpret Ω as a section of $\mathrm{End}(T^{1,0}M) \otimes \overline{\mathrm{End}(T^{1,0}M)}$:

$$\Omega_{i\bar{j}}{}^{\bar{l}k}(e_k \otimes \epsilon^i) \otimes \overline{(e_l \otimes \epsilon^j)}, \quad \Omega_{i\bar{j}}{}^{\bar{l}k} = \Omega_{i\bar{j}m\bar{n}}g^{m\bar{l}}g^{k\bar{n}},$$

where $\{e_i\}$ is a local frame of $T^{1,0}M$ and $\{\epsilon^i\}$ is the dual frame. Symmetries of Ω imply that this form is Hermitian, i.e., defines an Hermitian inner product on $(\mathrm{End}(T^{1,0}M))^* \simeq \mathrm{End}(T^{1,0}M)$. This interpretation of Ω will turn out to be very useful.

2.1.4 Curvature Positivity for Hermitian Vector Bundles

Now, we discuss positivity concepts for holomorphic vector bundles. Let us first recall the basic definition for line bundles. Consider a holomorphic line bundle $\mathcal{L} \to M$.

Definition 2.13. A holomorphic line bundle $\mathcal{L}$ is *very ample* if the space of global sections $H^0(M, \mathcal{L})$ defines an embedding $M \hookrightarrow \mathbb{P}(H^0(M, \mathcal{L})^*)$. Line bundle $\mathcal{L}$ is *ample*, if its positive tensor power is very ample.

Now, assume, that $\mathcal{L}$ is equipped with an Hermitian metric h.

Definition 2.14. A holomorphic Hermitian line bundle $(\mathcal{L}, h)$ is *positive*, if the curvature $\Theta^{\mathcal{L}} \in \Lambda^2(M, \mathbb{C})$ of the Chern connection D is *positive*, i.e.,

$$\Theta^{\mathcal{L}}(\xi, \bar{\xi}) > 0$$

for any nonzero $\xi \in T^{1,0}M$.

Example 2.15. Let (V, h_V) be a complex Hermitian vector space with a metric h_V. The line bundle $\mathcal{L} = \mathcal{O}(1)$ over $\mathbb{P}(V)$ is very ample with $H^0(\mathbb{P}(V), \mathcal{O}(1)) \simeq V^*$. Moreover, h_V induces a metric $h_{\mathcal{O}(1)}$ on $\mathcal{O}(1)$, which makes $(\mathcal{O}(1), h_{\mathcal{O}(1)})$ a positive Hermitian line bundle.

Example 2.15 implies that any ample line bundle admits a metric of positive curvature. Indeed, if $\mathcal{L}^{\otimes m}$ is a very ample line bundle on M, with the induced embedding $\iota_{\mathcal{L}^{\otimes m}} \colon M \to \mathbb{P}(V)$, then $\iota^*_{\mathcal{L}^{\otimes m}} h^{1/m}_{\mathcal{O}(1)}$ is a metric on $\mathcal{L}$ with a positive curvature. The converse of the above statement is the content of the Kodaira embedding theorem. Namely a holomorphic line bundle, admitting a metric with positive curvature is necessarily ample.

Now, assume that $\mathcal{E} \to M$ is a holomorphic vector bundle of rank $r \geqslant 1$. Originally, the following definition was given by Hartshorne for smooth projective varieties, however it makes sense for any compact complex manifold.

Definition 2.16 ([Har66])**.** For a holomorphic vector bundle $\mathcal{E} \to M$, consider the projectivization of its dual bundle (in other words, the bundle of hyperplanes in $\mathcal{E}$):

$$\pi \colon \mathbb{P}(\mathcal{E}^*) \to M.$$

Vector bundle $\mathcal{E}$ is *ample* if the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E}^*)}(1) \to \mathbb{P}(\mathcal{E}^*)$ is ample.

Remark 2.17. If $\mathcal{E}$ is a line bundle, then $\mathbb{P}(\mathcal{E}^*) = M$, and $\mathcal{O}_{\mathbb{P}(\mathcal{E}^*)}(1) \simeq \mathcal{E}$. Hence, the notion of ampleness for vector bundles is indeed a generalization of the ampleness for line bundles.

Next, we discuss various notions of curvature positivity for a holomorphic Hermitian vector bundle $(\mathcal{E}, h)$. In this part we follow [Dem12, Ch. VII, §6]. As before, denote by D the corresponding Chern connection and by $\Theta^{\mathcal{E}} \in \Lambda^{1,1}(M, \mathrm{End}(\mathcal{E}))$ its curvature. Pick a frame $\{e_\alpha\} = (e_1, \ldots, e_r)$ of $\mathcal{E}_m, m \in M$ and the dual frame $\{\epsilon^\beta\}$. Then

$$\Theta^{\mathcal{E}} = \Theta_{i\bar{j}\alpha}{}^{\beta} dz^i \wedge d\overline{z^j} \otimes \epsilon^\alpha \otimes e_\beta.$$

The Hermitian property of the Chern connection implies that

$$\Theta_{i\bar{j}\alpha\overline{\beta}} = \overline{\Theta_{j\bar{i}\beta\overline{\alpha}}}, \tag{2.7}$$

where $\Theta_{i\bar{j}\alpha\overline{\beta}} := \Theta_{i\bar{j}\alpha}{}^{\gamma} h_{\gamma\overline{\beta}}$.

Definition 2.18. The Chern curvature $\Theta^{\mathcal{E}}$ induces a pairing $\theta^{\mathcal{E}}$ on $T^{1,0}M \otimes \mathcal{E}$ defined by

$$\theta^{\mathcal{E}} = \Theta_{i\bar{j}\alpha\overline{\beta}}(dz^i \otimes \epsilon^\alpha) \otimes (\overline{dz^j \otimes \epsilon^\beta}),$$

so that for $\xi_1 \otimes v_1, \xi_2 \otimes v_2 \in T^{1,0}M \otimes \mathcal{E}$ we have

$$\theta^{\mathcal{E}}(\xi_1 \otimes v_1, \overline{\xi_2 \otimes v_2}) = h(\Theta^{\mathcal{E}}(\xi_1, \bar{\xi}_2)v_1, \bar{v}_2).$$

Identity (2.7) implies that $\theta^{\mathcal{E}}$ is an Hermitian form on $T^{1,0}M \otimes \mathcal{E}$.

Using the form $\theta^{\mathcal{E}}$ we can now introduce positivity concepts for $(\mathcal{E}, h)$.

Definition 2.19 (Curvature positivity notions)**.** Given a holomorphic Hermitian vector bundle $(\mathcal{E}, h)$ we say that it is

1. *Nakano positive* (see [Nak55]), if the Hermitian form $\theta^{\mathcal{E}}$ is positive definite on $T^{1,0}M \otimes \mathcal{E}$ (we write $\mathcal{E} >_{\mathrm{Nak}} 0$);
2. *Griffiths positive* (see [Gri69]), if the Hermitian form $\theta^{\mathcal{E}}$ is positive on all non-zero decomposable tensors $\{\xi \otimes v\} \subset T^{1,0}M \otimes \mathcal{E}$ (we write $\mathcal{E} >_{\mathrm{Gr}} 0$);
3. *m-positive*, where $1 \leqslant m \leqslant \min(r, \dim M)$, if the Hermitian form $\theta^{\mathcal{E}}$ is positive on all non-zero tensors of rank $\leqslant m$: $\{\sum_{i=1}^{m} \xi_i \otimes v_i\} \subset T^{1,0}M \otimes \mathcal{E}$ (we write $\mathcal{E} >_m 0$).

For all the concepts, the notions of negativity, semipositivity (=non-negativity), and semi-negativity (=non-positivity) are understood in the obvious way.

Clearly, if $\mathcal{E}$ is a line bundle, all three definitions coincide with the standard notion of positivity for line bundles (Definition 2.14). Three notions also coincide on a complex curve. In general, m-positivity interpolates between a weaker notion of Griffiths positivity ($m = 1$) and a stronger notion of Nakano positivity ($m = \min(r, \dim M)$).

Definition 2.20. We say that a bundle $(\mathcal{E}, h)$ is *dual*-Nakano (resp. Griffiths/m-) positive, if $(\mathcal{E}^*, h)$ is Nakano (resp. Griffiths/m-) *negative*.

Remark 2.21. Dual-Griffiths positivity is equivalent to Griffiths positivity [Dem12, Prop. 6.6], so there is no need to reserve a special name for it. The situation is more subtle for other notions. In particular, there exist examples of bundles, which are dual-Nakano positive, but not Nakano positive, e.g., the tangent bundle $T^{1,0}\mathbb{P}^n$ of a projective space $\mathbb{P}^n$, equipped with the Fubini-Studi metric.

Remark 2.22. Dual-Nakano positivity is equivalent to the positivity of the following Hermitian pairing on $T^{1,0}M \otimes \mathcal{E}^*$:

$$(T^{1,0}M \otimes \mathcal{E}^*) \otimes (\overline{T^{1,0}M \otimes \mathcal{E}^*}) \ni (u^i_\alpha, \overline{v^j_\beta}) \mapsto \Theta_{i\bar{j}}{}^{\bar{\beta}\alpha} u^i_\alpha \overline{v^j_\beta} \in \mathbb{C}, \tag{2.8}$$

where $\Theta_{i\bar{j}}{}^{\bar{\beta}\alpha} := \Theta_{i\bar{j}\gamma}{}^{\alpha} h^{\gamma\bar{\beta}}$. The pairing (2.8) for $(\mathcal{E}, h) = (T^{1,0}M, g)$ will play the crucial role in the formulation and the proof of our main results. In this case, $T^{1,0}M \otimes \mathcal{E}^* = \mathrm{End}(T^{1,0}M)$ is a Lie algebra, and the space of the Chern curvature tensors has a rich algebraic structure, which we investigate in the next section.

Unlike the situation with the line bundles, the relation between the ampleness of a higher rank vector bundle and the existence of an Hermitian metric, satisfying some positivity notion is not well-understood. In general, for a holomorphic Hermitian vector bundle $(\mathcal{E}, h)$ we have implications

$$\mathcal{E} >_{\mathrm{Nak}} 0 \Rightarrow \mathcal{E} >_m 0 \Rightarrow \mathcal{E} >_{\mathrm{Gr}} 0 \Rightarrow \mathcal{E} \text{ is ample.} \tag{2.9}$$

The converse of the last implication is the content of a well-known conjecture due to Griffiths.

Conjecture 2.23 (Griffiths Conjecture [Gri69]). *If $\mathcal{E} \to M$ is an ample[1] vector bundle, then $\mathcal{E}$ admits a Griffiths positive Hermitian metric.*

[1]In [Gri69], Griffiths uses the name *cohomologically positive* for ample vector bundles.

Remark 2.24. Kobayashi [Kob75] proved that ampleness of a vector bundle $\mathcal{E}$ is equivalent to the existence of a *Kobayashi negative Finsler metric on* $\mathcal{E}^*$. Hence, a possible approach to Griffiths conjecture would be to deform this Finsler metric into a genuine Hermitian metric of negative Griffiths curvature. Very recently, Wan [Wan18] proposed a geometric flow of Finsler metrics and studied preservation of the Kobayashi negativity under this flow.

An important feature of complex geometry is the existence of a deep relation between the Chern curvature of a bundle and the Chern curvatures of its sub/quotient bundles. Specifically, consider an exact sequence of holomorphic vector bundles on M

$$0 \to \mathcal{S} \xrightarrow{i} \mathcal{E} \xrightarrow{p} \mathcal{Q} \to 0.$$

Let $\mathcal{E}$ be equipped with an Hermitian metric h and assume that the metrics on $\mathcal{S}$ and $\mathcal{Q}$ are induced from h via identifications $\mathcal{S} \simeq \operatorname{im} i$, $\mathcal{Q} \simeq (\operatorname{im} i)^{\perp}$. We denote these metrics by i^*h and p_*h respectively. Define the *second fundamental form* $\beta \in \Lambda^{1,0}(M, \operatorname{Hom}(\mathcal{S}, \mathcal{Q}))$ by the identity

$$\beta_\xi(s) := p(\nabla^{\mathcal{E}}_\xi(i(s))),$$

where $\nabla^{\mathcal{E}}$ is the Chern connection on $\mathcal{E}$. The second fundamental form depends only on the value of a section at a given point; it vanishes precisely when the h-orthogonal splitting of $\mathcal{E}$ into $i(\mathcal{S})$ and $i(\mathcal{S})^{\perp}$ is holomorphic. Denote by $\beta^* \in \Lambda^{0,1}(M, \operatorname{Hom}(\mathcal{Q}, \mathcal{S}))$ the adjoint of β defined by the identity:

$$(p_*h)(\beta_\xi(s), \bar{q}) = (i^*h)(s, \overline{\beta^*_{\bar{\xi}}(q)}), \quad \xi \in T^{1,0}M, s \in \mathcal{S}, q \in \mathcal{Q}.$$

Fix a $\mathcal{C}^\infty$-isomorphism

$$\mathcal{S} \oplus \mathcal{Q} \simeq i(\mathcal{S}) \oplus (i(\mathcal{S}))^{\perp} = \mathcal{E}.$$

There are Chern curvature tensors: $\Theta^{\mathcal{S}} \in \Lambda^{1,1}(M, \operatorname{End}(\mathcal{S}))$, $\Theta^{\mathcal{E}} \in \Lambda^{1,1}(M, \operatorname{End}(\mathcal{E}))$, $\Theta^{\mathcal{Q}} \in \Lambda^{1,1}(M, \operatorname{End}(\mathcal{Q}))$, and we want to compare $\Theta^{\mathcal{E}}\big|_{\mathcal{S}}$ with $\Theta^{\mathcal{S}}$ and $\Theta^{\mathcal{E}}\big|_{\mathcal{Q}}$ with $\Theta^{\mathcal{Q}}$.

We have the following formula for the Chern connection $\nabla^{\mathcal{E}}$ in terms of the Chern connections of $\mathcal{S}$ and $\mathcal{Q}$:

$$\nabla^{\mathcal{E}} = \begin{pmatrix} \nabla^{\mathcal{S}} & -\beta^* \\ \beta & \nabla^{\mathcal{Q}} \end{pmatrix}.$$

Furthermore, the curvature of ∇^E with respect to this identification is given by (see [Dem12, Thm. 14.5])

$$\Theta^{\mathcal{E}} = \begin{pmatrix} \Theta^{\mathcal{S}} - \beta^* \wedge \beta & -(\beta^* \circ \nabla^{\mathcal{Q}} + \nabla^{\mathcal{S}} \circ \beta^*) \\ \beta \circ \nabla^{\mathcal{S}} + \nabla^{\mathcal{Q}} \circ \beta & \Theta^{\mathcal{Q}} - \beta \wedge \beta^* \end{pmatrix}. \tag{2.10}$$

Formula (2.10) has an important consequence.

Corollary 2.25. *Let $\xi, \eta \in T^{1,0}M$, $s_1, s_2 \in \mathcal{S}$ and $q_1, q_2 \in \mathcal{Q}$. Then*

$$\begin{aligned} h(\Theta^{\mathcal{E}}\big|_{\mathcal{S}}(\xi,\overline{\eta})s_1, \overline{s}_2) &= (i^*h)(\Theta^{\mathcal{S}}(\xi,\overline{\eta})s_1, \overline{s}_2) + (p_*h)(\beta_\xi(s_1), \overline{\beta_\eta(s_2)}), \\ h(\Theta^{\mathcal{E}}\big|_{\mathcal{Q}}(\xi,\overline{\eta})q_1, \overline{q}_2) &= (p_*h)(\Theta^{\mathcal{Q}}(\xi,\overline{\eta})q_1, \overline{q}_2) - (i^*h)(\beta^*_{\overline{\eta}}(q_1), \overline{\beta^*_{\overline{\xi}}(q_2)}). \end{aligned} \tag{2.11}$$

With Corollary 2.25 we make two important observations.

1. The curvature of $(\mathcal{S}, i^*h)$ does not exceed the curvature of $(\mathcal{E}, h)$ restricted to $\mathcal{S}$ in any of the Nakano/Griffiths/m-positivity senses;

2. The curvature of $(\mathcal{Q}, p_*h)$ is at least as positive as the the curvature of $(\mathcal{E}, h)$ restricted to $\mathcal{Q}$ in any of the *dual*-Nakano/Griffiths/m-positivity senses.

Corollary 2.25 is a reflection of an important principle: *curvature decreases in holomorphic subbundles and increases in quotient bundles* [GH94, Ch. 0 §5].

Recall that a holomorphic bundle $\mathcal{E}$ is *globally generated* by its sections, if the natural evaluation map $\mathrm{ev}\colon H^0(M, \mathcal{E}) \to \mathcal{E}$ is fiberwise surjective. Identity (2.11) implies that any globally generated holomorphic $\mathcal{E}$ can be equipped with a metric h, which makes $(\mathcal{E}, h)$ a dual-Nakano semipositive Hermitian bundle. Namely, it suffice to set $h = \mathrm{ev}_* h_0$ for an Hermitian metric h_0 on the vector space $H^0(M, \mathcal{E})$.

2.1.5 Space of Curvature Tensors

Let V be a complex vector space. We denote by $\overline{V}$ the underlying real vector space with a conjugate complex structure and by $\mathrm{Sym}^{1,1}(V)$ the subspace of $V \otimes \overline{V}$ spanned over $\mathbb{R}$ by all the elements of the form $v \otimes \overline{v}$, $v \in V$. In other words, $\mathrm{Sym}^{1,1}(V)$ is the set of (not necessary positive definite) Hermitian forms on V^*. Equip V with an Hermitian metric g and extend g to all associated tensor powers of V and $\overline{V}$. In this section, we denote by $\mathfrak{g} = \mathrm{End}(V)$ the endomorphism Lie algebra of V. Let $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{tr}}\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ be the trace pairing

$$\langle u, v\rangle_{\mathrm{tr}} := \mathrm{tr}(uv).$$

Definition 2.26. The space of *algebraic curvature tensors* on V is the vector space $\mathrm{Sym}^{1,1}(\mathfrak{g})$.

Pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{tr}}$ extends to a bilinear form on $\mathrm{Sym}^{1,1}(\mathfrak{g})$ in the obvious way:

$$\langle v\otimes\overline{v}, u\otimes\overline{u}\rangle_{\mathrm{tr}} := |\mathrm{tr}(uv)|^2.$$

Clearly, $\Omega\in\mathrm{Sym}^{1,1}(\mathfrak{g})$ represents a positive (resp. semipositive) Hermitian form on $\mathfrak{g}^*$ if and only if $\langle\Omega, u\otimes\overline{u}\rangle_{\mathrm{tr}}>0$ (resp. $\geqslant 0$) for any nonzero $u\in\mathfrak{g}$.

Remark 2.27. For $V=T^{1,0}M$, the space $\mathrm{Sym}^{1,1}(\mathfrak{g})$ models the space of Chern curvature tensors, represented as in (2.8). Unlike the Riemannian/Kähler setting, where one is interested only in the part of $\mathrm{Sym}^{1,1}(\mathfrak{g})$, satisfying the algebraic Bianchi identity, we consider the whole space $\mathrm{Sym}^{1,1}(\mathfrak{g})$, since the Chern curvature has less symmetries. In this case, $\Omega\in\mathrm{Sym}^{1,1}(\mathfrak{g})$ is semipositive precisely if and only if $(T^{1,0}M,g)$ is dual-Nakano semipositive.

There is a natural $\mathbb{R}$-linear *adjoint action* of $\mathfrak{g}$ on $\mathrm{Sym}^{1,1}(\mathfrak{g})$:

$$\mathrm{ad}_v(u\otimes\overline{u}) = [v,u]\otimes\overline{u}+u\otimes\overline{[v,u]},\quad v,u\in\mathfrak{g}.$$

For $\Omega\in\mathrm{Sym}^{1,1}(\mathfrak{g})$, let $\{v_i\}$ be an orthonormal basis of $\mathfrak{g}$, diagonalizing Ω with the real eigenvalues $\{\lambda_i\}$:

$$\Omega=\sum_i\lambda_i v_i\otimes\overline{v}_i.$$

We define two important quadratic operations on the space $\mathrm{Sym}^{1,1}(\mathfrak{g})$.

$\Omega^{\#}$: For $v_1\otimes\overline{w_1}, v_2\otimes\overline{w_2}\in\mathfrak{g}\otimes\overline{\mathfrak{g}}$ define

$$(v_1\otimes\overline{w_1})\#(v_2\otimes\overline{w_2}) = [v_1,v_2]\otimes\overline{[w_1,w_2]}. \tag{2.12}$$

This map gives rise to a bilinear operation $\#\colon \mathrm{Sym}^{1,1}(\mathfrak{g})\otimes\mathrm{Sym}^{1,1}(\mathfrak{g})\to\mathrm{Sym}^{1,1}(\mathfrak{g})$. Let $\Omega^{\#}:=\frac{1}{2}(\Omega\#\Omega)$ be the $\#$-square of Ω. In the basis $\{v_i\}$ the $\#$-square of Ω is given by

$$\Omega^{\#}=\sum_{i<j}\lambda_i\lambda_j[v_i,v_j]\otimes\overline{[v_i,v_j]}.$$

Ω^2: Metric g induces the isomorphism $\iota_g\colon\Omega\mapsto R^{\Omega}$, mapping Ω to the corresponding self-adjoint operator $R^{\Omega}\colon\mathfrak{g}\to\mathfrak{g}$. Define $\Omega^2:=\iota_g^{-1}((R^{\Omega})^2)$. In the basis $\{v_i\}$ the square of Ω is given by

$$\Omega^2:=\sum_i\lambda_i^2 v_i\otimes\overline{v}_i.$$

Note that Ω^2 is positive semidefinite, i.e., $\langle \Omega^2, u \otimes \overline{u}\rangle_{\mathrm{tr}} \geqslant 0$ for any $u \in \mathfrak{g}$. Moreover, $\langle \Omega^2, u \otimes \overline{u}\rangle_{\mathrm{tr}} = 0$ if and only if $u \in \ker \Omega$.

Operation $\Omega^\#$ was introduced by Hamilton in [Ham86], while studying the evolution equation for the Riemannian curvature tensor under the Ricci flow. The defining equation (2.12) makes sense for any Lie algebra $\mathfrak{g}$ and does not depend on the choice of metric on $\mathfrak{g}$. In [Ust17a], we used this operation for an arbitrary Lie algebra $\mathfrak{g}$ to study the HCF on complex homogeneous manifolds G/H. We discuss algebraic properties of operation $\#$ and its relation to the geometry of complex homogeneous spaces in Chapter 5.

The following proposition provides coordinate expressions for Ω^2 and $\Omega^\#$.

Proposition 2.28. *Let $\{e_m\}$ be a basis of V and $\{\epsilon^m\}$ be the dual basis. For an element $\Omega \in$* $\mathrm{Sym}^{1,1}(\mathfrak{g})$

$$\Omega = \Omega_{i\bar{j}}{}^{\bar{l}k}(e_k \otimes \epsilon^i) \otimes \overline{(e_l \otimes \epsilon^j)},$$

we have

$$(\Omega^\#)_{i\bar{j}}{}^{\bar{l}k} = \Omega_{p\bar{n}}{}^{\bar{l}k}\Omega_{i\bar{j}}{}^{\bar{n}p} - \Omega_{p\bar{j}}{}^{\bar{n}k}\Omega_{i\bar{n}}{}^{\bar{l}p},$$
$$(\Omega^2)_{i\bar{j}}{}^{\bar{l}k} = g^{m\bar{n}}g_{p\bar{s}}\Omega_{i\bar{n}}{}^{\bar{s}k}\Omega_{m\bar{j}}{}^{\bar{l}p}.$$

Proof. First, we compute $\Omega^\#$. For $e_m \otimes \epsilon^n$, $e_p \otimes \epsilon^s \in \mathfrak{g}$ we have

$$[e_m \otimes \epsilon^n, e_p \otimes \epsilon^s] = \delta_p^n e_m \otimes \epsilon^s - \delta_m^s e_p \otimes \epsilon^n.$$

Therefore

$$\begin{aligned}\Omega\#\Omega =& \Omega_{i\bar{j}}{}^{\bar{l}k}\Omega_{a\bar{b}}{}^{\bar{d}c}[e_k \otimes \epsilon^i, e_c \otimes \epsilon^a] \otimes \overline{[e_l \otimes \epsilon^j, e_d \otimes \epsilon^b]}\\ =& \Omega_{i\bar{j}}{}^{\bar{l}k}\Omega_{a\bar{b}}{}^{\bar{d}c}(\delta_c^i e_k \otimes \epsilon^a - \delta_k^a e_c \otimes \epsilon^i) \otimes \overline{(\delta_d^j e_l \otimes \epsilon^b - \delta_l^b e_d \otimes \epsilon^j)}.\end{aligned}$$

After expanding the Kronecker δ's we get the desired expression for $\Omega^\# = \frac{1}{2}\Omega\#\Omega$.

Now, we compute Ω^2. In coordinates, the operator $R^\Omega\colon \mathfrak{g} \to \mathfrak{g}$ is given by

$$R^\Omega(e_m \otimes \epsilon^p) = \Omega_{i\bar{j}}{}^{\bar{l}k}g^{p\bar{j}}g_{m\bar{l}}(e_k \otimes \epsilon^i).$$

Therefore

$$(R^\Omega)^2(e_m \otimes \epsilon^p) = \Omega_{n\bar{j}}{}^{\bar{l}s}\Omega_{i\bar{r}}{}^{\bar{q}k}g^{p\bar{j}}g_{m\bar{l}}g_{s\bar{q}}g^{n\bar{r}}(e_k \otimes \epsilon^i),$$

which implies the stated formula for Ω^2. $\square$

2.2 Hartshorne's and Frankel's Conjectures

In this section, we review Frankel's and Hartshorne's conjectures, characterizing the complex projective space $\mathbb{P}^n$ by positivity of its tangent bundle, and discuss various generalizations of these conjectures.

Recall that a Kähler manifold (M, g, J) has positive *holomorphic bisectional curvature*, if its tangent bundle $(T^{1,0}M, g)$ is Griffiths positive. In [Fra61], Frankel proposed the following conjecture.

Conjecture 2.29 (Frankel's conjecture)**.** *Every compact n-dimensional Kähler manifold of positive holomorphic bisectional curvature is biholomorphic to the complex projective space $\mathbb{P}^n$.*

By the results of Andreotti-Frankel this result was known for $n = 2$. For arbitrary n, in [Fra61], Frankel proved that manifolds, satisfying the assumptions of the conjecture, share certain cohomological properties with the projective spaces, namely, any two analytic submanifolds of complementary dimensions necessarily intersect. Finally, in [SY80] Siu and Yau resolved Frankel's conjecture in all dimensions. Proof of [SY80] is based on the cohomological characterization of the projective spaces due to Kobayashi, Ochiai [KO73] and uses energy minimizing harmonic maps $S^2 \to M$. An approach based solely on the Kähler-Ricci flow, was proposed by Chen, Sun, and Tian [CST09]. They gave an alternative proof of the Frankel's conjecture, based on the prior work on convergence of the Kähler-Ricci flow to Kähler-Ricci solitons and Kähler-Einstein metrics.

Partly motivated by the statement of Frankel's conjecture, Hartshorne [Har70] has suggested the following purely algebraic conjecture.

Conjecture 2.30 (Hartshorne's conjecture)**.** *Every irreducible n-dimensional non-singular projective variety with ample tangent bundle defined over an algebraically closed field* $\mathbf{k}$ *of characteristic* $\geqslant 0$ *is isomorphic to the projective space $\mathbb{P}^n$.*

Even over $\mathbb{C}$, Hartshorne's conjecture is stronger than the conjecture of Frankel. Indeed, as we have observed in (2.9), positivity of the holomorphic bisectional curvature implies the ampleness of $T^{1,0}M$, so Frankel's conjecture follows from Hartshorne's conjecture.

Hartshorne himself has proved the conjecture for surfaces [Har70]. Mabuchi [Mab78] proved the conjecture in dimension $n = 3$. Finally, Hartshorne's conjecture was completely proved by Mori [Mor79]. Interestingly, similarly to Siu and Yau, Mori studied rational curves $\mathbb{P}^1 \simeq S^2 \to M$ of minimal degree and their deformation spaces.

After almost simultaneous solutions to Frankel's and Hartshorne's conjectures, large amount of research has been driven by various generalizations of these results. One can try relaxing tangent bundle positivity conditions, in such a way, that there is still a hope for a complete classification of the underlying manifolds. On the differential-geometric side, it is natural to study Kähler manifolds with *semipositive* holomorphic bisectional curvature. This class is strictly larger, than just projective spaces, since it includes, e.g., products of complex tori and projective spaces. In fact, the Killing metric on a compact Hermitian symmetric space is an example of a Kähler metric with semipositive holomorphic bisectional curvature. The *generalized Frankel's conjecture* states that, morally, these are the only possible examples.

Conjecture 2.31 (Generalized Frankel's conjecture)**.** *Every compact n-dimensional Kähler manifold of semipositive bisectional curvature is isometric to the quotient of*

$$(\mathbb{C}^n, g^{\text{flat}}) \times (M_1, \kappa_1) \times \cdots \times (M_k, \kappa_k) \times (\mathbb{P}^{n_1}, g_1) \times \cdots \times (\mathbb{P}^{n_l}, g_l),$$

where $\{M_i\}$ are compact Hermitian symmetric spaces of rank $\geqslant 2$, g^{flat} is the Euclidean metric on $\mathbb{C}^n$, κ_i is the Killing metric on a symmetric space, and g_i is a Griffiths semipositive metric on a projective space.

In the case, when the manifold is assumed to be Kähler-Einstein, the conjecture was proved by Mok and Zhong [MZ86] by showing that the curvature tensor is parallel. The conjecture was proved by Cao and Chow [CC86] under a stronger curvature positivity assumption — semipositivity of dual-Nakano curvature. Shortly after, Mok [Mok88] has completely resolved the generalized Frankel's conjecture. His proof combines several ingredients: the regularization of the metric and the curvature by the Ricci flow, the theory of the *variety of minimal rational tangents*, developed by himself, and Berger's holonomy theorem [Ber55], which states that non-symmetric Kähler manifolds with positive Ricci curvature have restricted holonomy $U(n)$. The latter turned out to be the key step in many related uniformization results.

Recently, following the ideas of Brendle and Schoen, which were used in the classification of weakly 1/4-pinched Riemannian manifolds [BS08], Gu [Gu09] has substantially simplified Mok's proof of the generalized Frankel conjecture. Arguments of [BS08], and [Gu09] both rely on Berger's holonomy classification. These results illustrate that often Riemannian manifolds with a semipositive curvature in an appropriate sense, which do not admit a metric of a strictly positive curvature, are geometrically rigid, i.e., not only their topology is bounded, but also the underlying metric is uniquely defined.

Another possible way to weaken the assumption of Frankel's conjecture, is to use *holomorphic orthogonal bisectional curvature*. Recall that Ω has positive (resp. semipositive) holomorphic orthogonal bisectional curvature, if $\Omega(\xi, \bar{\xi}, \eta, \bar{\eta}) > 0$ (resp. $\geqslant 0$) whenever nonzero vectors $\xi, \eta \in T^{1,0}M$ are orthogonal to each other. It turns out, that (semi)positivity of holomorphic orthogonal bisectional curvature is not that different from Griffiths (semi)positivity. Namely, in 1992 in an unpublished work Cao and Hamilton observed that this positivity condition is preserved by the Kähler-Ricci flow [BEG13, Ch. 5]. Results of Chen [Che07], Gu, Zhang [GZ10], and Wilking [Wil13] on the behavior of this positivity condition under the Kähler-Ricci flow imply that only the projective spaces admit metrics of positive holomorphic orthogonal bisectional curvature. Recently, in [FLW17] authors modified the original argument of Siu and Yau to make it work under the assumption of the positivity of the holomorphic orthogonal bisectional curvature.

