Circle packings and Delaunay circle patterns for complex projective structures

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in collaboration with Jean-Marc Schlenker

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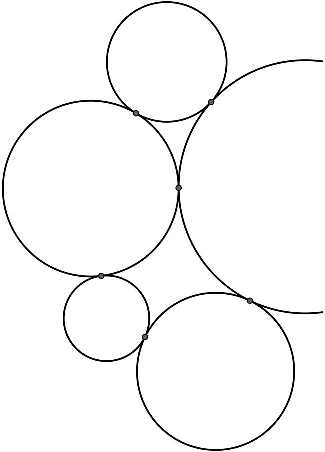

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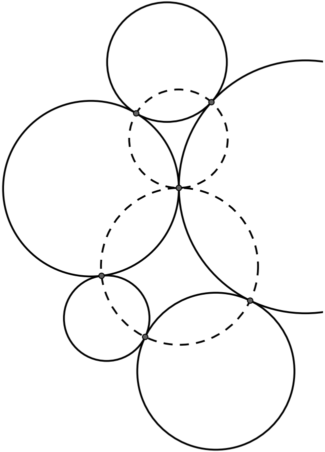

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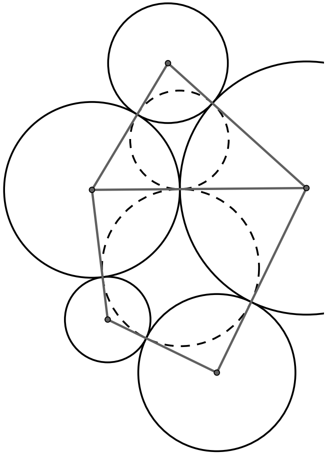

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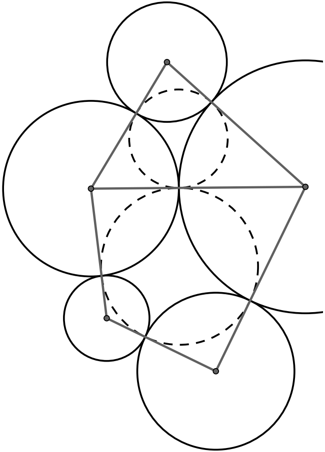

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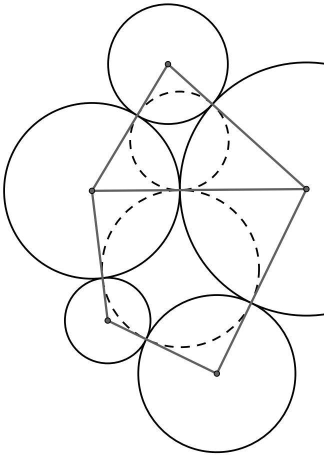
- ***Theorem (Brooks).*** The structures admitting a circle packing are dense in Teichmüller space.



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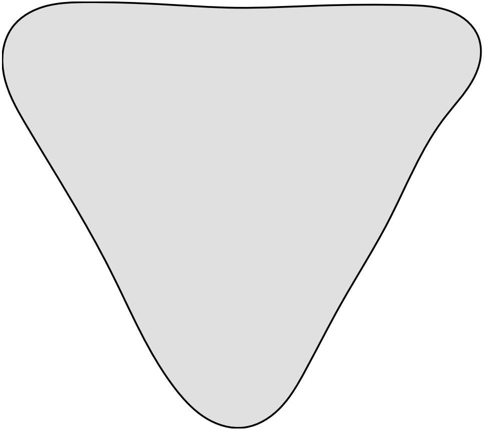

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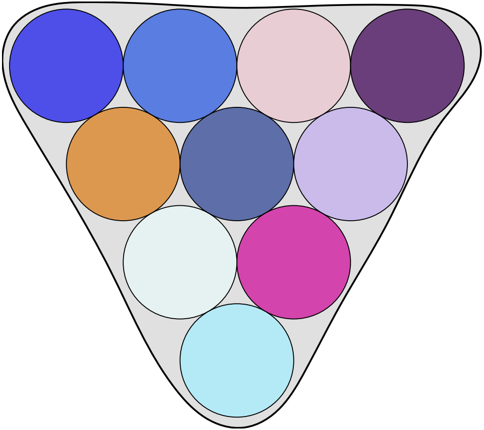

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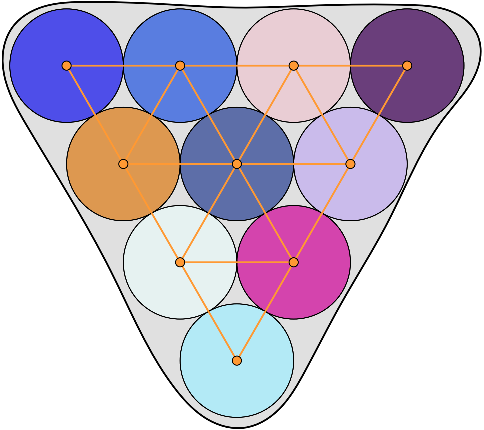

