Hyperbolic 3-manifolds with low cusp volume

Andrew Yarmola
University of Luxembourg
andrew.yarmola@uni.lu

in collaboration with David Gabai, Robert Haraway, Robert Meyerhoff, and Nathaniel Thurston

Geometric structures and representation varieties,
National University of Singapore, May 4, 2017

Finite volume hyperbolic 3-manifolds

- Let $X = \mathbb{H}^3/\Gamma$ where Γ is a *discrete torsion-free subgroup* of $\mathrm{PSL}(2,\mathbb{C})$ and X has *finite volume*.

Finite volume hyperbolic 3-manifolds

- Let $X = \mathbb{H}^3/\Gamma$ where Γ is a *discrete torsion-free subgroup* of $\mathrm{PSL}(2,\mathbb{C})$ and X has *finite volume*.
- X is the interior of a compact manifold whose boundary is a *union of tori* (possibly empty).

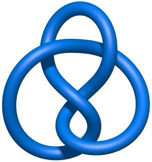
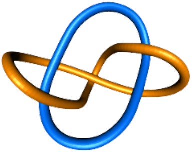


figure 8 knot



Whitehead link

Finite volume hyperbolic 3-manifolds

- Let $X = \mathbb{H}^3/\Gamma$ where Γ is a *discrete torsion-free subgroup* of $\mathrm{PSL}(2,\mathbb{C})$ and X has *finite volume*.
- X is the interior of a compact manifold whose boundary is a *union of tori* (possibly empty).
- A boundary torus $\mathfrak{c}$ is called a *cusp*.

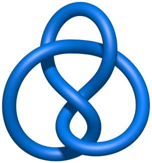
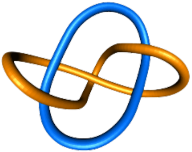


figure 8 knot



Whitehead link

Finite volume hyperbolic 3-manifolds

- Let $X = \mathbb{H}^3/\Gamma$ where Γ is a *discrete torsion-free subgroup* of $\mathrm{PSL}(2,\mathbb{C})$ and X has *finite volume*.
- X is the interior of a compact manifold whose boundary is a *union of tori* (possibly empty).
- A boundary torus $\mathfrak{c}$ is called a *cusp*.
- ***Theorem (Mostow).*** A complete finite volume hyperbolic structure on X is unique up to isometry.

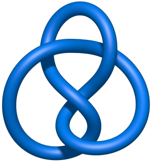
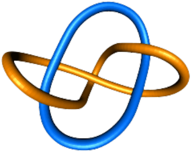


figure 8 knot



Whitehead link

Finite volume hyperbolic 3-manifolds

- Let $X = \mathbb{H}^3/\Gamma$ where Γ is a *discrete torsion-free subgroup* of $\mathrm{PSL}(2,\mathbb{C})$ and X has *finite volume*.
- X is the interior of a compact manifold whose boundary is a *union of tori* (possibly empty).
- A boundary torus $\mathfrak{c}$ is called a *cusp*.
- ***Theorem (Mostow).*** A complete finite volume hyperbolic structure on X is unique up to isometry.
- It follows that $\mathrm{vol}_{\mathbb{H}}(X)$ is *topological invariant.*

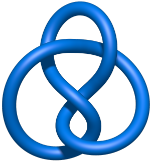
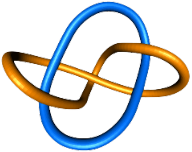


figure 8 knot



Whitehead link

Finite volume hyperbolic 3-manifolds

- Let $X = \mathbb{H}^3/\Gamma$ where Γ is a *discrete torsion-free subgroup* of $\mathrm{PSL}(2,\mathbb{C})$ and X has *finite volume*.
- X is the interior of a compact manifold whose boundary is a *union of tori* (possibly empty).
- A boundary torus $\mathfrak{c}$ is called a *cusp*.
- ***Theorem (Mostow).*** A complete finite volume hyperbolic structure on X is unique up to isometry.
- It follows that $\mathrm{vol}_{\mathbb{H}}(X)$ is *topological invariant.*
- ***Theorem (Thurston-Jørgensen).*** The subset of $\mathbb{R}$

  $$\{\mathrm{vol}_{\mathbb{H}}(X) \mid X \text{ is a complete hyperbolic 3-manifold}\}$$

  has order ω^ω and $X \mapsto \mathrm{vol}_{\mathbb{H}}(X)$ is *finite-to-one*.

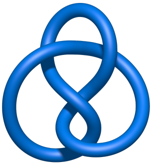
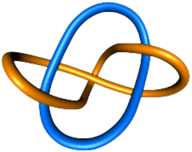


figure 8 knot



Whitehead link

Finite volume hyperbolic 3-manifolds

- Let $X = \mathbb{H}^3/\Gamma$ where Γ is a *discrete torsion-free subgroup* of $\mathrm{PSL}(2,\mathbb{C})$ and X has *finite volume*.
- X is the interior of a compact manifold whose boundary is a *union of tori* (possibly empty).
- A boundary torus $\mathfrak{c}$ is called a *cusp*.
- ***Theorem (Mostow).*** A complete finite volume hyperbolic structure on X is unique up to isometry.
- It follows that $\mathrm{vol}_{\mathbb{H}}(X)$ is *topological invariant.*
- ***Theorem (Thurston-Jørgensen).*** The subset of $\mathbb{R}$

  $$\{\mathrm{vol}_{\mathbb{H}}(X) \mid X \text{ is a complete hyperbolic 3-manifold}\}$$

  has order ω^ω and $X \mapsto \mathrm{vol}_{\mathbb{H}}(X)$ is *finite-to-one*.

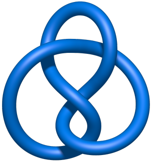
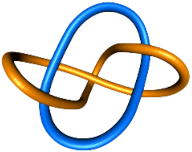


figure 8 knot



Whitehead link

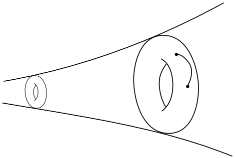
Maximal cusp neighborhoods

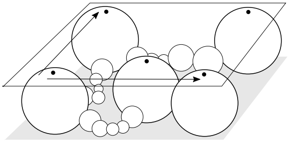
Maximal cusp neighborhoods

- A cusp $\mathfrak{c}$ corresponds to a conjugacy class of a *maximal* $\mathbb{Z} \oplus \mathbb{Z} \cong \Gamma_{\mathfrak{c}} \leq \Gamma$.