On the algebraic side, attempts to generalize the original Hartshorne's conjecture also led to new research directions, however a key uniformization question (Conjecture 2.33 below) is still largely open. In [CP91], Campana and Peternell study projective manifolds M with *numerically effective* (nef) tangent bundles. Recall, that this means that $c_1(\mathcal{L})$, where $\mathcal{L} = \mathcal{O}_{\mathbb{P}(T^*M)}(1)$, integrates nonnegatively over any holomorphic curve $C \subset \mathbb{P}(T^*M)$. The authors focus on projective threefolds M^3 and prove that any such M has an étale covering that is fibered over its Albanese variety with a fiber a rational homogeneous manifold. In the same paper, a similar theorem is conjectured to be true in higher dimensions.

In [DPS94], [DPS95], Demailly, Peternell, and Schneider further develop the tools for studying Kähler manifolds with nef tangent bundles. Recall that projective manifold M is *Fano*, if its anticanonical bundle K_M^{-1} is ample, or, equivalently, if it admits a Kähler metric with positive Ricci curvature. In [DPS94], the following structure result about the Albanese map on such M is proved.

Theorem 2.32 ([DPS94]). *Let M be a compact Kähler manifold with nef tangent bundle TM. Let $\widetilde{M}$ be a finite étale cover of maximum irregularity $q = q(\widetilde{M})$. Then*

1. $\pi_1(\widetilde{M}) = \mathbb{Z}^{2q}$.

2. *The Albanese map $\alpha\colon \widetilde{M} \to A(\widetilde{M})$ is a smooth fibration over a q-dimensional torus with nef relative tangent bundle.*

3. *The fibers F of α are Fano manifolds with nef tangent bundle (i.e., K_F^{-1} is ample, and TF is nef).*

The above theorem demonstrates, that the most essential step in the classification of Kähler manifolds with nef tangent bundles, is the classification of projective *Fano* manifolds with nef tangent bundle. The relevant part of the conjecture of [CP91] is the following statement.

Conjecture 2.33 (Campana-Peternell conjecture)**.** *Let M be a Fano manifold with nef tangent bundle. Then M is a rational homogeneous manifolds.*

In the last decades, Campana-Peternell conjecture has motivated large amount of research in algebraic geometry. Mok's variety of minimal rational tangents turned out particularly useful for proving certain partial results related to the conjecture [Mok02; Mok08; Hwa13]. For more details, we refer the reader to the survey [Mn+15] and references therein. Very recently, in [LOY18], the authors studied manifolds with *strictly nef* exterior powers $\Lambda^r(T^{1,0}M)$. They prove that such manifolds are rationally connected, and for $r = 1$ the only manifolds satisfying the corresponding positivity assumption are again the projective spaces. Rational homogeneous manifolds, are known to admits a Kähler-Einstein metric. By using a regularization theorem for closed positive (1,1)-currents, Demailly [Dem17] constructed *weakly almost Kähler-Einstein* metrics on a Fano manifold with nef tangent bundle.

Let us briefly discuss the relation between Campana-Peternell conjecture and the generalized Frankel's conjecture. It is known that any rational homogeneous manifold has a nef tangent bundle. Furthermore, there is a notion of nefness for holomorphic bundles on, not necessarily algebraic, compact complex manifolds (see [DPS94]), and it is known that on (M, g, J)

$$(TM, g) \geqslant_{\mathrm{Gr}} 0 \Rightarrow TM \text{ is nef.}$$

It implies that a Kähler manifold with a semipositive Griffiths curvature necessarily has a nef tangent bundle. Hence, for a Fano manifold, the assumption of the generalized Frankel Conjecture 2.31 implies the assumption of Campana-Peternell Conjecture 2.33. The converse implication does not hold. Indeed, by the generalized Frankel's conjecture, the only *Fano* examples of Kähler manifolds with $TM \geqslant_{\mathrm{Gr}} 0$ are rational symmetric spaces, and those are indeed rational homogeneous, however not all rational homogeneous manifolds are symmetric. A simplest example of a non-symmetric rational homogeneous manifold is the flag manifold

$$U(3)/T^3 \simeq GL_3(\mathbb{C})/B, \quad B = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \subset GL_3(\mathbb{C}).$$

This manifolds admits a metric with dual-Nakano semipositive curvature; it also admits a Kähler-Einstein metric, but by the discussion above they have to be different.

Large part of this thesis is motivated by the search for an appropriate differential-geometric substitute of Campana-Peternell Conjecture 2.33, which would provide a characterization for rational homogeneous manifolds. In Section 6.1, we propose differential-geometric notion of positivity for a tangent bundle on an Hermitian manifold (M, g, J), which, conjecturally characterize rational homogeneous manifolds among complex manifolds.

Chapter 3

Hermitian Curvature Flow

We start this chapter with reviewing various metric flows in Hermitian geometry. Focusing on a member of the family of Hermitian curvature flows, we compute the evolution equation of the Chern curvature. An important feature of this equation is that it takes a very clear algebraic form with the introduction of an auxiliary object — the torsion-twisted connection.

3.1 Geometric Flows in Hermitian Geometry

As we have discussed in the introduction, the Kähler-Ricci flow is a powerful tool for approaching various geometric classification problems in Kähler geometry. However, on a general Hermitian manifold (M, g, J) the Ricci curvature $\mathrm{Ric}(g)$ is not necessarily J-invariant, hence the Ricci flow does not preserve the Hermitian condition. This observation motivated different authors to introduce "Hermitian" modifications of the Kähler-Ricci flow. The idea is to use connections and tenors canonically attached to an Hermitian manifold, e.g., the Chern connection ∇, its curvature Ω, and torsion T, to define an evolution equation for a metric. Below we review several modifications of the Ricci flow in the realm of Hermitian manifolds. All these generalizations share a common property that they amount to the Kähler-Ricci flow if the initial data (M, g, J) is Kähler.

3.1.1 Chern-Ricci Flow

Recall that on a Kähler manifold (M, g, J, ω), the Ricci tensor $\mathrm{Ric}(X, Y)$ defines a (1,1) form $\rho = \sqrt{-1}\mathrm{Ric}_{i\bar{j}} dz^i \wedge d\overline{x^j}$, which is closed and represents the class $2\pi c_1(M)$ in cohomology. In local coordinates,

$$\rho_{i\bar{j}} = \sqrt{-1}\partial_i \overline{\partial_j} \log \det g,$$

and the Kähler-Ricci flow is given by the equation

$$\frac{dg_{i\bar{j}}}{dt} = \partial_i \overline{\partial_j} \log \det g. \tag{3.1}$$

In a series of papers [Gil11],[TW13],[GS15],[TW15] Gill, Tosatti, Weinkove, and Smith studied the flow given by the same equation on a general Hermitian manifold. Let us review some of their results.

Following the notations of Section 2.1.3, we can write (3.1) as

$$\frac{dg}{dt} = -S^{(1)}.$$

The Chern-Ricci contraction $S^{(1)}$ does not longer coincide with the Ricci curvature, but we still get the (1,1)-form $\sqrt{-1}S^{(1)}_{i\bar{j}} dz^i \wedge \overline{dx^j}$ representing the class $2\pi c_1(M)$ in the Bott-Chern cohomology. This flow is referred to as the Chern-Ricci flow. It was originally introduced by Gill in [Gil11], where he proved the following result:

Theorem 3.1 ([Gil11]). *If $c_1^{BC}(M) = 0$ then, for any initial metric $\omega_0 = \sqrt{-1}g_{i\bar{j}}dz^i \wedge d\overline{z^j}$, there exists a solution $\omega(t)$ to the Chern-Ricci flow* (3.1) *for all time and the metrics $\omega(t)$ converge smoothly as $t \to \infty$ to an Hermitian metric ω^∞ satisfying $\rho(\omega^\infty) = 0$.*

The above theorem is a direct generalization of the corresponding statement for the Kähler-Ricci flow, [Cao85]. In [TW15], Tosatti and Weinkove extend many results, known for the Kähler-Ricci flow, to the case of Chern-Ricci flow. Among other results, they obtain the maximal time existence:

Theorem 3.2 ([TW15, Thm. 1.2]). *There exists a unique maximal solution to the Chern-Ricci flow on $[0, T)$, where*

$$T = \sup\{t > 0 \mid \text{ there exists } \psi \in \mathcal{C}^\infty(M, \mathbb{R}) \text{ with } \omega_0 - tRic(\omega_0) + \sqrt{-1}\partial\overline{\partial}\psi > 0\}.$$

In [GS15], Gill and Smith gave an alternative description of the maximal time interval for the Chern-Ricci flow. They proved the scalar curvature blow-up result, which is analogous to the statement about the long time existence of the Kähler-Ricci flow.

Theorem 3.3 ([GS15, Thm. 1.1]). *Let M be a compact complex manifold of complex dimension n and ω_0 an Hermitian metric. Then $\omega(t)$ exists on the maximal interval $[0, T)$ and either $T = \infty$ or*

$$\limsup_{t \to T} (\sup_M (R(g(t))) = \infty,$$

where R denotes the scalar curvature of the Chern connection.

Recently, Yang [Yan16] via an explicit computation on Hopf manifolds, observed that the Chern-Ricci flow does not preserve Griffiths semipositivity. One of the principle motivations for the present thesis is to find a metric flow, which preserves some natural curvature positivity conditions on general non-Kähler manifolds, e.g., Griffiths/dual-Nakano semipositivity.

3.1.2 Family of Hermitian Curvature Flows

In [ST11], Streets and Tian defined a family of *Hermitian curvature flows* (HCF) on an arbitrary Hermitian manifold (M, g, J). Under this flow, the metric g is evolved according to the equation.

$$\frac{dg}{dt} = -S^{(2)} - Q, \tag{3.2}$$

where $S^{(2)}_{i\bar{j}} = g^{m\bar{n}} \Omega_{m\bar{n}i\bar{j}}$ is the *second Chern-Ricci curvature* of the Chern connection, and $Q = Q(T)$ is an arbitrary symmetric term of type (1,1), quadratic in T. In [ST11], the authors prove short time existence for this flow and derive basic long time blow-up and regularity properties. Let us review these results.

Proposition 3.4 ([ST11, Prop. 5.1]). *Given (M, g_0, J) a compact complex manifold, there exists a unique solution to HCF* (3.2) *with initial condition g_0 on $[0, \epsilon)$ for some $\epsilon > 0$.*

This proposition follows from the fact that the second order differential operator $\Phi \colon g \mapsto S^{(2)} + Q$ is strictly elliptic for any choice of Q.

Proposition 3.5 ([ST11, Prop. 5.2]). *Given (M, g_0, J) a compact complex manifold with Kähler metric g_0, let $g(s)$ denote the solution to HCF* (3.2) *with initial condition g_0, which exists on $[0, T)$. Then for all $t \in [0, T)$, $g(t)$ is Kähler and is a solution to the Kähler-Ricci flow.*

Similarly to other nonlinear parabolic evolution equations, HCF regularizes the initial data for any positive time $t > 0$:

Theorem 3.6 ([ST11, Thm. 7.3]). *Let $(M, g(t), J)$ be a solution to HCF for which the maximum principle holds. Then for each $\alpha > 0$ and every $m \in \mathbb{N}$ there exists a constant C_m depending only on $m, \dim M$, and $\max\{\alpha, 1\}$ such that if*

$$|\Omega|_{C^0(g(t))} \leqslant K, \quad |\nabla T|_{C^0(g(t))} \leqslant K, \quad |T|^2_{C^0(g(t))} \leqslant K$$

for all $x \in M$ and $t \in [0, \alpha/K]$, then

$$|\nabla^m \Omega|_{C^0(g(t))} \leqslant \frac{C_m K}{t^{m/2}}, \quad |\nabla^{m+1} T|_{C^0(g(t))} \leqslant \frac{C_m K}{t^{m/2}}.$$

As a corollary of this regularity result, we have a basic blow-up result about long-time existence.

Corollary 3.7. *There exists a constant $c = c(n)$ such that given (M, g, J) a complex manifold with Hermitian metric g, the solution $g(t)$ to HCF with initial condition g exists for*

$$t \in [0, c(n)/ \max(|\Omega|_{C^0(g(t))}, |\nabla T|_{C^0(g(t))}, |T|^2_{C^0(g(t))})].$$

Moreover the solution exists on a maximal time interval $[0, t_{\max})$, and if $t_{\max} < \infty$ then

$$\limsup_{t \to t_{\max}} \max(|\Omega|_{C^0(g(t))}, |\nabla T|_{C^0(g(t))}, |T|^2_{C^0(g(t))}) = \infty.$$

While these regularization and blow-up statements are similar to the ones for the standard Ricci flow, they require much stronger assumptions on the behavior of the metric along the flow (3.2). Namely, to deduce the long time existence, we have to control not only the norm of the Chern curvature tensor, but also the norms of the torsion and its covariant derivative. The expectation is that, under additional geometric assumptions on (M, g, J) and for an appropriate choice of the torsion terms in the family of flows (3.2), one will be able to significantly improve the results of Theorem 3.6 and Corollary 3.7, similarly to Theorems 3.2 and 3.3.

3.1.3 Specializations of the HCF Family

Hermitian curvature flows form a family of evolution equations for a metric, and various members of this family have different geometric and analytic properties. In this section we describe several specializations of the general HCF family (3.2).

In the original paper [ST11], Streets and Tian find a unique Hilbert-type functional $\mathcal{F}(g)$ which yields $S^{(2)}$ as the leading term in the Euler–Lagrange equation. Computing the variation of this functional along a one parameter family of metrics, authors arrive at a version of the HCF family given by

$$\frac{dg}{dt} = -S^{(2)} + \frac{1}{2}Q^1 - \frac{1}{4}Q^2 - \frac{1}{2}Q^3 + Q^4, \tag{3.3}$$

where

$$\begin{aligned}
Q^1_{i\bar{j}} &= g_{p\bar{s}} g^{m\bar{n}} T^p_{im} T^{\bar{s}}_{\bar{j}\bar{n}},\\
Q^2_{i\bar{j}} &= g^{m\bar{n}} g^{p\bar{s}} T_{mp\bar{j}} T_{\bar{n}\bar{s}i},\\
Q^3_{i\bar{j}} &= T^m_{im} T^{\bar{n}}_{\bar{j}\bar{n}},\\
Q^4_{i\bar{j}} &= \frac{1}{2} g^{p\bar{s}} (T^m_{pm} T_{\bar{s}\bar{j}i} + T^{\bar{n}}_{\bar{s}\bar{n}} T_{pi\bar{j}}).
\end{aligned}$$

Any critical metric for $\mathcal{F}(g)$ turns out to be scale-invariant under (3.3) (see [ST11, Prop. 3.3]).

An import class of Hermitian metrics between Kähler metrics and general Hermitian metrics, is the class of *pluriclosed metrics* (strongly Kähler with torsion).

Definition 3.8. An Hermitian metric on a complex manifold is called *pluriclosed*, if

$$\partial\overline{\partial}\omega = 0,$$

where ω is the fundamental (1,1)-form.

In [ST10], Streets and Tian observed that the flow

$$\frac{dg_{i\bar{j}}}{dt} = -S^{(2)}_{i\bar{j}} + Q^1_{i\bar{j}}$$

preserves the subspace of pluriclosed metrics. In a subsequent paper [ST13], they found a clear interpretation for this flow in terms of the Bismut-Ricci form on (M, g, J) and related the pluriclosed flow to the renormalization group flow in physics. This interpretation allows to prove that the pluriclosed flow is the gradient flow of the first eigenvalue of a certain Schrödinger operator. In further papers [ST12; Str17; AS17] authors found an interpretation of the pluriclosed flow in the context of *generalized geometry.*

In this thesis, we focus on a different member of the Hermitian curvature flow family. Namely, the main object of our study is the following specification of the HCF (3.2)

$$\boxed{\frac{dg_{i\bar{j}}(t)}{dt} = -S^{(2)}_{i\bar{j}} - Q_{i\bar{j}}} \tag{3.4}$$

where

$$S^{(2)}_{i\bar{j}} = g^{m\bar{n}}\Omega_{m\bar{n}i\bar{j}} \quad \text{and} \quad Q_{i\bar{j}} = \frac{1}{2}g^{m\bar{n}}g^{p\bar{s}}T_{pm\bar{j}}T_{\bar{s}\bar{n}i}$$

are the second Chern-Ricci curvature and a specific quadratic torsion term for $g = g(t)$. By Proposition 3.4, there exists a unique solution to equation (3.4) on a maximal time interval $[0, t_{\max})$. By slight abuse of notation, in what follows, this flow is referred to as the HCF. The choice of the torsion quadratic term $Q(T)$ is motivated by a very special evolution of the Chern curvature under this flow, which we compute further in this chapter.

Let us derive the coordinate expression for the HCF evolution term

$$\Psi_{i\bar{j}} = g^{m\bar{n}}\Omega_{m\bar{n}i\bar{j}} + \frac{1}{2}g^{m\bar{n}}g^{p\bar{s}}T_{mp\bar{j}}T_{\bar{n}\bar{s}i},$$

on (M, g, J). To simplify subsequent applications of this computation we provide a formula for the g-dual of $\Psi_{i\bar{j}}$

$$\Psi^{i\bar{j}} := g^{i\bar{n}}g^{m\bar{j}}\Psi_{m\bar{n}},$$

i.e., we use metric g to identify Ψ with a section of $\mathrm{Sym}^{1,1}(T^{1,0}M)$.

Proposition 3.9.

$$\Psi^{i\bar{j}} = g^{m\bar{n}}\partial_m\partial_{\bar{n}}g^{i\bar{j}} - \partial_m g^{i\bar{n}}\partial_{\bar{n}}g^{m\bar{j}}.$$

Proof. In local coordinates, the Christoffel symbols for the Chern connection on (TM, g) are

$$\Gamma^k_{ij} = g^{k\bar{l}}\partial_i g_{j\bar{l}}.$$

Hence for the Chern curvature we have:

$$g^{m\bar{l}}\Omega_{i\bar{j}m}{}^{k} = -g^{m\bar{l}}\partial_{\bar{j}}\left(g^{k\bar{n}}\partial_i g_{m\bar{n}}\right) = -g^{m\bar{l}}\partial_{\bar{j}}\left(-g^{k\bar{n}}g_{m\bar{s}}g_{p\bar{n}}\partial_i g^{p\bar{s}}\right) = g^{m\bar{l}}\partial_{\bar{j}}\left(g_{m\bar{s}}\partial_i g^{k\bar{s}}\right) =$$
$$= g^{m\bar{l}}\partial_{\bar{j}}g_{m\bar{s}}\partial_i g^{k\bar{s}} + \partial_i\partial_{\bar{j}}g^{k\bar{l}} = -g_{p\bar{s}}\partial_{\bar{j}}g^{p\bar{l}}\partial_i g^{k\bar{s}} + \partial_i\partial_{\bar{j}}g^{k\bar{l}}.$$

For the torsion tensor T^i_{mp} we compute:

$$T^i_{mp} = g^{i\bar{l}}(\partial_m g_{p\bar{l}} - \partial_p g_{m\bar{l}}) = g_{m\bar{l}}\partial_p g^{i\bar{l}} - g_{p\bar{l}}\partial_m g^{i\bar{l}}.$$

Therefore

$$\frac{1}{2}g^{m\overline{n}}g^{p\overline{s}}T^{i}_{mp}T^{\overline{j}}_{\overline{n}\overline{s}} = \frac{1}{2}g^{m\overline{n}}g^{p\overline{s}}(g_{m\overline{l}}\partial_p g^{i\overline{l}} - g_{p\overline{l}}\partial_m g^{i\overline{l}})(g_{k\overline{n}}\partial_{\overline{s}}g^{k\overline{j}} - g_{k\overline{s}}\partial_{\overline{n}}g^{k\overline{j}}) =$$
$$= g^{p\overline{s}}g_{k\overline{l}}\partial_p g^{i\overline{l}}\partial_{\overline{s}}g^{k\overline{j}} - \partial_p g^{i\overline{n}}\partial_{\overline{n}}g^{p\overline{j}}.$$

Using the above formulas together we get the expression for $\Psi^{i\overline{j}}$.

$$\Psi^{i\overline{j}} = g^{m\overline{n}}\partial_m\partial_{\overline{n}}g^{i\overline{j}} - \partial_p g^{i\overline{s}}\partial_{\overline{s}}g^{p\overline{j}}.$$

After relabeling the indices we get the stated formula. □

Combining Proposition 3.9 and the equation of the HCF flow (3.4), we get the following corollary.

Corollary 3.10. *Let (M, J, g_0) be an Hermitian manifold. Assume that $\tilde{g}(t) \in \mathrm{Sym}^{1,1}(TM)$ is a solution to the PDE on $M \times [0, t_{\max})$*

$$\begin{cases} \dfrac{d\tilde{g}^{i\overline{j}}}{dt} = \tilde{g}^{m\overline{n}}\partial_m\partial_{\overline{n}}\tilde{g}^{i\overline{j}} - \partial_m\tilde{g}^{i\overline{n}}\partial_{\overline{n}}\tilde{g}^{m\overline{j}}, \\ \tilde{g}(0) = g_0^{-1}, \end{cases} \tag{3.5}$$

such that $\tilde{g}^{i\overline{j}}(t)$ is positive definite for $t \in [0, t_{\max})$. Then $g(t) := \tilde{g}^{-1}(t)$ is the solution to the HCF on M.

An interesting feature of equation (3.5) is that its right-hand side depends only on $\tilde{g}$ and not on $\tilde{g}^{-1}$. In particular, there might exist a solution $\tilde{g}(t)$ starting with a degenerate or indefinite form. It would be interesting to find a geometric interpretation of such solution. However in this case, the equation is not parabolic anymore.

3.2 Evolution Equations for the Curvature under the HCF

In this section, we compute the evolution of the Chern curvature under the HCF (3.4). Let $\delta g = k$ be an arbitrary variation of the Hermitian metric. Let us compute the first variation of the Chern connection, torsion, and curvature. Note, that unlike ∇ itself, its variation $\delta\nabla$ is a tensor.

Proposition 3.11 ([ST11, Lemma 10.1], [Ust16, Prop. 2.1])**.** *Under the variation of the metric $\delta g = k$, the variation $\delta\nabla$ of the Chern connection is given by the formula*

$$(\delta\nabla)_{\overline{\xi}}\eta = (\delta\nabla)_{\xi}\overline{\eta} = 0, \quad g\big((\delta\nabla)_{\xi}\eta, \overline{\zeta}\big) = \nabla_{\xi}k(\eta, \overline{\zeta}).$$

Proof. To prove the first formula, we just notice that $\nabla_{\bar{\xi}}\eta = \bar{\partial}_{\bar{\xi}}\zeta$ is completely defined by the holomorphic structure on the bundle $T^{1,0}M$ and is independent of the choice of g.

To prove the second formula, let us take the variation of the identity

$$\xi \cdot g(\eta, \bar{\zeta}) = g(\nabla_\xi \eta, \bar{\zeta}) + g(\eta, \nabla_\xi \bar{\zeta}),$$

where $\xi, \eta, \zeta \in T^{1,0}M$.

$$\xi \cdot k(\eta, \bar{\zeta}) = k(\nabla_\xi \eta, \bar{\zeta}) + k(\eta, \nabla_\xi \bar{\zeta}) + g((\delta\nabla)_\xi \eta, \bar{\zeta}).$$

Collecting the expressions involving k on the one side, we get the desired identity. $\square$

Proposition 3.11 immediately imply the variation formula for the torsion.

Proposition 3.12 ([ST11, Lemma 10.4], [Ust16, Prop. 2.2])**.** *With the same notations as in Proposition 3.11, the variation of the torsion tensor $T(\xi, \eta)$ is given by the formula*

$$g((\delta T)(\xi, \eta), \bar{\zeta}) = \nabla_\xi k(\eta, \bar{\zeta}) - \nabla_\eta k(\xi, \bar{\zeta}).$$

Proposition 3.13 ([ST11, Lemma 10.2], [Ust16, Prop. 2.3])**.** *With the same notations as in Proposition 3.11, the variations of the (3,1) and (4,0) Chern curvatures are given by the formulas*

$$g\big((\delta\Omega)(\xi, \bar{\eta})\zeta, \bar{\nu}\big) = -\nabla_{\bar{\eta}} \nabla_\xi k(\zeta, \bar{\nu}) + \nabla_{\nabla_{\bar{\eta}}\xi} k(\zeta, \bar{\nu}),$$
$$(\delta\Omega)(\xi, \bar{\eta}, \zeta, \bar{\nu}) = k\big(\Omega(\xi, \bar{\eta})\zeta, \bar{\nu}\big) - \nabla_{\bar{\eta}} \nabla_\xi k(\zeta, \bar{\nu}) + \nabla_{\nabla_{\bar{\eta}}\xi} k(\zeta, \bar{\nu}).$$

Proof. Clearly the second formula follows from the first one. Before we start proving the first formula, note that for any $\zeta, \nu \in \Gamma(T^{1,0}M)$ we have $[\zeta, \bar{\nu}] = \nabla_\zeta \bar{\nu} - \nabla_{\bar{\nu}} \zeta$, since the $(1,1)$-part of the torsion vanishes. In particular, the $(1,0)$-part of $[\zeta, \bar{\nu}]$ is $-\nabla_{\bar{\nu}} \zeta$.

We have $\Omega(\xi,\overline{\eta})\zeta = [\nabla_\xi, \nabla_{\overline{\eta}}]\zeta - \nabla_{[\xi,\overline{\eta}]}\zeta$. Using the result of Proposition 3.11 we get

$$\begin{aligned}
&g\big((\delta\Omega)(\xi,\overline{\eta})\zeta,\overline{\nu}\big)\\
&\quad= g\big([(\delta\nabla)_\xi,\nabla_{\overline{\eta}}]\zeta,\overline{\nu}\big) + g\big([\nabla_\xi,(\delta\nabla)_{\overline{\eta}}]\zeta,\overline{\nu}\big) - g((\delta\nabla)_{[\xi,\overline{\eta}]}\zeta,\overline{\nu})\\
&\quad= g\big([(\delta\nabla)_\xi,\nabla_{\overline{\eta}}]\zeta,\overline{\nu}\big) + \nabla_{\nabla_{\overline{\eta}}\xi}k(\zeta,\overline{\nu})\\
&\quad= g\big((\delta\nabla)_\xi\nabla_{\overline{\eta}}\zeta,\overline{\nu}\big) - g\big(\nabla_{\overline{\eta}}(\delta\nabla)_\xi\zeta,\overline{\nu}\big) + \nabla_{\nabla_{\overline{\eta}}\xi}k(\zeta,\overline{\nu})\\
&\quad= \nabla_\xi k(\nabla_{\overline{\eta}}\zeta,\overline{\nu}) - \overline{\eta}\cdot g\big((\delta\nabla)_\xi\zeta,\overline{\nu}\big) + g\big((\delta\nabla)_\xi\zeta,\nabla_{\overline{\eta}}\overline{\nu}\big) + \nabla_{\nabla_{\overline{\eta}}\xi}k(\zeta,\overline{\nu})\\
&\quad= \nabla_\xi k(\nabla_{\overline{\eta}}\zeta,\overline{\nu}) - \overline{\eta}\cdot\nabla_\xi k(\zeta,\overline{\nu}) + \nabla_\xi k(\zeta,\nabla_{\overline{\eta}}\overline{\nu}) + \nabla_{\nabla_{\overline{\eta}}\xi}k(\zeta,\overline{\nu})\\
&\quad= -\nabla_{\overline{\eta}}\nabla_\xi k(\zeta,\overline{\nu}) + \nabla_{\nabla_{\overline{\eta}}\xi}k(\zeta,\overline{\nu}).
\end{aligned}$$

Here in the second equality we use the facts that $(\delta\nabla)_{\overline{\eta}}$ vanishes on (1,0)-vectors and that $(\delta\nabla)_{[\xi,\overline{\eta}]}\zeta = -(\delta\nabla)_{\nabla_{\overline{\eta}}\xi}\zeta$. □

Now, our goal is to derive the evolution equation for the Chern curvature under the HCF (3.4). The entire computation is based on Proposition 3.13 and uses solely Bianchi identities (2.5) and the commutation of covariant derivatives.

Proposition 3.14. *Assume that $g(t)$ solves the HCF* (3.4) *on* $[0;t_{\max})$. *Then the tensor* $\Omega(t) := \Omega^{g(t)}$ *evolves according to the equation*

$$\begin{aligned}
\frac{d}{dt}\Omega_{i\bar{j}k\bar{l}} = g^{m\bar{n}}\Big(&\nabla_m\nabla_{\bar{n}}\Omega_{i\bar{j}k\bar{l}} + T^{\bar{r}}_{\bar{n}\bar{j}}\nabla_m\Omega_{i\bar{r}k\bar{l}} + T^q_{mi}\nabla_{\bar{n}}\Omega_{q\bar{j}k\bar{l}}\\
&+ T^q_{mi}T^{\bar{r}}_{\bar{n}\bar{j}}\Omega_{q\bar{r}k\bar{l}} + g^{p\bar{s}}(T_{pm\bar{l}}\nabla_{\bar{n}}\Omega_{i\bar{j}k\bar{s}} + T_{\bar{s}\bar{n}k}\nabla_m\Omega_{i\bar{j}p\bar{l}}\\
&+ T_{pm\bar{l}}T^{\bar{r}}_{\bar{n}\bar{j}}\Omega_{i\bar{r}k\bar{s}} + T_{\bar{s}\bar{n}k}T^q_{mi}\Omega_{q\bar{j}p\bar{l}} + g^{q\bar{r}}T_{\bar{s}\bar{n}k}T_{qm\bar{l}}\Omega_{i\bar{j}p\bar{r}})\Big)\\
&+ g^{m\bar{n}}g^{p\bar{s}}(\Omega_{i\bar{j}m\bar{s}}\Omega_{p\bar{n}k\bar{l}} + \Omega_{m\bar{j}k\bar{s}}\Omega_{i\bar{n}p\bar{l}} - \Omega_{m\bar{j}p\bar{l}}\Omega_{i\bar{s}k\bar{n}} + \frac{1}{2}\nabla_i T_{pm\bar{l}}\nabla_{\bar{j}}T_{\bar{s}\bar{n}k})\\
&- g^{p\bar{s}}(S^{(2)}_{p\bar{l}} + Q_{p\bar{l}})\Omega_{i\bar{j}k\bar{s}} - g^{p\bar{s}}S^{(2)}_{p\bar{j}}\Omega_{i\bar{s}k\bar{l}} - g^{p\bar{s}}Q_{k\bar{s}}\Omega_{i\bar{j}p\bar{l}}.
\end{aligned}$$

Proof. The coordinate vector fields $e_i = \partial/\partial z^i$ are holomorphic, so we have $\nabla_i e_{\bar{j}} = \nabla_{\bar{j}}e_i = 0$. Hence Proposition 3.13 implies that

$$\frac{\partial}{\partial t}\Omega_{i\bar{j}k\bar{l}} = \nabla_{\bar{j}}\nabla_i\big(S^{(2)}_{k\bar{l}} + Q_{k\bar{l}}\big) - g^{p\bar{s}}(S^{(2)}_{p\bar{l}} + Q_{p\bar{l}})\Omega_{i\bar{j}k\bar{s}}.$$

We compute separately $\nabla_{\bar{j}}\nabla_i S^{(2)}_{k\bar{l}}$ and $\nabla_{\bar{j}}\nabla_i Q_{k\bar{l}}$.

Step 1. Compute $\nabla_{\bar{j}}\nabla_i S^{(2)}_{k\bar{l}}$.

