Computational techniques for hyperbolic 3-manifolds

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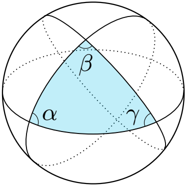
Homogeneous geometry in dimension 2



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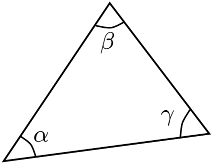

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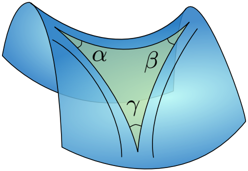

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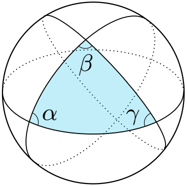

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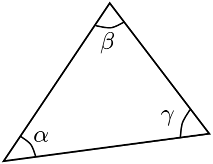

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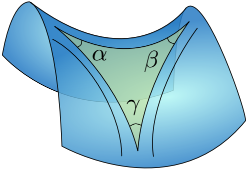

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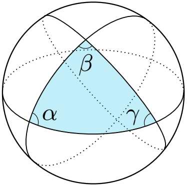

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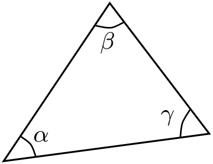

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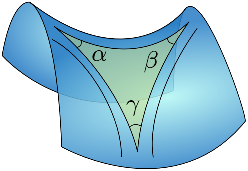

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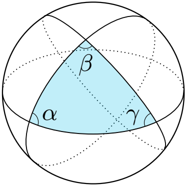

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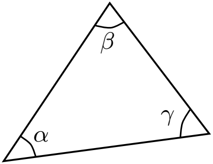

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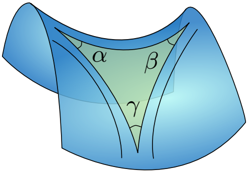

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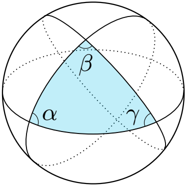

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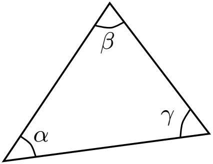

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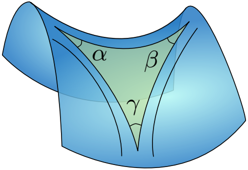

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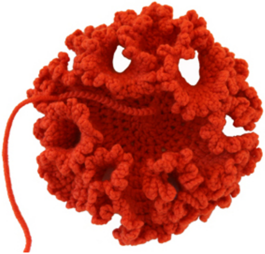

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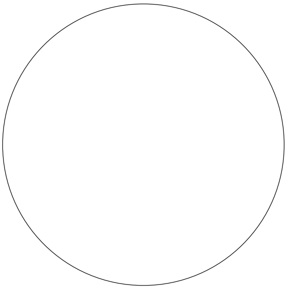

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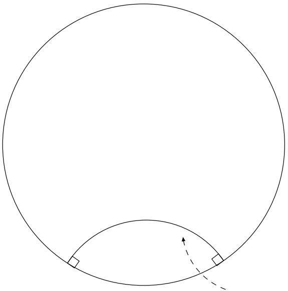

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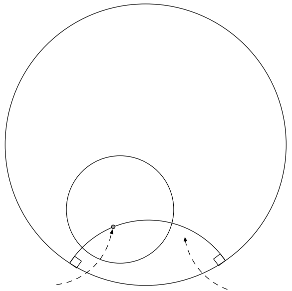

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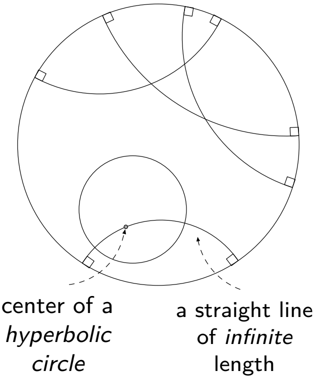

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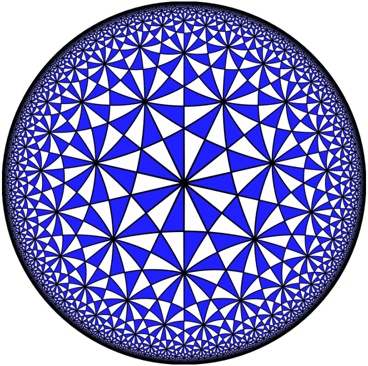

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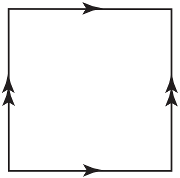

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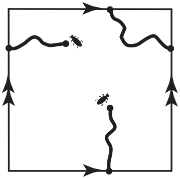

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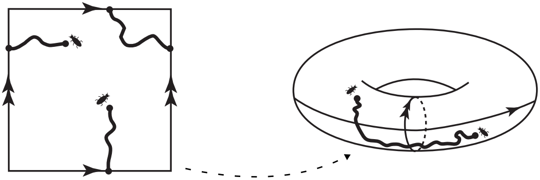