Maximal cusp neighborhoods

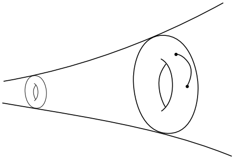
- A cusp $\mathfrak{c}$ corresponds to a conjugacy class of a *maximal* $\mathbb{Z} \oplus \mathbb{Z} \cong \Gamma_{\mathfrak{c}} \leq \Gamma$.
- $\Gamma_{\mathfrak{c}}$ is generated by two *parabolic* elements that *fix the same point* $z \in \hat{\mathbb{C}}$.

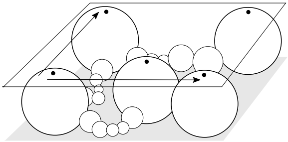




Maximal cusp neighborhoods

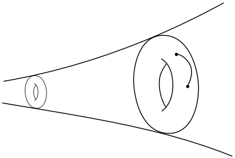
- A cusp $\mathfrak{c}$ corresponds to a conjugacy class of a *maximal* $\mathbb{Z} \oplus \mathbb{Z} \cong \Gamma_{\mathfrak{c}} \leq \Gamma$.
- $\Gamma_{\mathfrak{c}}$ is generated by two *parabolic* elements that *fix the same point* $z \in \hat{\mathbb{C}}$.
- For a horosphere H_z at z, $H_z/\Gamma_{\mathfrak{c}}$ is a *flat* torus.

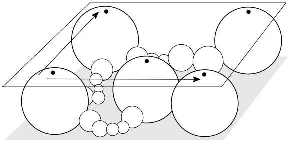




Maximal cusp neighborhoods

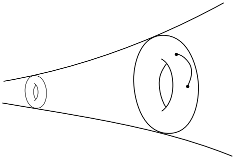
- A cusp $\mathfrak{c}$ corresponds to a conjugacy class of a *maximal* $\mathbb{Z} \oplus \mathbb{Z} \cong \Gamma_{\mathfrak{c}} \leq \Gamma$.
- $\Gamma_{\mathfrak{c}}$ is generated by two *parabolic* elements that *fix the same point* $z \in \hat{\mathbb{C}}$.
- For a horosphere H_z at z, $H_z/\Gamma_{\mathfrak{c}}$ is a *flat* torus.
- There exists a maximal embedded horoball neighborhood $B_{\mathfrak{c}}$ of $\mathfrak{c}$.

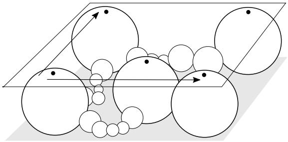




Maximal cusp neighborhoods

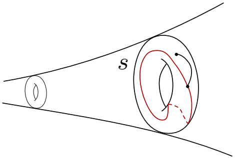
- A cusp $\mathfrak{c}$ corresponds to a conjugacy class of a *maximal* $\mathbb{Z} \oplus \mathbb{Z} \cong \Gamma_{\mathfrak{c}} \leq \Gamma$.
- $\Gamma_{\mathfrak{c}}$ is generated by two *parabolic* elements that *fix the same point* $z \in \hat{\mathbb{C}}$.
- For a horosphere H_z at z, $H_z/\Gamma_{\mathfrak{c}}$ is a *flat* torus.
- There exists a maximal embedded horoball neighborhood $B_{\mathfrak{c}}$ of $\mathfrak{c}$.
- If we look at all the lifts of $B_{\mathfrak{c}}$ in $\mathbb{H}^3$, we obtain a *horoball system* $\widetilde{B_{\mathfrak{c}}}$.

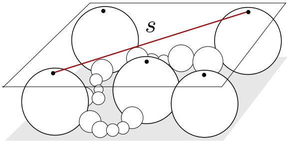




Maximal cusp neighborhoods

- A cusp $\mathfrak{c}$ corresponds to a conjugacy class of a *maximal* $\mathbb{Z} \oplus \mathbb{Z} \cong \Gamma_{\mathfrak{c}} \leq \Gamma$.
- $\Gamma_{\mathfrak{c}}$ is generated by two *parabolic* elements that *fix the same point* $z \in \hat{\mathbb{C}}$.
- For a horosphere H_z at z, $H_z/\Gamma_{\mathfrak{c}}$ is a *flat* torus.
- There exists a maximal embedded horoball neighborhood $B_{\mathfrak{c}}$ of $\mathfrak{c}$.
- If we look at all the lifts of $B_{\mathfrak{c}}$ in $\mathbb{H}^3$, we obtain a *horoball system* $\widetilde{B_{\mathfrak{c}}}$.
- Given a primitive element $s \in \Gamma_{\mathfrak{c}} \setminus \{id\}$, we think of it as a *slope* on the flat torus $\partial B_{\mathfrak{c}}$ and we measure length ℓ_s of the *geodesic representative* of s in the Euclidean metric on $\partial B_{\mathfrak{c}}$.





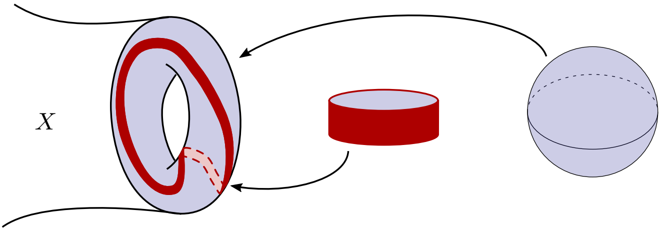
Dehn filling and drilling

Dehn filling and drilling

- *Dehn filling* is the topological process of gluing in a solid torus to a cusp. For every slope s on a cusp, there is a unique Dehn filling $X(s)$.

Dehn filling and drilling

- *Dehn filling* is the topological process of gluing in a solid torus to a cusp. For every slope s on a cusp, there is a unique Dehn filling $X(s)$.



