

Asymptotics of local height pairing

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The logarithmic coefficient is an algebraic local intersection index.

The asymptotic problem

S a smooth Riemann surface.

$\pi : X \rightarrow S$ a strongly semistable family of sm proj vars with rel dim d .

W, Z cycles of codim- m, n cycles, generically homologically triv. wrt S , with

$$m + n = d + 1.$$

For a singular fiber X_s , choose a local parameter t of $s \in S$ and

$$h(t) := \langle W_t, Z_t \rangle_\infty$$

is the Archimedean part of the Beilinson–Bloch height pairing.

Theorem 0.1 (assuming the Hodge conjecture)

$h(t) - \langle W, Z \rangle_{a,s} \log |t|$ extends continuously and is bounded near $t = 0$.

$\langle W, Z \rangle_{a,s}$ is Beilinson's local intersection index on the function-field model.

Two line bundles

Suppose π is smooth and W, Z properly intersect.

Bloch's motivic line bundle

$$\mathcal{L}_{\text{alg}} = \mathbb{E}_{W,Z}$$

from higher Chow groups, with a rational section

$$\sigma_{\text{alg}} \quad (\text{div}(\sigma_{\text{alg}}) = \pi_*(W \cdot Z)).$$

Yields $\langle W, Z \rangle_{a,s}$.

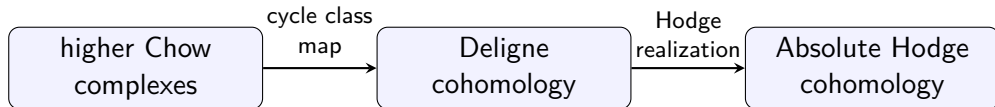
Hain's Hodge-theoretic "biextension" line bundle

$$\begin{aligned} H &:= R^{2m-1}\pi_*\mathbb{Z}(m)/\text{tors}, \\ J_H &:= \text{Ext}_{\text{VMHS}}^1(\mathbb{Z}, H) \quad (\text{intermediate extension}), \\ \nu = \text{AJ}(W) &\in J_H(S) \quad (\text{cf. } H^{2m-1}(X_s \setminus |W_s|, \mathbb{Z})), \\ \omega = \text{AJ}(Z) &\in J_{H^\vee}(S). \end{aligned}$$

Pulling back the Poincaré line bundle gives $\mathcal{L}_{\text{Hdg}}$, with a rational section

$$\begin{aligned} \sigma_{\text{Hdg}} &\quad (\text{cf. } H^{2m-1}(X_s \setminus |W_s|, |Z_s|, \mathbb{Z})), \\ h(t) &= \text{height}(\sigma_{\text{Hdg}}(t)). \end{aligned}$$

Comparison in the smooth case



The comparison theorem in the smooth case

$$\Phi: \mathcal{L}_{\text{alg}}^{\text{an}} \xrightarrow{\sim} \mathcal{L}_{\text{Hdg}}, \quad \Phi(\sigma_{\text{alg}}) = \sigma_{\text{Hdg}}.$$

At the singular fiber

Hodge conjecture

Modify W to $\widehat{W}$ by vertical cycles in singular fibers.

$\widehat{W}$ is homologically trivial on each component of the singular fibers.

Brosnan–Fang–Nie–Pearlstein

Replace Hodge theory by mixed Hodge modules.

Their theorem says the intermediate extension preserve the short exact sequences defining ν, ω .

By these, the intermediate extension also preserves the biextension defining σ_{Hdg} .