IMPROVED BOUNDS FOR AVERAGE BENDING ON THE CONVEX CORE OF A KLEINIAN GROUP

ANDREW YARMOLA

Abstract. Let $\Gamma \leq \text{Isom}^+(\mathbb{H}^3)$ be a finitely-generated Kleinian group and $N = \mathbb{H}^3/\Gamma$. In this note, we consider the case where the boundary $\partial C(N)$ of the convex core is non-empty and incompressible in $N$. Our main result improves bounds on the total bending of a geodesic arc on $\partial C(N)$ as a function of its length. Following the work of Bridgeman [Bri03], we use this to provide better bounds on the length and average bending of the measured lamination on $\partial C(N)$ and universal bounds on the optimal Lipschitz constant for a map from $\partial C(N)$ to the conformal structure at infinity $\partial_\infty N$. Additionally, we show that the shortest rectifiable path connecting four disjoint sequentially tangent hyperplanes in $\mathbb{H}^3$ is attained when the planes support four of the faces of a standard ideal octahedron.

This work formed part of the author’s thesis.

1. Introduction

Let $\Gamma \leq \text{Isom}^+(\mathbb{H}^3)$ be a finitely-generated Kleinian group, $N = \mathbb{H}^3/\Gamma$ and assume that the boundary $\partial C(N)$ of the convex core is non-empty and incompressible in $N$. Thurston [Thu91] showed that the path metric on $\partial C(N)$ is a complete hyperbolic metric. In addition to this structure, one may consider the conformal structure $\partial_\infty N = \Omega_\Gamma/\Gamma$, where $\Omega_\Gamma \subset \hat{\mathbb{C}}$ is the domain of discontinuity of $\Gamma$ as a subgroup of $\text{PSL}(2, \mathbb{C})$. Since $\partial C(N)$ is non-empty and incompressible in $N$, $\Omega_\Gamma$ is non-empty and its components are simply connected hyperbolic domains. Let $CH(\cdot)$ denote taking the convex hull in $\mathbb{H}^3$. There is a well-defined $\Gamma$-equivariant map $\tilde{r} : \Omega_\Gamma \to CH(\hat{\mathbb{C}} \setminus \Omega_\Gamma)$ given by letting $\tilde{r}(x)$ be the first point of intersection of a growing family of horoballs at $x \in \Omega_\Gamma$ with $CH(\hat{\mathbb{C}} \setminus \Omega_\Gamma)$. This map projects to the nearest point retraction $r : \partial_\infty N \to \partial C(N) = CH(\hat{\mathbb{C}} \setminus \Omega_\Gamma)/\Gamma$.

The hyperbolic structure on $\partial C(N)$ comes with a measured lamination $\mu_\Gamma$ on $\partial C(N)$, called the bending lamination. For any measured lamination $\mu$ on hyperbolic surface and $L > 0$, one defines the $L$-roundness $\|\mu\|_L$ as

$$\|\mu\|_L = \sup i(\alpha, \mu)$$

where the supremum is taken over all open geodesic arcs of length $L$. As seen in the work of Epstein, Marden, and Markovic [EMM04] and our previous work [BCY16] $L$-roundness for a measured lamination $\mu$ on $\mathbb{H}^2$ provides some control on the embeddedness of the associated pleating map $P_\mu : \mathbb{H}^2 \to \mathbb{H}^3$. Extending [BCY16 Theorem 3.1], we prove

In this note, we extend our previous results in [BCY16] to prove
Theorem 1.1. If $L \in (0, 2 \sinh^{-1}(2)]$, $\mu$ is a measured lamination on $\mathbb{H}^2$, and $P_{\mu}$ is an embedding, then $\|\mu\|_L \leq F(L)$ where

$$F(L) = \begin{cases} 
2 \cos^{-1}(-\sinh(L/2)) & \text{for } L \in [0, 2 \sinh^{-1}(1)] \\
3\pi - 2 \cos^{-1}\left(\left(\sqrt{\cosh(L)} - 1\right)/2\right) & \text{for } L \in (2 \sinh^{-1}(1), 2 \sinh^{-1}(2)]
\end{cases}$$

As a corollary of the proof of Theorem 1.1, we obtain

Corollary 1.1. The shortest rectifiable path $\alpha(t) \subset \mathbb{H}^3$ connecting four disjoint sequentially tangent hyperplanes in $\mathbb{H}^3$ has length $2 \sinh^{-1}(2)$ and is attained when the planes support four of the faces of a standard ideal octahedron.

Theorem 1.1 allows us to improve bounds by Bridgeman [Bri03, Theorem 1.2] on average bending and the Lipschitz constant for the homotopy inverse of the retraction map.

Theorem 1.2. There exist universal constants $K_0, K_1$ with $K_0 \leq 2.494$ and $K_1 \leq 3.101$ such that if $\Gamma \leq \text{Isom}^+(\mathbb{H}^3)$ is a finitely-generated Kleinian group, $N = \mathbb{H}^3/\Gamma$, and the boundary $\partial C(N)$ of the convex core is non-empty and incompressible in $N$, then

(i) if $\mu_\Gamma$ is the bending lamination of $\partial C(N)$, then

$$\ell_{\partial C(N)}(\mu_\Gamma) \leq K_0 \pi^2 |\chi(\partial C(N))|$$

(ii) for any closed geodesic $\alpha$ on $\partial C(N)$,

$$B_\Gamma(\alpha) = \frac{i(\alpha, \mu_\Gamma)}{\ell(\alpha)} \leq K_1$$

where $B_\Gamma(\alpha)$ is called the average bending of $\alpha$.