Dehn filling and drilling

- *Dehn filling* is the topological process of gluing in a solid torus to a cusp. For every slope s on a cusp, there is a unique Dehn filling $X(s)$.
- ***Theorem (Futer - Kalfagianni - Purcell).*** If $\ell_s > 2\pi$, then $X(s)$ admits a complete hyperbolic structure and

$$\left(1 - \frac{4\pi^2}{\ell_s^2}\right)^{3/2} \mathrm{vol}_{\mathbb{H}}(X) \leq \mathrm{vol}_{\mathbb{H}}(X(s)) < \mathrm{vol}_{\mathbb{H}}(X).$$

Dehn filling and drilling

- *Dehn filling* is the topological process of gluing in a solid torus to a cusp. For every slope s on a cusp, there is a unique Dehn filling $X(s)$.
- ***Theorem (Futer - Kalfagianni - Purcell).*** If $\ell_s > 2\pi$, then $X(s)$ admits a complete hyperbolic structure and

$$\left(1 - \frac{4\pi^2}{\ell_s^2}\right)^{3/2} \mathrm{vol}_{\mathbb{H}}(X) \leq \mathrm{vol}_{\mathbb{H}}(X(s)) < \mathrm{vol}_{\mathbb{H}}(X).$$

- Not only do we have control on volume, but this gives only *finitely many exceptional* s for which $X(s)$ is *not hyperbolic*.

Dehn filling and drilling

- *Dehn filling* is the topological process of gluing in a solid torus to a cusp. For every slope s on a cusp, there is a unique Dehn filling $X(s)$.
- ***Theorem (Futer - Kalfagianni - Purcell).*** If $\ell_s > 2\pi$, then $X(s)$ admits a complete hyperbolic structure and

$$\left(1 - \frac{4\pi^2}{\ell_s^2}\right)^{3/2} \mathrm{vol}_{\mathbb{H}}(X) \leq \mathrm{vol}_{\mathbb{H}}(X(s)) < \mathrm{vol}_{\mathbb{H}}(X).$$

- Not only do we have control on volume, but this gives only *finitely many exceptional* s for which $X(s)$ is *not hyperbolic*.
- *Dehn drilling* is the process of removing an embedded curve γ form X.

Dehn filling and drilling

- *Dehn filling* is the topological process of gluing in a solid torus to a cusp. For every slope s on a cusp, there is a unique Dehn filling $X(s)$.
- ***Theorem (Futer - Kalfagianni - Purcell).*** If $\ell_s > 2\pi$, then $X(s)$ admits a complete hyperbolic structure and

$$\left(1 - \frac{4\pi^2}{\ell_s^2}\right)^{3/2} \mathrm{vol}_{\mathbb{H}}(X) \leq \mathrm{vol}_{\mathbb{H}}(X(s)) < \mathrm{vol}_{\mathbb{H}}(X).$$

- Not only do we have control on volume, but this gives only *finitely many exceptional* s for which $X(s)$ is *not hyperbolic*.
- *Dehn drilling* is the process of removing an embedded curve γ form X.
- ***Theorem (Agol - Culler - Shalen).*** With X closed and not one of 5 manifolds, γ the *shortest* geodesic in X, then $X \smallsetminus \gamma$ admits a complete hyperbolic structure. With $B_{\mathfrak{c}_\gamma} \subset X \smallsetminus \gamma$ corresponding to γ,

$$\mathrm{vol}(B_{\mathfrak{c}_\gamma})/0.85328 \leq \mathrm{vol}_{\mathbb{H}}(X \smallsetminus \gamma) < 3.0177\, \mathrm{vol}_{\mathbb{H}}(X).$$

Dehn filling and drilling

- *Dehn filling* is the topological process of gluing in a solid torus to a cusp. For every slope s on a cusp, there is a unique Dehn filling $X(s)$.
- ***Theorem (Futer - Kalfagianni - Purcell).*** If $\ell_s > 2\pi$, then $X(s)$ admits a complete hyperbolic structure and

$$\left(1 - \frac{4\pi^2}{\ell_s^2}\right)^{3/2} \mathrm{vol}_{\mathbb{H}}(X) \leq \mathrm{vol}_{\mathbb{H}}(X(s)) < \mathrm{vol}_{\mathbb{H}}(X).$$

- Not only do we have control on volume, but this gives only *finitely many exceptional* s for which $X(s)$ is *not hyperbolic*.
- *Dehn drilling* is the process of removing an embedded curve γ form X.
- ***Theorem (Agol - Culler - Shalen).*** With X closed and not one of 5 manifolds, γ the *shortest* geodesic in X, then $X \smallsetminus \gamma$ admits a complete hyperbolic structure. With $B_{\mathfrak{c}_\gamma} \subset X \smallsetminus \gamma$ corresponding to γ,

$$\mathrm{vol}(B_{\mathfrak{c}_\gamma})/0.85328 \leq \mathrm{vol}_{\mathbb{H}}(X \smallsetminus \gamma) < 3.0177\,\mathrm{vol}_{\mathbb{H}}(X).$$

- Remark : There are very many people involved in obtaining these results. The number 0.85328 is the Böröczky constant for horoball packing.

Understanding all X with small $\mathrm{vol}(B_{\mathfrak{c}})$

Understanding all X with small $\mathrm{vol}(B_{\mathfrak{c}})$

Theorem (Gabai - Haraway - Meyerhoff - Thurston -Y.). *(Preliminary)*
Let X be an orientable finite volume hyperbolic 3-manifold and $\mathfrak{c}$ a cusp. Let $B_{\mathfrak{c}}$ be the maximal horoball neighborhood of $\mathfrak{c}$. If $\mathrm{vol}(B_{\mathfrak{c}}) \leq 2.62$ then X is a Dehn filling of one of the following 22 census (*parent*) manifolds.

m125	m129	m203	m295	m292	s443	s596	s647
s782	s780	s785	**s776**	s774	v2124	v2355	v2533
v2644	v2731	v3108	v3127	v3211	v3376		

Understanding all X with small $\mathrm{vol}(B_{\mathfrak{c}})$

Theorem (Gabai - Haraway - Meyerhoff - Thurston -Y.). *(Preliminary)* Let X be an orientable finite volume hyperbolic 3-manifold and $\mathfrak{c}$ a cusp. Let $B_{\mathfrak{c}}$ be the maximal horoball neighborhood of $\mathfrak{c}$. If $\mathrm{vol}(B_{\mathfrak{c}}) \leq 2.62$ then X is a Dehn filling of one of the following 22 census (*parent*) manifolds.

m125	m129	m203	m295	m292	s443	s596	s647
s782	s780	s785	**s776**	s774	v2124	v2355	v2533
v2644	v2731	v3108	v3127	v3211	v3376		

- The proof is computer assisted and relies on several rigorous validations.