Let us first modify term $\nabla_{\bar{j}}\nabla_i S^{(2)}_{k\bar{l}}$ by applying the second Bianchi identity.

$$\begin{aligned}\nabla_{\bar{j}}\nabla_i S^{(2)}_{k\bar{l}} &= g^{m\bar{n}}\nabla_{\bar{j}}\nabla_i \Omega_{m\bar{n}k\bar{l}} = g^{m\bar{n}}\nabla_{\bar{j}}(\nabla_m \Omega_{i\bar{n}k\bar{l}} + T^p_{mi}\Omega_{p\bar{n}k\bar{l}}) \\ &= g^{m\bar{n}}(\nabla_{\bar{j}}\nabla_m \Omega_{i\bar{n}k\bar{l}} + \nabla_{\bar{j}}T^p_{mi}\,\Omega_{p\bar{n}k\bar{l}} + T^p_{mi}\nabla_{\bar{j}}\Omega_{p\bar{n}k\bar{l}}).\end{aligned} \tag{3.6}$$

Next, we commute $\nabla_{\bar{j}}$ with ∇_m.

$$\begin{aligned}\nabla_{\bar{j}}\nabla_m \Omega_{i\bar{n}k\bar{l}} =& \nabla_m\nabla_{\bar{j}}\Omega_{i\bar{n}k\bar{l}} + \Omega_{m\bar{j}i}{}^{p}\Omega_{p\bar{n}k\bar{l}} + \Omega_{m\bar{j}\bar{n}}{}^{\bar{s}}\Omega_{i\bar{s}k\bar{l}} + \Omega_{m\bar{j}k}{}^{p}\Omega_{i\bar{n}p\bar{l}} + \Omega_{m\bar{j}\bar{l}}{}^{\bar{s}}\Omega_{i\bar{n}k\bar{s}} \\ =& \nabla_m\nabla_{\bar{j}}\Omega_{i\bar{n}k\bar{l}} + g^{p\bar{s}}\big(\Omega_{m\bar{j}i\bar{s}}\Omega_{p\bar{n}k\bar{l}} - \Omega_{m\bar{j}p\bar{n}}\Omega_{i\bar{s}k\bar{l}} + \Omega_{m\bar{j}k\bar{s}}\Omega_{i\bar{n}p\bar{l}} - \Omega_{m\bar{j}p\bar{l}}\Omega_{i\bar{n}k\bar{s}}\big).\end{aligned}$$

Now, we again apply the second Bianchi identity to the first term on the right hand side of the latter expression.

$$\begin{aligned}\nabla_m\nabla_{\bar{j}}\Omega_{i\bar{n}k\bar{l}} &= \nabla_m(\nabla_{\bar{n}}\Omega_{i\bar{j}k\bar{l}} + T^{\bar{s}}_{\bar{n}\bar{j}}\Omega_{i\bar{s}k\bar{l}}) \\ &= \nabla_m\nabla_{\bar{n}}\Omega_{i\bar{j}k\bar{l}} + \nabla_m T^{\bar{s}}_{\bar{n}\bar{j}}\Omega_{i\bar{s}k\bar{l}} + T^{\bar{s}}_{\bar{n}\bar{j}}\nabla_m\Omega_{i\bar{s}k\bar{l}}.\end{aligned}$$

Next, we use twice the first Bianchi identity

$$\begin{aligned}\nabla_{\bar{j}}T^p_{mi} &= \Omega^p_{i\bar{j}m} - \Omega^p_{m\bar{j}i} = g^{p\bar{s}}(\Omega_{i\bar{j}m\bar{s}} - \Omega_{m\bar{j}i\bar{s}}), \\ \nabla_m T^{\bar{s}}_{\bar{n}\bar{j}} &= \Omega^{\bar{s}}_{m\bar{n}\bar{j}} - \Omega^{\bar{s}}_{m\bar{j}\bar{n}} = g^{p\bar{s}}(\Omega_{m\bar{j}p\bar{n}} - \Omega_{m\bar{n}p\bar{j}}),\end{aligned}$$

and once the second Bianchi identity:

$$\nabla_{\bar{j}}\Omega_{p\bar{n}k\bar{l}} = \nabla_{\bar{n}}\Omega_{p\bar{j}k\bar{l}} + T^{\bar{s}}_{\bar{n}\bar{j}}\Omega_{p\bar{s}k\bar{l}}.$$

Collecting everything in (3.6) we get

$$\begin{aligned}\nabla_{\bar{j}}\nabla_i S^{(2)}_{k\bar{l}} &= g^{m\bar{n}}(\nabla_m\nabla_{\bar{n}}\Omega_{i\bar{j}k\bar{l}} + T^{\bar{s}}_{\bar{n}\bar{j}}\nabla_m\Omega_{i\bar{s}k\bar{l}} + T^p_{mi}\nabla_{\bar{n}}\Omega_{p\bar{j}k\bar{l}} + T^p_{mi}T^{\bar{s}}_{\bar{n}\bar{j}}\Omega_{p\bar{s}k\bar{l}}) \\ &\quad + g^{m\bar{n}}g^{p\bar{s}}(\Omega_{i\bar{j}m\bar{s}}\Omega_{p\bar{n}k\bar{l}} + \Omega_{m\bar{j}k\bar{s}}\Omega_{i\bar{n}p\bar{l}} - \Omega_{m\bar{j}p\bar{l}}\Omega_{i\bar{n}k\bar{s}}) - g^{p\bar{s}}S^{(2)}_{p\bar{j}}\Omega_{i\bar{s}k\bar{l}}.\end{aligned} \tag{3.7}$$

Step 2. Compute $\nabla_{\bar{j}}\nabla_i Q_{k\bar{l}}$.

$$\begin{aligned}2\nabla_{\bar{j}}\nabla_i Q_{k\bar{l}} &= g^{m\bar{n}}g^{p\bar{s}}\nabla_{\bar{j}}\nabla_i(T_{pm\bar{l}}T_{\bar{s}\bar{n}k}) \\ &= g^{m\bar{n}}g^{p\bar{s}}(\nabla_i T_{pm\bar{l}}\nabla_{\bar{j}}T_{\bar{s}\bar{n}k} + \nabla_{\bar{j}}T_{pm\bar{l}}\nabla_i T_{\bar{s}\bar{n}k} + T_{pm\bar{l}}\nabla_{\bar{j}}\nabla_i T_{\bar{s}\bar{n}k} + \nabla_{\bar{j}}\nabla_i T_{pm\bar{l}}T_{\bar{s}\bar{n}k}).\end{aligned}$$

We now compute $\nabla_{\bar{j}}\nabla_i T_{\bar{s}\bar{n}k}$ and $\nabla_{\bar{j}}\nabla_i T_{pm\bar{l}}$ using Bianchi identities.

$$\nabla_{\bar{j}}\nabla_i T_{\bar{s}\bar{n}k} = \nabla_{\bar{j}}(\Omega_{i\bar{n}k\bar{s}} - \Omega_{i\bar{s}k\bar{n}}) = \nabla_{\bar{n}}\Omega_{i\bar{j}k\bar{s}} + T^{\bar{r}}_{\bar{n}\bar{j}}\Omega_{i\bar{r}k\bar{s}} - \nabla_{\bar{s}}\Omega_{i\bar{j}k\bar{n}} - T^{\bar{r}}_{\bar{s}\bar{j}}\Omega_{i\bar{r}k\bar{n}}.$$

Using the fact $T_{pm\bar{l}}$ is anti-symmetric in m and p we get

$$\begin{aligned} g^{m\bar{n}}g^{p\bar{s}}T_{pm\bar{l}}\nabla_{\bar{j}}\nabla_i T_{\bar{s}\bar{n}k} =& g^{m\bar{n}}g^{p\bar{s}}T_{pm\bar{l}}(\nabla_{\bar{n}}\Omega_{i\bar{j}k\bar{s}} + T^{\bar{r}}_{\bar{n}\bar{j}}\Omega_{i\bar{r}k\bar{s}} - \nabla_{\bar{s}}\Omega_{i\bar{j}k\bar{n}} - T^{\bar{r}}_{\bar{s}\bar{j}}\Omega_{i\bar{r}k\bar{n}}) \\ =& 2g^{m\bar{n}}g^{p\bar{s}}T_{pm\bar{l}}(\nabla_{\bar{n}}\Omega_{i\bar{j}k\bar{s}} + T^{\bar{r}}_{\bar{n}\bar{j}}\Omega_{i\bar{r}k\bar{s}}). \end{aligned} \tag{3.8}$$

To compute $\nabla_{\bar{j}}\nabla_i T_{pm\bar{l}}$ we start with commuting derivatives.

$$\begin{aligned} \nabla_{\bar{j}}\nabla_i T_{pm\bar{l}} =& \nabla_i\nabla_{\bar{j}}T_{pm\bar{l}} + \Omega_{i\bar{j}p}{}^{q}T_{qm\bar{l}} + \Omega_{i\bar{j}m}{}^{q}T_{pq\bar{l}} + \Omega_{i\bar{j}\bar{l}}{}^{\bar{r}}T_{pm\bar{r}} \\ =& \nabla_i\nabla_{\bar{j}}T_{pm\bar{l}} + g^{q\bar{r}}(\Omega_{i\bar{j}p\bar{r}}T_{qm\bar{l}} - \Omega_{i\bar{j}m\bar{r}}T_{qp\bar{l}} - \Omega_{i\bar{j}q\bar{l}}T_{pm\bar{r}}). \end{aligned}$$

As in (3.8) we rewrite $g^{m\bar{n}}g^{p\bar{s}}\nabla_i\nabla_{\bar{j}}T_{pm\bar{l}}T_{\bar{s}\bar{n}k}$ and use the fact $T_{\bar{s}\bar{n}k}$ is anti-symmetric in $\bar{n}$ and $\bar{s}$.

$$\begin{aligned} & g^{m\bar{n}}g^{p\bar{s}}\nabla_{\bar{j}}\nabla_i T_{pm\bar{l}}T_{\bar{s}\bar{n}k} \\ &= g^{m\bar{n}}g^{p\bar{s}}T_{\bar{s}\bar{n}k}\big(2(\nabla_m\Omega_{i\bar{j}p\bar{l}} + T^q_{mi}\Omega_{q\bar{j}p\bar{l}}) + g^{q\bar{r}}(\Omega_{i\bar{j}p\bar{r}}T_{qm\bar{l}} - \Omega_{i\bar{j}m\bar{r}}T_{qp\bar{l}} - \Omega_{i\bar{j}q\bar{l}}T_{pm\bar{r}})\big) \\ &= g^{m\bar{n}}g^{p\bar{s}}T_{\bar{s}\bar{n}k}\big(2(\nabla_m\Omega_{i\bar{j}p\bar{l}} + T^q_{mi}\Omega_{q\bar{j}p\bar{l}}) + g^{q\bar{r}}(2\Omega_{i\bar{j}p\bar{r}}T_{qm\bar{l}} - \Omega_{i\bar{j}q\bar{l}}T_{pm\bar{r}})\big) \\ &= 2g^{m\bar{n}}g^{p\bar{s}}T_{\bar{s}\bar{n}k}\big(\nabla_m\Omega_{i\bar{j}p\bar{l}} + T^q_{mi}\Omega_{q\bar{j}p\bar{l}} + g^{q\bar{r}}\Omega_{i\bar{j}p\bar{r}}T_{qm\bar{l}}\big) - 2g^{q\bar{r}}\Omega_{i\bar{j}q\bar{l}}Q_{k\bar{r}}. \end{aligned} \tag{3.9}$$

Expressions (3.8), (3.9), and $g^{m\bar{n}}g^{p\bar{s}}(\nabla_i T_{pm\bar{l}}\nabla_{\bar{j}}T_{\bar{s}\bar{n}k} + \nabla_{\bar{j}}T_{pm\bar{l}}\nabla_i T_{\bar{s}\bar{n}k})$ together give

$$\begin{aligned} 2\nabla_{\bar{j}}\nabla_i Q_{k\bar{l}} =& g^{m\bar{n}}g^{p\bar{s}}(\nabla_i T_{pm\bar{l}}\nabla_{\bar{j}}T_{\bar{s}\bar{n}k} + \nabla_{\bar{j}}T_{pm\bar{l}}\nabla_i T_{\bar{s}\bar{n}k}) - 2g^{q\bar{r}}\Omega_{i\bar{j}q\bar{l}}Q_{k\bar{r}} \\ &+ 2g^{m\bar{n}}g^{p\bar{s}}\big(T_{pm\bar{l}}\nabla_{\bar{n}}\Omega_{i\bar{j}k\bar{s}} + T_{\bar{s}\bar{n}k}\nabla_m\Omega_{i\bar{j}p\bar{l}} \\ &+ T_{pm\bar{l}}T^{\bar{r}}_{\bar{n}\bar{j}}\Omega_{i\bar{r}k\bar{s}} + T_{\bar{s}\bar{n}k}T^q_{mi}\Omega_{q\bar{j}p\bar{l}} + g^{q\bar{r}}T_{\bar{s}\bar{n}k}T_{qm\bar{l}}\Omega_{i\bar{j}p\bar{r}}\big). \end{aligned} \tag{3.10}$$

Step 3. Collect all terms together.

Before we sum up terms $\nabla_{\bar{j}}\nabla_i S^{(2)}_{k\bar{l}}$ and $\nabla_{\bar{j}}\nabla_i Q_{k\bar{l}}$, let us note that

$$\begin{aligned} g^{m\bar{n}}g^{p\bar{s}}\nabla_{\bar{j}}T_{pm\bar{l}}\nabla_i T_{\bar{s}\bar{n}k} &= g^{m\bar{n}}g^{p\bar{s}}\nabla_{\bar{j}}T_{pm\bar{l}}\nabla_i T_{\bar{s}\bar{n}k} = g^{m\bar{n}}g^{p\bar{s}}(\Omega_{m\bar{j}p\bar{l}} - \Omega_{p\bar{j}m\bar{l}})(\Omega_{i\bar{n}k\bar{s}} - \Omega_{i\bar{s}k\bar{n}}) \\ &= g^{m\bar{n}}g^{p\bar{s}}(\Omega_{m\bar{j}p\bar{l}}\Omega_{i\bar{n}k\bar{s}} + \Omega_{p\bar{j}m\bar{l}}\Omega_{i\bar{s}k\bar{n}} - \Omega_{m\bar{j}p\bar{l}}\Omega_{i\bar{s}k\bar{n}} - \Omega_{p\bar{j}m\bar{l}}\Omega_{i\bar{n}k\bar{s}}) \\ &= 2g^{m\bar{n}}g^{p\bar{s}}(\Omega_{m\bar{j}p\bar{l}}\Omega_{i\bar{n}k\bar{s}} - \Omega_{m\bar{j}p\bar{l}}\Omega_{i\bar{s}k\bar{n}}). \end{aligned}$$

Equations (3.7), (3.10) with the term $-g^{p\bar{s}}(S^{(2)}_{p\bar{l}}+Q_{p\bar{l}})\Omega_{i\bar{j}k\bar{s}}$ give

$$\begin{aligned}\frac{d}{dt}\Omega_{i\bar{j}k\bar{l}} &= \nabla_{\bar{j}}\nabla_i\big(S^{(2)}_{k\bar{l}}+Q_{k\bar{l}}\big)-g^{p\bar{s}}(S^{(2)}_{p\bar{l}}+Q_{p\bar{l}})\Omega_{i\bar{j}k\bar{s}}\\ &= g^{m\bar{n}}\Big(\nabla_m\nabla_{\bar{n}}\Omega_{i\bar{j}k\bar{l}}+T^{\bar{r}}_{\bar{n}\bar{j}}\nabla_m\Omega_{i\bar{r}k\bar{l}}+T^q_{mi}\nabla_{\bar{n}}\Omega_{q\bar{j}k\bar{l}}\\ &+T^q_{mi}T^{\bar{r}}_{\bar{n}\bar{j}}\Omega_{q\bar{r}k\bar{l}}+g^{p\bar{s}}(T_{pm\bar{l}}\nabla_{\bar{n}}\Omega_{i\bar{j}k\bar{s}}+T_{\bar{s}\bar{n}k}\nabla_m\Omega_{i\bar{j}p\bar{l}}\\ &+T_{pm\bar{l}}T^{\bar{r}}_{\bar{n}\bar{j}}\Omega_{i\bar{r}k\bar{s}}+T_{\bar{s}\bar{n}k}T^q_{mi}\Omega_{q\bar{j}p\bar{l}}+g^{q\bar{r}}T_{\bar{s}\bar{n}k}T_{qm\bar{l}}\Omega_{i\bar{j}p\bar{r}})\Big)\\ &+g^{m\bar{n}}g^{p\bar{s}}(\Omega_{i\bar{j}m\bar{s}}\Omega_{p\bar{n}k\bar{l}}+\Omega_{m\bar{j}k\bar{s}}\Omega_{i\bar{n}p\bar{l}}-\Omega_{m\bar{j}p\bar{l}}\Omega_{i\bar{s}k\bar{n}}+\frac{1}{2}\nabla_iT_{pm\bar{l}}\nabla_{\bar{j}}T_{\bar{s}\bar{n}k})\\ &-g^{p\bar{s}}(S^{(2)}_{p\bar{l}}+Q_{p\bar{l}})\Omega_{i\bar{j}k\bar{s}}-g^{p\bar{s}}S^{(2)}_{p\bar{j}}\Omega_{i\bar{s}k\bar{l}}-g^{p\bar{s}}Q_{k\bar{s}}\Omega_{i\bar{j}p\bar{l}}.\end{aligned} \tag{3.11}$$

□

The evolution equation for Ω in the form (3.11) contains many different terms and is difficult to analyze directly. However, it turns out that it could be significantly simplified and reinterpreted in invariant coordinate-free terms, after applying the following tricks:

1. Rise the last two indices of $\Omega_{i\bar{j}k\bar{l}}$ via g and consider the Chern curvature tensor as a section of $\mathrm{Sym}^{1,1}(\mathrm{End}(T^{1,0}M))$;
2. Identify within equation (3.11) the Laplacian $\big(\Delta_D\Omega\big)_{i\bar{j}}^{\;\;\bar{l}k}$ of a wisely chosen connection D on $\mathrm{Sym}^{1,1}(\mathrm{End}(T^{1,0}M))$ (see Definition 2.6).

Connection D of part 2 comes from a connection ∇^T on TM, which we refer to as the *torsion-twisted* connection, and discuss in detail in the next section.

3.3 Torsion-twisted Connection

Let (M,g,J) be an Hermitian manifold with the Chern connection ∇. In this section we define a *torsion-twisted* connection ∇^T on the tangent bundle TM and study its properties.

3.3.1 Motivation

The main motivation for introducing the torsion-twisted connection is purely algebraic. We are aiming at finding a connection on the space of curvature tensors $\Lambda^{1,1}(M)\otimes\Lambda^{1,1}(M)$, such that its Laplacian absorbs all the first order derivatives of Ω in the evolution equation for Ω under the HCF (3.11).

Using the formula for the Chern Laplacian (2.3) $\Delta = \Delta_\nabla$, we note that the evolution term of equation (3.11) is of a form

$$\Delta\Omega_{i\bar{j}k\bar{l}} + g^{m\bar{n}}\big(T^{\bar{r}}_{\bar{n}\bar{j}}\nabla_m\Omega_{i\bar{r}k\bar{l}} + T^{q}_{mi}\nabla_{\bar{n}}\Omega_{q\bar{j}k\bar{l}} - g^{p\bar{s}}(T_{mp\bar{l}}\nabla_{\bar{n}}\Omega_{i\bar{j}k\bar{s}} + T_{\bar{n}\bar{s}k}\nabla_m\Omega_{i\bar{j}p\bar{l}})\big) + F_0, \tag{3.12}$$

where the term F_0 does not contain any derivatives of Ω. Therefore, by Lemma 2.7, if we set

$$A \in \Lambda^1\big(M, \mathrm{End}(\Lambda^{1,1}(M) \otimes \Lambda^{1,1}(M))\big)$$

to be a 1-form acting on $\Omega \in \Lambda^{1,1}(M) \otimes \Lambda^{1,1}(M)$ as

$$\begin{aligned} A_m\Omega_{i\bar{j}k\bar{l}} &:= T^{q}_{mi}\Omega_{q\bar{j}k\bar{l}} - g^{p\bar{s}}T_{mp\bar{l}}\Omega_{i\bar{j}k\bar{s}}, \\ A_{\bar{n}}\Omega_{i\bar{j}k\bar{l}} &:= T^{\bar{r}}_{\bar{n}\bar{j}}\Omega_{i\bar{r}k\bar{l}} - g^{p\bar{s}}T_{\bar{n}\bar{s}k}\Omega_{i\bar{j}p\bar{l}}, \end{aligned} \tag{3.13}$$

then the Laplacian $\Delta_{\nabla+A}$ of the connection $\nabla + A$ on $\Lambda^{1,1}(M) \otimes \Lambda^{1,1}(M)$ will pick up all the derivatives in equation (3.12), and the evolution term will take form

$$\Delta_{\nabla+A}\Omega_{i\bar{j}k\bar{l}} + \widetilde{F_0}$$

for some new zero-order term $\widetilde{F_0}$.

Equation (3.13) suggests that we should consider the following two connections on TM (see [Ust16, Def. 3.5], [Ust17b, §1]).

Definition 3.15 (Torsion-twisted connections)**.** Define $\nabla^T, \nabla^{T^\star}$ to be two *torsion-twisted* connections on TM given by the identities

$$\begin{aligned} \nabla^T_X Y &= \nabla_X Y - T(X, Y), \\ \nabla^{T^\star}_X Y &= \nabla_X Y + g(Y, T(X,\,\cdot\,))^\star, \end{aligned}$$

where $X, Y, Z \in TM$, ∇ is the Chern connection, and $\star\colon T^*M \to TM$ is the isomorphism induced by g. Equivalently, in the coordinates, for a vector field $\xi = \xi^p\frac{\partial}{\partial z^p}$ one has

$$\begin{aligned} &\nabla^T_i\xi^p = \nabla_i\xi^p - T^p_{ij}\xi^j, \quad \nabla^T_{\bar{j}}\xi^p = \nabla_{\bar{j}}\xi^p, \\ &\nabla^{T^\star}_i\xi^p = \nabla_i\xi^p, \quad \nabla^{T^\star}_{\bar{j}}\xi^p = \nabla_{\bar{j}}\xi^p + g^{p\bar{s}}T_{\bar{j}\bar{s}k}\xi^k. \end{aligned}$$

As usual, we extend ∇^T and $\nabla^{T^\star}$ to the connections on all vector bundles associated with $T^{1,0}M$ via the Leibniz rule.

Remark 3.16. Both connections ∇^T and $\nabla^{T^\star}$ do not preserve g, unless (M, g, J) is Kähler. However, it is easy to check that g considered as a section of $T^*M \otimes T^*M$ is parallel with respect to the connection $\nabla^T \otimes \mathrm{id} + \mathrm{id} \otimes \nabla^{T^\star}$, i.e., for any vector fields X, Y, Z, we have

$$X \cdot g(Y, Z) = g(\nabla^T_X Y, Z) + g(Y, \nabla^{T^\star}_X Z).$$

In other words, $\nabla^{T^\star}$ is *dual conjugate* to ∇^T via g.

With the use of Definition 3.15, connection $\nabla + A$ on $\Lambda^{1,1}(M) \otimes \Lambda^{1,1}(M)$ can be interpreted as the connection acting as ∇^T on the first two arguments and acting as $\nabla^{T^\star}$ on the last two arguments. Taking into account Remark 3.16, we observe that

$$((\nabla + A)\Omega)_{i\bar{j}k\bar{l}} = g_{k\bar{s}} g_{p\bar{l}} (\nabla^T \Omega)_{i\bar{j}}{}^{\bar{s}p},$$

and

$$(\Delta_{\nabla + A}\Omega)_{i\bar{j}k\bar{l}} = g_{k\bar{s}} g_{p\bar{l}} (\Delta_{\nabla^T} \Omega)_{i\bar{j}}{}^{\bar{s}p}.$$

These identities indicate that, from the point of view of the HCF, it is more natural to consider the Chern curvature Ω as a section of $\mathrm{Sym}^{1,1}(\mathrm{End}(T^{1,0}M)) \subset \mathrm{End}(T^{1,0}M) \otimes \mathrm{End}(T^{0,1}M)$, by rising the last two indices of $\Omega_{i\bar{j}k\bar{l}}$. Then the connection $\nabla + A$ on $\Lambda^{1,1}(M) \otimes \Lambda^{1,1}(M)$ defined via equation (3.13) pulls back to the connection ∇^T on $\mathrm{End}(T^{1,0}M) \otimes \mathrm{End}(T^{0,1}M)$. Below we deal only with the connection ∇^T and refer to it as *the* torsion-twisted connection.

3.3.2 Curvature of the Torsion-twisted Connection

In this section we provide some basic properties of the torsion-twisted connection ∇^T, and compute its curvature.

First, we note that ∇^T is compatible with the holomorphic structure on $T^{1,0}M$, i.e., its (0,1) type part coincides with $\bar{\partial}$. Indeed, since the (1,1) type part of the torsion tensor of a Chern connection vanishes, we have

$$\nabla^T_{\bar{\eta}} \xi = \nabla_{\bar{\eta}} \xi - T(\xi, \bar{\eta}) = \nabla_{\bar{\eta}} \xi = \bar{\partial}_{\bar{\eta}} \xi.$$

In particular, any ∇^T-parallel section of a holomorphic vector bundle associated to $T^{1,0}M$ is automatically holomorphic. Let Ω_{∇^T} denote the curvature of the torsion-twisted connection.

Proposition 3.17 (Torsion-twisted curvature)**.** *Ω_{∇^T} is of the type $(2,0)+(1,1)$ and has components*

$$\Omega_{\nabla^T}(\xi,\overline{\eta})\zeta = \Omega(\zeta,\overline{\eta})\xi,$$
$$\Omega_{\nabla^T}(\xi,\eta)\zeta = \nabla_\zeta T(\xi,\eta),$$

where $\xi,\eta,\zeta \in T^{1,0}M$.

Proof. The torsion-twisted connection is holomorphic, therefore the type (0,2) part of curvature vanishes: ${\Omega_{\nabla^T}}^{(0,2)} = \overline{\partial}^2 = 0$.

For a vector $w \in T_{\mathbb{C}}M$, define an endomorphism

$$T_w \colon T_{\mathbb{C}}M \to T_{\mathbb{C}}M, \quad v \mapsto T(w,v).$$

Since the (1,1) part of T vanishes, for $\xi \in T^{1,0}M$, the endomorphism T_ξ is zero on $T^{0,1}M$ and maps $T^{1,0}M$ into $T^{1,0}M$. Now, assume that ξ,η,ζ are local coordinate holomorphic vector fields. Then for the type (1,1) part we have

$$\begin{aligned}\Omega_{\nabla^T}(\xi,\overline{\eta})\zeta &= [\nabla_\xi - T_\xi, \nabla_{\overline{\eta}} - T_{\overline{\eta}}]\zeta = \big(\Omega(\xi,\overline{\eta}) - [\nabla_\xi, T_{\overline{\eta}}] + [\nabla_{\overline{\eta}}, T_\xi] + [T_\xi, T_{\overline{\eta}}]\big)\zeta \\ &= \Omega(\xi,\overline{\eta})\zeta + \nabla_{\overline{\eta}}(T_\xi\zeta) = \Omega(\xi,\overline{\eta})\zeta + (\nabla_{\overline{\eta}}T)(\xi,\zeta),\end{aligned}$$

where we use the facts that vector fields ξ and $\overline{\eta}$ commute, and that $\nabla_{\overline{\eta}}$ annihilates ξ and ζ. By the first Bianchi identity, the final expression equals $\Omega(\zeta,\overline{\eta})\xi$.

Similarly, we compute the (2,0) part of the curvature Ω_{∇^T}, using the fact that the (2,0) part of the Chern curvature vanishes.

$$\begin{aligned}\Omega_{\nabla^T}(\xi,\eta)\zeta &= [\nabla_\xi - T_\xi, \nabla_\eta - T_\eta]\zeta = \big([\nabla_\xi,\nabla_\eta] + [\nabla_\eta, T_\xi] - [\nabla_\xi, T_\eta] + [T_\xi, T_\eta]\big)\zeta \\ &= \nabla_\eta(T(\xi,\zeta)) - T(\xi,\nabla_\eta\zeta) - \nabla_\xi(T(\eta,\zeta)) + T(\eta,\nabla_\xi\zeta) + T(\xi,T(\eta,\zeta)) - T(\eta,T(\xi,\zeta)) \\ &= \nabla_\xi T(\zeta,\eta) + \nabla_\eta T(\xi,\zeta) + T(\nabla_\eta\xi - \nabla_\xi\eta,\zeta) + T(\xi,T(\eta,\zeta)) - T(\eta,T(\xi,\zeta)) \\ &= \nabla_\xi T(\zeta,\eta) + \nabla_\eta T(\xi,\zeta) + T(T(\eta,\xi),\zeta) + T(T(\zeta,\eta),\xi) + T(T(\xi,\zeta),\eta) = \nabla_\zeta T(\xi,\eta),\end{aligned}$$

where in the last identity we use the (3,0) type part of the first Bianchi identity. □

Remark 3.18. By the previous proposition, endomorphism $u \in \mathrm{End}(T^{1,0}M)$ lies in the kernel of the bilinear form $\langle \Omega, \cdot \otimes \overline{\cdot} \rangle_{\mathrm{tr}}$ if and only if $\mathrm{tr}(u \circ \Omega_{\nabla^T}(\xi,\overline{\eta})) = 0$ for any $\xi,\eta \in T^{1,0}M$.

Corollary 3.19 (Torsion-twisted Ricci form)**.** *The Ricci form of the torsion twisted connection*

$$\rho_{\nabla^T} := \sqrt{-1}\mathrm{tr}_{\mathrm{End}(T^{1,0}M)}\Omega_{\nabla^T} \in \Lambda^2(M,\mathbb{C})$$

is of type $(2,0)+(1,1)$ *and has components*

$$(\rho_{\nabla^T})^{(1,1)} = \sqrt{-1}\Omega_{k\bar{j}i}{}^{k} dz^i \wedge d\overline{z^j},$$
$$(\rho_{\nabla^T})^{(2,0)} = \frac{\sqrt{-1}}{2}\nabla_k T^k_{ij} dz^i \wedge dz^j.$$

Definition 3.20 (Holonomy group of ∇^T)**.** Define $\mathrm{Hol}^0_{\nabla^T} \subset GL(T^{1,0}M)$ to be the restricted holonomy group of ∇^T and denote by $\mathfrak{hol}^{\mathbb{R}}_{\nabla_T} \subset \mathrm{End}(T^{1,0}M)$ its Lie algebra. Let

$$\mathfrak{hol}_{\nabla^T} := \mathrm{span}_{\mathbb{C}}(\mathfrak{hol}^{\mathbb{R}}_{\nabla_T} \subset \mathrm{End}(T^{1,0}M))$$

be the $\mathbb{C}$-span of $\mathfrak{hol}^{\mathbb{R}}_{\nabla_T}$ in $\mathrm{End}(T^{1,0}M)$.