(iii) there exists a $(1 + K_1)$-Lipschitz map $s : \partial C(N) \rightarrow \partial_\infty N$ that is a homotopy inverse to the nearest point retraction $r : \partial_\infty N \rightarrow \partial C(N)$.

The previous best constants in [Bri03 Theorem 1.2] were $K_0 \leq 2.8396$ and $K_1 \leq 3.4502$.

2. Background

2.1. Kleinian Groups and Convex Hulls. Let $\Gamma \leq \text{Isom}^+(\mathbb{H}^3)$ be a discrete torsion free subgroup. Define the limit set of $\Gamma$ to be $\Lambda_\Gamma = \overline{\Gamma x} \cap \partial_\infty \mathbb{H}^3$ for any $x \in \mathbb{H}^3$. This definition is independent of the choice of $x$. We say that $\Gamma$ is a Kleinian group if $\Lambda_\Gamma$ contains at least 3 points. The set $\Omega(\Gamma) = \partial_\infty \mathbb{H}^3 \setminus \Lambda_\Gamma$ is called the domain of discontinuity of $\Gamma$. It can be equivalently defined as the largest open subset in $\partial_\infty \mathbb{H}^3$ where $\Gamma$ acts properly discontinuously.

The convex hull $CH(X)$ of a closed set $X \subset \partial_\infty \mathbb{H}^3$ is smallest convex subset of $\mathbb{H}^3$ such that $CH(X) \cap \partial_\infty \mathbb{H}^3 = X$. We require that $X$ contain more than two points. For a Kleinian group $\Gamma$, the convex hull of $\Gamma$ is $CH(\Lambda_\Gamma)$ and the convex hull of $\mathbb{H}^3/\Gamma$ is $CH(\Lambda_\Gamma)/\Gamma$, which is the smallest $\pi_1$-injective convex submanifold.
Laminations.

2.2. A hyperbolic domain \( \Omega \) in \( \hat{\mathbb{C}} \) is a connected open set such that \( \hat{\mathbb{C}} \setminus \Omega \) is at least 3 points. In particular, when one identifies \( \partial_{\infty} \mathbb{H}^3 \cong \hat{\mathbb{C}} \), a connected component of \( \Omega(\Gamma) \) for a Kleinian group \( \Gamma \) is a hyperbolic domain. Let \( X = \hat{\mathbb{C}} \setminus \Omega \). Epstein and Marden \[EMS7\] show that if \( X \) is not contained in a circle, then \( CH(X) \) has non empty interior and a well defined boundary, denoted Dome(\( \Omega \)) = \( \partial CH(X) \). If \( X \) lies in a circle, then \( CH(X) \) lies in a hyperbolic plane and is bounded by a countable collection of complete geodesics. In this setting, Dome(\( \Omega \)) is defined as the double \( CH(X) \) along those geodesics.

Points on Dome(\( \Omega \)) can be connected by rectifiable paths along Dome(\( \Omega \)) and so it inherits a path metric from \( \mathbb{H}^3 \). Thurston \[Thu91\] showed that this path metric is, in fact, a complete hyperbolic metric. Further, he demonstrates that the covering map \( \mathbb{H}^2 \to \text{Dome}(\Omega) \) as a very specific structure that we now describe.

2.2. Laminations. A geodesic lamination on \( \mathbb{H}^2 \) is a closed subset \( \lambda \subset \mathcal{G}(\mathbb{H}^2) \) which does not contain any intersecting geodesics. It can be realized on \( \mathbb{H}^2 \) as a closed set foliated by complete geodesics and therefore the elements of \( \lambda \) are called leaves. A measured lamination \( \mu \) on \( \mathbb{H}^2 \) is a non-negative countably additive measure \( \mu \) on \( \mathcal{G}(\mathbb{H}^2) \) supported on a geodesic lamination. A geodesic arc \( \alpha \) in \( \mathbb{H}^2 \) is said to be transverse to \( \mu \), if it is transverse to every geodesic in \( \text{supp}(\mu) \). Whenever \( \alpha \) is transverse to \( \mu \), we define

\[
i(\mu, \alpha) = \mu \left( \{ \gamma \in \mathcal{G}(\mathbb{H}^2) \mid \gamma \cap \alpha \neq \emptyset \} \right).
\]

If \( \alpha \) is not transverse to \( \mu \), then it is contained in a geodesic of \( \text{supp}(\mu) \) and we let \( i(\mu, \alpha) = 0 \).

Given a measured lamination \( \mu \) on \( \mathbb{H}^2 \), we may construct a pleated plane \( P_\mu : \mathbb{H}^2 \to \mathbb{H}^3 \), well-defined up to post-composition with elements of \( \text{Isom}^+(\mathbb{H}^3) \). \( P_\mu \) is an isometry on the components of \( \mathbb{H}^2 \setminus \text{supp}(\mu) \), which are called flats. If \( \mu \) is a finite-leaved lamination, then \( P_\mu \) is simply obtained by bending, consistently rightward, by the angle \( \mu(\{l\}) \) along each leaf \( l \) of \( \mu \). Since any measured lamination is a limit of finite-leaved laminations, one may define \( P_\mu \) in general by taking limits (see \[EMS7\] Theorem 3.11.9).

**Lemma 2.1.** \[EMS7\] If \( \Omega \) is a hyperbolic domain, there is a lamination \( \mu \) on \( \mathbb{H}^2 \) such that \( P_\mu \) is a locally isometric covering map with image Dome(\( \Omega \)).