Understanding all X with small $\mathrm{vol}(B_{\mathfrak{c}})$

Theorem (Gabai - Haraway - Meyerhoff - Thurston -Y.). *(Preliminary)*
Let X be an orientable finite volume hyperbolic 3-manifold and $\mathfrak{c}$ a cusp. Let $B_{\mathfrak{c}}$ be the maximal horoball neighborhood of $\mathfrak{c}$. If $\mathrm{vol}(B_{\mathfrak{c}}) \leq 2.62$ then X is a Dehn filling of one of the following 22 census (*parent*) manifolds.

m125	m129	m203	m295	m292	s443	s596	s647
s782	s780	s785	**s776**	s774	v2124	v2355	v2533
v2644	v2731	v3108	v3127	v3211	v3376		

- The proof is computer assisted and relies on several rigorous validations.
- There are *infinitely many* pairs $(X, \mathfrak{c})$ with $\mathrm{vol}(B_{\mathfrak{c}}) \leq 2.62$.

Understanding all X with small $\mathrm{vol}(B_{\mathfrak{c}})$

Theorem (Gabai - Haraway - Meyerhoff - Thurston -Y.). *(Preliminary)*
Let X be an orientable finite volume hyperbolic 3-manifold and $\mathfrak{c}$ a cusp. Let $B_{\mathfrak{c}}$ be the maximal horoball neighborhood of $\mathfrak{c}$. If $\mathrm{vol}(B_{\mathfrak{c}}) \leq 2.62$ then X is a Dehn filling of one of the following 22 census (*parent*) manifolds.

m125	m129	m203	m295	m292	s443	s596	s647
s782	s780	s785	**s776**	s774	v2124	v2355	v2533
v2644	v2731	v3108	v3127	v3211	v3376		

- The proof is computer assisted and relies on several rigorous validations.
- There are *infinitely many* pairs $(X, \mathfrak{c})$ with $\mathrm{vol}(B_{\mathfrak{c}}) \leq 2.62$.
- *In practice*, given $(X, \mathfrak{c})$, I can tell you which of the 22 manifolds is a parent.

Understanding all X with small $\mathrm{vol}(B_{\mathfrak{c}})$

Theorem (Gabai - Haraway - Meyerhoff - Thurston -Y.). *(Preliminary)*
Let X be an orientable finite volume hyperbolic 3-manifold and $\mathfrak{c}$ a cusp. Let $B_{\mathfrak{c}}$ be the maximal horoball neighborhood of $\mathfrak{c}$. If $\mathrm{vol}(B_{\mathfrak{c}}) \leq 2.62$ then X is a Dehn filling of one of the following 22 census (*parent*) manifolds.

m125	m129	m203	m295	m292	s443	s596	s647
s782	s780	s785	**s776**	s774	v2124	v2355	v2533
v2644	v2731	v3108	v3127	v3211	v3376		

- The proof is computer assisted and relies on several rigorous validations.
- There are *infinitely many* pairs $(X, \mathfrak{c})$ with $\mathrm{vol}(B_{\mathfrak{c}}) \leq 2.62$.
- *In practice*, given $(X, \mathfrak{c})$, I can tell you which of the 22 manifolds is a parent.
- The list can be made smaller as s776 (the only 3-cusped parent manifold) is a parent for some of the other 22.

Understanding all X with small $\mathrm{vol}(B_{\mathfrak{c}})$

Theorem (Gabai - Haraway - Meyerhoff - Thurston -Y.). *(Preliminary)*
Let X be a finite volume hyperbolic 3-manifold and $\mathfrak{c}$ a cusp. If $\mathrm{vol}(B_{\mathfrak{c}}) \leq 2.62$ then X is a Dehn filling of one of the following 22 census manifolds.

m125	m129	m203	m295	m292	s443	s596	s647
s782	s780	s785	**s776**	s774	v2124	v2355	v2533
v2644	v2731	v3108	v3127	v3211	v3376		

- Using Agol-Culler-Shalen and Böröczky's horoball packing density, we chose the bound 2.62 to identify all *closed* X with $\mathrm{vol}_{\mathbb{H}}(X) < 1.01749$. These should be the Weeks manifold, Vol_2 and Vol_3. Requires rigorous volume estimates and Futer-Kalfagianni-Purcell.

Understanding all X with small $\mathrm{vol}(B_{\mathfrak{c}})$

Theorem (Gabai - Haraway - Meyerhoff - Thurston -Y.). *(Preliminary)*
Let X be a finite volume hyperbolic 3-manifold and $\mathfrak{c}$ a cusp. If $\mathrm{vol}(B_{\mathfrak{c}}) \leq 2.62$ then X is a Dehn filling of one of the following 22 census manifolds.

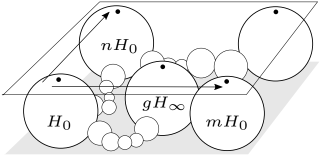
m125	m129	m203	m295	m292	s443	s596	s647
s782	s780	s785	**s776**	s774	v2124	v2355	v2533
v2644	v2731	v3108	v3127	v3211	v3376		

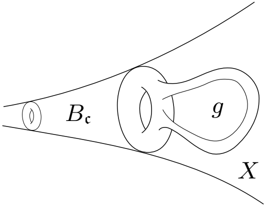
- Using Agol-Culler-Shalen and Böröczky's horoball packing density, we chose the bound 2.62 to identify all *closed* X with $\mathrm{vol}_{\mathbb{H}}(X) < 1.01749$. These should be the Weeks manifold, Vol_2 and Vol_3. Requires rigorous volume estimates and Futer-Kalfagianni-Purcell.
- Agol shows that one cusped manifolds with more than 8 exceptional slopes have $\mathrm{vol}(B_{\mathfrak{c}}) < 2.572$. Rigorous bounds on slope length and the 2π-Theorem should show that $\mathbb{S}^3 \smallsetminus \{\text{figure 8 knot}\}$ is one of only two manifolds with 10 exceptional slopes (maximum).