3.3.3 Reinterpreting the Evolution Equations for the Curvature

It turns out that with the use of the torsion-twisted connection, the evolution equation for Ω under the HCF takes particularly simple and clear form. To deduce this expression, we rewrite (3.11) as the evolution equation for $\Omega = \Omega_{i\bar{j}}{}^{\bar{l}k}$. By (3.11), we have

$$\begin{aligned}\frac{d}{dt}\Omega_{i\bar{j}k\bar{l}} = g^{m\bar{n}}\Big(&\nabla_m\nabla_{\bar{n}}\Omega_{i\bar{j}k\bar{l}} + T^{\bar{r}}_{\bar{n}\bar{j}}\nabla_m\Omega_{i\bar{r}k\bar{l}} + T^q_{mi}\nabla_{\bar{n}}\Omega_{q\bar{j}k\bar{l}}\\ &+ T^q_{mi}T^{\bar{r}}_{\bar{n}\bar{j}}\Omega_{q\bar{r}k\bar{l}} + g^{p\bar{s}}(T_{pm\bar{l}}\nabla_{\bar{n}}\Omega_{i\bar{j}k\bar{s}} + T_{\bar{s}\bar{n}k}\nabla_m\Omega_{i\bar{j}p\bar{l}}\\ &+ T_{pm\bar{l}}T^{\bar{r}}_{\bar{n}\bar{j}}\Omega_{i\bar{r}k\bar{s}} + T_{\bar{s}\bar{n}k}T^q_{mi}\Omega_{q\bar{j}p\bar{l}} + g^{q\bar{r}}T_{\bar{s}\bar{n}k}T_{qm\bar{l}}\Omega_{i\bar{j}p\bar{r}})\Big)\\ &+ g^{m\bar{n}}g^{p\bar{s}}(\Omega_{i\bar{j}m\bar{s}}\Omega_{p\bar{n}k\bar{l}} + \Omega_{m\bar{j}k\bar{s}}\Omega_{i\bar{n}p\bar{l}} - \Omega_{m\bar{j}p\bar{l}}\Omega_{i\bar{s}k\bar{n}} + \frac{1}{2}\nabla_i T_{pm\bar{l}}\nabla_{\bar{j}}T_{\bar{s}\bar{n}k})\\ &- g^{p\bar{s}}(S^{(2)}_{p\bar{l}} + Q_{p\bar{l}})\Omega_{i\bar{j}k\bar{s}} - g^{p\bar{s}}S^{(2)}_{p\bar{j}}\Omega_{i\bar{s}k\bar{l}} - g^{p\bar{s}}Q_{k\bar{s}}\Omega_{i\bar{j}p\bar{l}}.\end{aligned}$$

Therefore for $\Omega_{i\bar{j}}{}^{\bar{l}k} = \Omega_{i\bar{j}a\bar{b}}g^{a\bar{l}}g^{k\bar{b}}$ we have

$$\begin{aligned}\frac{d}{dt}\Omega_{i\bar{j}}{}^{\bar{l}k} = g^{m\bar{n}}\Big(&\nabla_m\nabla_{\bar{n}}\Omega_{i\bar{j}}{}^{\bar{l}k} + T^{\bar{r}}_{\bar{n}\bar{j}}\nabla_m\Omega_{i\bar{r}}{}^{\bar{l}k} + T^q_{mi}\nabla_{\bar{n}}\Omega_{q\bar{j}}{}^{\bar{l}k}\\ &+ T^q_{mi}T^{\bar{r}}_{\bar{n}\bar{j}}\Omega_{q\bar{r}}{}^{\bar{l}k} + T^k_{pm}\nabla_{\bar{n}}\Omega_{i\bar{j}}{}^{\bar{l}p} + T^{\bar{l}}_{\bar{s}\bar{n}}\nabla_m\Omega_{i\bar{j}}{}^{\bar{s}k}\\ &+ T^k_{pm}T^{\bar{r}}_{\bar{n}\bar{j}}\Omega_{i\bar{r}}{}^{\bar{l}p} + T^{\bar{l}}_{\bar{s}\bar{n}}T^q_{mi}\Omega_{q\bar{j}}{}^{\bar{s}k} + T^{\bar{l}}_{\bar{s}\bar{n}}T^k_{qm}\Omega_{i\bar{j}}{}^{\bar{s}q}\Big)\\ &+ \Omega_{i\bar{j}}{}^{\bar{n}p}\Omega_{p\bar{n}}{}^{\bar{l}k} + g_{p\bar{s}}g^{m\bar{n}}\Omega_{m\bar{j}}{}^{\bar{l}p}\Omega_{i\bar{n}}{}^{\bar{s}k} - \Omega_{m\bar{j}}{}^{\bar{s}k}\Omega_{i\bar{s}}{}^{\bar{l}m} + \frac{1}{2}g^{m\bar{n}}g^{p\bar{s}}\nabla_i T^k_{pm}\nabla_{\bar{j}}T^{\bar{l}}_{\bar{s}\bar{n}}\\ &- g^{p\bar{s}}S^{(2)}_{p\bar{j}}\Omega_{i\bar{s}}{}^{\bar{l}k} + g^{p\bar{l}}S^{(2)}_{p\bar{s}}\Omega_{i\bar{j}}{}^{\bar{s}k}.\end{aligned} \tag{3.14}$$

Most of the terms in (3.14) come from the torsion-twisted Laplacian:

Lemma 3.21. *The torsion-twisted Laplacian of the Chern curvature tensor* $\Omega_{i\bar{j}}{}^{\bar{l}k}$ *equals*

$$\begin{aligned}\Delta_{\nabla^T}\Omega_{i\bar{j}}{}^{\bar{l}k} = g^{m\bar{n}}\Big(&\frac{1}{2}\nabla_m\nabla_{\bar{n}}\Omega_{i\bar{j}}{}^{\bar{l}k} + \frac{1}{2}\nabla_{\bar{n}}\nabla_m\Omega_{i\bar{j}}{}^{\bar{l}k}\\ &+ T^{\bar{r}}_{\bar{n}\bar{j}}\nabla_m\Omega_{i\bar{r}}{}^{\bar{l}k} + T^q_{mi}\nabla_{\bar{n}}\Omega_{q\bar{j}}{}^{\bar{l}k}\\ &+ T^q_{mi}T^{\bar{r}}_{\bar{n}\bar{j}}\Omega_{q\bar{r}}{}^{\bar{l}k} + T^k_{pm}\nabla_{\bar{n}}\Omega_{i\bar{j}}{}^{\bar{l}p} + T^{\bar{l}}_{\bar{s}\bar{n}}\nabla_m\Omega_{i\bar{j}}{}^{\bar{s}k}\\ &+ T^k_{pm}T^{\bar{r}}_{\bar{n}\bar{j}}\Omega_{i\bar{r}}{}^{\bar{l}p} + T^{\bar{l}}_{\bar{s}\bar{n}}T^q_{mi}\Omega_{q\bar{j}}{}^{\bar{s}k} + T^{\bar{l}}_{\bar{s}\bar{n}}T^k_{qm}\Omega_{i\bar{j}}{}^{\bar{s}q}\\ &+ \frac{1}{2}(\nabla_{\bar{n}}T^p_{mi}\Omega_{p\bar{j}}{}^{\bar{l}k} - \nabla_{\bar{n}}T^k_{mp}\Omega_{i\bar{j}}{}^{\bar{l}p} + \nabla_m T^{\bar{s}}_{\bar{n}\bar{j}}\Omega_{i\bar{s}}{}^{\bar{l}k} - \nabla_m T^{\bar{l}}_{\bar{n}\bar{s}}\Omega_{i\bar{j}}{}^{\bar{s}k})\Big).\end{aligned}$$

Proof. Applying Lemma 2.7 to $\nabla^T = \nabla - T$, we obtain

$$\Delta_{\nabla^T}\Omega = (\Delta - 2\mathrm{tr}_g(T\circ\nabla) + \mathrm{tr}_g(T\circ T) - \operatorname{div} T)\Omega,$$

where for $w \in T_{\mathbb{C}}M$ endomorphism $T_w\colon T_{\mathbb{C}}M \to T_{\mathbb{C}}M, v \mapsto T(w,v)$ is defined on $T_{\mathbb{C}}M$ as in Proposition 3.17 and is extended to the space of curvature type tensors $\mathrm{End}(T^{1,0}M)\otimes\mathrm{End}(T^{0,1}M)$ as a derivation. Then we have

$$\begin{aligned}\Delta\Omega_{i\bar{j}}{}^{\bar{l}k} &= g^{m\bar{n}}\frac{1}{2}(\nabla_m\nabla_{\bar{n}}\Omega_{i\bar{j}}{}^{\bar{l}k} + \nabla_{\bar{n}}\nabla_m\Omega_{i\bar{j}}{}^{\bar{l}k}),\\ -2\mathrm{tr}_g(T\circ\nabla)\Omega_{i\bar{j}}{}^{\bar{l}k} &= g^{m\bar{n}}(T^{\bar{r}}_{\bar{n}\bar{j}}\nabla_m\Omega_{i\bar{r}}{}^{\bar{l}k} + T^q_{mi}\nabla_{\bar{n}}\Omega_{q\bar{j}}{}^{\bar{l}k} - T^k_{mp}\nabla_{\bar{n}}\Omega_{i\bar{j}}{}^{\bar{l}p} - T^{\bar{l}}_{\bar{n}\bar{s}}\nabla_m\Omega_{i\bar{j}}{}^{\bar{s}k}),\\ \mathrm{tr}_g(T\circ T)\Omega_{i\bar{j}}{}^{\bar{l}k} &= g^{m\bar{n}}(T^q_{mi}T^{\bar{r}}_{\bar{n}\bar{j}}\Omega_{q\bar{r}}{}^{\bar{l}k} - T^k_{mp}T^{\bar{r}}_{\bar{n}\bar{j}}\Omega_{i\bar{r}}{}^{\bar{l}p} - T^{\bar{l}}_{\bar{n}\bar{s}}T^q_{mi}\Omega_{q\bar{j}}{}^{\bar{s}k} + T^{\bar{l}}_{\bar{n}\bar{s}}T^k_{mq}\Omega_{i\bar{j}}{}^{\bar{s}q}),\\ -(\operatorname{div} T)\Omega_{i\bar{j}}{}^{\bar{l}k} &= \frac{1}{2}g^{m\bar{n}}\Big(\nabla_{\bar{n}}T^p_{mi}\Omega_{p\bar{j}}{}^{\bar{l}k} - \nabla_{\bar{n}}T^k_{mp}\Omega_{i\bar{j}}{}^{\bar{l}p} + \nabla_m T^{\bar{s}}_{\bar{n}\bar{j}}\Omega_{i\bar{s}}{}^{\bar{l}k} - \nabla_m T^{\bar{l}}_{\bar{n}\bar{s}}\Omega_{i\bar{j}}{}^{\bar{s}k}\Big).\end{aligned}$$

Collecting the four summands together, we get the stated identity. ☐

With the use of the Laplacian Δ_{∇^T} of ∇^T and its curvature Ω_{∇^T} (Proposition 3.17), one gets a simple evolution equation for the Chern curvature $\Omega_{i\bar{j}}{}^{\bar{l}k}$ under the HCF. Before we formulate it, let us introduce a final bit of notation. The curvature of the torsion-twisted connection Ω_{∇^T} is a section of $\Lambda^2(M,\mathbb{C})\otimes\mathrm{End}(T^{1,0}M)$. Using the metric contraction on $\Lambda^2(M,\mathbb{C})\otimes\overline{\Lambda^2(M,\mathbb{C})}$, we can take the trace of

$$\Omega_{\nabla^T}\otimes\overline{\Omega_{\nabla^T}} \in \Lambda^2(M,\mathbb{C})\otimes\mathrm{End}(T^{1,0}M)\otimes\overline{\Lambda^2(M,\mathbb{C})}\otimes\overline{\mathrm{End}(T^{1,0}M)}$$

and define a tensor

$$\mathrm{tr}_{\Lambda^2(M)}(\Omega_{\nabla^T}\otimes\overline{\Omega_{\nabla^T}}) \in \mathrm{End}(T^{1,0}M)\otimes\mathrm{End}(T^{0,1}M).$$

An important property of the contraction $\mathrm{tr}_{\Lambda^2(M)}(\Omega_{\nabla^T}\otimes\overline{\Omega_{\nabla^T}})$ is that it is always a semipositive element of $\mathrm{Sym}^{1,1}(\mathrm{End}(T^{1,0}M)) \subset \mathrm{End}(T^{1,0}M) \otimes \mathrm{End}(T^{0,1}M)$.

At this point, we are ready to formulate the evolution equation for Ω in invariant terms. This equation is the key ingredient, required for formulating and proving the maximum principles for Ω.

Proposition 3.22. *Under the HCF, $\Omega \in \mathrm{Sym}^{1,1}(\mathrm{End}(T^{1,0}M)) \subset \mathrm{End}(T^{1,0}M) \otimes \mathrm{End}(T^{0,1}M)$ evolves by equation*

$$\boxed{\frac{d}{dt}\Omega = \Delta_{\nabla^T}\Omega + \Omega^{\#} + \mathrm{tr}_{\Lambda^2(M)}(\Omega_{\nabla^T} \otimes \overline{\Omega}_{\nabla^T}) + \mathrm{ad}_u\Omega} \tag{3.15}$$

where $u \in \mathrm{End}(T^{1,0}M)$ is given by $u_i^j = -\frac{1}{2}\Omega_{i\bar{n}}{}^{\bar{n}j}$.

Proof. Combining, equation (3.14) and Lemma 3.21, we get

$$\begin{aligned}
\frac{d}{dt}\Omega_{i\bar{j}}{}^{\bar{l}k} &= \Delta_{\nabla^T}\Omega_{i\bar{j}}{}^{\bar{l}k} + \frac{1}{2}g^{m\bar{n}}[\nabla_m, \nabla_{\bar{n}}]\Omega_{i\bar{j}}{}^{\bar{l}k} \\
&+ \Omega_{i\bar{j}}{}^{\bar{n}p}\Omega_{p\bar{n}}{}^{\bar{l}k} + g_{p\bar{s}}g^{m\bar{n}}\Omega_{m\bar{j}}{}^{\bar{l}p}\Omega_{i\bar{n}}{}^{\bar{s}k} - \Omega_{m\bar{j}}{}^{\bar{s}k}\Omega_{i\bar{s}}{}^{\bar{l}m} \\
&+ \frac{1}{2}g^{m\bar{n}}g^{p\bar{s}}\nabla_i T^k_{pm}\nabla_{\bar{j}}T^{\bar{l}}_{\bar{s}\bar{n}} - g^{p\bar{s}}S^{(2)}_{p\bar{j}}\Omega_{i\bar{s}}{}^{\bar{l}k} + g^{p\bar{l}}S^{(2)}_{p\bar{s}}\Omega_{i\bar{j}}{}^{\bar{s}k} \\
&- \frac{1}{2}(\nabla_{\bar{n}}T^p_{mi}\Omega_{p\bar{j}}{}^{\bar{l}k} - \nabla_{\bar{n}}T^k_{mp}\Omega_{i\bar{j}}{}^{\bar{l}p} + \nabla_m T^{\bar{s}}_{\bar{n}\bar{j}}\Omega_{i\bar{s}}{}^{\bar{l}k} - \nabla_m T^{\bar{l}}_{\bar{n}\bar{s}}\Omega_{i\bar{j}}{}^{\bar{s}k}).
\end{aligned}$$

By Proposition 2.28, $(\Omega^{\#})_{i\bar{j}}{}^{\bar{l}k} = \Omega_{i\bar{j}}{}^{\bar{n}p}\Omega_{p\bar{n}}{}^{\bar{l}k} - \Omega_{m\bar{j}}{}^{\bar{s}k}\Omega_{i\bar{s}}{}^{\bar{l}m}$.

Next, by a direct application of the first Bianchi identity,

$$\begin{aligned}
&\frac{1}{2}g^{m\bar{n}}[\nabla_m, \nabla_{\bar{n}}]\Omega_{i\bar{j}}{}^{\bar{l}k} - \frac{1}{2}(\nabla_{\bar{n}}T^p_{mi}\Omega_{p\bar{j}}{}^{\bar{l}k} - \nabla_{\bar{n}}T^k_{mp}\Omega_{i\bar{j}}{}^{\bar{l}p} \\
&+ \nabla_m T^{\bar{s}}_{\bar{n}\bar{j}}\Omega_{i\bar{s}}{}^{\bar{l}k} - \nabla_m T^{\bar{l}}_{\bar{n}\bar{s}}\Omega_{i\bar{j}}{}^{\bar{s}k}) - g^{p\bar{s}}S^{(2)}_{p\bar{j}}\Omega_{i\bar{s}}{}^{\bar{l}k} + g^{p\bar{l}}S^{(2)}_{p\bar{s}}\Omega_{i\bar{j}}{}^{\bar{s}k} \\
&= \frac{1}{2}\left(-\Omega_{i\bar{n}}{}^{\bar{n}p}\Omega_{p\bar{j}}{}^{\bar{l}k} - \Omega_{m\bar{j}}{}^{\bar{s}m}\Omega_{i\bar{s}}{}^{\bar{l}k} + \Omega_{m\bar{s}}{}^{\bar{l}m}\Omega_{i\bar{j}}{}^{\bar{s}k} + \Omega_{p\bar{n}}{}^{\bar{n}k}\Omega_{i\bar{j}}{}^{\bar{l}p}\right) \\
&= (\mathrm{ad}_u\Omega)_{i\bar{j}}{}^{\bar{l}k}.
\end{aligned}$$

for $u_i^p = -\frac{1}{2}\Omega_{i\bar{n}}{}^{\bar{n}p}$.

Finally, the formulas for Ω_{∇^T} (Proposition (3.17)) imply that the terms

$$g_{p\bar{s}}g^{m\bar{n}}\Omega_{m\bar{j}}{}^{\bar{l}p}\Omega_{i\bar{n}}{}^{\bar{s}k} \text{ and } \frac{1}{2}g^{m\bar{n}}g^{p\bar{s}}\nabla_i T^k_{pm}\nabla_{\bar{j}}T^{\bar{l}}_{\bar{s}\bar{n}}$$

add up to $\mathrm{tr}_{\Lambda^2(M)}(\Omega_{\nabla^T} \otimes \overline{\Omega_{\nabla^T}})$. □

Remark 3.23. Equation (3.15) is a direct generalization of the evolution of the Riemannian curvature under the Ricci/Kähler-Ricci flow. If the underlying Hermitian manifold (M, g, J) is Kähler, the torsion-twisted curvature Ω_{∇^T} coincides with Ω, the Laplacian Δ_{∇^T} is the Riemannian Laplacian, and $-2u$ is the Ricci operator, so (3.15) takes a familiar form

$$\frac{d}{dt}\Omega = \Delta\Omega + \Omega^{\#} + \Omega^2 + \mathrm{ad}_u\Omega,$$

where one can get rid of the last term $\mathrm{ad}_u\Omega$ by using the *moving frame trick*, [Ham86].

Chapter 4

Preservation of Curvature Positivity along the HCF

This is the core chapter of the thesis. We start with reviewing Hamilton's maximum principle for tensors and reprove it in a slightly more general setup of non-Riemannian connections. Next, for every $GL(T^{1,0}M)$-invariant subset $S \subset \mathrm{End}(T^{1,0}M)$ and a nice invariant function $F \colon \mathrm{End}(T^{1,0}M) \to \mathbb{R}$, we define a convex subset $C(S, F) \subset \mathrm{Sym}^{1,1}(\mathrm{End}(T^{1,0}M))$ of Chern curvature tensors. Using Hamilton's maximum principle, we prove that the HCF preserves all sets $C(S, F)$. Furthermore, following the framework of Brendle and Schoen, we prove a version of the strong parabolic maximum principle for $C(S, F)$. Finally, we provide a list of concrete examples of curvature positivity conditions, preserved by the HCF, which includes dual-Nakano semipositivity, Griffiths semipositivity, semipositivity ot the second scalar curvature. The presentation of this chapter follows [Ust17b].

4.1 Refined Hamilton's Maximum Principle

In this section we prove a modification of Hamilton's maximum principle for tensors. In special cases it was used in [Ham82] to control Ricci positivity in dimension 3 and in [Ban84] to prove that semipositivity of the bisectional holomorphic curvature is preserved. After a general formulation by Hamilton in [Ham86], maximum principle for tensor was used as a key step in many applications of the Ricci flow [Ham86; CC86; Mok88; Cao92; BW08; BS08; BS09; Gu09; GZ10; Wil13]. Let us start with recalling this principle in its original form.

Let M be a closed smooth manifold with a Riemannian metric g and let $\mathcal{E} \to M$ be a vector bundle equipped with a metric h and a metric connection $\nabla^{\mathcal{E}}$. With the use of the Levi-Civita connection we extend $\nabla^{\mathcal{E}}$ to a connection on $\Lambda^1(M) \otimes \mathcal{E}$. Denote by $\Delta^{\mathcal{E}} \colon \mathcal{C}^\infty(M, \mathcal{E}) \to \mathcal{C}^\infty(M, \mathcal{E})$ the Laplacian of $\nabla^{\mathcal{E}}$ (see Definition 2.6):

$$\Delta^{\mathcal{E}} s = \operatorname{tr}_g(\nabla^{\mathcal{E}} \circ \nabla^{\mathcal{E}}(s)).$$

Let $\varphi \colon \mathcal{E} \to T^{\mathrm{vert}}\mathcal{E} \simeq \mathcal{E}$ be a smooth vertical vector field on the total space of $\mathcal{E}$. We are interested in the short-time behavior of the solutions to a nonlinear parabolic equation for $f \in C^\infty(M \times [0, \epsilon), \mathcal{E})$

$$\frac{df}{dt} = \Delta^{\mathcal{E}} f + \varphi(f), \tag{4.1}$$

where the background data $(h, g, \nabla^{\mathcal{E}}, \varphi)$ is allowed to depend smoothly on t.

Let $Y \subset \mathbb{R}^N$ be a closed convex subset. Recall the definition of a support functional.

Definition 4.1. A *support functional* for a closed convex set $Y \in \mathbb{R}^N$ at a boundary point $y \in \partial Y$ is a linear function

$$\alpha \colon \mathbb{R}^N \to \mathbb{R},$$

such that $\langle \alpha, y \rangle \geqslant \langle \alpha, y' \rangle$ for any $y' \in Y$. The set of support functionals at $y \in \partial Y$ forms a nonempty closed convex cone in $(\mathbb{R}^N)^*$. The set of support functionals of the unit length (with respect to an underlying metric on $\mathbb{R}^N$) will be denoted $\mathcal{S}_y$.

Let $X \subset \mathcal{E}$ be a subset of the total space of $\mathcal{E}$ satisfying the following properties

(P1) X is closed;

(P2) the fiber $X_m = X \cap \mathcal{E}_m$ over any $m \in M$ is convex;

(P3) X is invariant under the parallel transport induced by $\nabla^{\mathcal{E}}$;

(P4) For any boundary point $x \in \partial X_m$ and any support functional $\alpha \in \mathcal{S}_x \subset \mathcal{E}_m^*$ at x, we have $\langle \alpha, \varphi(x) \rangle \leqslant 0$.

Assume that the initial data f_0 lies in X, that is $f_0(m) \in X_m$ for any $m \in M$. Hamilton's maximum principle states that the set X is invariant under the PDE (4.1), i.e., $f(m,t)$ remains in X_m for $t > 0$, as long, as the equation is solvable. Specifically, slightly rephrasing [Ham86], we have the following results.

Theorem 4.2. *If for every fiber $\mathcal{E}_m$, $m \in M$, the solutions of the ODE*

$$\frac{df}{dt} = \varphi(f)$$

remain in $X_m \subset \mathcal{E}_m$, then the solutions of the PDE (4.1) *also remain in X.*

Lemma 4.3. *For a closed convex subset $X_m \subset \mathcal{E}_m$ the solution of the ODE*

$$\frac{df}{dt} = \varphi(f)$$

remains in X_m if and only if X_m satisfies property (P4).

The proof of both results is based on the fact that the invariance of X under PDE/ODE is equivalent to the invariance of all the 'halfspaces' $\{x \in X_m \mid \langle \alpha, x\rangle \leqslant \langle \alpha, f\rangle\}$ for all $f \in \partial X_m$, $\alpha \in \mathcal{S}_f$, $m \in M$.

In what follows, we will need these results in a slightly more general setup. Namely, we

(a) do not assume that connection $\nabla^{\mathcal{E}}$ preserves the bundle metric h;

(b) in the definition of $\Delta^{\mathcal{E}}$, allow to use any (not necessarily the Levi-Civita) connection ∇^{TM} on TM to extend $\nabla^{\mathcal{E}}$ to a connection on $\Lambda^1(M) \otimes \mathcal{E}$.

Necessity of this generalization comes from the presence of the Laplacian Δ_{∇^T} of a non-metric connection in equation (3.15). These modifications do not affect neither the setup nor the original proof of Lemma 4.3, since it depends only on the properties of X in each individual fiber X_m. Hence, only the proof of Theorem 4.2 requires modifications.

Proof of Theorem 4.2 in a general setup. We will go over the Hamilton's proof of the theorem and point the steps requiring the invariance of h under $\nabla^{\mathcal{E}}$. In each case, we provide the necessary modifications to drop this assumption. As in Hamilton's proof, we will use the basic theory of differential inequalities for Lipschitz functions [Ham86, §3].

Denote by $|\cdot|$ the length function induced by h on $\mathcal{E}$ and $\mathcal{E}^*$. Let $f(m,0) = f_0(m)$ be the initial data with $f_0(m) \in X_m$ for any $m \in M$. Let $f(m,t)$ be the solution to the PDE (4.1) on $[0,\epsilon]$ and denote by $B_R = \{e \in \mathcal{E} \mid |e| \leqslant R\}$ the disk bundle of radius R in $\mathcal{E}$.

Step 1. We modify X and φ so that X becomes compact and φ becomes compactly supported. To do that, pick R large enough such that for any $m \in M$, $t \in [0,\epsilon]$ we have $f(m,t) \in B_R$ and

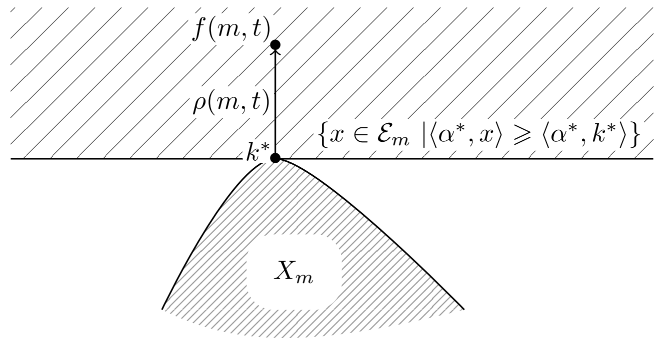

Figure 4.1: Definition of $\rho(m,t)$.

$k^* \in B_R$, where k^* is the point of ∂X_m closest to $f(m,t)$. Consider $\widetilde{X} = X \cap B_{3R}$ and multiply $\varphi(f)$ by a cutoff function, which is supported on B_{3R} and equals 1 on B_{2R}. Clearly, if the solution of a new equation on $[0,\epsilon]$ stays in $\widetilde{X}$, then the solution of the initial equation stays in X. From now X is compact.

Remark 4.4. Unlike the situation in the original proof, with the above modification, the set X does not remain invariant under $\nabla^{\mathcal{E}}$, since h and B_R are not preserved by $\nabla^{\mathcal{E}}$. However, we still have the following local invariance property on $X \cap B_R$

(P3*) There exists $\delta = \delta(g,h,R) > 0$ such that for any path $\gamma(\tau) \in M, \tau \in [0,1]$ of length $< \delta$ and any $s \in \partial X_{\gamma(0)} \cap B_R$ the $\nabla^{\mathcal{E}}$-parallel transport of s along γ lies in $\partial X_{\gamma(1)}$. Moreover, this parallel transport carries support functionals to support functionals.

Step 2. For a fixed $m \in M$ and $t \in [0,\epsilon]$ define

$$\rho(m,t) = \sup\{\langle \alpha, f(m,t) - k\rangle\},$$

where the supremum is taken over $k \in \partial X_m$, $\alpha \in \mathcal{S}_k$ (i.e., α is a support functional at k and $|\alpha| = 1$). Since the domain of this supremum is compact, it is attained at some $\alpha = \alpha^*$, $k = k^*$. If $f(m,t) \notin X_m$, then $\rho(m,t)$ equals the distance from $f(m,t)$ to ∂X_m (see Figure 4.1). Otherwise, if $f(m,t) \in X_m$, then $\rho(m,t)$ equals the negative distance from $f(m,t)$ to ∂X_m. By our choice of R, point k^* lies in $X_m \cap B_R$.

Now, define

$$\widehat{\rho}(t) = \sup_{m \in M} \rho(m,t).$$

Function $\widehat{\rho}(t)$ is Lipschitz, and $\widehat{\rho}(t) \leqslant 0$ (resp. < 0) if and only $f \in X$ (resp. f belongs to the interior of X). Therefore, to prove Theorem 4.2 it is enough to prove that $\widehat{\rho}(t) \leqslant 0$, provided $\widehat{\rho}(0) \leqslant 0$. We claim that there exists a constant $C > 0$ such that

$$\frac{d\widehat{\rho}}{dt} \leqslant C|\widehat{\rho}(t)|.$$

To prove the claim we plug in the definition of $\widehat{\rho}$ and use [Ham86, Lemma 3.5]:

$$\frac{d\widehat{\rho}}{dt} \leqslant \sup\Big\{\frac{d}{dt}\langle\alpha, f(m,t) - k\rangle\Big\},$$

where supremum is taken over $m \in M$, $k \in \partial X_m$, $\alpha \in \mathcal{S}_k$ such that the maximum of $\langle\alpha, f(m,t) - k\rangle$ is attained, i.e., $\langle\alpha, f(m,t) - k\rangle = \widehat{\rho}(t)$. Together with the equation for f this gives

$$\frac{d\widehat{\rho}}{dt} \leqslant \sup\{\langle\alpha, \Delta^{\mathcal{E}} f + \varphi(f)\rangle\} = \sup\{\langle\alpha, \Delta^{\mathcal{E}} f\rangle + \langle\alpha, \varphi(f)\rangle\}.$$

We claim that both summands could be bounded from above by $C|\widehat{\rho}(t)|$ for some constant $C > 0$.

Step 2a. Let $\{e_i\}$ be a g-orthonormal frame of T_mM. Define $\gamma_i(\tau), i = 1, \dots \dim M$ to be the geodesic path of connection ∇^{TM} in the direction e_i and denote by D_i the covariant derivative along γ_i'. Then

$$\Delta^{\mathcal{E}} = \sum_{i=1}^{\dim M} D_i^2.$$

We extend vectors $k \in \mathcal{E}_m, \alpha \in \mathcal{E}_m^*$ along each of the paths γ_i by $\nabla^{\mathcal{E}}$-parallel transport. By property (P3*), in a small neighborhood of $m \in M$ we still have $k \in \partial X$, and α is a support functional at k. Note, however, that in order to get an element in $\mathcal{S}_k$ over a point $m_0 \neq m$, we need to normalize α, since $\nabla^{\mathcal{E}}$ does not preserve metric h; so $\alpha/|\alpha| \in \mathcal{S}_k$. Since point $m \in M$, and the corresponding vectors $k \in \partial X_m$, $\alpha \in \mathcal{S}_k$ are chosen in such a way that $\langle\alpha, f(m,t) - k\rangle$ attains its maximum — $\widehat{\rho}(t)$, the function $\Phi_i(\tau) := \langle\alpha/|\alpha|, f(\gamma_i(\tau), t) - k\rangle$ is maximal at $\tau = 0$. Therefore

$$0 = \Phi_i'(0) = (D_i|\alpha|^{-1})\widehat{\rho}(t) + D_i\langle\alpha, f - k\rangle;$$

$$0 \geqslant \Phi_i''(0) = (D_i^2|\alpha|^{-1})\widehat{\rho}(t) + 2(D_i|\alpha|^{-1})D_i\langle\alpha, f - k\rangle + \langle\alpha, D_i^2 f\rangle.$$

With the use of the first equation we can rewrite the inequality as

$$\langle\alpha, D_i^2 f\rangle \leqslant -(D_i^2|\alpha|^{-1})\widehat{\rho}(t) + 2(D_i|\alpha|^{-1})^2\widehat{\rho}(t).$$

Let C' be an upper bound for $\left|2(D_i|\alpha|^{-1})^2 - D_i^2|\alpha|^{-1}\right|$ over $m \in M$, $\alpha \in \{\alpha \in \mathcal{E}_m^* \mid |\alpha| = 1\}$,

$e_i \in \{v \in T_m M \mid |v| = 1\}$. Summing the inequality above over $i = 1, \ldots, \dim M$, we deduce the inequality

$$\langle \alpha, \Delta^{\mathcal{E}} f \rangle \leqslant C|\widehat{\rho}(t)|,$$

for $C = C' \dim M$, as required.

Remark 4.5. In the original proof, the bundle connection $\nabla^{\mathcal{E}}$ preserves h, hence for the $\nabla^{\mathcal{E}}$-parallel extension of α we have $|\alpha| \equiv 1$, so $\alpha \in \mathcal{S}_k$. In particular, we could take $C = 0$.

Step 2b. Recall that $\alpha \in \mathcal{S}_k$, therefore by property (P4) $\langle \alpha, \varphi(k) \rangle \leqslant 0$. Hence we have

$$\langle \alpha, \varphi(f) \rangle = \langle \alpha, \varphi(k) \rangle + \langle \alpha, \varphi(f) - \varphi(k) \rangle \leqslant \langle \alpha, \varphi(f) - \varphi(k) \rangle \leqslant C|f - k| = C|\widehat{\rho}(t)|,$$

where C is a generic constant bounding the norm of the derivative of $\varphi \colon \mathcal{E} \to \mathcal{E}$.