2.3. Pleated Planes. For any point \( x \in \text{Dome}(\Omega) \), a support plane \( P \) at \( x \) is a totally geodesic plane through \( x \) which is disjoint from the interior of the convex hull of \( \hat{\mathbb{C}} \setminus \Omega \). At least one support plane exists at every point \( x \in \text{Dome}(\Omega) \) and \( \text{Dome}(\Omega) \cap P \) is either a geodesic line with endpoints in \( \partial \Omega \), called a bending line, or a flat, which is the convex hull.
of a subset of $\partial P$ containing at least 3 points. The boundary geodesics of a flat will also be called bending lines. Support planes come with a preferred normal direction pointing away from $CH(\mathbb{C} \setminus \Omega)$. The closure of the complement of $\mathbb{H}^3 \setminus P$ that lies in this direction is called the associated half space, denoted $H_P$. A detailed discussion and proofs on these facts can be found in [EMS7].

For a curve $\alpha : (a, b) \to \text{Dome}(\Omega)$, it is natural to consider the space of support planes at each point $\alpha(t)$. A theorem of Kulkarni and Pinkall [KP94] asserts that the space of support planes to $\text{Dome}(\Omega)$ is an $\mathbb{R}$-tree in the induced path metric from $\mathcal{P}(\mathbb{H}^3)$ whenever $\Omega$ is a simply connected hyperbolic domain. Recall that an $\mathbb{R}$-tree is a simply connected, geodesic metric space such that for any two points there is a unique embedded arc connecting them. Therefore, dual to any rectifiable path $\alpha : (a, b) \to \text{Dome}(\Omega)$, there is a continuous path $P_t : (c, d) \to \mathcal{P}(\mathbb{H}^3)$ and a map $p : (c, d) \to (a, b)$ such that $P_t$ is a support plane at $\alpha(p(t))$. It also follows that we can define terminal support planes on the ends of $\alpha$ by $P_a = \lim_{t\to c^+} P_t$ and $P_b = \lim_{t\to d^-} P_t$.

Epstein and Marden further show that for every point $x \in \text{Dome}(\Omega)$, there is a neighborhood $W \subset \mathbb{H}^3$ of $x$ such that if $l_1, l_2$ are bending lines that meet $W$, then any support plane that meets $l_1$ intersects all support planes that meet $l_2$ [EMS7, Lemma 1.8.3]. The transverse intersection of two support planes $P, Q$ is called a ridge line. Notice that if two support planes $P, Q$ intersect, they either do so at a ridge line or $P = Q$. If $P = Q$ and the interiors of $H_P, H_Q$ are not equal, then $\hat{\mathbb{C}} \setminus \Omega$ is contained in a the circle $\partial P$.

The exterior angle, denoted $\angle_{\text{ext}}(P, Q)$, between two intersecting or tangent support planes is the angle between their normal vectors at any point of intersection or tangency. We define the interior angle by $\angle_{\text{int}}(P, Q) = \pi - \angle_{\text{ext}}(P, Q)$.

Let $\mu$ be the measured lamination on $\text{Dome}(\Omega)$ such that $P_{\mu} : \mathbb{H}^2 \to \text{Dome}(\Omega)$ is the pleated plane. By a transverse geodesic arc $\alpha : (a, b) \to \text{Dome}(\Omega)$, we will mean arc such that $P_{\alpha}^{-1}(\alpha)$ is a geodesic arc in $\mathbb{H}^2$ and transverse to $\text{supp}(\mu)$. We say the terminal support planes $P_a, P_b$ form a roof over $\alpha$ if the interiors of the associated half spaces $H_t$ intersects $H_a$ for all $t$. Roofs play an important role in approximating the bending along $\alpha$.

**Lemma 2.2.** (Lemmas 4.1 and 4.2 [BC03]) Let $\mu$ be the measured lamination on $\text{Dome}(\Omega)$. If $\alpha : (a, b) \to \text{Dome}(\Omega)$ is a transverse geodesic arc such that the terminal support planes $P_a, P_b$ form a roof over $\alpha$ then $\angle_{\text{int}}(P, Q) \leq \angle_{\text{ext}}(P, Q) = \pi - \angle_{\text{int}}(P, Q)$.

**Lemma 2.3.** Let $\Omega$ be a simply connected hyperbolic domain and $\alpha : (a, b) \to \text{Dome}(\Omega)$ a transverse geodesic arc. If the interiors of the terminal half spaces $H_a, H_b$ intersect, then $P_a$ and $P_b$ form a roof over $\alpha$.

**Proof.** Intuitively, this is a consequence of the fact that support planes can’t form “loops” when $\Omega$ is simply connected. Recall that the space of support planes to $\text{Dome}(\Omega)$ is an $\mathbb{R}$-tree. Since $\alpha$ is geodesic, the of support planes $P_t$ to $\alpha$ must be embedded, and therefore the unique path between $P_a$ and $P_b$. As the interiors of $H_a, H_b$ intersect, either $P_a = P_b$.
or $P_a, P_b$ intersect at a ridge line $\ell_r$. In the former case, it follows that $P_t = P_a = P_b$ is constant and therefore $H_t = H_a$ for all $t$.

In the later case, consider the path $\beta$ which goes from $\alpha(a)$ to $\ell_r$ along $P_a$ and from $\ell_r$ to $\alpha(b)$ along $P_b$. We can project $\beta$ to $r(\beta) \subset \text{Dome}(\Omega)$. Since $P_t$ is the unique path connecting $P_a$ to $P_b$, it follows that the path of support planes along $r(\beta)$ must fun over all of $P_t$. By construction, every support plane to $r(\beta)$ must contain the ridge line $\ell_r$. Thus, the interiors of $H_t$ and $H_a$ intersect for all $t$ and $P_a, P_b$ is a roof over $\alpha$.