Bicuspid subgroup

Bicuspid subgroup

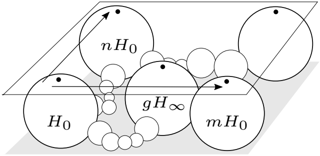
- Conjugate so that $\Gamma_{\mathfrak{c}}$ is generated by
$$m = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad n = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}.$$

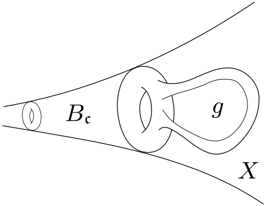




Bicuspid subgroup

- Conjugate so that $\Gamma_{\mathfrak{c}}$ is generated by
$m = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $n = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$.
- A point of tangency for $B_{\mathfrak{c}}$ corresponds to an element $g \in \Gamma$ with $g = \begin{pmatrix} PSi & -i/S \\ Si & 0 \end{pmatrix}$.

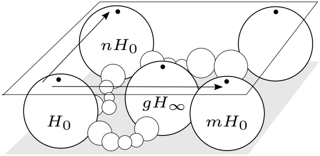


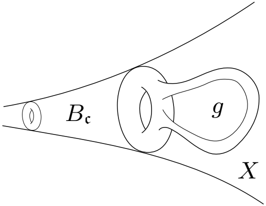


Bicuspid subgroup

- Conjugate so that $\Gamma_{\mathfrak{c}}$ is generated by
$m = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $n = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$.
- A point of tangency for $B_{\mathfrak{c}}$ corresponds to an element $g \in \Gamma$ with $g = \begin{pmatrix} PSi & -i/S \\ Si & 0 \end{pmatrix}$.
- The *bicuspid subgroup* of $(X, \mathfrak{c})$ is

$$Q_{\mathfrak{c}} = Q(L, P, S) = \langle m, n, g \rangle \leq \mathrm{PSL}(2, \mathbb{C})$$



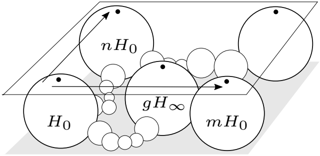


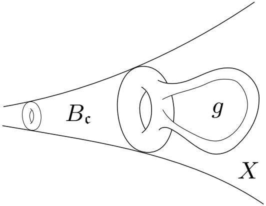
Bicuspid subgroup

- Conjugate so that $\Gamma_{\mathfrak{c}}$ is generated by $m = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $n = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$.
- A point of tangency for $B_{\mathfrak{c}}$ corresponds to an element $g \in \Gamma$ with $g = \begin{pmatrix} PSi & -i/S \\ Si & 0 \end{pmatrix}$.
- The *bicuspid subgroup* of $(X, \mathfrak{c})$ is

$$Q_{\mathfrak{c}} = Q(L, P, S) = \langle m, n, g \rangle \le \mathrm{PSL}(2, \mathbb{C})$$

- ***Theorem (Agol).*** If $\mathrm{vol}(B_{\mathfrak{c}}) < \pi$ then $[\Gamma : Q_{\mathfrak{c}}] < \infty$ and there exists a non-trivial word $w(m, n, g) = id$.



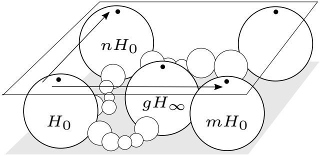


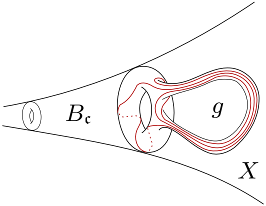
Bicuspid subgroup

- Conjugate so that $\Gamma_{\mathfrak{c}}$ is generated by
$m = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $n = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$.
- A point of tangency for $B_{\mathfrak{c}}$ corresponds to an element $g \in \Gamma$ with $g = \begin{pmatrix} PSi & -i/S \\ Si & 0 \end{pmatrix}$.
- The *bicuspid subgroup* of $(X, \mathfrak{c})$ is

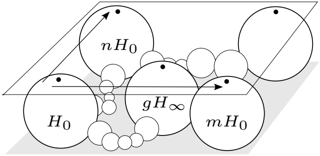
$$Q_{\mathfrak{c}} = Q(L, P, S) = \langle m, n, g \rangle \leq \mathrm{PSL}(2, \mathbb{C})$$

- ***Theorem (Agol).*** If $\mathrm{vol}(B_{\mathfrak{c}}) < \pi$ then $[\Gamma : Q_{\mathfrak{c}}] < \infty$ and there exists a non-trivial word $w(m, n, g) = id$.
- We call the word $w(m, n, g)$ a *variety word*.





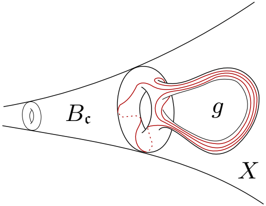
Bicuspid subgroup



- Conjugate so that $\Gamma_{\mathfrak{c}}$ is generated by $m = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $n = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$.
- A point of tangency for $B_{\mathfrak{c}}$ corresponds to an element $g \in \Gamma$ with $g = \begin{pmatrix} PSi & -i/S \\ Si & 0 \end{pmatrix}$.
- The *bicuspid subgroup* of $(X, \mathfrak{c})$ is

$$Q_{\mathfrak{c}} = Q(L, P, S) = \langle m, n, g \rangle \leq \mathrm{PSL}(2, \mathbb{C})$$

- ***Theorem (Agol).*** If $\mathrm{vol}(B_{\mathfrak{c}}) < \pi$ then $[\Gamma : Q_{\mathfrak{c}}] < \infty$ and there exists a non-trivial word $w(m, n, g) = id$.
- We call the word $w(m, n, g)$ a *variety word*.
- The *length* $\ell_g(w) = \#$ of g and g^{-1}'s.



Parameter space search

Parameter space search

Theorem. If $\mathrm{vol}(B_{\mathfrak{c}}) < 2.62$ then $Q_{\mathfrak{c}}$ admits one of 85 variety words w_i. Furthermore, $\ell_g(w_i) \leq 7$ for all i.