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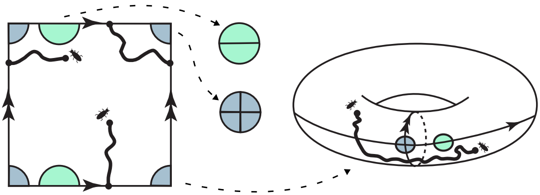

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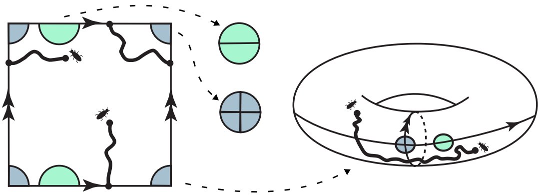

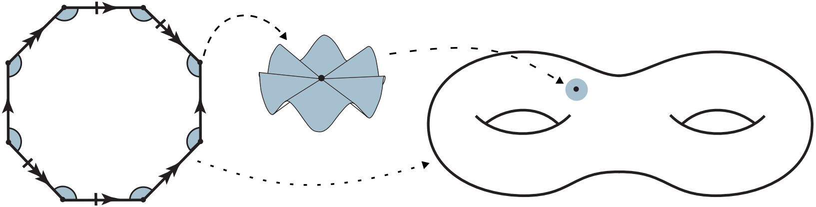

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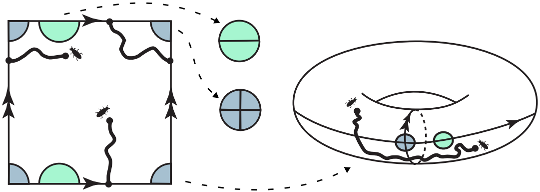

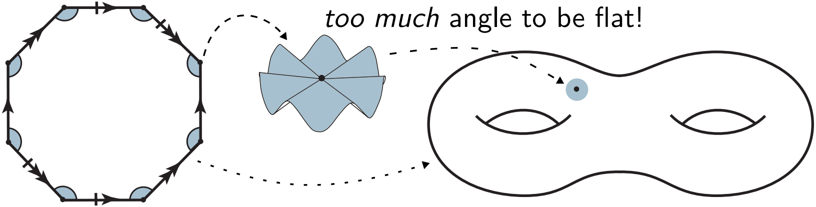

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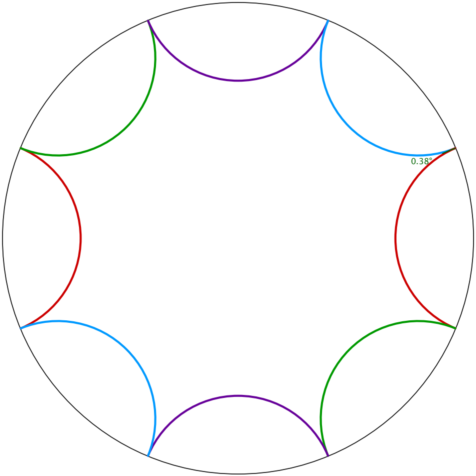

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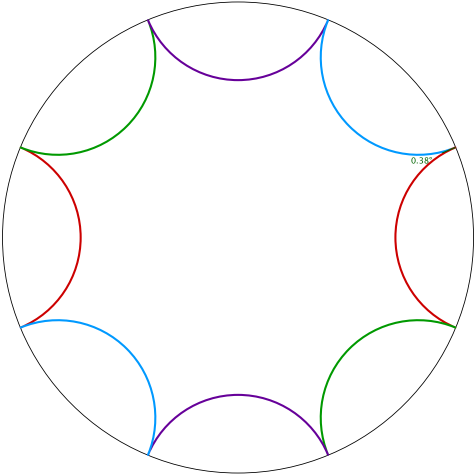

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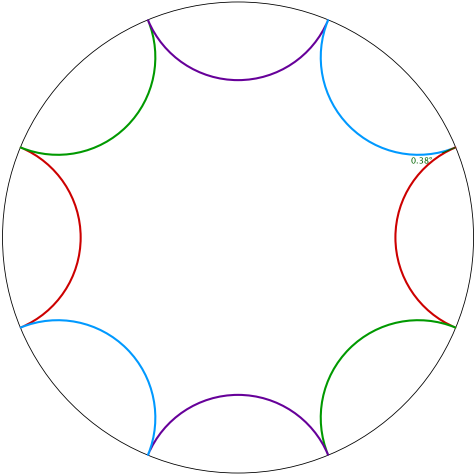

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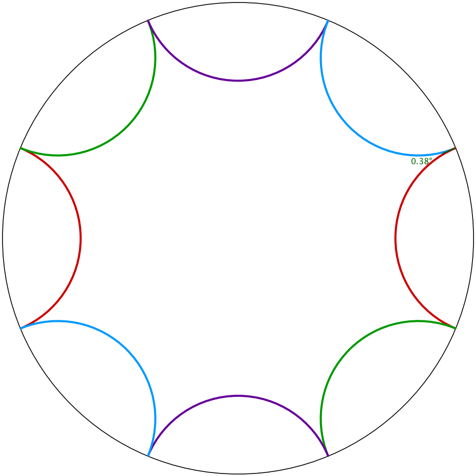
- Notice that the sides are still *straight* line segments.