2.4. $L$-roundness. For a measured lamination $\mu$ on $\mathbb{H}^2$, Epstein, Marden and Markovic [EMM04] defined the roundness of $\mu$ to be $||\mu|| = \sup i(\mu, \alpha)$ where the supremum is taken over all open unit length geodesic arcs in $\mathbb{H}^2$. The roundness bounds the total bending of $P_\mu$ on any segment of length 1 and is closely related to average bending, which was introduced earlier by Bridgeman [Bri98].

In our work, we consider the $L$-roundness of a measured lamination for any $L > 0$

$$||\mu||_L = \sup i(\alpha, \mu)$$

where now the supremum is taken over all open geodesic arcs of length $L$ in $\mathbb{H}^2$. We note that the supremum over open geodesic arcs of length $L$, is the same as that over half open geodesic arcs of length $L$.

3. Improved Upper Bound on $L$-Roundness for Embedded Pleated Planes

In this section, we adapt the techniques of [Bri03] to obtain an improved bound on the $L$-roundness of an embedded pleated plane. As it appears here, Theorem 1.1 is an extended version of our work in [BCY16 Theorem 3.1].

**Theorem 1.1.** If $L \in (0, 2 \sinh^{-1}(2)]$, $\mu$ is a measured lamination on $\mathbb{H}^2$, and $P_\mu$ is an embedding, then $||\mu||_L \leq F(L)$ where

$$F(L) = \begin{cases} 
2 \cos^{-1}(\sinh(L/2)) & \text{for } L \in [0, 2 \sinh^{-1}(1)] \\
3\pi - 2 \cos^{-1}\left(\left(\sqrt{\cosh(L)} - 1\right)/2\right) & \text{for } L \in (2 \sinh^{-1}(1), 2 \sinh^{-1}(2)]
\end{cases}$$

The proof relies on a careful analysis of minimal lengths of arcs joining a sequence of 3 or 4 pleated planes. We present these arguments as Lemmas 3.1 [3.2] [3.4]

**Lemma 3.1.** Let $P_0, P_1, P_2$ be planes in $\mathbb{H}^3$ with boundary circles $C_i \in \partial_\infty \mathbb{H}^3$. Assume that $C_0 \cap C_2 = \{a\}$, $a \notin C_1$, and the minor angles $\angle_m(C_0, C_1) = \angle_m(C_1, C_2) = \theta < \pi/2$. If $\alpha : [0, 1] \to \mathbb{H}^3$ is a rectifiable path with $\alpha(0) \in P_0, \alpha(1) \in P_2$ and $\alpha(t_1) \in P_1$ for some $t_1 \in (0, 1)$. Then,

$$\ell(\alpha) \geq 2 \sinh^{-1}(\cos \theta).$$

**Proof.** Since $a \notin C_1$, there is a plane $T \subset \mathbb{H}^3$ perpendicular to all $P_i$. Let $\lambda_t = T \cap P_t$. Take $\pi$ to be the nearest point projection of $\alpha$ onto $T$. Since nearest point projections shrink
distances, \( \ell(\alpha) \geq \ell(\pi) \). In addition, as \( T \) is perpendicular to \( P_i \), we have \( \pi(0) \in \lambda_0, \pi(1) \in \lambda_2 \) and \( \pi(t_1) \in \lambda_1 \). We can identify \( T \) with the Poincare disk and conjugate \( \lambda_i \) as in Figure 1.

By symmetry, the shortest curve connecting \( \lambda_0 \) to \( \lambda_2 \) via \( \lambda_1 \) is the symmetric piecewise geodesic \( \beta \) depicted in Figure 1. Let \( x \) be the sub-arc of \( \lambda_1 \) between \( \lambda_1 \cap \lambda_2 \) and \( \lambda_1 \cap \beta \).

Then, one may apply hyperbolic trigonometry formulae [Bea95, Theorem 7.9.1] and [Bea95, Theorem 7.11.2] to obtain

\[
\sinh(x) \tan \theta = 1 \quad \text{and} \quad \sinh(\ell(\beta)/2) = \sinh(x) \sin \theta.
\]

Therefore,

\[
\ell(\alpha) \geq \ell(\beta) \geq 2 \sinh^{-1}(\cos \theta).
\]

**Lemma 3.2.** Let \( P_0, P_1, P_2, P_3 \) be planes in \( \mathbb{H}^3 \) with boundary circles \( C_i \in \partial_{\infty} \mathbb{H}^3 \). Assume

(i) \( P_0 \cap P_2 = P_1 \cap P_3 = P_0 \cap P_3 = \emptyset \)
(ii) \( C_0 \cap C_3 = \{a\} \) and \( C_1 \cap C_2 = \{b\} \)
(iii) \( a \notin C_1 \cup C_2 \) and \( b \notin C_0 \cup C_3 \).
(iv) let \( \eta_i \) be normal directions to \( C_i \) such that \( \eta_0, \eta_3 \) point away from each other and \( \eta_1, \eta_2 \) point toward each other, then \( \angle(\eta_0, \eta_1) = \angle(\eta_2, \eta_3) = \theta < \pi/2 \).