Parameter space search

Theorem. If $\mathrm{vol}(B_{\mathfrak{c}}) < 2.62$ then $Q_{\mathfrak{c}}$ admits one of 85 variety words w_i. Furthermore, $\ell_g(w_i) \le 7$ for all i.

- Our goal is to understand the collection of all $(P, S, L) \in \mathbb{C}^3$ such that $Q(P, S, L)$ is discrete, torsion-free, and $\mathrm{vol}(B_{\mathfrak{c}}) = |S^2|\,\mathrm{im}(L)/2 \le 2.62$.

Parameter space search

Theorem. If $\mathrm{vol}(B_{\mathfrak{c}}) < 2.62$ then $Q_{\mathfrak{c}}$ admits one of 85 variety words w_i. Furthermore, $\ell_g(w_i) \leq 7$ for all i.

- Our goal is to understand the collection of all $(P, S, L) \in \mathbb{C}^3$ such that $Q(P, S, L)$ is discrete, torsion-free, and $\mathrm{vol}(B_{\mathfrak{c}}) = |S^2| \, \mathrm{im}(L)/2 \leq 2.62$.
- Conjugation and reflection arguments allow us to restrict ourselves to a compact parameter space $\mathcal{P} \subset \mathbb{C}^3$ that contains all Q of interest.

Parameter space search

Theorem. If $\mathrm{vol}(B_{\mathfrak{c}}) < 2.62$ then $Q_{\mathfrak{c}}$ admits one of 85 variety words w_i. Furthermore, $\ell_g(w_i) \leq 7$ for all i.

- Our goal is to understand the collection of all $(P, S, L) \in \mathbb{C}^3$ such that $Q(P, S, L)$ is discrete, torsion-free, and $\mathrm{vol}(B_{\mathfrak{c}}) = |S^2| \, \mathrm{im}(L)/2 \leq 2.62$.
- Conjugation and reflection arguments allow us to restrict ourselves to a compact parameter space $\mathcal{P} \subset \mathbb{C}^3$ that contains all Q of interest.
- We choose $\mathcal{P}$ to be 6-dim box with side ratios $(2^{5/6}, 2^{4/6}, \ldots, 2^{1/6}, 1)$.

Parameter space search

Theorem. If $\mathrm{vol}(B_{\mathfrak{c}}) < 2.62$ then $Q_{\mathfrak{c}}$ admits one of 85 variety words w_i. Furthermore, $\ell_g(w_i) \leq 7$ for all i.

- Our goal is to understand the collection of all $(P, S, L) \in \mathbb{C}^3$ such that $Q(P, S, L)$ is discrete, torsion-free, and $\mathrm{vol}(B_{\mathfrak{c}}) = |S^2| \, \mathrm{im}(L)/2 \leq 2.62$.
- Conjugation and reflection arguments allow us to restrict ourselves to a compact parameter space $\mathcal{P} \subset \mathbb{C}^3$ that contains all Q of interest.
- We choose $\mathcal{P}$ to be 6-dim box with side ratios $(2^{5/6}, 2^{4/6}, \ldots, 2^{1/6}, 1)$.
- By cutting each dimension in half, we can encode sub-boxes in binary.

Parameter space search

Theorem. If $\mathrm{vol}(B_{\mathfrak{c}}) < 2.62$ then $Q_{\mathfrak{c}}$ admits one of 85 variety words w_i. Furthermore, $\ell_g(w_i) \leq 7$ for all i.

- Our goal is to understand the collection of all $(P, S, L) \in \mathbb{C}^3$ such that $Q(P, S, L)$ is discrete, torsion-free, and $\mathrm{vol}(B_{\mathfrak{c}}) = |S^2| \, \mathrm{im}(L)/2 \leq 2.62$.
- Conjugation and reflection arguments allow us to restrict ourselves to a compact parameter space $\mathcal{P} \subset \mathbb{C}^3$ that contains all Q of interest.
- We choose $\mathcal{P}$ to be 6-dim box with side ratios $(2^{5/6}, 2^{4/6}, \ldots, 2^{1/6}, 1)$.
- By cutting each dimension in half, we can encode sub-boxes in binary.
- We *eliminate* (sub)-boxes that *contain no discrete torsion-free points*.

Parameter space search

Theorem. If $\mathrm{vol}(B_{\mathfrak{c}}) < 2.62$ then $Q_{\mathfrak{c}}$ admits one of 85 variety words w_i. Furthermore, $\ell_g(w_i) \leq 7$ for all i.

- Our goal is to understand the collection of all $(P, S, L) \in \mathbb{C}^3$ such that $Q(P, S, L)$ is discrete, torsion-free, and $\mathrm{vol}(B_{\mathfrak{c}}) = |S^2|\,\mathrm{im}(L)/2 \leq 2.62$.
- Conjugation and reflection arguments allow us to restrict ourselves to a compact parameter space $\mathcal{P} \subset \mathbb{C}^3$ that contains all Q of interest.
- We choose $\mathcal{P}$ to be 6-dim box with side ratios $(2^{5/6}, 2^{4/6}, \ldots, 2^{1/6}, 1)$.
- By cutting each dimension in half, we can encode sub-boxes in binary.
- We *eliminate* (sub)-boxes that *contain no discrete torsion-free points*.
- Our approach is to find a *killer word* $w_{kill}(m, n, g)$ for a box $\mathcal{B}$ such that
  1. $w_{kill}(m, n, g) \cdot H_\infty \cap H_\infty \neq \emptyset$ over all of $\mathcal{B}$
  2. $w_{kill}(m, n, g) \neq id$ over all of $\mathcal{B}$

Parameter space search

Theorem. If $\mathrm{vol}(B_{\mathfrak{c}}) < 2.62$ then $Q_{\mathfrak{c}}$ admits one of 85 variety words w_i. Furthermore, $\ell_g(w_i) \leq 7$ for all i.