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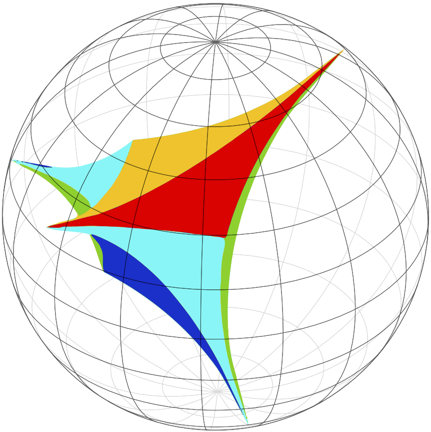

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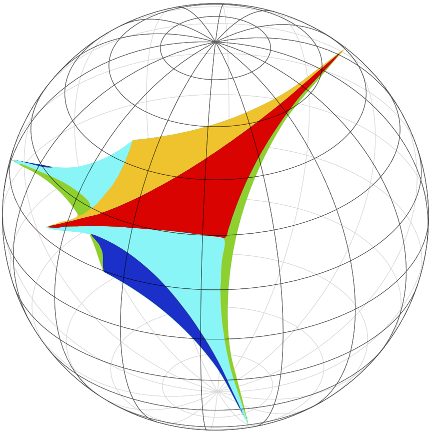

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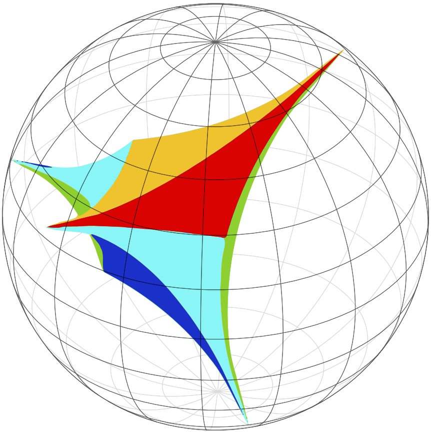
- The angle condition becomes more complicated – we now get equations called the *Poincaré Polyhedron Theorem* and the *Thurston gluing equations*.
- Given a *combinatorial* gluing, we can use computational techniques to find solutions for the shapes of these polyhedra.



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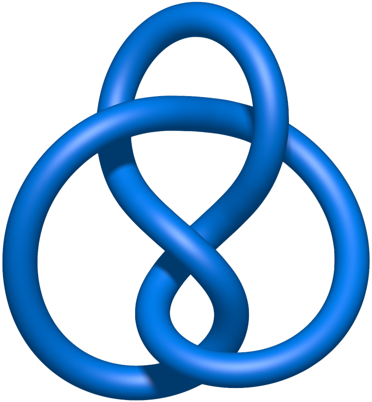

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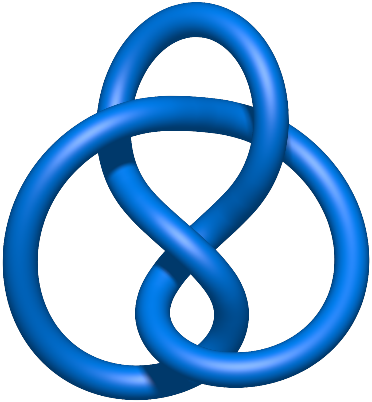

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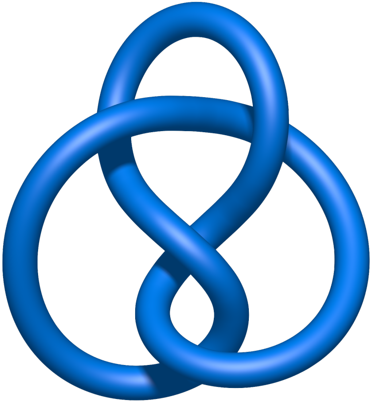
- We look at the complement in the 3-sphere – this is $\mathbb{R}^3$ plus a point at infinity.
- Going in the reverse direction, we see that for some 3-manifolds (given combinatorially) it is possible to find a (complete) hyperbolic structure. It so happens that when the structure is *finite volume*, it is *unique* (for the given combinatorial gluing).



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- Broadly speaking, the Geometrization Theorem states that 3-manifolds can be (uniquely) *cut* along surfaces such that the remaining pieces admit one of 8 *model geometries*, with hyperbolic geometry being very prevalent.

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