Computational techniques for hyperbolic 3-manifolds

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Homogeneous geometry in dimension 2

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Hyperbolic Space

Homogeneous geometries

Hyperbolic 3-manifolds

<table>
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<th>Poincaré disk model</th>
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This saddle-shaped geometry is called hyperbolic geometry. To work with it, we need a model where we can draw "straight" lines and measure length and angles. Let $H^2$ be the interior of the unit disk – the circle will be "at infinity." Hyperbolic straight lines are circular arcs perpendicular to the "circle at infinity." Hyperbolic circles look like regular circles, but their centers are elsewhere. Hyperbolic triangles look very much like the triangles on the saddle.
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- Notice that the sides are still straight line segments.
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- Given a combinatorial gluing, we can use computational techniques to find solutions for the shapes of these polyhedra.
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- We look at the complement in the 3-sphere – this is $\mathbb{R}^3$ plus a point at infinity.
- Going in the reverse direction, we see that for some 3-manifolds (given combinatorially) it is possible to find a (complete) hyperbolic structure. It so happens that when the structure is *finite volume*, it is *unique* (for the given combinatorial gluing).
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- Broadly speaking, the Geometrization Theorem states that 3-manifolds can be (uniquely) cut along surfaces such that the remaining pieces admit one of 8 model geometries, with hyperbolic geometry being very prevalent.
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