If \( \alpha : [0, 1] \to \mathbb{H}^3 \) is a rectifiable path with \( \alpha(0) \in P_0, \alpha(1) \in P_3, \alpha(t_1) \in P_1, \) and \( \alpha(t_2) \in P_2 \) for some \( t_1, t_2 \in (0, 1) \) with \( t_1 < t_2 \). Then,

\[
\ell(\alpha) \geq \cosh^{-1}\left(\frac{1 + 2 \cos \theta}{2}\right).
\]

**Proof.** Let \( \rho_i \) denote the reflection across \( P_i \), and \( \rho_{i,j} = \rho_i \circ \rho_j \). Since \( \alpha \) is supported by the planes \( P_i \), we may look at pieces of \( \alpha \) under a series of reflections. In particular, consider the curve

\[
\beta = \alpha[0, t_1] \cup \rho_1(\alpha[t_1, t_2]) \cup \rho_{1,2}(\alpha[t_2, 1]).
\]
Notice that $\beta$ is a curve from $P_0$ to $\rho_{1,2}(P_3)$ and $\ell(\alpha) = \ell(\beta)$. Our goal is now to find a lower bound for $\ell(\beta)$ in terms of $\theta$.

By construction, $\beta$ is longer than the geodesic from $P_0$ to $\rho_{1,2}(P_3)$. Notice that this geodesic intersects $P_1$ and $\rho_{1}(P_2)$, so after reflecting some pieces, it satisfies the assumptions of the Lemma. Let $T$ be the hyperplane going through the Euclidean centers of $C_0$ and $\rho_{1,2}(C_3)$. Since the geodesic between $P_0$ and $\rho_{1,2}(P_3)$ is unique, it must lie in $T$. Refer to Figure 2 for the generic configuration.

We need to say a few words about the validity of Figure 2 for our computations. Conjugating, we can map the points $a \to 0$ and $b \to \infty$. It follows from (ii) and (iii) that $C_1, C_2$ are parallel lines and $C_0, C_3$ are circles in the plane. Assumptions (i) and (iv) also guarantee that, maybe after flipping, $0 \leq \phi \leq \pi/2$. It is straightforward to check that assumption (iv) on a choice of normal directions guarantees that $\theta$ is correctly labeled in Figure 2.

Identify $T$ with $\mathbb{U}^2$ so that the center of $C_0$ corresponds to 0. We compute the distance between the two disjoint geodesics $\lambda = T \cap P_0$ and $\gamma = T \cap \rho_{1,2}(P_3)$. Let $z_1 < z_2 < z_3 < z_4$, $z_i \in \mathbb{R} \subset \partial T$ be the points $\partial \lambda \cup \partial \gamma$. We can use the standard cross ration to compute

$$\left(\frac{z_1 - z_3}{z_1 - z_4}; \frac{z_2 - z_4}{z_2 - z_3}\right) = \coth^2 \left(\frac{1}{2} d_H(\lambda, \gamma)\right) > 0$$

$$d_H(\lambda, \gamma) = \log \left(\frac{\sqrt{\left(\frac{z_1 - z_3}{z_1 - z_4}\right) \left(\frac{z_2 - z_4}{z_2 - z_3}\right) + 1}}{\sqrt{\left(\frac{z_1 - z_3}{z_1 - z_4}\right) \left(\frac{z_2 - z_4}{z_2 - z_3}\right) - 1}}\right).$$

Let $r_0, r_1, \phi, \theta$ be as in Figure 2 and normalize the diagram as shown. By directly constructing a diagram from our parameters, one checks that a configuration satisfies our assumptions if and only if

$$0 \leq \theta < \pi/2 \quad \text{and} \quad 0 \leq \phi \leq \pi/2$$

$$0 \leq r_i + r_i \cos \theta \leq 1 \text{ for } i = 0, 1$$

$$1 = (r_0 + r_1) (\cos \theta + \cos \phi)$$
To evaluate the cross ratio, let $z_1 = -r_0, z_2 = r_0, z_3 = c - r_0$, and $z_4 = c + r_0$, where $c$ is the distance between the Euclidean centers of $C_0$ and $\rho_{1,2}(C_3)$. Computing, we have

$$c^2 = (r_0 + r_1)^2 \sin^2 \phi + (2 - (r_0 + r_1) \cos \phi)^2 = 4 - 4(r_0 + r_1) \cos \phi + (r_0 + r_1)^2.$$  

The cross ratio of these points is then

$$x = (-r_0, c - r_1; r_0, c + r_1) = \frac{(r_0 - r_1)^2 - c^2}{(r_0 + r_1)^2 - c^2} = 1 + \frac{r_0 r_1}{1 - (r_0 + r_1) \cos \phi} = 1 + \frac{r_0 r_1}{(r_0 + r_1) \cos \theta}.$$  

Therefore,

$$\ell(\alpha) \geq d_\mathcal{H}(P_0, \rho_{1,2}(P_3)) \geq \inf_{r_0, r_1, \phi} \log \left( \frac{\sqrt{x} + 1}{\sqrt{x} - 1} \right) = \inf_{r_0, r_1, \phi} \log \left( 1 + \frac{2}{\sqrt{x} - 1} \right).$$

Since $\log (1 + 2/(\sqrt{x} - 1))$ is a decreasing function of $x$, our goal is to maximize $x$ over all allowable configurations with fixed $0 \leq \theta < \pi/2$. Our parameter conditions imply

$$0 \leq r_i + r_i \cos \theta \leq 1, \quad \frac{1}{1 + \cos \theta} \leq (r_0 + r_1), \quad \text{and} \quad (r_0 + r_1) \leq \frac{1}{\cos \theta}.$$  

Since $0 \leq \theta \leq \pi/2$, it is easy to see that this region is a triangle in the $(r_0, r_1)$-plane bounded by $r_i = 1/(1 + \cos \theta)$ and $(r_0 + r_1) = 1/(1 + \cos(\theta))$, see Figure 3.