Step 3. Lipschitz function $\widehat{\rho}(t)$ has the initial condition $\widehat{\rho}(0) \leqslant 0$ and satisfies differential inequality $d\widehat{\rho}/dt \leqslant C|\widehat{\rho}|$. By a general result [Ham86], it implies $\widehat{\rho}(t) \leqslant 0$ for $t \geqslant 0$. This is equivalent to the required invariance: $f(m, t) \in X_m$ for any $m \in M$, $t \geqslant 0$. This proves Theorem 4.2.

Note that with the same reasoning, we can prove a bit more. If $\widehat{\rho}(0) < 0$, then $\widehat{\rho}(t) \leqslant \widehat{\rho}(0)e^{-Ct} < 0$ for $t \geqslant 0$. Therefore, if $f(m, 0)$ lies in the interior of X for all $m \in M$, then the same is true for $f(m, t)$, $t > 0$. So, the interior of X is also preserved by the PDE (4.1). $\square$

Theorem 4.2 allows to construct many invariant sets X for certain PDEs of the form (4.1). In practice, it is easy to construct subset $X \subset \mathcal{E}$, satisfying conditions (P1), (P2), (P3) (i.e., closed, convex, and invariant under a parallel transport), while condition (P4) is the most essential and difficult to meet. In the next section we apply these results to the evolution equation for the Chern curvature under the HCF.

4.2 Invariant Sets of Curvature Operators

By the philosophy of Hamilton's maximum principle, in order to find invariant sets for a heat-type equation $\frac{df}{dt} = \Delta f + \varphi(f)$ one has to study an ODE $\frac{df}{dt} = \varphi(f)$. Following this idea, in the present section we start with studying an ODE

$$\frac{d\Omega}{dt} = \Omega^{\#} + \mathrm{tr}_{\Lambda^2(M)}(\Omega_{\nabla^T} \otimes \overline{\Omega}_{\nabla^T}) + \mathrm{ad}_v \Omega,$$

on the space of algebraic curvature tensors $\mathrm{Sym}^{1,1}(\mathfrak{g})$, $\mathfrak{g} = \mathrm{End}(V)$ given by the zero-order part of the evolution equation for $\Omega^{g(t)}$ under the HCF (equation (3.15)),

$$\frac{d}{dt}\Omega = \Delta_{\nabla^T}\Omega + \Omega^{\#} + \mathrm{tr}_{\Lambda^2(M)}(\Omega_{\nabla^T} \otimes \overline{\Omega}_{\nabla^T}) + \mathrm{ad}_u\Omega.$$

We construct a family of invariant sets for this ODE by verifying property (P4).

4.2.1 ODE-invariant Sets

As in Section 2.1.5, let $V = \mathbb{C}^n$, $\mathfrak{g} = \mathrm{End}(V)$, and $G = GL(V)$. Consider $\Omega(t) \in \mathrm{Sym}^{1,1}(\mathfrak{g})$ evolving according to an ODE

$$\frac{d\Omega}{dt} = \Omega^{\#} + AA^* + \mathrm{ad}_v\Omega, \tag{4.2}$$

where

1. $AA^* = \sum_i a_i \otimes \overline{a_i}$, for some collection of vectors $\{a_i \in \mathfrak{g}\}$;

2. v is an element of $\mathfrak{g}$.

Both v and A are allowed to depend on time. Following the notations of Section 4.1, denote $\varphi(\Omega) := \Omega^{\#} + AA^* + \mathrm{ad}_v\Omega$. We describe a family of convex subsets of $\mathrm{Sym}^{1,1}(\mathfrak{g})$, for which we are aiming to prove the invariance under (4.2), and, eventually, the invariance under the HCF.

Let $S \subset \mathfrak{g}$ be a subset invariant under the adjoint action of $G = GL(V)$ and let $F\colon \mathfrak{g} \to \mathbb{R}$ be a continuous function satisfying the following properties.

($\star_1$) F is $\mathrm{Ad}\,G$-invariant. Since diagonalizable matrices are dense in $\mathfrak{g}$, F can be though of as a symmetric function in the eigenvalues $\{\mu_i\}$ of $s \in \mathfrak{g}$;

($\star_2$) For any sequence $s_i \in \mathfrak{g}$ and any $\lambda_i \searrow 0$, such that $\lambda_i s_i \to s$, there exists a finite limit

$$\lim_{i\to\infty} F(s_i)\lambda_i^2,$$

and its value depends only on s. We denote this limit by $F_\infty(s)$.

Definition 4.6. Continuous function $F\colon \mathfrak{g} \to \mathbb{R}$ satisfying properties ($\star_1$) and ($\star_2$) is called *nice*. There is a function $F_\infty\colon \mathfrak{g} \to \mathbb{R}$ attached to any nice function.

Examples of nice F are $F(s) = a|\mathrm{tr} s|^2 + b$ with the corresponding limit $F_\infty(s) = a|\mathrm{tr} s|^2$ and $F(s) = \sum_{\mu_i \in \mathrm{spec}(s)} |\mu_i|$ with $F_\infty(s) \equiv 0$. In most of the examples below the only reasonable choice is $F \equiv 0$.

Given a tuple (S, F), we define a subset of $\mathrm{Sym}^{1,1}(\mathfrak{g})$:

$$C(S, F) := \{\Omega \in \mathrm{Sym}^{1,1}(\mathfrak{g}) \mid \langle \Omega, s \otimes \overline{s} \rangle_{\mathrm{tr}} \geqslant F(s) \text{ for all } s \in S\}.$$

As the intersection of closed halfspaces, the set $C(S, F)$ is closed and convex. Since S and F are $\mathrm{Ad}\, G$-invariant, set $C(S, F)$ is also invariant under the induced action of G. We claim that $C(S, F)$ is preserved by ODE (4.2).

Theorem 4.7. *The set $C(S, F) \subset \mathrm{Sym}^{1,1}(\mathfrak{g})$ is closed, convex, and satisfies property (P4) for ODE* (4.2)*. In particular, by Lemma 4.3 set $C(S, F)$ is invariant under this ODE.*

Let us first prove a lemma. In this lemma we do not assume that F is continuous. Its proof follows the lines of [Wil13].

Lemma 4.8. *If $\Omega \in C(S, F)$ and $u \in S$ is such that $\langle \Omega, u \otimes \overline{u} \rangle_{\mathrm{tr}} = F(u)$, then $\langle \Omega^\# + AA^* + \mathrm{ad}_v \Omega, u \otimes \overline{u} \rangle_{\mathrm{tr}} \geqslant 0$.*

Proof. First, note that $\langle AA^*, u \otimes \overline{u} \rangle_{\mathrm{tr}} = \sum_i |\mathrm{tr}(a_i u)|^2 \geqslant 0$. Hence, it remains to prove that $\langle \Omega^\# + \mathrm{ad}_v \Omega, u \otimes \overline{u} \rangle_{\mathrm{tr}} \geqslant 0$. We claim, that

(C1) $\langle \mathrm{ad}_v \Omega, u \otimes \overline{u} \rangle_{\mathrm{tr}} = 0$,

(C2) $\langle \Omega^\#, u \otimes \overline{u} \rangle_{\mathrm{tr}} \geqslant 0$.

Proof of both claims is based on the variation of $\langle \Omega, u \otimes \overline{u} \rangle_{\mathrm{tr}}$ in u.

Fix an element $x \in \mathfrak{g}$ and let $u(\tau) = \exp(\tau\, \mathrm{ad}_x) u$ be the orbit of u induced by a 1-parameter subgroup of $\mathrm{Ad}\, G$. By the invariance of S under the adjoint action, we have $u(\tau) \in S$. Therefore by the definition of $C(S, F)$, the function

$$\Psi(\tau) := \langle \Omega, u(\tau) \otimes \overline{u(\tau)} \rangle_{\mathrm{tr}}$$

is bounded below by $F(u)$ and, by our choice of u, attains this minimum at $\tau = 0$. It follows that $\Psi'(0) = 0$ and $\Psi''(0) \geqslant 0$. Specifically,

$$\begin{aligned} &\langle \Omega, \mathrm{ad}_x u \otimes \overline{u} + u \otimes \overline{\mathrm{ad}_x u} \rangle_{\mathrm{tr}} = 0, \\ &\langle \Omega, \mathrm{ad}_x u \otimes \overline{\mathrm{ad}_x u} \rangle_{\mathrm{tr}} + \langle \Omega, \mathrm{ad}_x \mathrm{ad}_x u \otimes \overline{u} \rangle_{\mathrm{tr}} + \langle \Omega, u \otimes \overline{\mathrm{ad}_x \mathrm{ad}_x u} \rangle_{\mathrm{tr}} \geqslant 0. \end{aligned} \tag{4.3}$$

The first identity is equivalent to $\langle \mathrm{ad}_x\Omega, u\otimes\overline{u}\rangle_{\mathrm{tr}} = 0$ for all $x \in \mathfrak{g}$, implying the vanishing of (C1). Now let us prove (C2). After summing up the second line of (4.3) for x and $\sqrt{-1}x$ we arrive at

$$\langle \Omega, \mathrm{ad}_x u \otimes \overline{\mathrm{ad}_x u}\rangle_{\mathrm{tr}} \geqslant 0.$$

This inequality holds for any $x \in \mathfrak{g}$, thus Hermitian form $Q^{\Omega}(x,\overline{x}) := \langle \Omega, \mathrm{ad}_x u \otimes \overline{\mathrm{ad}_x u}\rangle_{\mathrm{tr}} = \langle \Omega, \mathrm{ad}_u x \otimes \overline{\mathrm{ad}_u x}\rangle_{\mathrm{tr}}$ is positive semidefinite.

Next, we choose an adopted basis $\{v_i\}_{i=1}^N$ of $\mathfrak{g}$ as follows.

1. Fix a complement of $\ker \mathrm{ad}_u$ in $\mathfrak{g}$:
$$\mathfrak{g} = \ker \mathrm{ad}_u \oplus W.$$
2. Let $\{v_i\}_{i=r+1}^N$ be a basis of $\ker \mathrm{ad}_u$, so $Q^{\Omega}(v_i, \overline{\cdot}) = 0$ for any $r+1 \leqslant i \leqslant N$.
3. Fix a basis of V, i.e., identify $\mathfrak{g} = \mathrm{End}(V)$ with the space of complex $n\times n$ matrices, and define an inner Hermitian product on W
$$H(\,\cdot\,, \overline{\cdot}) = \mathrm{tr}((\mathrm{ad}_u \cdot)(\mathrm{ad}_u \cdot)^*),$$
where $b^* = \overline{b}^{\dagger}$ is the transposed conjugate of b in this fixed basis.
4. On W we have a positive definite Hermitian form H and a positive semidefinite Hermitian form $Q^{\Omega}|_W$. Let $v_1, \ldots, v_r$ be an H-orthonormal basis of W, which diagonalizes $Q^{\Omega}|_W$:
$$Q^{\Omega}(v_i, \overline{v}_j) = \delta_{ij}\mu_i, \quad H(v_i, \overline{v}_j) = \delta_{ij}, \quad 1 \leqslant i, j \leqslant r.$$

Let $\Omega = \sum_{i,j=1}^N a_{i\bar{j}} v_i \otimes \overline{v_j}$ be the expression for Ω in this basis. Then

$$\Omega^{\#} = \frac{1}{2}\sum_{i,j,k,l=1}^{N} a_{i\bar{j}} a_{k\bar{l}} [v_i, v_k] \otimes \overline{[v_j, v_l]}.$$

Therefore

$$
\begin{aligned}
\langle \Omega^{\#}, u \otimes \overline{u} \rangle_{\mathrm{tr}} &= \frac{1}{2} \sum_{i,j,k,l=1}^{N} a_{i\bar{j}} a_{k\bar{l}} \mathrm{tr}(u[v_i, v_k]) \otimes \overline{\mathrm{tr}(u[v_j, v_l])} \\
&= \frac{1}{2} \sum_{i,j,k,l=1}^{r} a_{i\bar{j}} a_{k\bar{l}} \mathrm{tr}(v_i[u, v_k]) \otimes \overline{\mathrm{tr}(v_j[u, v_l])} \\
&= \frac{1}{2} \sum_{k,l=1}^{r} \Big(a_{k\bar{l}} \sum_{i,j=1}^{r} a_{i\bar{j}} \mathrm{tr}(v_i[u, v_k]) \otimes \overline{\mathrm{tr}(v_j[u, v_l])} \Big) \\
&= \frac{1}{2} \sum_{k,l=1}^{r} a_{k\bar{l}} Q^{\Omega}(v_k, \overline{v_l}) = \frac{1}{2} \sum_{k=1}^{r} a_{k\bar{k}} \mu_k.
\end{aligned}
$$

It remains to show that $a_{k\bar{k}} \geqslant 0$. Indeed,

$$
0 \leqslant Q^{\Omega}((\mathrm{ad}_u v_k)^*, \overline{(\mathrm{ad}_u v_k)^*}) = \sum_{i,j=1}^{r} a_{i\bar{j}} \mathrm{tr}(\mathrm{ad}_u v_i (\mathrm{ad}_u v_k)^*) \overline{\mathrm{tr}(\mathrm{ad}_u v_j (\mathrm{ad}_u v_k)^*)}
$$
$$
= \sum_{i,j=1}^{r} a_{i\bar{j}} H(v_i, \overline{v}_k) \overline{H(v_j, \overline{v}_k)} = a_{k\bar{k}}.
$$

Hence $a_{k\bar{k}} \geqslant 0$, and $\langle \Omega^{\#}, u \otimes \overline{u} \rangle_{\mathrm{tr}} = \sum_k a_{k\bar{k}} \mu_k \geqslant 0$, as required. □

Proof of Theorem 4.7. We prove property (P4) for ODE (4.2) and convex set $C(S,F)$. Let $\overline{S}$ be the closure of S. Clearly $C(S,F) = C(\overline{S}, F)$, so without loss of generality we can assume that $S = \overline{S}$. Take a point at the boundary of $C(S,F)$:

$$
y \in \partial C(S,F).
$$

We want to describe the set of support functionals for $C(S,F)$ at y. Let α be such a functional and take any $w \in \mathrm{Sym}^{1,1}(\mathfrak{g})$ such that $\langle \alpha, w \rangle > 0$. Since α is a support functional, for any $\theta > 0$ we have $y + \theta w \notin C(S,F)$, i.e., there exists $s \in S$ (depending on w and θ) such that

$$
\langle y + \theta w, s \otimes \overline{s} \rangle_{\mathrm{tr}} < F(s).
$$

Let $\theta_i \searrow 0$ be a monotonically decreasing sequence of real numbers. Choose $s_i \in S$ such that the inequality above holds. Since $y \in C(S,F)$, we have

$$
F(s_i) + \langle \theta_i w, s_i \otimes \overline{s}_i \rangle_{\mathrm{tr}} \leqslant \langle y + \theta_i w, s_i \otimes \overline{s}_i \rangle_{\mathrm{tr}} < F(s_i).
$$

There are two options.

1. Some subsequence of $|s_i|$ stays bounded. Then after passing to a subsequence we may assume

that $s_i \to s \in S$. In this case we have

$$\langle w, s \otimes \overline{s}\rangle_{\mathrm{tr}} \leqslant 0, \quad \langle y, s \otimes \overline{s}\rangle_{\mathrm{tr}} = F(s), \text{ for some } s \in S.$$

2. $|s_i| \to \infty$. Then after passing to a subsequence we may assume that for some $\lambda_i \searrow 0$ the sequence $\lambda_i s_i$ converges to an element s in the set:

$$\partial_\infty S := \{Y \in \mathfrak{g} \mid \text{there exists } \lambda_i \searrow 0, s_i \in S \text{ with } \lambda_i s_i \to Y\}.$$

   This set is called *the boundary of S at infinity.* From the definition of F_∞ it is clear that $\Omega \in C(\partial_\infty S, F_\infty)$. In this case we have

$$\langle w, s \otimes \overline{s}\rangle_{\mathrm{tr}} \leqslant 0, \quad \langle y, s \otimes \overline{s}\rangle_{\mathrm{tr}} = F_\infty(s), \text{ for some } s \in \partial_\infty S.$$

Inequality $\langle w, s \otimes \overline{s}\rangle_{\mathrm{tr}} \leqslant 0$ is valid in both cases for any w such that $\langle \alpha, w\rangle > 0$. Therefore α cannot be separated in $\mathrm{Sym}^{1,1}(\mathfrak{g})^*$ by a hyperplane $\langle \alpha, w\rangle = 0$ from the set of linear functionals on $\mathrm{Sym}^{1,1}(\mathfrak{g})$

$$\mathcal{F}_y := \mathcal{F}_y^b \cup \mathcal{F}_y^\infty,$$

where

$$\mathcal{F}_y^b = \{-\langle\,\cdot\,, s \otimes \overline{s}\rangle_{\mathrm{tr}} \mid s \in S \text{ s.t. } \langle y, s \otimes \overline{s}\rangle_{\mathrm{tr}} = F(s)\},$$

$$\mathcal{F}_y^\infty = \{-\langle\,\cdot\,, s \otimes \overline{s}\rangle_{\mathrm{tr}} \mid s \in \partial_\infty S \text{ s.t. } \langle y, s \otimes \overline{s}\rangle_{\mathrm{tr}} = F_\infty(s)\}.$$

Hence, α lies in the convex cone spanned by the elements of $\mathcal{F}_y$:

$$\alpha \in \mathrm{Cone}(\mathcal{F}_y).$$

Thus, in order to verify property (P4), we need to check that $\langle \alpha, \varphi(\Omega)\rangle \leqslant 0$ for all α in $\mathcal{F}_y$. For $\alpha \in \mathcal{F}_y^b$ this is exactly the statement of Lemma 4.8. For $\alpha \in \mathcal{F}_y^\infty$ this is the statement of Lemma 4.8 applied to $C(\partial_\infty S, F_\infty)$ (at this point continuity of F_∞ is not required). This proves property (P4), and, by Lemma 4.3, the invariance of $C(S, F)$ under ODE (4.2). □

Remark 4.9. The reason why we have to consider the inequalities produced by $(\overline{S}, F)$ and $(\partial_\infty S, F_\infty)$, is that they provide the closure of the set of inequalities defining $C(S, F)$ in 'the space of all linear inequalities' on $\mathrm{Sym}^{1,1}(\mathfrak{g})$.

4.2.2 PDE-invariant Sets

Let $g(t)$ be a solution to the HCF on (M, g, J). With the results of the previous subsection, we turn back to PDE (3.15) satisfied by the Chern curvature tensor $\Omega^{g(t)}$:

$$\frac{d\Omega}{dt} = \Delta_{\nabla^T}\Omega + \Omega^{\#} + \mathrm{tr}_{\Lambda^2(M)}(\Omega_{\nabla^T} \otimes \overline{\Omega_{\nabla^T}}) + \mathrm{ad}_v\Omega. \tag{4.4}$$

First, we note that the term

$$\mathrm{tr}_{\Lambda^2(M)}(\Omega_{\nabla^T} \otimes \overline{\Omega_{\nabla^T}})$$

is of the form AA^* (see (4.2)) for

$$A = \{\Omega_{\nabla^T}(e_i, e_j), 1 \leqslant i < j \leqslant \dim_{\mathbb{R}} M \mid \{e_i\} \text{ is an orthonormal basis of } TM\}.$$

Hence, the zero order part of the PDE for Ω is a specialization of the right hand side of ODE (4.2).

Let $V = \mathbb{C}^{\dim M}$, $G = GL(V)$, and denote by

$$P \to M$$

the principle G-bundle of complex frames in $T^{1,0}M$. For any G-space X, denote by $X \times_G P$ the associated fiber bundle. The set $C(S, F) \times_G P \subset \mathrm{Sym}^{1,1}(\mathrm{End}(T^{1,0}M))$ is closed, convex, and satisfies (P4). It also satisfies property (P3), since $C(S, F) \subset \mathrm{End}(V)$ is invariant under the adjoint action of $GL(V)$, and therefore $C(S, F) \times_G P$ is invariant under the action of the holonomy group of any principle G-connection in P. Hence, we can apply Theorem 4.2 and conclude that $C(S, F) \times_G P$ is invariant under (4.4), proving the following theorem.

Theorem 4.10 (Lie-algebraic curvature conditions preserved by the HCF)**.** *Consider an* $\mathrm{Ad}\,G$*-invariant subset* $S \subset \mathrm{End}(V)$ *and a nice function* $F\colon \mathrm{End}(V) \to \mathbb{R}$*. Let* $g = g(t)$ *be a solution to the HCF* (3.4) *on an Hermitian manifold* (M, g, J) *for* $t \in [0, t_{\max})$*. Assume that* $\Omega^{g(0)}$ *satisfies* $C(S, F)$*, i.e.,*

$$\Omega^{g(0)} \in C(S, F) \times_G P.$$

Then the same holds for all $t \in [0, t_{\max})$*.*

In Section 4.4 below, we provide some concrete examples of conditions $C(S, F)$, which are preserved by the HCF.

4.3 Strong Maximum Principle of Brendle and Schoen

Hamilton's maximum principle in the form of Theorem 4.2 is a version of a *weak* parabolic maximum principle, i.e., a statement about preservation of a non-strict inequality along a heat-type flow. In many settings a *strong* maximum principle is satisfied. That is a statement characterizing solutions $f(t)$ of (4.1), which meet the boundary of a preserved set X at some $t > 0$. For example, in [Ham86], Hamilton has proved the following lemma.

Lemma 4.11 ([Ham86, pp. Lm. 8.2])**.** *Let B be a symmetric bilinear form on M. Suppose B satisfies a heat equation*

$$\frac{dB}{dt} = \Delta B + \varphi(B),$$

where the matrix $\varphi(B) \geqslant 0$ for all $B \geqslant 0$. Then if $B \geqslant 0$ at time $t = 0$, it remains so for $t > 0$. Moreover there exists an interval $0 < t < t_{\max}$ on which the rank of B is constant and the null space of B is invariant under parallel translation and invariant in time and also lies in the null space of $\varphi(M)$.

Lemma 4.11 provides a very strong restriction on the null space of a semipositive bilinear form, satisfying a heat equation. In our situation, we could apply it to $\Omega^{g(t)}$, which evolves by equation (3.15) and satisfies $C(S, F)$ at $t = 0$, where $S = \mathrm{End}(V)$, and $F \equiv 0$.

It turns out, a slightly weaker statement is true for any $C(S, F)$. Below we describe a version of the strong maximum principle for Theorem 4.10.

Theorem 4.12 (Strong maximum principle for Ω)**.** *Consider an $\mathrm{Ad}\, G$-invariant subset $S \subset \mathrm{End}(V)$ and a nice function $F \colon \mathrm{End}(V) \to \mathbb{R}$. Let $g = g(t)$ be a solution to the HCF (3.4) on an Hermitian manifold (M, g, J). Assume that $\Omega^{g(0)}$ satisfies $C(S, F)$. Then for any $t > 0$ the set*

$$N(t) := \{s \in S \times_G P \mid \langle \Omega^{g(t)}, s \otimes \bar{s} \rangle_{\mathrm{tr}} = F(s)\}$$

is preserved by the ∇^T-parallel transport. Moreover, if $s \in N(t)$, then the 2-form $\mathrm{tr}(s \circ (\Omega_{\nabla^T})(\,\cdot\,,\cdot\,)) \in \Lambda^2(M, \mathbb{C})$ vanishes, or, equivalently,

(a) $s^i_k \Omega_{i\bar{j}}{}^{\bar{l}k} = 0$, *in particular,* $F(s) = 0$;

(b) $s^i_k \nabla_i T^k_{jl} = 0$.

This theorem is an extension of Brendle and Schoen's [BS08] strong maximum principle, which was originally proved for the isotropic curvature evolved under the Ricci flow. A general argument in the context of the Ricci flow was given by Wilking [Wil13, A.1]. A similar proof with minor modifications works for the HCF. In [Ust16, Th. 5.2] this argument was used in the case of Griffiths positivity.

Proof. We may assume that S is an orbit of the $\mathrm{Ad}\, G$-action on $\mathrm{End}(V)$, otherwise, we decompose S into separate orbits and prove the result for each orbit independently. In this case, S is a smooth G-homogeneous space, and $S \times_G P$ is a smooth fiber bundle over M.

The idea is to treat $N(t)$ as the zero set of the function

$$\Phi(t, \cdot)\colon S \times_G P \to \mathbb{R}, \quad \Phi(t,s) = \langle \Omega^{g(t)}, s \otimes \overline{s} \rangle_{\mathrm{tr}} - F(s)$$

and to prove a certain differential inequality for Φ, which makes it possible to apply Proposition 4 of [BS08].

First, note that by assumption, function $\Phi(0,s)$ is non-negative on $S \times_G P$, and by Theorem 4.10 the same holds for $t > 0$. By the evolution equation for Ω, (4.4), function $\Phi\colon [0, t_{\max}) \times (S \times_G P) \to \mathbb{R}$ satisfies equation

$$\frac{d\Phi(t,s)}{dt} = \Delta_h \Phi(t,s) + \langle \Omega^{\#}, s \otimes \overline{s} \rangle_{\mathrm{tr}} + |\mathrm{tr}(s \circ (\Omega_{\nabla^T}))|^2_{\Lambda^2(M)} + \langle \mathrm{ad}_v \Omega, s \otimes \overline{s} \rangle_{\mathrm{tr}}, \tag{4.5}$$

where Δ_h is the horizontal Laplacian of the connection ∇^T, defined by the identity

$$\Delta_h \Phi(t,s) = \langle \Delta_{\nabla^T} \Omega, s \otimes \overline{s} \rangle.$$

The horizontal Laplacian can be computed as follows. Let $\{X_i\}$ be a $g(t)$-orthonormal collection of vector fields in a neighbourhood of $m \in M$. Denote $Y_i := \nabla_{X_i} X_i$ and let $\{\widehat{X_i}\}$, $\{\widehat{Y_i}\}$ be the ∇^T-horizontal lifts of these vector fields to $S \times_G P$. Then $\Delta_h \Phi(s,t) := \sum_i (\widehat{X_i} \cdot \widehat{X_i} \cdot \Phi - \widehat{Y_i} \cdot \Phi)$.

The summand $|\mathrm{tr}(s \circ (\Omega_{\nabla^T}))|^2_{\Lambda^2(M)}$ in (4.5) is clearly non-negative. Using the same bounds for the remaining terms, as in [Wil13, Th. A.1], we conclude, that on a small relatively compact coordinate neighbourhood of $(t,s) \in (0, t_{\max}) \times (S \times_G P)$, for a large positive constant k

$$\begin{aligned} \left| \frac{d\Phi}{dt} \right| &\leqslant k |D\Phi|, \\ (Y_i \cdot \Phi) &\leqslant k |D\Phi|, \\ \langle \Omega^{\#}, s \otimes \overline{s} \rangle &\geqslant k \inf \{ \frac{d^2}{d\tau^2}\Big|_{\tau=0} \Phi(t, \exp(\tau \mathrm{ad}_x) s) \mid x \in \mathrm{End}(V), |x| \leqslant 1 \}, \end{aligned}$$

where $D\Phi$ is the gradient of Φ with respect to some fixed background metric on $S \times_G P$. Therefore, for a large positive constant K we have

$$\sum_i \widehat{X_i}\cdot\widehat{X_i}\cdot\Phi \leqslant -K \inf_{|x|\leqslant 1}(D^2\Phi(t,s))(\mathrm{ad}_x(s),\mathrm{ad}_x(s)) + K|D\Phi|,$$

where $D^2\Phi$ is the Hessian of Φ.

At this point, we can use

Proposition 4.13 ([BS08, Prop. 4]). *Let $\mathcal{U}$ be an open subset of $\mathbb{R}^n$ and let $X_1,\dots,X_m$ be smooth vector fields on $\mathcal{U}$. Assume that $u:\mathcal{U}\to\mathbb{R}$ is a non-negative smooth function satisfying*

$$\sum_{j=1}^m (D^2u)(X_j,X_j) \leqslant -K \inf_{|\xi|\leqslant 1}(D^2u)(\xi,\xi) + K|Du| + Ku,$$

where K is a positive constant. Let $F=\{x\in\mathcal{U}\colon u(x)=0\}$. Finally, let $\gamma:[0,1]\to\Omega$ be a smooth path such that $\gamma(0)\in F$ and $\gamma'(s)=\sum_{j=1}^m f_j(s)X_j(\gamma(s))$, where $f_1,\dots,f_m\colon[0,1]\to\mathbb{R}$ are smooth functions. Then $\gamma(s)\in F$ for all $s\in[0,1]$.

Applying Proposition 4.13, to the function Φ and the vector fields $\widehat{X_i}$, we conclude that the zero set of Φ is invariant under the flow generated by the vector fields $\widehat{X_i}$. Since these fields span the ∇^T-horizontal subspaces of the fiber bundle $S\times_G P\to M$, we obtain that zeros of Φ are invariant under the ∇^T-parallel transport.

Now, let us prove (a) and (b). Let $s\in S\times_G P$ be a zero of $\Phi(t_0,\cdot)$ for some $t_0>0$. Function $\Phi(t,\cdot)$ attains a local minimum at s, $t=t_0$, therefore $\partial_t\Phi(t_0,s)=0$. As we have proved in Theorem 4.10, all summands on the right hand side of (4.5) are non-negative, therefore they must vanish. In particular, $\mathrm{tr}(s\circ(\Omega_{\nabla^T}))=0$, as stated. By the formula for the curvature of the torsion-twisted connection (Proposition 3.17), we conclude that (a) $s^i_k\Omega_{i\bar{j}}{}^{\bar{l}k}=0$; (b) $s^i_k\nabla_i T^k_{jl}=0$. ☐

Theorem 4.12 implies a more familiar version of the strong maximum principle.

Corollary 4.14. *Consider an $\mathrm{Ad}\,G$-invariant subset $S\subset\mathrm{End}(V)$ and a nice function $F\colon\mathrm{End}(V)\to\mathbb{R}$. Let $g=g(t)$ be a solution to the HCF (3.4) on an Hermitian manifold (M,g,J) for $t\in[0,t_{\max})$. Assume that $\Omega^{g(0)}$ satisfies $C(S,F)$, and that there exists $m_0\in M$ such that $\Omega^{g(0)}$ satisfies strict inequalities, defining $C(\overline{S},F)$ and $C(\partial_\infty S,F_\infty)$, at m_0:*

$$\begin{aligned}&\langle\Omega^{g(0)}_{m_0}, s\otimes\overline{s}\rangle_{\mathrm{tr}} > F(s),\ \textit{for any } s\in(\overline{S}\times_G P)_{m_0},\\ &\langle\Omega^{g(0)}_{m_0}, s\otimes\overline{s}\rangle_{\mathrm{tr}} > F_\infty(s),\ \textit{for any } s\in(\partial_\infty S\times_G P)_{m_0}.\end{aligned}\tag{4.6}$$

Then for any $t\in(0,t_{\max})$, inequalities (4.6) hold everywhere on M.

Proof. **Step 1.** We claim that Ω_m lies in the interior of $(C(S,F)\times_G P)_m$ if and only if Ω_m satisfies inequalities (4.6). Indeed, if Ω lies on the boundary of $(C(S,F)\times_G P)_{m_0}$, then, following the proof of Theorem 4.7, we can find a support functional of the form

$$\langle\,\cdot\,, s\otimes s\rangle_{\mathrm{tr}},\quad s\in((\overline{S}\cup\partial_\infty S)\times_G P)_{m_0},$$

such that in (4.6) we have equality.

Conversely, if for some $s\in((\overline{S}\cup\partial_\infty S)\times_G P)_m$ we have equality in (4.6), then there is $y\in \mathrm{Sym}^{1,1}(\mathrm{End}(T^{1,0}_m M))$ arbitrary close to Ω_m, such that $y\notin(C(S,F)\times_G P)_m$. So, $\Omega_m\in(\partial C(S,F)\times_G P)_m$.