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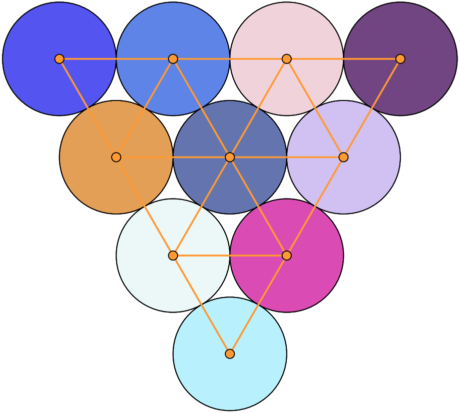

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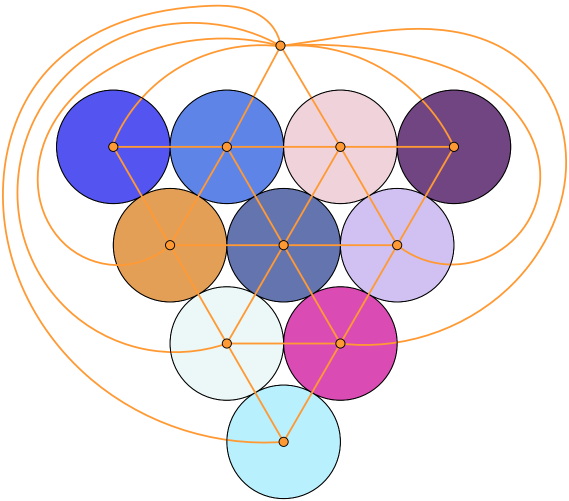

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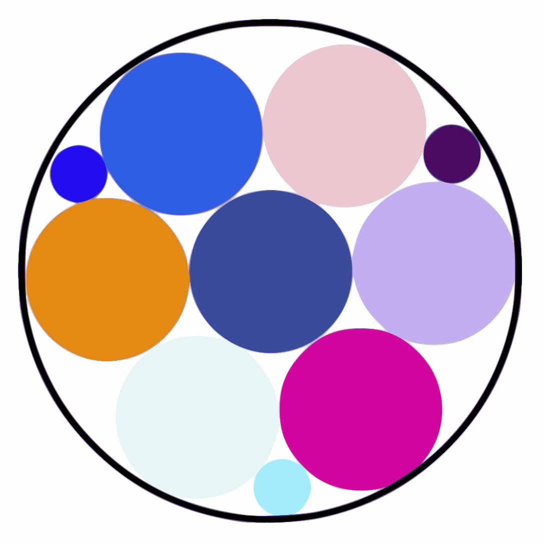

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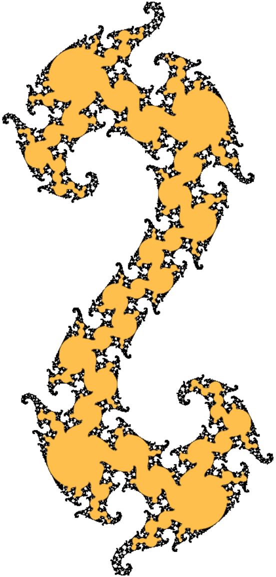

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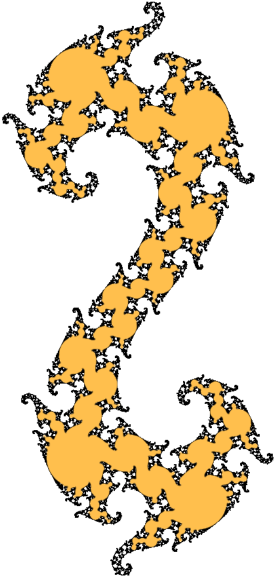

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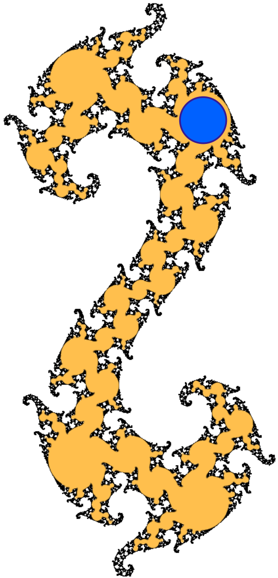

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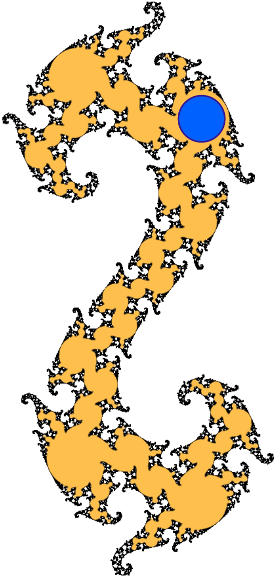
- When the covering map $\widetilde{S} \to S$ is injective on D, we call D an *embedded disk*.