![Figure 3. Constraints for maximizing $x = 1 + \frac{r_0 r_1}{(r_0 + r_1) \cos \theta}$ in Lemma 3.2](image)

We also have

$$\frac{\partial x}{\partial r_i} = \frac{r_i^2}{(r_i + r_j)^2} > 0 \text{ for } r_i, r_j > 0 \text{ where } \{i, j\} = \{0, 1\},$$

so the maximum value of $x$ is attained on the boundary of our triangle. On the edges corresponding to $r_i = 1/(1 + \cos \theta)$, we get a maximum when $r_0 = r_1 = 1/(1 + \cos \theta)$. For the edge corresponding to $(r_0 + r_1) = 1/(1 + \cos(\theta))$, we have a maximum at $r_0 = r_1 = 1/(2 + 2 \cos \theta)$. Of these two points, $x$ has the largest value at the former, so

$$\sup_{r_0, r_1, \phi} x = x \mid_{r_i = 1/(1 + \cos \theta)} = 1 + \frac{1}{2(1 + \cos \theta) \cos \theta}.$$  

Lastly, note that using $\cosh(z) = (e^z + e^{-z})/2$, we have

$$\cosh \left( \log \left( \frac{\sqrt{x} + 1}{\sqrt{x} - 1} \right) \right) = \frac{1}{2} \left( \frac{\sqrt{x} + 1}{\sqrt{x} - 1} + \frac{\sqrt{x} - 1}{\sqrt{x} + 1} \right) = \frac{x + 1}{x - 1}.$$
Our desired results follows,
\[ \ell(\alpha) \geq \inf_{r_0, r_1, \phi} \log \left( \frac{\sqrt{x} + 1}{\sqrt{x} - 1} \right) = \inf_{r_0, r_1, \phi} \cosh^{-1} \left( \frac{x + 1}{x - 1} \right) = \cosh^{-1} \left( (2 \cos \theta + 1)^2 \right). \]

**Corollary 3.3.** The shortest rectifiable path \( \alpha(t) \subset \mathbb{H}^3 \) connecting four sequentially tangent hyperplanes in \( \mathbb{H}^3 \) has length \( 2 \sinh^{-1}(2) \) and is attained when the planes support four of the faces of a standard ideal octahedron.

**Proof.** If \( \theta = 0 \), then the geodesic we have find in Lemma 3.2 has length
\[ \cosh^{-1} \left( (2 \cos(0) + 1)^2 \right) = \cosh^{-1}(9) = 2 \sinh^{-1}(2). \]
The critical values of \( r_0, r_1 \) were \( r_i = 1/(1 + \cos \theta) = 1/2 \), so \( 1 = (r_0 + r_1)(\cos \theta + \cos \phi) = 1 + \cos \phi \) and \( \phi = \pi/2 \). This configuration and the other four planes supporting a standard ideal octahedron are shown in Figure 4.

**Figure 4.** The supporting planes of a standard ideal octahedron in Cor 3.3

Next, we prove a slight generalization of Lemma 3.2 where we replace the tangency of \( P_0 \) and \( P_3 \) for another condition.

**Lemma 3.4.** Let \( P_0, P_1, P_2, P_3 \) be planes in \( \mathbb{H}^3 \) with boundary circles \( C_i \in \partial_{\infty} \mathbb{H}^3 \). Assume

(i) \( P_0 \cap P_2 = P_1 \cap P_3 = \emptyset \)
(ii) \( C_1 \cap C_2 = \{b\} \) and \( b \notin C_0 \cup C_3 \).
(iii) let \( P_*, \) be the unique plane between \( P_1 \) and \( P_2 \) tangent to \( P_3 \), then \( \partial P_* \cap C_0 \neq \emptyset \)
(iv) let \( \eta_i \) be normal directions to \( C_i \) such that \( \eta_0, \eta_3 \) point away from each other and \( \eta_1, \eta_2 \) point toward each other, then \( \angle(\eta_0, \eta_1) = \angle(\eta_2, \eta_3) = \theta < \pi/2 \).

If \( \alpha : [0, 1] \rightarrow \mathbb{H}^3 \) is a rectifiable path with \( \alpha(0) \in P_0, \alpha(1) \in P_3, \alpha(t_1) \in P_1, \) and \( \alpha(t_2) \in P_2 \) for some \( t_1, t_2 \in (0, 1) \) with \( t_1 < t_2 \). Then,
\[ \ell(\alpha) \geq \cosh^{-1} \left( (2 \cos \theta + 1)^2 \right). \]
Proof. We will reduce to the case of Lemma 3.2 as follows. We can conjugate $b \to \infty$ and build as similar diagram with $C_3$ “below” $C_0$ as before, except they may no longer be tangent. Condition (iii) implies that there is some “slide” of $P_0$ along $P_\star$ to a plane $P'_0$ that is tangent to $P_3$, see Figure 5.

Notice that the “slide” operation does not change radii of the circles in our configuration. In the proof of Lemma 3.2, the cross ratio was given as

$$x = \frac{(r_0 - r_1)^2 - c_1^2}{(r_0 + r_1)^2 - c_2^2}$$

This function is decreasing in $c$, so if we replace $c$ with the shorter $c'$ as in Figure 5. This gives a larger value of $x$ and, therefore, a shorter geodesic. Thus, we replace $P_0$ with $P'_0$ and apply Lemma 3.2.