Step 2. Run the HCF for small time $t_\varepsilon>0$ such that $\Omega^{g(t)}_{m_0}$ still lies in the interior of $(C(S,F)\times_G P)_{m_0}$ for $t\in(0,t_\varepsilon)$. Fix any $t\in(0,t_\varepsilon)$. By Theorem 4.12 the set $N(t)$ corresponding to $(\overline{S},F)$ is invariant under ∇^T-parallel transport. At the same time, $N(t)_{m_0}$ is empty, therefore for any $m\in M$ the set $N(t)_m$ is empty as well, i.e.,

$$\langle\Omega^{g(t_\varepsilon)}, s\otimes\overline{s}\rangle_{\mathrm{tr}}>F(s)$$

for any $s\in\overline{S}$ everywhere on M. Applying the same reasoning to $(\partial_\infty S, F_\infty)$, we conclude that $\langle\Omega^{g(t_\varepsilon)}, s\otimes\overline{s}\rangle_{\mathrm{tr}}>F_\infty(s)$ for any $s\in\partial_\infty S$ everywhere on M.

Step 3. We have proved that $\Omega^{g(t)}$, lies in the interior of $C(S,F)\times_G P$ for $t\in(0,t_\epsilon)$. By the proof of Hamilton's maximum principle, Ω remains in the interior of the convex set $C(S,F)\times_G P$ for all $t\in(0,t_{\max})$. □

4.4 Examples

Let us provide some specific examples of curvature conditions preserved by the HCF. In most of the examples below $F\equiv 0$.

Example 4.15 (Dual-Nakano semipositivity). Recall that the Chern curvature Ω considered as a section of $\mathrm{Sym}^{1,1}(\mathrm{End}(T^{1,0}M))$ is dual-Nakano semipositive, if it represents a semipositive pairing on $\mathrm{End}(T^{1,0}M)^*$, see Definitions 2.19 and 2.20. Choose $S=\mathrm{End}(T^{1,0}M)$. Then the cone $C(S,0)$ is the set of dual-Nakano semipositive curvature tensors. By Theorem 4.10 this set is preserved by the HCF.

Manifolds admitting a Dual-Nakano semipositive Hermitian metrics are rather scarce. However, such metrics exist on all complex homogeneous manifolds. We will discuss these metrics in more detail in Chapter 5.

Example 4.16 (Griffiths semipositivity)*.* Now, we demonstrate preservation of Griffiths semipositivity under the HCF. It was first proved in [Ust16] by adopting the arguments of Mok [Mok88] and Bando [Ban84], who proved the corresponding statement for the Kähler-Ricci flow. We deduce preservation of Griffiths semipositivity as a particular case of Theorem 4.10.

Chern curvature Ω is Griffiths semipositive if and only if $\Omega \in C(S, 0)$, where

$$S = \{u \in \mathrm{End}(T^{1,0}M) \mid \mathrm{rank}(u) = 1\}.$$

This set is clearly $\mathrm{Ad}G$-invariant, therefore Theorem 4.10 applies, and Griffiths semipositivity is preserved under the HCF.

Example 4.17 (Semipositivity of holomorphic orthogonal bisectional curvature)*.* Analogously to the Kähler situation, the Chern curvature tensor $\Omega = \Omega_{i\bar{j}k\bar{l}}$ is said to have a semipositive *holomorphic orthogonal bisectional curvature*, if for any $\xi, \eta \in T^{1,0}M$ s.t $g(\xi, \overline{\eta}) = 0$

$$\Omega(\xi, \bar{\xi}, \eta, \overline{\eta}) \geqslant 0.$$

Chern curvature Ω considered as a section of $\mathrm{Sym}^{1,1}(\mathrm{End}(T^{1,0}M))$ has semipositive holomorphic orthogonal bisectional curvature if and only if $\Omega \in C(S, 0)$, where

$$S = \{u \in \mathrm{End}(T^{1,0}M) \mid \mathrm{rank}(u) = 1,\ \mathrm{tr}(u) = 0\}.$$

The set S is again $\mathrm{Ad}G$-invariant, and applying Theorem 4.10, we get preservation of this curvature semipositivity condition under the HCF.

Example 4.18 (Dual-m-semipositivity)*.* For

$$S = \{u \in \mathrm{End}(T^{1,0}M) \mid \mathrm{rank}(u) = m\},$$

the cone $C(S, F)$ consists of the dual-m-semipositive curvature tensors, so this condition is also preserved under the HCF.

Example 4.19 (Lower bounds on the second scalar curvature)*.* It is well-known that under the Ricci flow, the lower bound on the scalar curvature is improved, unless the manifold is Ricci-flat. It turns out that the second scalar curvature under the HCF satisfies similar monotonicity. Namely, take $S = \{\mathrm{Id}\} \in \mathrm{End}(T^{1,0}M)$. Then for any $q \in \mathbb{R}$, $F \equiv q$ the condition $C(S,F)$ is preserved under the HCF. In particular, the infimum of $\langle \Omega, \mathrm{Id} \otimes \overline{\mathrm{Id}} \rangle_{\mathrm{tr}} = \Omega_{i\bar{j}}{}^{\bar{j}i} = \widehat{\mathrm{sc}}$ is nondecreasing.

The same result can be obtained without invoking Hamilton's maximum principle for tensors. Indeed, after contracting equation (4.4), we get

$$\frac{d\,\widehat{\mathrm{sc}}}{dt} = \Delta\widehat{\mathrm{sc}} + |\rho_{\nabla^T}|^2,$$

where $\rho_{\nabla^T} \in \Lambda^2(M,\mathbb{C})$ is the Chern-Ricci form of the torsion-twisted connection (see Corollary 3.19). The zero-order expression on the right-hand side is semipositive, and, by the standard maximum principle for parabolic equations, the quantity $\inf_M \widehat{\mathrm{sc}}$ is nondecreasing in t.

Chapter 5

The Hermitian Curvature Flow on Complex Homogeneous Manifolds

In this chapter, we focus on the behavior of the HCF (3.4)

$$\frac{dg_{i\bar{j}}}{dt} = -g^{m\bar{n}}\Omega_{m\bar{n}i\bar{j}} - \frac{1}{2}g^{m\bar{n}}g^{p\bar{s}}T_{mp\bar{j}}T_{\bar{n}\bar{s}i} \tag{5.1}$$

on a (not necessary compact) complex homogeneous manifold $M = G/H$ acted on by a connected complex group G with the isotropy subgroup H. There are several reasons why these manifolds are of a special interest for us.

1. Homogeneous manifolds form a rich family for which one can explicitly compute certain metrics and analyze behavior of a metric flow. For example, in [GP10; Lau13; BL17] authors study the Ricci flow on Lie groups and homogeneous manifolds; in [Bol16; FV15] authors consider the *pluriclosed flow*, on compact homogeneous surfaces and solvmanifolds, respectively. We expect that our computations will shed some light on the geometric nature of the HCF (5.1).

2. Non-symmetric rational homogeneous manifolds G/P, where G is a reductive algebraic group and P is its parabolic subgroup, are projective manifolds, however, there are 'natural' Hermitian metrics induced by the Killing metric of the compact form of G, which are typically non-Kähler (see [Yan94] for an explicit computation on a complete flag manifold F_3). It is interesting to analyze whether our metric flow distinguishes these special metrics.

3. In Chapter 4, we, in particular, proved that the HCF preserves dual-Nakano/Griffiths semi-positivity of the initial metric. Homogeneous manifolds equipped with submersion metrics (see Definition 5.6 below) are essentially the only known source of such examples.

Most of the results in this chapter can be found in [Ust17a].

5.1 Complex Homogeneous Manifolds

Throughout this chapter, for an homogeneous manifold $M = G/H$ we denote by $\mathfrak{g}$ and $\mathfrak{h}$ the complex Lie algebras of G and H, with the Lie bracket of $\mathfrak{g}$ written as $[\,\cdot\,,\,\cdot\,]$. Let us recall some basic facts from the structure theory of complex homogeneous spaces. For more details we refer the reader to [Huc90]. Given a homogeneous manifold G/H, we define the *isotropy representation* of H (resp. $\mathfrak{h}$) in $\mathfrak{g}/\mathfrak{h}$ by restricting the adjoint representation Ad (resp. ad) and considering the induced action on $\mathfrak{g}/\mathfrak{h}$:

$$\sigma_{\mathrm{Ad}}\colon H \to GL(\mathfrak{g}/\mathfrak{h}), \quad \sigma_{\mathrm{ad}}\colon \mathfrak{h} \to \mathrm{End}(\mathfrak{g}/\mathfrak{h}).$$

This representations are not necessarily faithful. The structure group of the tangent bundle $T^{1,0}M$ can be reduced to the image of H in $GL(\mathfrak{g}/\mathfrak{h})$.

Remark 5.1. In the next chapter, we prove that, for a large class of natural metrics on M, the image $\sigma_{\mathrm{ad}}(\mathfrak{h}) \subset \mathrm{End}(\mathfrak{g}/\mathfrak{h}) \simeq \mathrm{End}(T^{1,0}_{[eH]}M)$ coincides with the Lie algebra of the torsion-twisted holonomy group, $\mathfrak{hol}_{\nabla^T}$ (see Definition 3.20 and Corollary 6.8).

Consider

$$N := \mathrm{Stab}_G(\mathfrak{h}) = \{\gamma \in G \mid \mathrm{Ad}_\gamma(\mathfrak{h}) = \mathfrak{h}\}.$$

Remark 5.2. We have $v \in \mathrm{Lie}(N) \subset \mathfrak{g}$ if and only if, $[v] \in \mathfrak{g}/\mathfrak{h}$ is annihilated by the image of the isotropy representation σ_{ad}.

Clearly $H \subset N$, so there is a fibration

$$G/H \to G/N.$$

If $N = H$, then G/H belongs to a very special class of *rational homogeneous manifolds*, i.e., it is projective and birational to the projective space. One can think of G/N as the Ad_G-orbit of $\mathfrak{h}$ inside the complex Grassmanian $\mathrm{Gr}_{\mathbb{C}}(\dim\mathfrak{g}, \dim\mathfrak{h})$. Fibration $G/H \to G/N$ is called *Tits fibration.*

Theorem 5.3 ([Huc90, Thm. 1.2]). *Let M be a connected compact complex manifold which is homogeneous with respect to the action of a connected complex Lie group G. Let $M = G/H \to G/N =: Q$ be the Tits fibration. Then*

1. *The fiber $F = N/H$ is a connected complex parallelizable manifold.*

2. *The base Q is a rational homogeneous manifold.*

Remark 5.4. Compact complex parallelizable and rational homogeneous manifolds have explicit descriptions:

- Any compact complex parallelizable manifold is of the form K/Γ, where K is a complex Lie group, and Γ is its discrete subgroup.

- Any rational homogeneous manifold is of the form S/P, where S is complex semi-simple group, and $P \subset S$ is a parabolic subgroup (i.e., contains the maximal connected solvable subgroup of S).

Now, we turn our attention to metric structures on complex homogeneous manifolds. Let $\mathcal{M}(M)$ be the infinite-dimensional space of Hermitian metrics on M modulo isometry. There are two distinguished subspaces of $\mathcal{M}(M)$.

Definition 5.5 (Invariant metrics). Define $\mathcal{M}^{\mathrm{inv}}(M)$ to be the space of *G-invariant Hermitian metrics* on M. These metrics are in one-to-one correspondence with Ad_H-invariant Hermitian metrics on the vector space $\mathfrak{g}/\mathfrak{h}$, see [KN63, p. X.3]:

$$\mathcal{M}^{\mathrm{inv}}(M) \longleftrightarrow \{\mathrm{Ad}_H\text{-invariant metrics on } \mathfrak{g}/\mathfrak{h}\}\,.$$

Clearly, the set $\mathcal{M}^{\mathrm{inv}}(M)$ is preserved by any metric flow of the form $\frac{dg}{dt} = R(g)$, where R is a tensor field, such that its value at $m \in M$ is determined by the germ of g at m, provided that the solution to the flow is unique. Hence, any such 'natural' metric flow defines an ODE on the finite dimensional space $\mathcal{M}^{\mathrm{inv}}(M)$.

The study of invariant metrics on homogeneous manifolds is a classical subject of differential geometry [Bes87]. These metrics provide a "hands-on" construction of a multitude of explicit Riemannian manifolds with certain prescribed geometric properties. One can then translate the differential-geometric problems into the questions of basic linear algebra and representation theory.

However, on a general homogeneous manifold, $\mathcal{M}^{\mathrm{inv}}(M)$ could be empty. Below we will be interested in a different class of metrics, which is always nonempty for any complex homogeneous manifold.

Definition 5.6 (Submersion metrics)**.** Let h be an Hermitian metric on the Lie algebra $\mathfrak{g}$ of a complex Lie group G. Metric h defines a unique right-invariant Hermitian metric on G, such that its restriction to $T^{1,0}_e G \simeq \mathfrak{g}$ coincides with h. Then there is a unique induced metric on G/H, which turns the projection $G \to G/H$ into an Hermitian submersion. We define $\mathcal{M}^{\mathrm{sub}}(M)$ to be the space of *submersion metrics*. We have a map

$$\{\text{Hermitian metrics on } \mathfrak{g}\}/\mathrm{Ad}_G \longrightarrow \mathcal{M}^{\mathrm{sub}}(M).$$

There is an alternative description of submersion metrics. The holomorphic tangent bundle of a complex homogeneous manifold M is globally generated by $\mathfrak{g} \subset H^0(M, T^{1,0}M)$ via the infinitesimal evaluation map. Hence, any Hermitian metric h on the vector space $\mathfrak{g}$ induces a metric on its quotient $T^{1,0}M$. Note that typically the submersion Hermitian metrics are not G-homogeneous.

In general, there is no reason for the set $\mathcal{M}^{\mathrm{sub}}(M)$ to be invariant under a metric flow. However, it turns out that the HCF (5.1) preserves $\mathcal{M}^{\mathrm{sub}}(M)$ (see Theorem 5.18 below). In some sense this is an expected result, since the HCF preserves Griffiths/dual-Nakano semipositivity (see Theorem 4.10, and examples in Section 4.4) and the metrics in $\mathcal{M}^{\mathrm{sub}}(M)$ are the only known dual-Nakano semipositive metrics on a general complex homogeneous manifold (see Proposition 5.7 below). What is somewhat less expected, is that there exists an ODE on $\mathrm{Sym}^{1,1}(\mathfrak{g})$ — the set of Hermitian metrics on $\mathfrak{g}^*$, which induces the HCF on all G-homogeneous manifolds *independently of the isotropy subgroup H*.

The evolution term of this ODE is given by a generalization of a familiar operation # and turns out to be very similar to the ODE defined by the zero-order part of the evolution equation for the Riemannian curvature (resp. Chern curvature) under the Ricci flow (resp. HCF). Namely, in the case of the Ricci flow the curvature operator $R \in \mathrm{Sym}^2(\mathfrak{so}(n))$ evolves (in the moving frame) according to the equation

$$\frac{dR}{dt} = \Delta R + R^2 + R^{\#}$$

and the relevant ODE is

$$\frac{dR}{dt} = R^2 + R^{\#}.$$

For the HCF on the set of submersion metrics $\mathcal{M}^{\mathrm{sub}}(M)$, the ODE for $B \in \mathrm{Sym}^{1,1}(\mathfrak{g})$ takes form

$$\frac{dB}{dt} = B^{\#}.$$

This is a Riccati-type system of equations, as the evolution term $B^{\#}$ is quadratic in B.

In a special case when G is the complexification of a compact simple Lie group, we use the ODE for $B \in \mathrm{Sym}^{1,1}(\mathfrak{g})$ to construct scale-static solutions to the HCF on any homogeneous manifold of G. Such a metric corresponds to the Killing metric of the compact real form of G and can be thought of as the HCF-analogue of the Einstein metric. We explicitly solve the ODE in some simple examples (see Example 5.20 for the diagonal Hopf surface and Example 5.26 for the Iwasawa threefold) and formulate conjectural pinching behavior of this differential equation (Conjecture 5.21). It seems that at this point we need a better understanding of the algebraic properties of the operation $\#$ to resolve the conjecture and to understand the behavior of the HCF on $\mathcal{M}^{\mathrm{sub}}(G/H)$ for a general G .

In Section 5.2.3, we make the first step towards this understanding and study the blow-up behavior of ODE $\frac{dB}{dt} = B^{\#}$ on an arbitrary complex Lie group. Namely, we analyze how the growth rate of a solution $B(t)$ depends on the algebraic properties of $\mathfrak{g}$ (see Theorems 5.24 and 5.25).

5.2 Hermitian Curvature Flow of Submersion Metrics

5.2.1 Curvature of Submersion Metrics

In this section, we explicitly compute the Chern curvature of a submersion metric on a complex homogeneous manifold $M = G/H$. The short exact sequence of complex vector spaces

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$$

defines a short exact sequence of holomorphic vector bundles on M. Specifically, at $[\gamma H] \in M$ we have

$$0 \to \mathrm{Ad}_\gamma \mathfrak{h} \xrightarrow{i} \mathfrak{g} \xrightarrow{p} T^{1,0}_{[\gamma H]} M \to 0,$$

where $\mathfrak{g} \to M$ is a trivial bundle, and $\mathrm{Ad}(\mathfrak{h}) := \{\mathrm{Ad}_\gamma \mathfrak{h}\}_{[\gamma H] \in M}$ is its subbundle. The second fundamental form of this exact sequence

$$\beta \in \Lambda^{1,0}(M, \mathrm{Hom}(\mathrm{Ad}(\mathfrak{h}), T^{1,0} M))$$

can be computed as follows. For a fixed $w \in \mathfrak{h}$ and $\xi \in T^{1,0}M$, let $v \in \mathfrak{g}$ be a lift of ξ, i.e., $p(v) = \xi$. Consider a curve

$$t \mapsto [\exp(tv)\gamma H] \in M,$$

and a section of $\mathrm{Ad}(\mathfrak{h})$ over this curve

$$t \mapsto \mathrm{Ad}_{\exp(tv)\gamma} w = \mathrm{Ad}_{\exp(tv)} \mathrm{Ad}_\gamma w.$$

Using the flat connection on $\mathfrak{g}$, we find

$$\beta_\xi(\mathrm{Ad}_\gamma w) = p\left(\frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\exp(tv)} \mathrm{Ad}_\gamma w \right) = p([v, \mathrm{Ad}_\gamma w]).$$

It is easy to see that the expression on the right hand side does not depend on the lift of $\xi \in T^{1,0}_{[\gamma H]}M$ to $v \in \mathfrak{g}$.

Let $h \in \mathrm{Sym}^{1,1}(\mathfrak{g}^*)$ be an Hermitian metric on the Lie algebra of G. Then we have a submersion metric on M, which we denote by $g = p_* h$. Metric h also defines a metric $i^* h$ on $\mathrm{Ad}(\mathfrak{h})$, and the latter, together with $p_* h$, yield the adjoint of the second fundamental form

$$\beta^* \in \Lambda^{0,1}(M, \mathrm{Hom}(T^{1,0}M, \mathrm{Ad}_\gamma(\mathfrak{h}))).$$

According to the computation of Section 2.1.4 (see Corollary 2.25), the Chern curvature Ω of $(M, p_* h, J)$, considered as a section of $\Lambda^{1,1}(M) \otimes \Lambda^{1,1}(M)$, is given by

$$(p_* h)(\Omega(\xi, \overline{\eta})\zeta, \overline{\nu}) = (i^* h)(\beta^*_{\overline{\eta}}(\zeta), \overline{\beta^*_{\overline{\xi}}(\nu)}). \tag{5.2}$$

One obtains a much cleaner expression for the Chern curvature, if, following the convention adopted throughout this thesis, treats Ω as a section of $\mathrm{Sym}^{1,1}(\mathrm{End}(T^{1,0}M))$. In this case, (5.2) transforms into

$$\Omega = \mathrm{tr}_{\mathrm{Ad}_\gamma \mathfrak{h}}(\beta \otimes \overline{\beta}) := \sum_{i=1}^{\dim \mathfrak{h}} \beta(w_i) \otimes \overline{\beta(w_i)}, \tag{5.3}$$

where $\{w_i\}$ is an $(i^* h)$-orthonormal basis of $\mathrm{Ad}_\gamma \mathfrak{h}$.

Observe that by formula (5.3), the Chern curvature $\Omega \in \mathrm{Sym}^{1,1}(\mathrm{End}(T^{1,0}M))$ is semipositive in the dual-Nakano sense (this fact was already discussed at the end of Section 2.1.4). Now, let us compute the kernel of Ω at $[eH] \in G/H$. Endomorphism $u \in \mathrm{End}(T^{1,0}_{[eH]}M) \simeq \mathrm{End}(\mathfrak{g}/\mathfrak{h})$ lies in the kernel of

$$\langle \Omega, \cdot \otimes \overline{\cdot} \rangle_{\mathrm{tr}}$$

if an only if $\mathrm{tr}(\beta(w) \circ u) = 0$ for any $w \in \mathfrak{h}$. By the formula for β, this is equivalent to

$$\mathrm{tr}(\sigma_{\mathrm{ad}}(w) \circ u) = 0, \ \forall w \in \mathfrak{h}.$$

In particular, since β does not depend on the choice of metric on $\mathfrak{g}$, the kernel of $\langle \Omega, \cdot \otimes \overline{\cdot} \rangle_{\mathrm{tr}}$ also depends only on the pair $(\mathfrak{g}, \mathfrak{h})$ itself. We collect the observations above in the following proposition.

Proposition 5.7 (The kernel of Ω for a submersion metric)**.** *Let $M = G/H$ be a complex homogeneous manifold, equipped with a submersion metric $g = p_* h$, for $h \in \mathrm{Sym}^{1,1}(\mathfrak{g}^*)$. Then*

1. *the Chern curvature Ω of (M, g, J) is dual-Nakano semipositive;*
2. *the kernel K of $\langle \Omega, \cdot \otimes \overline{\cdot} \rangle_{\mathrm{tr}}$ at $[eH]$ is independent of h and is given by the* tr*-orthogonal complement of the image of isotropy representation $\sigma_{\mathrm{ad}}(\mathfrak{h}) \subset \mathrm{End}(\mathfrak{g}/\mathfrak{h})$:*

$$K = \{u \in \mathrm{End}(\mathfrak{g}/\mathfrak{h}) \mid \mathrm{tr}(\sigma_{\mathrm{ad}}(w) \circ u) = 0 \ \forall w \in \mathfrak{h}\}.$$

5.2.2 ODE on the Symmetric Square of a Lie Algebra

In this section, we explicitly compute the HCF on a complex homogeneous manifold, equipped with a submersion metric $g = p_* h$, $h \in \mathrm{Sym}^{1,1}(\mathfrak{g}^*)$. Denote by $\{s_\alpha\}$ a basis of holomorphic vector fields induced by one-parameter subgroups of G, i.e., $s_\alpha = p(e_\alpha)$, where $\{e_\alpha\}$ is the basis of $\mathfrak{g}$, and

$$p \colon \mathfrak{g} \to T^{1,0}M$$

is the evaluation map.

Let $s_\alpha = a^i_\alpha \, \partial/\partial z^i$ be the local coordinate expression for the vector field s_α, and denote $a^{\bar{i}}_{\bar{\alpha}} := \overline{a^i_\alpha}$. Functions a^i_α are holomorphic. In the coordinates, the submersion metric $g = g_{i\bar{j}}$ is given by the expression

$$g_{i\bar{j}} = \left(a^i_\alpha a^{\bar{j}}_{\bar{\beta}} h^{\alpha\bar{\beta}} \right)^{-1}. \tag{5.4}$$

Indeed, $a^i_\alpha a^{\bar{j}}_{\bar{\beta}} h^{\alpha\bar{\beta}}$ is the Hermitian metric on $\Lambda^{1,0}(M)$ induced by the inclusion $p^* \colon \Lambda^{1,0}(M) \to \mathfrak{g}^*$, and the Hermitian metric $g_{i\bar{j}}$ on $T^{1,0}M$ is its inverse.

Proposition 5.8. *Let $(G/H, g, J)$ be a complex homogeneous manifold, with a submersion metric $g = p_* h$, $h \in \mathrm{Sym}^{1,1}(\mathfrak{g}^*)$. Let $\{e_\alpha\}_{\alpha=1}^m$ be an h-orthonormal frame of $\mathfrak{g}$, with $s_\alpha := p_*(e_\alpha)$. Then* *the evolution term for the HCF considered as a section of* $\mathrm{Sym}^{1,1}(T^{1,0}M)$ *is given by*

$$\Psi(g) = \frac{1}{2}\sum_{\alpha,\beta=1}^{m} [s_\alpha, s_\beta] \otimes \overline{[s_\alpha, s_\beta]},$$

where $[\,\cdot\,,\,\cdot\,]$ *is the commutator of vector fields on* $M = G/H$.

Proof. First, we recall that by Proposition 3.9, the evolution term $\Psi(g)$ for a general metric g is

$$\Psi^{i\bar{j}} = g^{m\bar{n}}\partial_m\partial_{\bar{n}}g^{i\bar{j}} - \partial_m g^{i\bar{n}}\partial_{\bar{n}}g^{m\bar{j}}.$$

Next, since vector fields $s_\alpha = a^i_\alpha \partial/\partial z^i$ are holomorphic, all the derivatives $\overline{\partial} a^i_\alpha$, $\partial a^{\bar{j}}_{\bar{\beta}}$ vanish. Using this fact and the coordinate expression for Ψ, with the metric $g^{i\bar{j}}$ as in (5.4), we find

$$\begin{aligned}
\Psi(g)^{i\bar{j}} &= g^{m\bar{n}}\partial_m\partial_{\bar{n}}g^{i\bar{j}} - \partial_m g^{i\bar{n}}\partial_{\bar{n}}g^{m\bar{j}} \\
&= h^{\gamma\bar{\delta}}a^m_\gamma a^{\bar{n}}_{\bar{\delta}} h^{\alpha\bar{\beta}}\partial_m a^i_\alpha \partial_{\bar{n}} a^{\bar{j}}_{\bar{\beta}} - h^{\gamma\bar{\delta}}\partial_m a^i_\gamma a^{\bar{n}}_{\bar{\delta}} h^{\alpha\bar{\beta}} a^m_\alpha \partial_{\bar{n}} a^{\bar{j}}_{\bar{\beta}} \\
&= h^{\alpha\bar{\beta}}h^{\gamma\bar{\delta}}\big(a^m_\gamma\partial_m a^i_\alpha \cdot a^{\bar{n}}_{\bar{\delta}}\partial_{\bar{n}} a^{\bar{j}}_{\bar{\beta}} - a^m_\alpha\partial_m a^i_\gamma \cdot a^{\bar{n}}_{\bar{\delta}}\partial_{\bar{n}} a^{\bar{j}}_{\bar{\beta}}\big) \\
&= \frac{1}{2}h^{\alpha\bar{\beta}}h^{\gamma\bar{\delta}}\big(a^m_\gamma\partial_m a^i_\alpha - a^m_\alpha\partial_m a^i_\gamma\big)\cdot\big(a^{\bar{n}}_{\bar{\delta}}\partial_{\bar{n}} a^{\bar{j}}_{\bar{\beta}} - a^{\bar{n}}_{\bar{\beta}}\partial_{\bar{n}} a^{\bar{j}}_{\bar{\delta}}\big).
\end{aligned} \tag{5.5}$$

In the last equality we used the fact that the whole expression is symmetric under the change $(\alpha\bar{\beta}) \leftrightarrow (\gamma\bar{\delta})$. The last two multiples in the last expressions are exactly the i and $\bar{j}$ coordinates of the Lie brackets $[a^m_\gamma\partial/\partial z^m, a^m_\alpha\partial/\partial z^m]$ and $[a^{\bar{n}}_{\bar{\delta}}\partial/\partial\overline{z^n}, a^{\bar{n}}_{\bar{\beta}}\partial/\partial\overline{z^n}]$. Since $\{e_\alpha\}_{\alpha=1}^m$ is an h-orthonormal basis of $\mathfrak{g}$, we get the stated formula. $\square$

The expression (5.5) for $\Psi(g)$ suggests that the HCF with $g_0 = p_* h \in \mathcal{M}^{\mathrm{sub}}(M)$ is governed by the Lie algebra structure on the space of holomorphic vector fields $H^0(M, T^{1,0}M)$. To study this relation, we introduce an algebraic operation, generalizing operation # on the space of the Chern curvature tensors (Section 2.1.5).

Definition 5.9 (Operation #)**.** Let $\mathfrak{g}_\mathbb{R}$ be a real Lie algebra. Define a symmetric bilinear $\mathrm{ad}\,\mathfrak{g}_\mathbb{R}$-invariant operation

$$\#\colon \mathfrak{g}_\mathbb{R}^{\otimes 2} \otimes \mathfrak{g}_\mathbb{R}^{\otimes 2} \to \mathfrak{g}_\mathbb{R}^{\otimes 2},$$

by the formula

$$(v_1 \otimes v_2)\#(w_1 \otimes w_2) = [v_1, w_1] \otimes [v_2, w_2].$$

If we choose a basis $\{e_\alpha\}$ of $\mathfrak{g}_\mathbb{R}$ and denote by $c^\gamma_{\alpha\beta}$ its structure constants, then for $B = \{B^{\alpha\beta}\}$,

$D = \{D^{\alpha\beta}\}$, $B, D \in \mathfrak{g}_{\mathbb{R}}^{\otimes 2}$

$$(B\#D)^{\alpha\beta} = c^{\alpha}_{\epsilon\delta} c^{\beta}_{\gamma\theta} B^{\epsilon\gamma} D^{\delta\theta}.$$

Clearly, the operation $\#$ preserves the parity of the decomposition $\mathfrak{g}_{\mathbb{R}}^{\otimes 2} = \mathrm{Sym}^2(\mathfrak{g}_{\mathbb{R}}) \oplus \Lambda^2(\mathfrak{g}_{\mathbb{R}})$, i.e., defines the maps

$$\#\colon \mathrm{Sym}^2(\mathfrak{g}_{\mathbb{R}}) \otimes \mathrm{Sym}^2(\mathfrak{g}_{\mathbb{R}}) \to \mathrm{Sym}^2(\mathfrak{g}_{\mathbb{R}}),$$

$$\#\colon \Lambda^2(\mathfrak{g}_{\mathbb{R}}) \otimes \Lambda^2(\mathfrak{g}_{\mathbb{R}}) \to \mathrm{Sym}^2(\mathfrak{g}_{\mathbb{R}}),$$

$$\#\colon \Lambda^2(\mathfrak{g}_{\mathbb{R}}) \otimes \mathrm{Sym}^2(\mathfrak{g}_{\mathbb{R}}) \to \Lambda^2(\mathfrak{g}_{\mathbb{R}}).$$

Now, let $\mathfrak{g}$ be a complex Lie algebra. Similarly to the real case, we denote by the symbol $\#$ the map

$$\#\colon (\mathfrak{g} \otimes \bar{\mathfrak{g}}) \otimes (\mathfrak{g} \otimes \bar{\mathfrak{g}}) \to (\mathfrak{g} \otimes \bar{\mathfrak{g}}),$$

$$(v_1 \otimes \overline{v}_2)\#(w_1 \otimes \overline{w}_2) = [v_1, w_1] \otimes [\overline{v}_2, \overline{w}_2].$$

As in the real case, $\#$ preserves the parity of $\mathfrak{g} \otimes \bar{\mathfrak{g}}$, in particular, it induces a bilinear map on the set of Hermitian elements of $\mathfrak{g} \otimes \bar{\mathfrak{g}}$

$$\#\colon \mathrm{Sym}^{1,1}(\mathfrak{g}) \otimes \mathrm{Sym}^{1,1}(\mathfrak{g}) \to \mathrm{Sym}^{1,1}(\mathfrak{g}).$$

We will write $B^{\#}$ for $\frac{1}{2}B\#B$.

Remark 5.10. The operation $\#$ was introduced by Hamilton in the context of the Ricci flow, see [Ham86]. Definition 5.9 differs from the one of Hamilton in several aspects:

1. originally $\#$ was defined only for the real Lie algebra $\mathfrak{so}(n)$, while our definition makes sense for any Lie algebra;

2. Hamilton used $\#$ only for the part of $\mathrm{Sym}^2(\mathfrak{so}(n))$ satisfying the first Bianchi identity;

3. Hamilton used the Killing metric to interpret $\#$ as a bilinear operator on the space self-adjoint operators.

This operation and its algebraic properties play the key role in the characterization of compact manifolds with 2-positive curvature operator [BW08]. For an arbitrary real metric Lie algebra this operation was also considered by Wilking in [Wil13, §3].