- Our goal is to understand the collection of all $(P, S, L) \in \mathbb{C}^3$ such that $Q(P, S, L)$ is discrete, torsion-free, and $\mathrm{vol}(B_{\mathfrak{c}}) = |S^2|\,\mathrm{im}(L)/2 \leq 2.62$.
- Conjugation and reflection arguments allow us to restrict ourselves to a compact parameter space $\mathcal{P} \subset \mathbb{C}^3$ that contains all Q of interest.
- We choose $\mathcal{P}$ to be 6-dim box with side ratios $(2^{5/6}, 2^{4/6}, \ldots, 2^{1/6}, 1)$.
- By cutting each dimension in half, we can encode sub-boxes in binary.
- We *eliminate* (sub)-boxes that *contain no discrete torsion-free points*.
- Our approach is to find a *killer word* $w_{kill}(m, n, g)$ for a box $\mathcal{B}$ such that
  1. $w_{kill}(m, n, g) \cdot H_\infty \cap H_\infty \neq \emptyset$ over all of $\mathcal{B}$
  2. $w_{kill}(m, n, g) \neq id$ over all of $\mathcal{B}$
- ***Lemma.*** Let $V_w = \{w(m, n, g) = id\} \subset \mathcal{P}$, then there is a computable neighborhood $N_w \supset V_w$, such that $N_w \setminus V_w$ contains no discrete points.

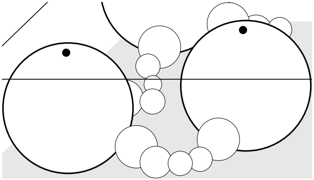
Horoball systems and necklaces

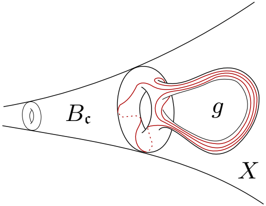
Horoball systems and necklaces

- A variety word w corresponds to a *necklace* η_w of horoballs in $\widetilde{B_\mathfrak{c}} \subset \mathbb{H}^3$ of $\ell_g(w)$ balls.

Horoball systems and necklaces

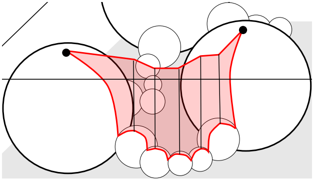
- A variety word w corresponds to a *necklace* η_w of horoballs in $\widetilde{B_{\mathfrak{c}}} \subset \mathbb{H}^3$ of $\ell_g(w)$ balls.

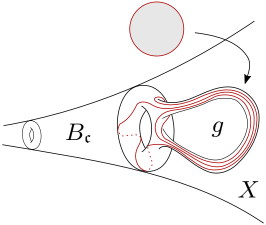




Horoball systems and necklaces

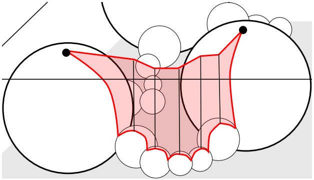
- A variety word w corresponds to a *necklace* η_w of horoballs in $\widetilde{B_{\mathfrak{c}}} \subset \mathbb{H}^3$ of $\ell_g(w)$ balls.
- A manifold obtained by attaching a disk along w is called a *necklace manifold* K_w.

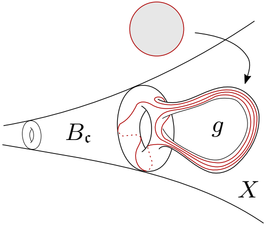




Horoball systems and necklaces

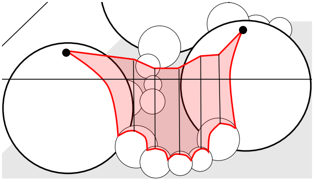
- A variety word w corresponds to a *necklace* η_w of horoballs in $\widetilde{B_{\mathfrak{c}}} \subset \mathbb{H}^3$ of $\ell_g(w)$ balls.
- A manifold obtained by attaching a disk along w is called a *necklace manifold* K_w.
- If we can find an *embedded* $K_w \subset X$ and K_w admits a *complete* hyperbolic structure with 2 cusps, then X is a Dehn filling of K_w.

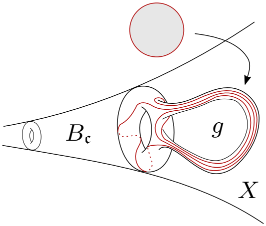




Horoball systems and necklaces

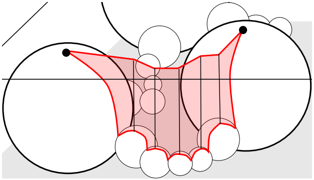
- A variety word w corresponds to a *necklace* η_w of horoballs in $\widetilde{B_\mathfrak{c}} \subset \mathbb{H}^3$ of $\ell_g(w)$ balls.
- A manifold obtained by attaching a disk along w is called a *necklace manifold* K_w.
- If we can find an *embedded* $K_w \subset X$ and K_w admits a *complete* hyperbolic structure with 2 cusps, then X is a Dehn filling of K_w.
- *Idea:* If w is "simple" enough, then a disk can always be attached in $X \setminus \overline{B_\mathfrak{c}}$.

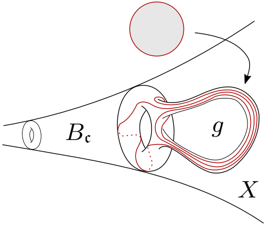




Horoball systems and necklaces

- A variety word w corresponds to a *necklace* η_w of horoballs in $\widetilde{B_{\mathfrak{c}}} \subset \mathbb{H}^3$ of $\ell_g(w)$ balls.
- A manifold obtained by attaching a disk along w is called a *necklace manifold* K_w.
- If we can find an *embedded* $K_w \subset X$ and K_w admits a *complete* hyperbolic structure with 2 cusps, then X is a Dehn filling of K_w.
- *Idea:* If w is "simple" enough, then a disk can always be attached in $X \setminus \overline{B_{\mathfrak{c}}}$.
- The groups $\pi_1(K_{w_i}) = \langle m, n, g \mid [m,n], w_i \rangle$ can be shown to be hyperbolic using John Berge's program Heegard. Note : *this is not enough to give Dehn filling.* In practice, however, this recovers the parent.





Necklace ≤ 7 manifolds

Necklace ≤ 7 manifolds

- ***Theorem.*** If $\mathrm{vol}(B_{\mathfrak{c}}) \leq 2.62$, there is an *embedded* ≤ 7-necklace manifold K in X.