Proof of Theorem 1.1. Fix $L \in (0, 2 \sinh^{-1}(2)]$. If we fix $\|\mu\|_L$, then for every $\epsilon > 0$, we can find a geodesic arc $\alpha : (0, 1) \to P_\mu$ with $\ell(\alpha) = L$ such that $\|\mu\|_L - \epsilon < i(\alpha, \mu) \leq \|\mu\|_L$. Let $\{P_t\}$ for $t \in [0, 1]$ denote the path of support planes to $\alpha$ and $p : [0, 1] \to [0, 1]$ be such that $P_1$ is a support plane at $\alpha(p(t))$. Here, we take $P_0 = \lim_{t \to 0^+} P_t$ and $P_1 = \lim_{t \to 1^-} P_t$. We will divide our argument into cases via bounds on $\|\mu\|_L$.

Case $\|\mu\|_L \leq \pi$. This is the trivial case as $0 \leq L$ implies $\|\mu\|_L \leq \pi = F(0) \leq F(L)$.

Case $\pi < \|\mu\|_L \leq 2\pi$. Fix $\epsilon > 0$ small enough and $\alpha$ of length $L$ such that

$$\pi < \|\mu\|_L - \epsilon < i(\alpha, \mu) \leq \|\mu\|_L \leq 2\pi.$$ 

Let $2\theta = 2\pi - \|\mu\|_L + \epsilon < \pi$, then by assumption $2\pi - 2\theta < i(\alpha, \mu)$. As the interior angle between $P_0$ and $P_t$ decreases continuously, it follows from the roof property (Lemma 2.2) that there must be a $t_1$ such that $\angle_{int}(P_0, P_{t_1}) = \theta$ as $i(\alpha, \mu) > \pi - \theta$. Similarly, there must

![Figure 5. The “slide” move of $P_0$ to $P'_0$ in Lemma 3.4. Notice that the Euclidean length $c \geq c'$.](image-url)
be at \( t_2 \) such that \( \angle_{\text{int}}(P_1, P_2) = \theta \) as \( i(\alpha, \mu) > 2\pi - 2\theta \). Notice that \( P_0 \cap P_2 = \emptyset \), as otherwise either they form a roof over \( \alpha \) by Lemma \( 2.3 \) and \( i(\alpha, \mu) \leq \pi \), a contraction.

![Diagram](image)

**Figure 6.** The “grow” move of \( P_0 \) to \( P'_0 \) in **Case** \( \pi < \|\mu\|_L \leq 2\pi \) of Theorem \( 1.1 \)

Since \( 2\theta < \pi \), our planes \( P_0, P_1, P_2 \) almost satisfy the conditions of Lemma \( 3.1 \). By mapping \( P_1 \) to a vertical plane in the upper half space model for \( \mathbb{H}^3 \), we easy see that we can “grow” \( P_0 \) to a plane \( P'_0 \) that is tangent to \( P_2 \) while keeping the interior angle with \( P_1 \) equal to \( \theta \), see Figure 6. The plane \( P'_0 \) is not a support plane, but a sub-arc of \( \alpha[p(0), p(t_2)] \) joins it to \( P_{t_2} \). Therefore, the shortest curve between \( P'_0 \) and \( P_{t_2} \) with a point on \( P_t \) is shorter than \( \alpha \). We apply Lemma \( 3.1 \) to \( P'_0, P_t, P_{t_2} \) and see

\[
L \geq 2 \sinh^{-1}(\cos \theta) \quad \Rightarrow \quad \cos^{-1}(\sinh(L/2)) \leq \theta.
\]

\[
\|\mu\|_L = 2\pi - 2\theta + \epsilon \leq 2\pi - 2 \cos^{-1}(\sinh(L/2)) + \epsilon = 2 \cos^{-1}(-\sinh(L/2)) + \epsilon.
\]

Since \( \epsilon > 0 \) can be taken arbitrarily small and \( F(L) \) is an increasing function, \( \|\mu\|_L \leq F(L) \).

**Case** \( 2\pi < \|\mu\|_L \leq 3\pi \). Fix \( \epsilon > 0 \) small enough and \( \alpha \) of length \( L \) such that

\[
2\pi < \|\mu\|_L - \epsilon < i(\alpha, \mu) \leq \|\mu\|_L \leq 3\pi.
\]

Let \( 2\theta = 3\pi - \|\mu\|_L + \epsilon < \pi \), then by assumption \( 3\pi - 2\theta < i(\alpha, \mu) \). As before, since the interior angle decreases and we cannot violate the roof property, there exists \( t_1 \) such that \( \angle_{\text{int}}(P_0, P_{t_1}) = \theta \), the smallest \( t_2 \) such that \( \angle_{\text{int}}(P_{t_1}, P_{t_2}) = 0 \), and \( t_3 \) such that \( \angle_{\text{int}}(P_{t_2}, P_{t_3}) = \theta \).

We want to modify our set of planes slightly to satisfy the assumptions of Lemma \( 3.4 \). We see that \( P_0 \cap P_{t_2} = P_{t_1} \cap P_{t_3} = P_0 \cap P_{t_3} = \emptyset \) by Lemma \( 2.3 \) as any roofs over subarcs of \( \alpha \) would decrease its bending. Let \( P_t \) be the unique plane between \( P_{t_1} \) and \( P_{t_2} \) that is tangent to \( P_{t_3} \). If \( P_t \cap P_0 = \emptyset \), we can then “grow” \( P_0 \) to \( P'_0 \) so that \( P'_0 \cap P_t \neq \emptyset \), see Figure 7. As before, \( P'_0 \) is joined to \( P_{t_3} \) by a sub-arc of \( \alpha[p(0), p(t_3)] \). As \( \theta < \pi/2 \), all the assumptions of Lemma \( 3.4 \) are satisfied, so we have

\[
L \geq \cos^{-1}\left(\frac{(2\cos \theta + 1)^2}{2}\right) \quad \Rightarrow \quad \cos^{-1}\left(\frac{\sqrt{\cosh(L) - 1}/2}{2}\right) \leq \theta.
\]

\[
\|\mu\|_L = 3\pi - 2\theta + \epsilon \leq 3\pi - 2 \cos^{-1}\left(\frac{\sqrt{\cosh(L) - 1}/2}{2}\right) + \epsilon.
\]
Since $\epsilon > 0$ can be taken arbitrarily small, $\|\mu\|_L \leq F(L)$.