Remark 5.11. Operation $\#$ is natural, i.e., if $\rho\colon \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras, then $\rho(B)\#\rho(D) = \rho(B\#D)$ for any $B, D \in \mathrm{Sym}^{1,1}(\mathfrak{g})$.

The following lemma easily follows from the definition by considering a basis of $\mathfrak{g}$ which diagonalizes the two forms.

Lemma 5.12. *Let $\mathfrak{g}$ be a real (resp. complex) Lie algebra. Assume that forms $B, D \in \mathrm{Sym}^2(\mathfrak{g})$ (resp. $B, D \in \mathrm{Sym}^{1,1}(\mathfrak{g})$) are symmetric (resp. Hermitian) positive definite. Then $B\#D$ is positive semidefinite with $\ker(B\#D) = \mathrm{Ann}([\mathfrak{g},\mathfrak{g}]) \subset \mathfrak{g}^*$.*

Example 5.13. Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(2)$ and denote by $\langle\ ,\ \rangle$ the invariant metric on $\mathfrak{g}_{\mathbb{R}}$ normalized in such a way that $\langle [e_1, e_2], e_3\rangle = \pm 1$ for any orthonormal triple e_1, e_2, e_3. Take $B_{\mathbb{R}} \in \mathrm{Sym}^2(\mathfrak{g}_{\mathbb{R}})$ and chose a $\langle\ ,\ \rangle$-orthonormal basis e_1, e_2, e_3, which diagonalizes $B_{\mathbb{R}}$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Then

$$\begin{aligned} B_{\mathbb{R}}^{\#} &= 1/2(\lambda_1 e_1 \otimes e_1 + \lambda_2 e_2 \otimes e_2 + \lambda_3 e_3 \otimes e_3)\#(\lambda_1 e_1 \otimes e_1 + \lambda_2 e_2 \otimes e_2 + \lambda_3 e_3 \otimes e_3) \\ &= (\lambda_2\lambda_3 e_1 \otimes e_1 + \lambda_1\lambda_3 e_2 \otimes e_2 + \lambda_1\lambda_2 e_3 \otimes e_3) \end{aligned}$$

is diagonalized in the same basis with the eigenvalues $\lambda_2\lambda_3, \lambda_1\lambda_3, \lambda_1\lambda_2$. In particular, if $B_{\mathbb{R}}$ is proportional to the dual of the metric $\langle\ ,\ \rangle$ then so is $B_{\mathbb{R}}^{\#}$.

Example 5.14. Let $\mathfrak{g}_{\mathbb{R}}$ be a compact simple Lie algebra with an invariant metric $\langle\ ,\ \rangle$. Let $B_{\mathbb{R}} = \langle\ ,\ \rangle^{-1}$ be the dual of $\langle\ ,\ \rangle$, i.e., for an orthonormal basis $e_1, \dots, e_m$ let $B_{\mathbb{R}} = \sum e_i \otimes e_i$. Then $B_{\mathbb{R}}^{\#}$ is proportional to $B_{\mathbb{R}}$ with a positive factor.

Indeed, since $\mathfrak{g}_{\mathbb{R}}$ is simple, $[\mathfrak{g}_{\mathbb{R}}, \mathfrak{g}_{\mathbb{R}}] = \mathfrak{g}_{\mathbb{R}}$, and by Lemma 5.12 both $B_{\mathbb{R}}$ and $B_{\mathbb{R}}^{\#}$ are positive definite $\mathrm{ad}\,\mathfrak{g}_{\mathbb{R}}$-invariant elements in $\mathrm{Sym}^2(\mathfrak{g}_{\mathbb{R}})$. As $\mathfrak{g}_{\mathbb{R}}$ is simple, such an element is unique up to multiplication by a positive constant, so $B_{\mathbb{R}}^{\#} = \lambda B_{\mathbb{R}}$, $\lambda > 0$.

We expect that proportionality $B^{\#} = \lambda B$ characterizes the $\mathrm{ad}\,\mathfrak{g}_{\mathbb{R}}$-invariant positive definite forms on any compact simple Lie algebra.

Question 5.15. *Let $\mathfrak{g}_{\mathbb{R}}$ be a compact simple real Lie algebra with an invariant metric $\langle\ ,\ \rangle$. Let $B \in \mathrm{Sym}^2(\mathfrak{g}_{\mathbb{R}})$ be a positive definite form such that $B^{\#} = \lambda B$. Is B^{-1} proportional to $\langle\ ,\ \rangle$?*

Remark 5.16. If $\mathfrak{g}_{\mathbb{R}}$ is a real Lie algebra equipped with an invariant metric $\langle\ ,\ \rangle$ extended in an obvious way to all tensor products of $\mathfrak{g}_{\mathbb{R}}$, then a trilinear form

$$P(B, C, D) := \langle B\#C, D\rangle, \quad B, C, D \in {\mathfrak{g}_{\mathbb{R}}}^{\otimes 2}.$$

is totally symmetric. It follows from the coordinate expression for $B\#C$ through the structure constants and the fact that in any $\langle\ ,\ \rangle$-orthonormal basis the structure constants are totally skew-symmetric.

Remark 5.17. If $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$ is the complexification of a real Lie algebra, then a real symmetric form $B_{\mathbb{R}} \in \mathrm{Sym}^2(\mathfrak{g}_{\mathbb{R}})$ defines an Hermitian form $\varphi(B_{\mathbb{R}}) \in \mathrm{Sym}^{1,1}(\mathfrak{g})$: in the basis $\{e_\alpha\}$ of $\mathfrak{g}_{\mathbb{R}}$ and the corresponding basis of $\mathfrak{g}$ this form is given by $\varphi(B_{\mathbb{R}})_{\alpha\overline{\beta}} := (B_{\mathbb{R}})_{\alpha\beta}$. It is clear that $\varphi(B_{\mathbb{R}})\#\varphi(B_{\mathbb{R}}) = \varphi(B_{\mathbb{R}}\#B_{\mathbb{R}})$.

Let us turn back to a complex homogeneous manifold $M = G/H$, equipped with a submersion metric $g = p_*h$, $h \in \mathrm{Sym}^{1,1}(\mathfrak{g}^*)$. Note that if a group K acts on $M = G/H$ and actions of G and K commute, then the metric g is K-invariant. We will use this observation in Example 5.20 below.

With Proposition 5.8 and Definition 5.9 we can reduce the HCF on a complex homogeneous manifold (M, g_0, J) to an ODE for $h^{-1} = B \in \mathrm{Sym}^{1,1}(\mathfrak{g})$.

Theorem 5.18 (HCF of a submersion metric)**.** *Let $M = G/H$ be a complex homogeneous manifold equipped with a submersion Hermitian metric $g_0 = p_*h_0 \in \mathcal{M}^{\mathrm{sub}}(M)$, where $h_0 \in \mathrm{Sym}^{1,1}(\mathfrak{g}^*)$. Let $B(t)$ be the solution to the ODE*

$$\begin{cases} \dfrac{dB}{dt} = B^{\#}, \\ B(0) = h_0^{-1}. \end{cases} \tag{5.6}$$

Then $g(t) = p_*(B(t)^{-1})$ solves the HCF on (M, g_0, J). In particular $g(t) \in \mathcal{M}^{\mathrm{sub}}(M)$.

Proof. If $B(t)$ solves (5.6), then by Proposition 5.8 the Hermitian metric $\tilde{g}(t)$ on $\Lambda^{1,0}(M)$, induced from $B(t) \in \mathrm{Sym}^{1,1}(\mathfrak{g})$ via the map $\Lambda^{1,0}(M) \to \mathfrak{g}^*$, satisfies the partial differential equation

$$\begin{cases} \dfrac{d\tilde{g}}{dt} = \Psi(\tilde{g}^{-1}), \\ \tilde{g}(0) = g_0^{-1}, \end{cases}$$

where, as in Proposition 3.9, Ψ is identified with a section of $\mathrm{Sym}^{1,1}(T^{1,0}M)$. Hence $g(t) = \tilde{g}(t)^{-1}$ is the solution to the HCF on (M, g_0, J). $\square$

Surprising consequence of this theorem is that ODE (5.6) gives solutions to the HCF on all G-homogeneous manifolds $M = G/H$ equipped with a submersion metric independently of the isotropy subgroup H.

Example 5.19. Let $G = SL(2, \mathbb{C})$. The Lie algebra of G has the compact real form $\mathfrak{su}(2)$, i.e., $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{su}(2) \otimes \mathbb{C}$. Assume that $B_0 \in \mathrm{Sym}^{1,1}(\mathfrak{sl}(2, \mathbb{C}))$ corresponds to $B_{\mathbb{R}} \in \mathrm{Sym}^{1,1}(\mathfrak{su}(2))$ (see

Remark 5.17). Then ODE (5.6) reduces to the equation for $B_{\mathbb{R}} \in \mathrm{Sym}^2(\mathfrak{su}(2))$

$$\frac{dB_{\mathbb{R}}}{dt} = B_{\mathbb{R}}^{\#}.$$

Let $\langle\ ,\ \rangle$ be a positive definite multiple of the Killing form of $\mathfrak{su}(2)$. Assume that $\langle\ ,\ \rangle$ is normalized in such way that for any orthonormal basis e_1, e_2, e_3 we have $\langle [e_1, e_2], e_3\rangle = \pm 1$.

Let e_1, e_2, e_3 be an orthonormal basis of $\mathfrak{su}(2)$ diagonalizing $B_{\mathbb{R}}$. Denote the eigenvalues of $B_{\mathbb{R}}$ with respect to $\langle\ ,\ \rangle$ by $\lambda_1, \lambda_2, \lambda_3$. In this basis the evolution equation takes the form

$$\begin{cases} \dfrac{d\lambda_1}{dt} = \lambda_2\lambda_3 \\ \dfrac{d\lambda_2}{dt} = \lambda_1\lambda_3 \\ \dfrac{d\lambda_3}{dt} = \lambda_1\lambda_2. \end{cases} \tag{5.7}$$

These equations imply that $\det B_{\mathbb{R}} = \lambda_1\lambda_2\lambda_3$ satisfies

$$\frac{d \det B_{\mathbb{R}}}{dt} = (\lambda_2\lambda_3)^2 + (\lambda_1\lambda_3)^2 + (\lambda_1\lambda_2)^2 \geqslant 3(\det B_{\mathbb{R}})^{4/3}.$$

So $\det B_{\mathbb{R}}(t) \geqslant 1/(C-t)^3$ for some $C > 0$, and the solution of (5.7) blows up as $t \to t_{\max} < \infty$. Moreover, for any $i, j \in \{1, 2, 3\}$

$$\frac{d}{dt}(\lambda_i^2 - \lambda_j^2) = 0,$$

hence all $\lambda_i \to +\infty$ as $t \to t_{\max}$, $i \in \{1, 2, 3\}$ and $\lambda_i/\lambda_j \to 1$. It follows that $B_{\mathbb{R}}(t)$ pinches towards the (dual of the) Killing form:

$$B_{\mathbb{R}}(t)/|B_{\mathbb{R}}(t)|_\infty \to \langle\ ,\ \rangle^{-1} = e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3.$$

Example 5.20 (Diagonal Hopf surface). Diagonal Hopf surface is the quotient $M = \big(\mathbb{C}^2 \backslash (0,0)\big)/\Gamma$, where the generator of $\Gamma \simeq \mathbb{Z}$ acts as $(z^1, z^2) \mapsto (\mu z^1, \mu z^2)$ for some $\mu \in \mathbb{C}$ with $|\mu| > 1$. M is a compact complex manifold diffeomorphic to $S^3 \times S^1$. Note that the natural action of $SL(2, \mathbb{C})$ on $\mathbb{C}^2$ commutes with Γ, hence descends to the transitive action on M.

As in Example 5.19, any element $B_{\mathbb{R}} \in \mathrm{Sym}^2(\mathfrak{su}(2))$ defines a metric g_0 on M. By the computation of Example 5.19 and Theorem 5.18, under the HCF this metric converges after normalization to the metric $g_\infty = p_*(\langle\ ,\ \rangle)$.

In coordinates z^1, z^2 on $\mathbb{C}^2\backslash\{0\} \subset \mathbb{C}^2$ this metric is given by

$$g_\infty^{i\bar{j}} = \frac{1}{|z|^4}(\delta^{ij}|z|^2 + \bar{z}^i z^j).$$

Example 5.19 demonstrates the expected behavior of the ODE (5.6) for any complex Lie group G with a simple compact real form. Namely, assume that $\mathfrak{g} = \mathfrak{g}_\mathbb{R} \otimes \mathbb{C}$ is the complexification of a simple compact real Lie algebra $\mathfrak{g}_\mathbb{R}$ with an invariant metric $\langle\ ,\ \rangle$. Let us denote by

$$\kappa \in \mathrm{Sym}^{1,1}(\mathfrak{g})$$

the element corresponding to $\langle\ ,\ \rangle^{-1} \in \mathrm{Sym}^2(\mathfrak{g}_\mathbb{R})$ (as in Remark 5.17). We refer to κ as the Killing form. For such G and κ we propose the following conjecture.

Conjecture 5.21. *Let $B(t)$ be the solution to the ODE* (5.6) *on the maximal time interval* $[0, t_{\max})$*. Then there exists $\gamma \in G$, $\lambda \in \mathbb{R}$ such that $B(t)$ pinches towards $\lambda \mathrm{Ad}_\gamma(\kappa)$:*

$$B(t)/|B(t)|_\infty \to \lambda \mathrm{Ad}_\gamma(\kappa),\ \ t \to t_{\max}.$$

Let complex Lie group G and $\kappa \in \mathrm{Sym}^{1,1}(\mathfrak{g})$ be as above. The following result demonstrates that the submersion metric induced by κ on a G-homogeneous manifold, is HCF-Einstein, i.e., scale-static under the flow. This observation provides some evidence for Conjecture 5.21 to be true.

Theorem 5.22. *Let G be the complexification of a simple compact Lie group. Let $M = G/H$ be a complex homogeneous manifold equipped with the Hermitian metric $g_0 = p_*(\kappa^{-1})$. Then g_0 is scale-static under the HCF, i.e., $\Psi(g_0) = \lambda g_0$ for some positive constant λ.*

Proof. In Example 5.14 we observed that for $B_\mathbb{R} = \langle\ ,\ \rangle^{-1}$

$$B_\mathbb{R}^\# = \lambda B_\mathbb{R}.$$

Hence for $\kappa = \varphi(B_\mathbb{R})$ (see Remark 5.17) we have $\kappa^\# = \lambda\kappa$. This fact together with Theorem 5.18 imply that g_0 is scale static under the HCF. $\square$

5.2.3 Blow-up Behavior of the ODE

In this section, we study the HCF on a complex Lie group G, equipped with a submersion metric $g_0 \in \mathcal{M}^{\mathrm{sub}}(G)$ (g_0 is identified with its restriction to $\mathfrak{g} \simeq T_{\mathrm{id}}G$). By Theorem 5.18 the HCF reduces to the ODE for $B(t) \in \mathrm{Sym}^{1,1}(\mathfrak{g})$ with $B_0 = g_0^{-1}$

$$\begin{cases} \dfrac{dB}{dt} = B^{\#}, \\ B(0) = B_0. \end{cases} \tag{5.8}$$

It turns out, that the growth rate of a solution $B(t)$ is completely determined by the algebraic properties of the underlying Lie algebra. Namely, $B(t)$ has polynomial, exponential growth, or a finite time blow-up, depending on whether $\mathfrak{g}$ is nilpotent, solvable, or admits a semisimple quotient.

Before we state and prove the results on the growth rate of a solution to (5.8) let us make the following elementary observation.

Proposition 5.23. *Let $B(t), D(t) \in \mathrm{Sym}^{1,1}(\mathfrak{g})$ be the solutions to* (5.8) *with the initial conditions $B(0) = B_0$ and $D(0) = D_0$ such that $B_0 \geqslant D_0 \geqslant 0$. Then for $t \geqslant 0$*

$$B(t) \geqslant D(t).$$

Proof. We claim that for $B, D \in \mathrm{Sym}^{1,1}(\mathfrak{g})$ if $B \geqslant D \geqslant 0$, then $B^{\#} \geqslant D^{\#}$. Indeed, let $\{e_i\}$ be a basis diagonalizing simultaneously B and D:

$$B = \sum a_i e_i \otimes \bar{e}_i, \quad k = \sum b_i e_i \otimes \bar{e}_i$$

with $a_i \geqslant b_i \geqslant 0$. Then $B^{\#} = \sum_{i,j} a_i a_j [e_i, e_j] \otimes \overline{[e_i, e_j]} \geqslant \sum_{i,j} b_i b_j [e_i, e_j] \otimes \overline{[e_i, e_j]} = D^{\#}$.

Now, fix a background positive-definite form $I \in \mathrm{Sym}^{1,1}(\mathfrak{g})$, and define

$$\rho(t) = \sup_{\xi \in \mathfrak{g}^*,\ I(\xi, \bar{\xi}) = 1} \big(D(t)(\xi, \bar{\xi}) - B(t)(\xi, \bar{\xi}) \big) \tag{5.9}$$

We have $\rho(0) \leqslant 0$, $D(t) \leqslant B(t) + \rho(t) I$. In particular, by the claim above, $D^{\#}(t) \leqslant B^{\#}(t) + \rho(t) I \# B + \rho^2(t) I^{\#}$. As in the proof of Hamilton's maximum principle,

$$\frac{d\rho}{dt} \leqslant \sup\big(D^{\#}(t)(\xi, \bar{\xi}) - B^{\#}(t)(\xi, \bar{\xi})\big),$$

where the supremum is taken over all such ξ that in (5.9) the supremum is achieved. Therefore, on any fixed time interval with $|\rho(t)|$ bounded, we have

$$\frac{d\rho}{dt} \leqslant C_1 \rho(t) + C_2 \rho^2(t) \leqslant C |\rho(t)|$$

for some constant C. Hence $\rho(t) \leqslant 0$, provided $\rho(0) \leqslant 0$. $\square$

Theorem 5.24. *For a complex Lie algebra $\mathfrak{g}$ and a positive definite Hermitian form $B_0 \in \mathrm{Sym}^{1,1}(\mathfrak{g})$ let $B(t)$ be the solution to the ODE* (5.8) *on the maximal time interval $[0; t_{\max})$, $0 < t_{\max} \leqslant +\infty$. Then the following are equivalent:*

1. *$\mathfrak{g}$ is a nilpotent Lie algebra;*

2. *for any initial data B_0, the solution $B(t)$ has at most* polynomial growth, *i.e., $t_{\max} = +\infty$, and there exists a polynomial p such that*

$$B(t) < p(t)B_0;$$

3. *for some initial data B_0 the solution $B(t)$ has* subexponential growth, *i.e., $t_{\max} = +\infty$, and for any $\epsilon > 0$ there exists $T_\epsilon > 0$ such that for $t > T_\epsilon$*

$$B(t) < e^{\epsilon t}B_0.$$

Proof. We prove implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

$1 \Rightarrow 2$. By Ado's theorem for nilpotent Lie algebras [Hoc66] there exits a faithful representation of $\mathfrak{g}$ into some $\mathfrak{gl}(V)$ such that $\mathfrak{g}$ acts by nilpotent endomorphisms. With the use of the basic theory of Lie algebras [Hum73, §3.3] one can assume that the image of this representation lies in $\mathfrak{n}(n)$ — the Lie algebra of strictly upper-triangular $n \times n$ matrices:

$$\rho\colon \mathfrak{g} \to \mathfrak{n}(n).$$

We extend ρ to a map $\rho\colon \mathrm{Sym}^{1,1}(\mathfrak{g}) \to \mathrm{Sym}^{1,1}(\mathfrak{n}(n))$ in the obvious way.

Let $\{E_{i,j} | 1 \leqslant i < j \leqslant n\}$ be the elementary matrices spanning $\mathfrak{n}(n)$. We fix a collection of positive real numbers $\{f_0^{(k)}\}_{k=1}^{n-1}$ such that the Hermitian form $f_0 \in \mathrm{Sym}^{1,1}(\mathfrak{n}(n))$

$$f_0 := \sum_{1 \leqslant i < j \leqslant n} f_0^{(j-i)} E_{i,j} \otimes \overline{E_{i,j}}$$

is greater than $\rho(B_0)$. Consider the solution $f(t) \in \mathrm{Sym}^{1,1}(\mathfrak{n}(n))$ to the ODE

$$\frac{df}{dt} = f^{\#}$$

with the initial condition $f(0) = f_0$. After expanding the definition of $f^{\#}$ we see that this ODE is equivalent to a system of $n-1$ scalar equations

$$\frac{df^{(k)}}{dt} = \frac{1}{2}\sum_{j=1}^{k-1} f^{(j)} f^{(k-j)}, \quad k = 1, \dots, n-1.$$

Solving these equations inductively for $k = 1, \dots, n-1$ we get $f^{(k)}(t) = p_{k-1}(t)$, where $p_{k-1}(t)$ is a polynomial of degree $(k-1)$.

Hermitian forms $\rho(B(t))$ and $f(t)$ satisfy the same ODE with the initial conditions $\rho(B_0) < f_0$. Therefore Proposition 5.23 implies that $\rho(B(t)) \leqslant f(t)$. Since $f(t)$ has polynomial growth, we get

$$\rho(B(t)) < p(t) f_0$$

for some polynomial $p(t)$. Finally, using the fact that ρ is faithful and $B_0 \in \mathrm{Sym}^{1,1}(\mathfrak{g})$ is positive definite, we find a constant C such that

$$B(t) < Cp(t)B_0.$$

As $B(t)$ is bounded on any interval $[0, t_{\max})$, the solution extends to the whole $[0; +\infty)$.

$2 \Rightarrow 3$. Is trivially true.

$3 \Rightarrow 1$. Assume that 1 does not hold, and $\mathfrak{g}$ is not nilpotent. Then by Engel's theorem for some $x \in \mathfrak{g}$ the operator ad_x is not nilpotent. Hence the map $\mathrm{ad}_x \colon \mathfrak{g} \to \mathfrak{g}$ has non-zero eigenvalue λ:

$$[x, y] = \lambda y, y \neq 0.$$

Consider $f_0 := a_0 x \otimes \overline{x} + b_0 y \otimes \overline{y} \in \mathrm{Sym}^{1,1}(\mathfrak{g})$. Note that $f_0^{\#} = |\lambda|^2 a_0 b_0 y \otimes \overline{y}$, hence for the functions $a(t), b(t)$ with $a(0) = a_0$, $b(0) = b_0$ satisfying

$$\frac{da}{dt} = 0, \quad \frac{db}{dt} = |\lambda|^2 ab$$

the form $f(t) = a(t) x \otimes \overline{x} + b(t) y \otimes \overline{y}$ solves the ODE (5.8). Explicitly these functions are given by $a(t) = a_0$, $b(t) = b_0 e^{|\lambda|^2 a_0 t}$. If positive numbers a_0, b_0 are small enough, one has $f_0 < B_0$, hence by Proposition 5.23 $f(t) < B(t)$. Therefore $B(t)$ cannot have subexponential growth. Contradiction. $\square$

Theorem 5.25. *For a complex Lie algebra $\mathfrak{g}$ and a positive definite Hermitian form $B_0 \in \mathrm{Sym}^{1,1}(\mathfrak{g})$, let $B(t)$ be the solution to the ODE* (5.8) *on the maximal time interval* $[0; t_{\max})$*, $0 < t_{\max} \leqslant +\infty$.*

Then the following are equivalent:

1. $\mathfrak{g}$ *is a solvable Lie algebra;*

2. *for any initial data* B_0*, the solution* $B(t)$ *has at most* exponential growth*, i.e.,* $t_{\max} = +\infty$*, and there exist constants* C, K *such that*

$$B(t) < Ce^{Kt} B_0;$$

3. *for some initial data* B_0*, the solution* $B(t)$ *exists on* $[0, +\infty)$*, i.e.,* $t_{\max} = +\infty$*.*

Proof. The proof is essentially analogous to Theorem 5.24. We prove implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

$1 \Rightarrow 2$. By Ado's theorem there exits a faithful representation of $\mathfrak{g}$ into some $\mathfrak{gl}(V)$, and by Lie's theorem one can assume that the image of this representation lies in $\mathfrak{b}(n)$ — the Borel subalgebra of $\mathfrak{gl}(V)$, consisting of upper-triangular $n \times n$ matrices:

$$\rho \colon \mathfrak{g} \to \mathfrak{b}(n).$$

Let $\{E_{i,j} | 1 \leqslant i \leqslant j \leqslant n\}$ be the elementary matrices spanning $\mathfrak{b}(n)$. We fix a collection of positive real numbers $\{f_0^{(k)}\}_{k=0}^{n-1}$ such that the Hermitian form $f_0 \in \mathrm{Sym}^{1,1}(\mathfrak{b}(n))$

$$f_0 := \sum_{1 \leqslant i \leqslant j \leqslant n} f_0^{(j-i)} E_{i,j} \otimes \overline{E_{i,j}}$$

is greater than $\rho(B_0)$. Consider the solution $f(t) \in \mathrm{Sym}^{1,1}(\mathfrak{b}(n))$ to the ODE (5.8) with the initial condition $f(0) = f_0$. After expanding the definition of $f^{\#}$ we see that this ODE is equivalent to a system of $n - 1$ scalar equations

$$\frac{df^{(k)}}{dt} = \frac{1}{2} \sum_{j=0}^{k} f^{(j)} f^{(k-j)}, \quad k = 1, \ldots, n-1$$

with $f^{(0)}(t) \equiv f_0^{(0)}$. Solving these equations inductively for $k = 1, \ldots, n-1$ we get $f^{(k)}(t) = q_k(e^{f_0^{(0)} t})$, where $q_k(t)$ is a polynomial of degree k with $q_k(0) = 0$.

Hermitian forms $\rho(B(t))$ and $f(t)$ solve the same ODE with the initial conditions $\rho(B_0) < f_0$. Therefore Proposition 5.23 implies that $\rho(B(t)) \leqslant f(t)$. Since $f(t)$ has exponential growth, we get

$$\rho(B(t)) < C_0 e^{Kt} f_0$$

for some constants C_0, K. Finally, using the fact that ρ is faithful and $B_0 \in \mathrm{Sym}^{1,1}(\mathfrak{g})$ is positive definite, we find a constant C such that

$$B(t) < Ce^{Kt}B_0.$$

As $B(t)$ is bounded on any interval $[0, t_{\max})$, the solution extends to the whole $[0; +\infty)$.

$2 \Rightarrow 3$. Is trivially true.

$3 \Rightarrow 1$. Assume that 1 does not hold, and $\mathfrak{g}$ is not solvable. Denote by $Rad(\mathfrak{g})$ the maximal solvable ideal. Then the quotient $\mathfrak{g}/Rad(\mathfrak{g})$ is semisimple Lie algebra and has a simple summand $\mathfrak{g}_0$. So there is a surjective homomorphism onto a simple Lie algebra.

$$\rho\colon \mathfrak{g} \to \mathfrak{g}_0.$$

As in the set-up for Conjecture 5.21 let $\kappa \in \mathrm{Sym}^{1,1}(\mathfrak{g}_0)$ be a positive-definite Hermitian form corresponding to the Killing metric of the compact real form of $\mathfrak{g}_0$. Then according to Example 5.14 and Theorem 5.22 $\kappa^{\#} = \lambda\kappa$ for some $\lambda > 0$.

Now, let $B(t) \in \mathrm{Sym}^{1,1}(\mathfrak{g})$ be a solution to the ODE (5.8) defined on $[0, +\infty)$. Choose $\epsilon_0 > 0$ such that $\epsilon_0\kappa < \rho(B(0))$. If $\epsilon(t)$ satisfies the equation

$$\frac{d\epsilon}{dt} = \lambda\epsilon^2, \quad \epsilon(0) = \epsilon_0,$$

then $f(t) = \epsilon(t)\kappa$ solves the ODE (5.8) for $f(t) \in \mathrm{Sym}^{1,1}(\mathfrak{g}_0)$ with the initial data $f(0) = \epsilon_0\kappa$. Explicitly we have

$$\epsilon(t) = \frac{\epsilon_0}{1 - \epsilon_0\lambda t}.$$

On the one hand we have the solution $f(t)$ to (5.8) blowing up at the finite time $t = (\epsilon_0\lambda)^{-1}$, on the other hand $f(0) < \rho(B(0))$, hence by Proposition 5.23 for any $t \geqslant 0$

$$f(t) < \rho(B(t)).$$

Contradiction with the finiteness of $B(t)$ for all $t \in [0, +\infty)$. □

Example 5.26 (Iwasawa manifold). Let G be the 3-dimensional complex Heisenberg group

$$G := \left\{ \left. \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \right| a, b, c \in \mathbb{C} \right\}.$$

and $\Gamma \subset G$ its discrete subgroup, consisting of matrices with $a, b, c \in \mathbb{Z}[i]$. The quotient $M = G/\Gamma$ is the *Iwasawa manifold*. M is a compact complex parallelizable manifold, i.e., the base of its Tits fibration is a point (see Theorem 5.3). M does not admit any Kähler metric; in fact, since M is not formal, there is no complex structure on M admitting a Kähler metric.

The Lie algebra of G is $\mathfrak{g} = \mathrm{span}(\partial_a, \partial_b, \partial_c) \simeq \mathfrak{n}(3)$. Consider $g_0 = p_*(B_0^{-1})$, where $B_0 \in \mathrm{Sym}^{1,1}(\mathfrak{g})$. Denote by $B(t)$ the solution to the ODE (5.8). Theorem 5.24 provides an explicit expression for $B(t)$ and, in particular, implies that $B(t)$ polynomially blows up as $t \to \infty$. In fact, since $B^{\#}$ is proportional to $\partial_b \otimes \overline{\partial_b}$ for any $B \in \mathrm{Sym}^{1,1}(\mathfrak{g})$, we see that $\partial_b \otimes \overline{\partial_b}$ is the only coordinate of $B(t)$, which blows up. For the solution $g(t) = p_*(B(t)^{-1})$ to the HCF this means that as $t \to \infty$

$$g(t)(\partial_b, \partial_b) \to 0, \quad g(t)|_{\mathrm{span}(\partial_a, \partial_c)} \equiv g(0)|_{\mathrm{span}(\partial_a, \partial_c)}.$$

To get a geometric picture, consider the Gromov-Hausdorff limit of $(G/\Gamma, g(t))$. It is easy to see that the projection onto coordinates a and c defines a holomorphic fibration

$$\pi\colon G/\Gamma \to \mathbb{C}/\mathbb{Z}[i] \times \mathbb{C}/\mathbb{Z}[i].$$

The fibers of π are the orbits of the flow generated by $\mathbb{C} \cdot \partial_c$. The limiting behavior of $g(t)$ implies that in the Gromov-Hausdorff limit, the fibers with the submersion metric uniformly collapse to a point as $t \to +\infty$ and G/Γ collapses to the product of elliptic curves:

$$(G/\Gamma, g(t)) \underset{GH}{\to} (\mathbb{C}/\mathbb{Z}[i] \times \mathbb{C}/\mathbb{Z}[i], g(0)|_{\mathrm{span}(\partial_a, \partial_c)}).$$

Using the computations of Theorem 5.24 one can show that the HCF exhibits a similar behavior on all complex nilmanifolds of the form G/Γ, where G is a complex nilpotent group, and $\Gamma \subset G$ is a cocompact lattice.

Chapter 6

Applications

In this chapter, we discuss applications of the results of Chapters 4 and 5. First, motivated by the preservation of the curvature positivity under the HCF (Theorem 4.10), we formulate a weak differential-geometric version of the Campana-Peternell conjecture. We use the HCF to make some initial progress in approaching this conjecture. Next, we use Theorem 4.10 to study geometric properties of the HCF on general Hermitian manifolds.