Necklace ≤ 7 manifolds

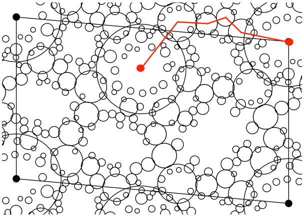
- ***Theorem.*** If $\mathrm{vol}(B_{\mathfrak{c}}) \leq 2.62$, there is an *embedded* ≤ 7-necklace manifold K in X.
- We show that ≤ 7 necklaces are *unknotted, unblocked*, and *unlinked*. For $n = 6$ unblocked is done by Adams-Knudson.

Necklace ≤ 7 manifolds

- ***Theorem.*** If $\mathrm{vol}(B_{\mathfrak{c}}) \leq 2.62$, there is an *embedded* ≤ 7-necklace manifold K in X.
- We show that ≤ 7 necklaces are *unknotted*, *unblocked*, and *unlinked*. For $n = 6$ unblocked is done by Adams-Knudson.
- One call ways find a *full* ≤ 7 necklace structure – the boundary of the attached disk cuts $\partial B_{\mathfrak{c}}$ into disks.

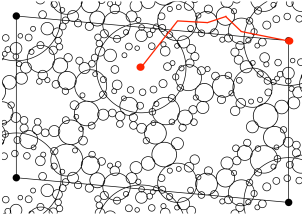
Necklace ≤ 7 manifolds

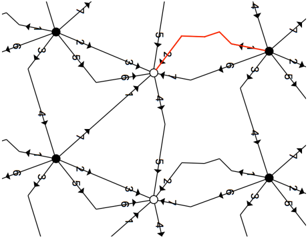
- ***Theorem.*** If $\mathrm{vol}(B_{\mathfrak{c}}) \leq 2.62$, there is an *embedded* ≤ 7-necklace manifold K in X.
- We show that ≤ 7 necklaces are *unknotted, unblocked*, and *unlinked*. For $n = 6$ unblocked is done by Adams-Knudson.
- One call ways find a *full* ≤ 7 necklace structure – the boundary of the attached disk cuts $\partial B_{\mathfrak{c}}$ into disks.



Necklace ≤ 7 manifolds

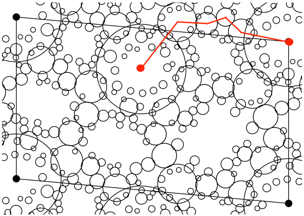
- ***Theorem.*** If $\mathrm{vol}(B_{\mathfrak{c}}) \leq 2.62$, there is an *embedded* ≤ 7-necklace manifold K in X.
- We show that ≤ 7 necklaces are *unknotted, unblocked*, and *unlinked*. For $n = 6$ unblocked is done by Adams-Knudson.
- One call ways find a *full* ≤ 7 necklace structure – the boundary of the attached disk cuts $\partial B_{\mathfrak{c}}$ into disks.

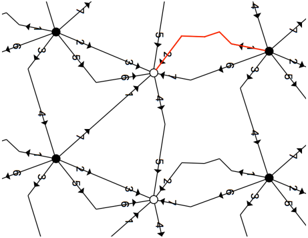




Necklace ≤ 7 manifolds

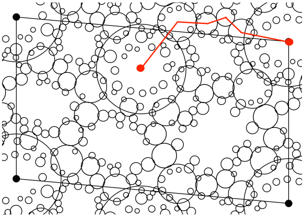
- ***Theorem.*** If $\mathrm{vol}(B_{\mathfrak{c}}) \leq 2.62$, there is an *embedded* ≤ 7-necklace manifold K in X.
- We show that ≤ 7 necklaces are *unknotted, unblocked*, and *unlinked*. For $n = 6$ unblocked is done by Adams-Knudson.
- One call ways find a *full* ≤ 7 necklace structure – the boundary of the attached disk cuts $\partial B_{\mathfrak{c}}$ into disks.
- Full n-necklace manifolds arise as gluings of n-*dipyramids* – suspension of a regular n-gon.

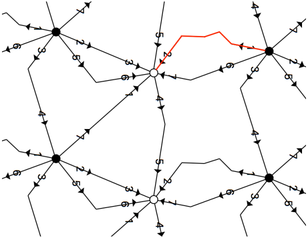


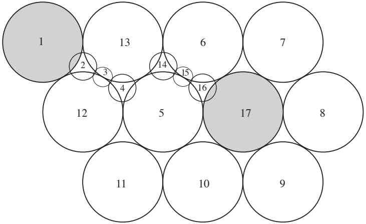


Necklace ≤ 7 manifolds

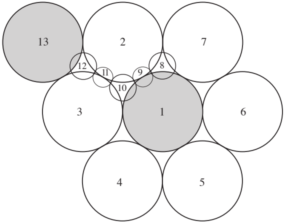
- ***Theorem.*** If $\mathrm{vol}(B_\mathfrak{c}) \leq 2.62$, there is an *embedded* ≤ 7-necklace manifold K in X.
- We show that ≤ 7 necklaces are *unknotted, unblocked*, and *unlinked*. For $n = 6$ unblocked is done by Adams-Knudson.
- One call ways find a *full* ≤ 7 necklace structure – the boundary of the attached disk cuts $\partial B_\mathfrak{c}$ into disks.
- Full n-necklace manifolds arise as gluings of n-*dipyramids* – suspension of a regular n-gon.
- ***Theorem.*** The full ≤ 7-necklace manifolds that are embeddable into hyperbolic 3-manifolds are the ones on our list.



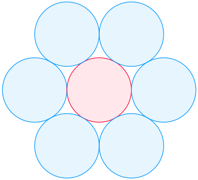




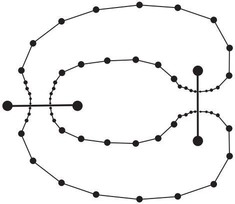
Knotted 18-necklace whose core is the trefoil. Horoball 18 is at infinity.



Knotted 14-necklace with an unknotted core. Horoball 14 is at infinity.



A 6-necklace blocked by red and infinity horoballs.



Borromean linking of unblocked and unknotted necklace.

Thank you!