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To complete the Conjecture, one would need to show that

- $\mathcal{C}_\tau$ is a manifold,
- f_τ is locally injective,
- $\mathcal{C}_\tau$ is connected.

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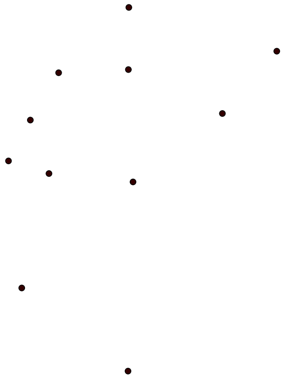

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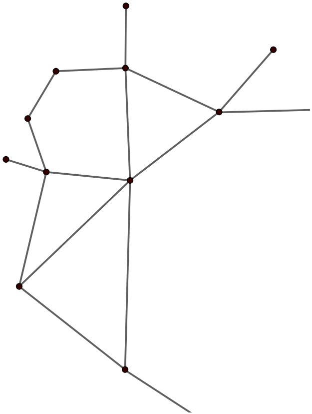

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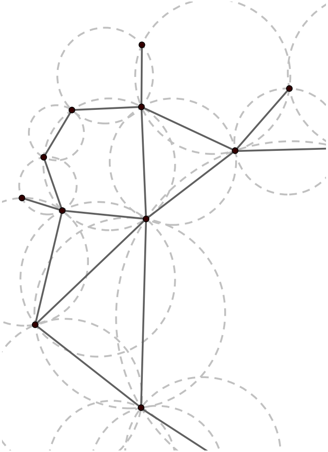

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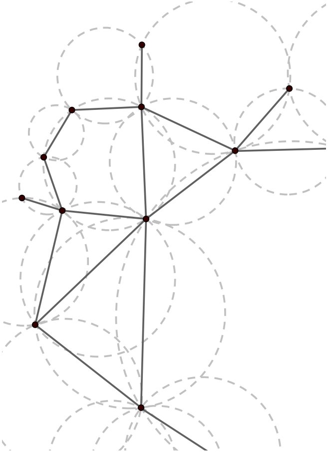

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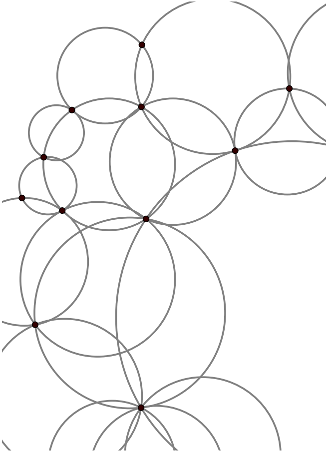

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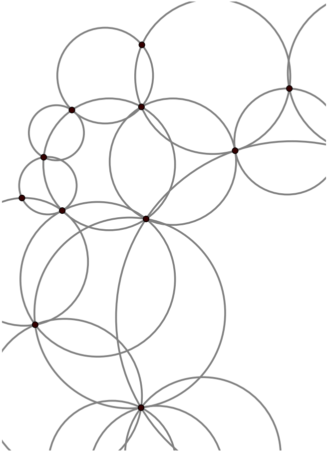

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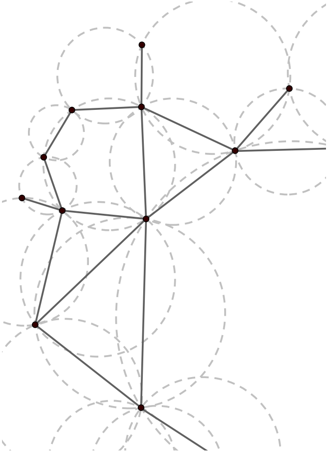

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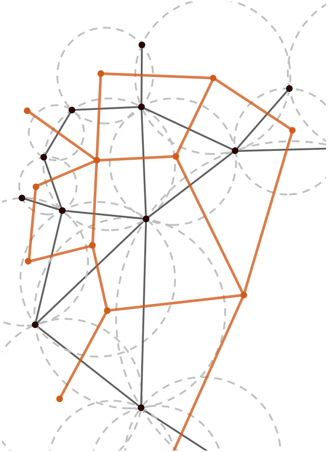