6.1 Weak Campana-Peternell Conjecture

6.1.1 Formulation

Recall that Campana-Peternell conjecture states, that any Fano manifold M with nef tangent bundle $T^{1,0}M$ is rational homogeneous. Motivated by the interplay between Frankel's and Hartshorne's conjectures, we propose the following conjecture.

Conjecture 6.1. *Any complex Fano manifold which admits an Hermitian metric of Griffiths/dual-Nakano semipositive curvature must be isomorphic to a rational homogeneous space.*

Explicit computations of Chapter 5 suggest a more refined version of the conjecture.

Conjecture 6.2 (Weak Campana-Peternell conjecture)**.** *Let (M, J) be a compact complex manifold, such that $M \not\simeq \mathbb{P}^n$, $M \not\simeq X \times Y$, $\dim X, \dim Y > 0$. Let M be equipped with an Hermitian metric, such that its Chern curvature dual-Nakano/Griffiths semipositive. Assume additionally that the first*

Chern-Ricci form

$$\rho = \frac{\sqrt{-1}}{2\pi}\Omega_{i\bar{j}}{}^{\bar{n}m} g_{m\bar{n}} dz^i \wedge d\overline{z^j}$$

is positive. Then (M, g, J) is isometric to a rational homogeneous manifold $(G/H, p_*h, J)$, with p_*h — a submersion metric induced from $h \in \mathrm{Sym}^{1,1}(\mathfrak{g}^*)$.

Proposition 6.3. *Any rational homogeneous manifold $(G/H, p_*h, J)$ satisfies the assumptions of the weak Campana-Peternell conjecture.*

Proof. By the results of Section 5.2.1, the Chern curvature of $(G/H, p_*h, J)$ equals

$$\mathrm{tr}_{\mathrm{Ad}(\mathfrak{h})}(\beta \otimes \overline{\beta}),$$

where $\beta \in \Lambda^{1,0}(M, \mathrm{Hom}(\mathrm{Ad}(\mathfrak{h}), T^{1,0}M))$ is the second fundamental form of the extension

$$0 \to \mathrm{Ad}(\mathfrak{h}) \to \mathfrak{g} \to T^{1,0}M \to 0.$$

In particular, $(G/H, p_*h, J)$ has a dual-Nakano semipositive curvature.

Now, prove the positivity of the Chern-Ricci form. Let $\xi \in T^{1,0}_m(G/H)$ be a vector in the kernel of ρ. For simplicity, assume $m = [eH]$. There is an exact sequence

$$0 \to \mathfrak{h} \xrightarrow{i} \mathfrak{g} \xrightarrow{p} T^{1,0}_{[eH]}(G/H) \to 0,$$

and by the formula for Ω, (5.3),

$$\rho(\xi, \overline{\xi}) = |\beta_\xi|^2_{\mathrm{Hom}(\mathfrak{h}, \mathfrak{g}/\mathfrak{h})}.$$

Pick $v \in \mathfrak{g}$, such that $\xi = p(v) \in \mathfrak{g}/\mathfrak{h}$. Then ξ lies in the kernel of ρ if and only if, for every $w \in \mathfrak{h}$

$$\beta_\xi(w) = p([v, w]) = 0.$$

This is equivalent to

$$[v, \mathfrak{h}] \subset \mathfrak{h},$$

i.e., $\exp(v) \in G$ normalizes $\mathfrak{h}$. But, by the structure result for complex homogeneous manifolds (Theorem 5.3), any rational homogeneous manifold G/H coincides with the base of its Tits fibration G/N. Therefore, the normalizer of H in G coincides with H itself, so $\exp(v) \in H$, and $v \in \mathfrak{h}$. Hence $0 = p(v) = \xi \in \mathfrak{g}/\mathfrak{h}$. □

The relation between Campana-Peternell conjecture and its weak version is similar to the relation between Hartshorne's and Frankel's conjecture. Namely, they characterize the same set of projective manifolds, while the assumption on the algebraic side (Hartshorne/Campana-Peternell) is weaker, than the assumption on the differential-geometric side (Frankel/weak Campana-Peternell):

$$\exists \text{ Kähler metric } g, \text{ s.t. } (T^{1,0}M, g) >_{\mathrm{Gr}} 0 \implies T^{1,0}M \text{ is ample}$$
$$\exists \text{ Hermitian metric } g, \text{ s.t. } (T^{1,0}M, g) \geqslant_{\mathrm{Gr}} 0 \implies T^{1,0}M \text{ is nef .}$$

In a certain sense, the weak Campana-Peternell conjecture is analogous to the generalized Frankel's conjecture. Namely, if we assume that M is not a product of two manifolds of smaller dimension and is not isomorphic to the projective space, then, under the corresponding curvature semipositivity assumption, we have a *rigidity result* for the metric:

1. According to the generalized Frankel's conjecture, the metric is uniquely determined by the underlying manifold, which is a rational symmetric space of rank $\geqslant 2$.

2. According to the weak Campana-Peternell conjecture, the metric is expected to belong to a finite-dimensional space of submersion metrics.

6.1.2 Uniformization of Griffiths Quasipositive Hermitian Manifolds

We use the strong maximum principle in the form of Corollary 4.14 together with Mori's solution to Hartshorne's conjecture to prove the following uniformization result.

Theorem 6.4. *Let (M, g, J) be a compact complex n-dimensional Hermitian manifold such that its Griffith curvature is* quasipositive, *i.e.,*

1. *the Chern curvature Ω^g is Griffiths semipositive;*

2. *Ω^g_m is Griffiths positive at some point $m \in M$.*

Then M is biholomorphic to the projective space $\mathbb{P}^n$.

Proof. Let $g(t)$ be a solution to the HCF on (M, g, J) with the initial data $g(0) = g$. By Corollary 4.14, applied to

$$S = \{u \in \mathrm{End}(V) \mid \mathrm{rank}(u) = 1\}, \quad F \equiv 0,$$

for any $t > 0$, Hermitian manifold $(M, g(t), J)$ has Griffiths positive Chern curvature. In particular, the first Chern-Ricci form

$$\rho = \sqrt{-1}\mathrm{tr}_{\mathrm{End}(T^{1,0}M)}\Omega, \quad \rho_{i\bar{j}} := \sqrt{-1}\Omega_{i\bar{j}k}{}^{k},$$

which represents (up to a factor of 2π) the first Chern class of the anticanonical bundle $-K_M$, is strictly positive. Therefore $-K_M$ is ample, and M is projective.

On the other hand, strict Griffiths-positivity of Ω, implies the ampleness of $T^{1,0}M$ (see Section 2.1.4). Therefore, we can apply the result of Mori [Mor79] and conclude that M is isomorphic (as a smooth projective manifold over $\mathbb{C}$) to the projective space $\mathbb{P}^n$. □

Remark 6.5. The use of the HCF is essential for the proof Theorem 6.4, since a priori, we know neither (1) whether M is algebraic, nor (2) if it admits an Hermitian metric of positive Griffiths curvature. From the assumptions on M it follows only that $-K_M$ is nef and big. Using a recent result of Yang [Yan17, Thm. 1.2], we can conclude that $-K_M$ is ample, under an additional assumption that M admits a Kähler metric.

It would be interesting to give a direct proof of Theorem 6.4, independently of Mori's results, e.g., along the lines of [CST09], where the authors used the limit of the Kähler-Ricci flow starting with a Griffiths positive metric, to prove that the underlying manifold is $\mathbb{P}^n$. However, such a proof will require substantially better understanding of the analytical properties of the HCF flow, than we have at the moment.

6.1.3 Geometry of Dual-Nakano Semipositive Hermitian Manifolds

In this section, (M, g, J) is an Hermitian manifold with a dual-Nakano semipositive curvature. We will not be able to resolve the weak Campana-Peternell conjecture, but we use the HCF to recover certain geometric structures on M, which are similar to the ones of complex homogeneous spaces.

Let $g(t), t \in [0, t_{\max})$ be the solution to the HCF on (M, g, J). Now, fix $t \in (0, t_{\max})$. Applying Theorem 4.12, we conclude that Hermitian form $\Omega^{g(t)} \in \mathrm{Sym}^{1,1}(\mathrm{End}(T^{1,0}M))$ has constant rank, and the kernel of $\langle \Omega, \cdot \otimes \bar{\cdot} \rangle_{\mathrm{tr}}$:

$$K := \{v \in \mathrm{End}(T^{1,0}M) \mid \langle \Omega, v \otimes \bar{\cdot} \rangle_{\mathrm{tr}} = 0\},$$

is invariant under the torsion-twisted parallel transport. The next theorem shows, that K is closely tied with the holonomy algebra of ∇^T and is not arbitrary.

Theorem 6.6. *Let $g(t)$ be the solution to the HCF on (M, g, J). Assume that $g(0)$ is dual-Nakano semipositive. Then for $t > 0$, at any point $m \in M$, subspace K is the* tr*-orthogonal complement of the restricted torsion-twisted holonomy subalgebra:*

$$K = \{v \in \mathrm{End}(T^{1,0}M) \mid \mathrm{tr}(v \circ w) = 0 \ \forall \ w \in \mathfrak{hol}_{\nabla^T}\}. \tag{6.1}$$

Proof. Let us denote the space on the right hand side of (6.1) by L. By Ambrose-Singer holonomy theorem [AS53], the complex holonomy Lie algebra at $m \in M$ is given by

$$(\mathfrak{hol}_{\nabla^T})_m = \mathrm{span}_{\mathbb{C}}\{\Gamma(\gamma)^*(\Omega_{\nabla^T}(X,Y))_{m'}\},$$

where the span is taken over all points $m' \in M$, all paths $\gamma\colon [0,1] \to M$ with the endpoints m, m', all tangent vectors $X, Y \in T_{m'}M$, and $\Gamma(\gamma)^*\colon \mathrm{Sym}^{1,1}(\mathrm{End}(T^{1,0}M))_{m'} \to \mathrm{Sym}^{1,1}(\mathrm{End}(T^{1,0}M))_m$ denotes the torsion-twisted parallel transport along γ.

The explicit formula for the (1,1)-type part of Ω_{∇^T} (Proposition 3.17) implies, that, if $\mathrm{tr}(v \circ w) = 0$ for any $w \in \mathfrak{hol}_{\nabla^T}$, then

$$\mathrm{tr}(v \circ \Omega_{\nabla^T}(\xi, \overline{\eta})) = 0 \text{ for any } \xi, \eta \in T^{1,0}M.$$

The latter is equivalent to $v \in K$, therefore, we have

$$L \subset K.$$

Conversely, consider $v \in K$ over $m \in M$. By Theorem 4.12, $\mathrm{tr}(v \circ \Omega_{\nabla^T}(X,Y)_m) = 0$ for any $X, Y \in T_mM$. Moreover, as K is invariant under the torsion-twisted parallel transport, we also have

$$\mathrm{tr}(\Gamma^*(\gamma)v \circ \Omega_{\nabla^T}(X,Y)_{m'}) = 0,$$

for any $m' \in M$, $X, Y \in T_{m'}M$, and a path γ from m to m'. Equivalently,

$$\mathrm{tr}(v \circ \Gamma^*(\gamma^{-1})(\Omega_{\nabla^T}(X,Y)_{m'})) = 0.$$

Therefore, by Ambrose-Singer theorem, $K \subset L$. ∎

Theorem 6.6 provides strong obstructions on the geometry of M. Since we are assuming that M is not isomorphic to $\mathbb{P}^n$, it cannot admit a dual-Nakano positive metric, therefore K is necessarily nonempty. By the theorem, it means that the $\mathbb{C}$-span of the torsion-twisted holonomy Lie algebra is necessarily a proper Lie subalgebra of $\mathrm{End}(T^{1,0}M)$:

$$\mathfrak{hol}_{\nabla^T} \subsetneq \mathrm{End}(T^{1,0}M).$$

In the Riemannian case, in a similar situation, application of Berger's holonomy theorem implies that either the underlying manifold locally splits as a product, or it is a symmetric space (see, e.g., [Mok88], [Gu09], [BS08]). In our case, the situation is much more subtle, since connection ∇^T is not metric and has nonzero torsion. Neither de Rham decomposition for the reducible Riemannian holonomy, nor Berger's holonomy theorem work in such generality. However, it is plausible that, under our assumptions on (M, g, J), connection ∇^T carries some additional special features. A potential approach to the weak Campana-Peternell conjecture is to answer the following question.

Question 6.7. *What are possible torsion-twisted holonomy groups on a non-homogeneous Hermitian manifold (M, g, J)?*

An answer to this question would provide an *Hermitian* generalization of the Berger holonomy theorem.

If we apply Theorem 6.6 to a homogeneous manifold equipped with a submersion metric, we obtain the following corollary.

Corollary 6.8. *Let $M = G/H$ be a complex homogeneous space with a submersion metric $g = p_* h_0$ and the isotropy representation*

$$\sigma_{\mathrm{ad}} \colon \mathfrak{h} \to \mathrm{End}(\mathfrak{g}/\mathfrak{h}).$$

Then for the complex holonomy Lie algebra of the torsion-twisted connection $\mathfrak{hol}_{\nabla^T} \subset \mathrm{End}(T^{1,0}M) \simeq \mathrm{End}(\mathfrak{g}/\mathfrak{h})$ we have

$$\mathfrak{hol}_{\nabla^T} = \sigma_{\mathrm{ad}}(\mathfrak{h}).$$

Proof. Of course, this corollary can be proved by a straightforward computation, since the torsion twisted connection of $(T(G/H), p_* h_0)$ can be found explicitly. Below, however, we give a slightly indirect argument based on the HCF.

For simplicity, we do all computations on $T^{1,0}_{[eH]}M \simeq \mathfrak{g}/\mathfrak{h}$. Let, as before, $K \subset \mathrm{End}(\mathfrak{g}/\mathfrak{h})$ be the kernel of $\langle \Omega^g, \cdot \otimes \bar{\cdot} \rangle_{\mathrm{tr}}$. Consider $h(t) \in \mathrm{Sym}^{1,1}(\mathfrak{g}^*)$, $t \in (-\epsilon, 0]$, such that, for $h^{-1} \in \mathrm{Sym}^{1,1}(\mathfrak{g})$, we have

$$\frac{d(h^{-1}(t))}{dt} = (h^{-1})^{\#}, \quad h(0) = h_0.$$

Then, by Theorem 5.18, $g(t) = p_* h(t)$ is the solution to the HCF on G/H, therefore, we can apply Theorem 6.6 to $g = p_* h_0$ and conclude that

$$K = \{v \in \mathrm{End}(\mathfrak{g}/\mathfrak{h}) \mid \mathrm{tr}(v \circ w) = 0 \ \forall\ w \in \mathfrak{hol}_{\nabla^T}\}.$$

Similarly, by Proposition 5.7,

$$K = \{v \in \mathrm{End}(\mathfrak{g}/\mathfrak{h}) \mid \mathrm{tr}(\sigma_{\mathrm{ad}}(w) \circ v) = 0 \ \forall w \in \mathfrak{h}\}.$$

Therefore, both $\sigma_{\mathrm{ad}}(\mathfrak{h})$ and $\mathfrak{hol}_{\nabla^T}$ represent the tr-orthogonal complement of K. □

Remark 6.9. Given an Hermitian manifold (M, g, J) with a dual-Nakano semipositive metric, we can apply Hamilton's Lemma 4.11 to $\Omega^{g(t)}$. Therefore, we conclude that for $t > 0$ the holonomy Lie algebra $\mathfrak{hol}_{\nabla^T}$ of the torsion-twisted connection is constant in t. This observation agrees with the fact that on a complex homogeneous manifold, $\mathfrak{hol}_{\nabla^T}$ coincides with the image of the isotropy representation $\sigma_{\mathrm{ad}}(\mathfrak{h})$, and the latter is clearly independent of the metric.

6.2 Monotonicity under the Hermitian Curvature Flow

6.2.1 Non-decreasing Scalar Quantities

Theorem 4.10 allows to produce many monotonic quantities for the HCF on (M, g, J). Let $S \subset \mathrm{End}(V)$ be a closed scale-invariant, $\mathrm{Ad}\, G$-invariant subset. Define

$$\mu(S, g) := \max\{\mu \in \mathbb{R} \mid \Omega \text{ satisfies } C(S, F), \text{ where } F(s) = \mu|\mathrm{tr} s|^2\} \in \mathbb{R} \cup \{\pm\infty\},$$

where $\max\{\varnothing\} := -\infty$.

Proposition 6.10. *Let $g = g(t)$ be the solution to the HCF on (M, g, J). Then for any $S \subset \mathrm{End}(V)$ as above the quantity $\mu(S, g(t))$ is non-decreasing along the HCF. Moreover, $\mu(S, g(t)) > \mu(S, g(0))$ for $t > 0$, unless $\mu(S, g(0)) \in \{-\infty, 0, +\infty\}$.*

Proof. If $\Omega^{g(0)}$ satisfies $C(S, \mu|\mathrm{tr} s|^2)$, then by Theorem 4.10, $\Omega^{g(t)}$ also satisfies $C(S, \mu|\mathrm{tr} s|^2)$ for $t > 0$. Hence $\mu(S, g(t)) \geqslant \mu(S, g(0))$, and we have non-strict monotonicity.

Now, assume $\mu(S, g(0)) \notin \{-\infty, 0, +\infty\}$, but $\mu(S, g(t)) = \mu(S, g(0)) = \mu$. Therefore for any $\varepsilon_i > 0$ we have $\Omega^{g(t)} \notin C(S, (\mu - \varepsilon_i)|\mathrm{tr} s|^2)$. Letting $\varepsilon_i \searrow 0$, we conclude that $\Omega^{g(t)}$ hits the boundary of $C(S, \mu|\mathrm{tr}s|^2)$ (here we are using the closedness and scale-invariance of S). Therefore, in notations of Theorem 4.12, $N(t, m) \neq \varnothing$. By Theorem 4.12 (a), it can happen only if $\mu = 0$. $\square$

A similar statement is valid for other one-parametric monotonic family of functions $F(s)$.

For $S = \{\lambda\,\mathrm{Id} \mid \lambda \in \mathbb{R}\}$ Proposition 6.10 gives the monotonicity of the lower bound for $\widehat{\mathrm{sc}}$, see Example 4.19.

6.2.2 HCF-periodic Metrics

There is an essential difficulty in using a geometric flow in classification problems, if there exist stationary *breathers* — non-fixed periodic solutions to the flow. In the Kähler setting, there are no stationary breathers for the Kähler-Ricci, and the metrics fixed by the Kähler-Ricci flow are tautologically the Ricci flat (Calabi-Yau) metrics. At the first glance, the situation with the HCF is much more subtle, since the vanishing of the evolution term $S^{(2)} + Q$ for the HCF, does not have any clear cohomological interpretation. At the moment, we cannot completely exclude periodic solutions for the HCF. However, a possible existence of an HCF-periodic solution on (M, g, J) puts strong geometric obstructions on the geometry of M. In particular, similarly to the Kähler situation, we can conclude that $K_{\widetilde{M}}$ is trivial, and $c_1(M) \in H^2(M, \mathbb{Z})$ is torsion.

Theorem 6.11. *If a compact (M, g, J) admits a periodic solution to the HCF, then*

1. *the torsion-twisted Ricci form ρ_{∇^T} vanishes;*
2. *the restricted torsion-twisted holonomy group is a subgroup of $SL(T^{1,0}M)$;*
3. *the universal cover of M (denote it $\widetilde{M}$) admits a nowhere vanishing holomorphic volume form.*

Proof. The vanishing of ρ_{∇^T} implies that there is the inclusion of the restricted holonomy group

$$\mathrm{Hol}^0_{\nabla^T} \subset SL(T^{1,0}M),$$

and, using the ∇^T-parallel transport, we can construct a nowhere vanishing holomorphic section of $K_{\widetilde{M}}$ on $\widetilde{M}$. Hence it remains to prove that $\rho_{\nabla^T} = 0$.

Consider a periodic solution to the HCF on (M, g, J). Contracting the evolution equation for

$\Omega \in \mathrm{Sym}^{1,1}(\mathrm{End}(T^{1,0}M))$ twice, we obtain a parabolic equation for $\widehat{\mathrm{sc}}$:

$$\frac{d\,\widehat{\mathrm{sc}}}{dt} = \Delta\widehat{\mathrm{sc}} + |\rho_{\nabla^T}|^2.$$

Using a standard strong maximum principle, we conclude that $\inf_M \widehat{\mathrm{sc}}$ is strictly increasing in time, unless $\rho_{\nabla^T} = 0$ everywhere on M. Therefore, since the solution to the HCF is periodic, the quantity $\inf_M \widehat{\mathrm{sc}}$ cannot be strictly increasing in time, and we indeed have $\rho_{\nabla^T} = 0$. □

Remark 6.12. Triviality of $K_{\widetilde{M}}$ is stronger than just vanishing of $c_1(M) \in H^2(M, \mathbb{C})$. For example, Calabi-Eckman complex structures on $S^3 \times S^3$ have holomorphically non-trivial canonical bundle, while $c_1 = 0$, see, e.g., discussion in [BDV09, §2].

For the next proposition we need to introduce the following notion. An Hermitian manifold (M, g, J) is *balanced*, if $d\omega^{n-1} = 0$; it is *conformally balanced*, if the manifold $(M, e^{\varphi}g, J)$ is balanced for some function φ. Let M be a manifold, satisfying the assumptions of Theorem 6.11. Assume additionally, that the full holonomy group preserves the volume form:

$$\mathrm{Hol}_{\nabla^T} \subset SL(T^{1,0}M),$$

i.e., M admits a ∇^T-parallel holomorphic volume form $\Psi \in \Lambda^{n,0}(M)$. By Theorem 6.11 this happens, e.g., if $\pi_1(M) = 1$. In this case, we can prove even stronger restrictions on the geometry of M.

Proposition 6.13. *Given (M, g, J), $\Psi \in \Lambda^{n,0}(M)$ as above, M is conformally balanced with the conformal factor*

$$e^{\varphi} = |\Psi|^{\frac{2}{n-1}}.$$

Proof. The statement of the proposition is equivalent to the vanishing

$$d(|\Psi|^2\omega^{n-1}) = 0. \tag{6.2}$$

Given $\xi \in T^{1,0}M$, let us compute $\xi \cdot |\Psi|^2$:

$$\begin{aligned}
\xi \cdot |\Psi|^2 &= g(\nabla_\xi \Psi, \overline{\Psi}) = g((\nabla_\xi - \nabla^T_\xi)\Psi, \overline{\Psi}) \\
&= -\xi^k T^{i_s}_{k i_p} \Psi_{i_1 i_2 \ldots i_{(p-1)} i_s i_{(p+1)} \ldots i_n} \overline{\Psi_{j_1 j_2 \ldots j_n}} g^{i_1 \bar{j}_1} \ldots g^{i_n \bar{j}_n} \\
&= -\xi^k T^p_{kp} |\Psi|^2 = -\langle \theta^{1,0}, \xi \rangle |\Psi|^2,
\end{aligned}$$

where $\theta^{1,0} = (\theta^{1,0})_k dz^k = T^p_{kp} dz^k \in \Lambda^{1,0}(M)$ is the (1,0) part of the Lee form $\theta = \mathrm{tr}_\omega d\omega$. Therefore

$$\partial(|\Psi|^2\omega^{n-1}) = \partial|\Psi|^2 \wedge \omega^{n-1} + |\Psi|^2 \wedge \partial\omega^{n-1} = |\Psi|^2(-\theta^{1,0} \wedge \omega^{n-1} + \partial\omega^{n-1}) = 0,$$

where in the last step we used the characteristic identity for the Lee form:

$$\theta \wedge \omega^{n-1} = d\omega^{n-1}.$$

□

Remark 6.14. Equation (6.2) is very similar to the *dilatino* equation of the Strominger system [GF16]:

$$d(|\Psi|\,\omega^{n-1}) = 0.$$

It is still an open question, whether the HCF admits non-trivial, i.e., not stationary, periodic solutions.

Question 6.15. *Is it true that if $g = g(t)$ is a periodic solution to the HCF on (M, g, J), then $g(t)$ is a stationary solution, i.e., $g \equiv g(0)$?*

Theorem 6.11 motivates us to formulate the following problem.

Problem 6.16. *Let (M, g, J, Ψ) be a conformally balanced Hermitian manifold with Ψ — a trivialization of the canonical bundle. When does M admit an HCF-stationary metric? By Theorem 6.11, such a metric necessarily will have $\rho_{\nabla^T} = 0$, and Ψ will be ∇^T-parallel.*

This problem is a non-Kähler version of Calabi's conjecture. Namely, if the underlying manifold (M, J) admits a Kähler metric ω, then by Yau's theorem [Yau77], there exists a unique Kähler metric $\omega_\varphi = \omega + i\partial\overline{\partial}\varphi$, such that $Ric(\omega_\varphi) = 0$. Of course, in this case the torsion vanishes, and all four Chern-Ricci curvatures coincide and equal zero.

Chapter 7

Conclusion

Recently, the study of metric flows in the Hermitian setting started gaining an increasing amount of interest and is forming a new actively evolving field. The main perspective of the present thesis is that these flows could and should be applied to approach various uniformization problems in complex geometry.

Following the familiar route of the development of the Ricci flow, we identified a member (the HCF) of a general Hermitian curvature flows family, which preserves many natural curvature (semi)positivity conditions in Hermitian geometry. To analyze the HCF, we introduced a new geometric concept, the *torsion-twisted connection* ∇^T, which is canonically attached to any Hermitian manifold. With the use of the torsion-twisted connection, we were able to find a clear evolution equation for the Chern curvature Ω under the HCF and managed to apply a fundamental technical tool — a refinement of Hamilton's maximum principle for tensors, to Ω.

Complex manifolds, admitting Hermitian metrics of *positive* curvature, are well-understood due to the solutions to Frankel's and Hartshorne's conjectures. Therefore the *strong* maximum principle for the Chern curvature evolving under the HCF provides an approach for studying manifolds admitting Hermitian metrics of *semipositive* curvature. Study of such manifolds fits well into uniformization conjectures in complex and algebraic geometry. Motivated by the algebraic Campana-Peternell conjecture and by the fact that the Griffiths/dual-Nakano semipositivity is preserved under the HCF, we proposed a weak differential-geometric Campana-Peternell conjecture. The HCF allows to make some initial progress towards this conjecture and suggests further possible generalizations. Explicitly computing the HCF on the complex homogeneous manifolds, equipped with submersion metric, we obtained additional evidence supporting the weak Campana-Peternell conjecture.

Since the study of the HCF has been just started, there are still many basic question to answer. While some of these questions make sense for any member of the general HCF family, our findings suggest that some of them might become more accessible for the specific HCF, which is considered in this thesis. Let us review some of the possible directions for the future research.

Streets and Tian [ST11] proved, that if $t_{\max}$ is the maximal time such that there exists a solution to the HCF on $[0, t_{\max})$, then $\limsup_{t \to t_{\max}} \max\{|\Omega|, |T|, |\nabla T|\} = +\infty$. This basic blow-up is quite impractical to use for analyzing the long-time existence of the flow. Therefore it is important to answer the following question.

Problem 7.1. *Is it possible to prove the existence of the HCF up to time $t_{\max}$ by controlling less geometric quantities, than the full norms of Ω, T, ∇T?*

By Theorem 6.11, the existence of a periodic solution on (M, g, J) puts severe constraints on the geometry of manifold M. This motivates the following problem.

Problem 7.2. *Does there exist a non-trivial, i.e., non-stationary, periodic solution to the HCF on some (M, g, J)? What can be said about HCF-static metrics?*

Important analytical tool in studying Ricci flow is the monotonicity of $\mathcal{F}$ and $\mathcal{W}$ functionals. It would be very helpful to have such functionals for the HCF, or its modifications. In particular, existence of such functional will most likely help ruling out breathers.

Problem 7.3. *Find analogues of $\mathcal{F}$ and $\mathcal{W}$ functionals for the HCF.*

While analysis of a long-time behavior of an Hermitian flow on a general Hermitian manifold might be challenging, sometimes it is reasonable to focus on Hermitian metrics satisfying some partial 'integrability condition', e.g., pluriclosed metrics ($\partial\overline{\partial}\omega = 0$), Gauduchon metrics ($\partial\overline{\partial}\omega^{n-1} = 0$), balanced metrics ($\partial^*\omega = 0$), conformally balanced metrics ($\partial^*\omega = \overline{\partial}\varphi$), locally conformally Kähler metrics.

Problem 7.4. *Does there exist any partial metric integrability condition, preserved by the HCF?*

The evolution equation for Ω under the HCF (Proposition 3.22) and the strong maximum principle (Theorem 4.12) suggest that the geometry of the HCF has strong ties with the properties of the torsion-twisted connection ∇^T and its holonomy group. It would be interesting to find a direct geometric interpretation of ∇^T and to address differential-geometric questions concerning the torsion-twisted holonomy Lie algebra $\mathfrak{hol}_{\nabla^T}$. In the Riemannian setting, the restricted holonomy of the Levi-Civita connection belongs to a very small list, unless the underlying manifolds is locally symmetric (see [Ber55]). We proved that on a homogeneous manifold $(G/H, p_*h, J)$, the torsion-twisted holonomy Lie algebra coincides with the image of the isotropy representation. One could expect that similarly to the Riemannian case, $\mathfrak{hol}_{\nabla^T}$ belongs to a small list, unless the underlying manifold is locally homogeneous.

Problem 7.5. *Does there exist Berger-type classification of the torsion-twisted holonomy groups* Hol_{∇^T}*? What can be said about manifold* (M, g, J)*, if* Hol_{∇^T} *is reducible? 'small'?*

We proved that the HCF of a submersion metric on a complex homogeneous manifold G/H is induced by an ODE on $\mathrm{Sym}^{1,1}(\mathfrak{g})$. Conjectural behavior of the HCF on complex homogeneous manifolds of simple reductive algebraic groups suggests a pinching of this ODE towards the Killing form (Conjecture 5.21). This is a purely algebraic problem, which could be approached by carefully studying the interplay between the ODE and the decomposition of $\mathrm{Sym}^{1,1}(\mathfrak{g})$ into the irreducible subrepresentations of G.

Problem 7.6. *Let* $\mathfrak{g}$ *be a simple complex Lie algebra,* $B(0) \in \mathrm{Sym}^{1,1}(\mathfrak{g})$*,* $B(0) > 0$*. Denote by* $B(t) \in \mathrm{Sym}^{1,1}(\mathfrak{g})$ *the solution to the ODE* $\partial_t B = B^{\#}$*. Then there exists* $\gamma \in G$*, such that* $B(t)$ *pinches towards* $\mathrm{Ad}_g(\kappa)$*, where* κ *is the Killing metric of the compact form of* $\mathfrak{g}$*.*

The coordinate expression for the HCF evolution term

$$\Psi(g)^{i\bar{j}} = g^{m\bar{n}}\partial_m\partial_{\bar{n}}g^{i\bar{j}} - \partial_m g^{i\bar{n}}\partial_{\bar{n}}g^{m\bar{j}}$$

indicates that Ψ is a natural nonlinear second order differential operator $\Lambda^{1,1}(TM) \to \Lambda^{1,1}(TM)$. An answer to the following question might shed the light on the geometric nature of the HCF.

Problem 7.7. *Does* Ψ *fit into an hierarchy of natural differential operators, e.g., similar to the Schouten-Nijenhuis bracket* $\Lambda^*(TM) \otimes \Lambda^*(TM) \to \Lambda^*(TM)$*?*

Another specific direction concerns the study of the HCF on complex surfaces. Many computations substantially simplify in $\dim_{\mathbb{C}} = 2$ and make it easier to study the long-time behavior of the flow.

Problem 7.8. *Investigate the HCF on complex surfaces.*

Purportedly, if we understand singularity formation under the HCF in complex dimension 2 well enough, we might try using this flow to develop an Hermitian version of *analytical minimal model* program (see [ST17]) and to study class VII surfaces, by analyzing the blow-up behavior of the HCF. Another possible direction of research concerns questions on the existence of integrable complex structures. Supposedly, with a complete understanding of limiting behavior of Hermitian flows, we may be able to rule out existence of complex structures on certain topological manifolds.

Solutions to the problems above will build a foundation for further applications of the HCF.

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