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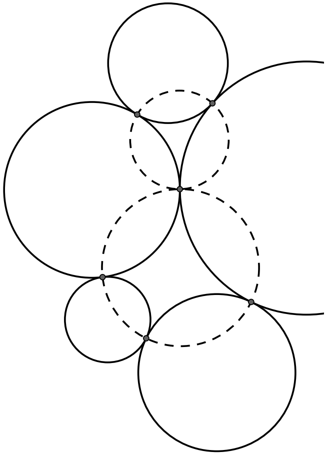

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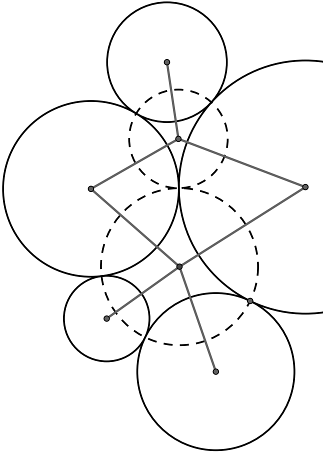

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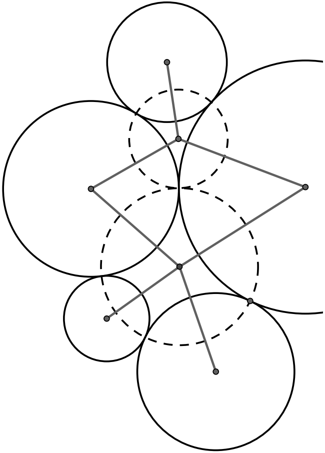
- The *nerve* of $\mathcal{D}$ is the dual $\eta^*_{\mathcal{D}}$ of $\eta_{\mathcal{D}} = \eta$.
- To get a Delaunay circle pattern from a circle packing, one just needs to add the dual disks.
- At points of $\mathcal{V}(\eta_{\mathcal{D}})$, we can measure angles between outward normals of overlapping disks to define the *angle function* $\theta_{\mathcal{D}} : \mathcal{E}(\eta^*_{\mathcal{D}}) \to (0, \pi)$.



Properness for Delaunay circle patterns

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Theorem (Schlenker–Y.). Let (η, θ) be an admissible pair, then the forgetful map $f_{\eta,\theta} : \mathcal{C}_{\eta,\theta} \to \mathcal{T}$ is proper.

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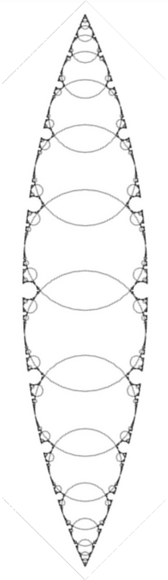

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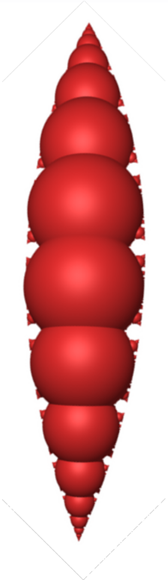

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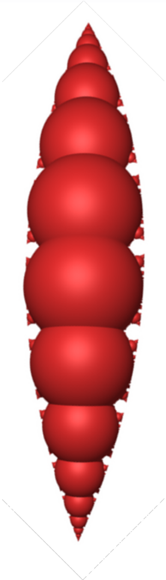

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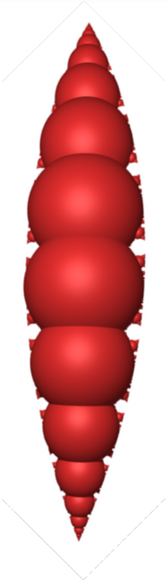

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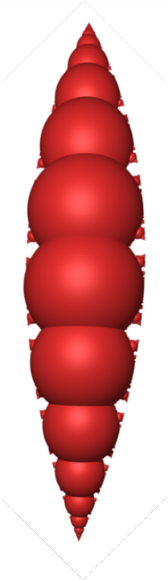

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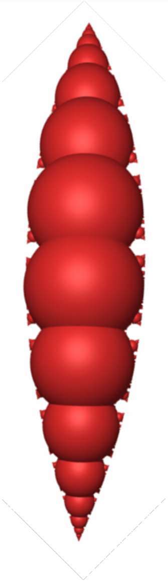
- ***Lemma.*** For an admissible pair (η, θ), the set of all μ_σ with σ appearing in $\mathcal{C}_{\eta,\theta}$ is is pre-compact in $\mathcal{ML}$.
- Let $K \subset \mathcal{T}$ be compact, then results on grafting and this Lemma show that set of σ appearing in $f_{\eta,\theta}^{-1}(K)$ is pre-compact.



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- This Lemma and compactness properties of the bundle of round disks on $\mathcal{C}$ let us conclude that $f_{\eta,\theta}^{-1}(K)$ is pre-compact.

Thank you!