**Case $\|\mu\|_L > 3\pi$.** We can choose $\alpha$ of length $L$ such that $i(\alpha, \mu) > 3\pi$. As before, we find the smallest $t_1, t_2, t_3$ (in that order) such that $\angle_{int}(P_0, P_{t_1}) = \angle_{int}(P_{t_1}, P_{t_2}) = \angle_{int}(P_{t_2}, P_{t_3}) = 0$. Notice that $P_{t_3}$ is *not* the terminal support plane for $\alpha$, as $i(\alpha, \mu) > 3\pi$. After a possible “grow” move, this configuration corresponds to the case of Lemma 3.4 with $\theta = 0$. This, however, implies

$$L > \cosh^{-1} \left( (2 \cos(0) + 1)^2 \right) = \cosh^{-1}(9) = 2 \sinh^{-1}(2)$$

which contradicts the fact that we fixed $L \in (0, 2 \sinh^{-1}(2)]$. \qed

### 4. Improved Bounds on Average Bending and Lipschitz Constants

In this section, we improve the Lipschitz and average bending bounds of [Bri03, Theorem 1.2].

**Theorem 1.2.** There exist universal constants $K_0, K_1$ with $K_0 \leq 2.494$ and $K_1 \leq 3.101$ such that if $\Gamma \leq \text{Isom}^+(\mathbb{H}^3)$ is a finitely-generated Kleinian group, $N = \mathbb{H}^3/\Gamma$, and the boundary $\partial C(N)$ of the convex core is non-empty and incompressible in $N$, then

(i) if $\mu_\Gamma$ is the bending lamination of $\partial C(N)$, then

$$\ell_{\partial C(N)}(\mu_\Gamma) \leq K_0 \pi^2 |\chi(\partial C(N))|$$

(ii) for any closed geodesic $\alpha$ on $\partial C(N)$,

$$B_\Gamma(\alpha) = \frac{i(\alpha, \mu_\Gamma)}{\ell(\alpha)} \leq K_1$$

where $B_\Gamma(\alpha)$ is called the average bending of $\alpha$. 

---

**Figure 7.** The “grow” move of $P_0$ to $P'_0$ in Case $2\pi < \|\mu\|_L \leq 3\pi$ of Theorem 1.1.
(iii) there exists a \((1 + K_1)\)-Lipschitz map \(s : \partial C(N) \to \partial \infty N\) that is a homotopy inverse to the nearest point retraction \(r : \partial \infty N \to \partial C(N)\).

**Proof.** Our result is a direct generalization of [Bri03] by using our function \(F(L)\) from Theorem 1.1. We provide an outline of the proof.

Let \(\delta\) be a geodesic arc on \(P_{\mu_t}\) and fix \(L \in (0, 2 \sinh^{-1}]\). Set \([x]\) to be the least integer \(\geq x\). By subdividing \(\delta\) into arcs or length \(\leq L\), we see

\[
B_\Gamma(\delta) \leq \frac{\|\mu_t\|_L}{\ell(\delta)} \left[ \frac{\ell(\delta)}{L} \right] \leq \frac{\|\mu_t\|_L}{\ell(\delta)} \left( \frac{\ell(\delta)}{L} + 1 \right) = \frac{\|\mu_t\|_L}{L} \left( 1 + \frac{L}{\ell(\delta)} \right) \leq \frac{F(L)}{L} \left( 1 + \frac{L}{\ell(\delta)} \right)
\]

For an infinite length geodesic \(\beta\) on \(P_{\mu_t}\) and a point \(x \in \beta\), let \(\beta^t_x\) denote the sub-arc centered at \(x\) of length \(2t\). One can define average bending for \(\beta\) as

\[
B_\Gamma(\beta_x) = \limsup_{t \to \infty} B_\Gamma(\beta^t_x).
\]

In [Bri98], Bridgeman shows that this notion is well defined and independent of \(x\). In particular, by taking \(\ell(\delta) \to \infty\) in the bound on \(B_\Gamma(\delta)\), we see that for any infinite length geodesic \(\beta\) on \(P_{\mu_t}\),

\[
B_\Gamma(\beta) \leq \frac{F(L)}{L} \text{ for all } L \in (0, 2 \sinh^{-1}(2)].
\]

Let

\[
K_1 = \min \left[ \left( \frac{3\pi - 2 \cos^{-1} \left( \sqrt{\frac{\cosh(L)}{2}} - \frac{1}{2} \right) }{L} \right) \right] \text{ over } L \in (2 \arcsinh(1), 2 \sinh^{-1}(2)]
\]

Then, \(B_\Gamma(\beta) \leq K_1 \approx 3.101\), where the minimum is attained at \(L \approx 2.74104\).

For a closed geodesic \(\alpha\) on \(\partial C(N)\), let \(\tilde{\alpha} \subset P_{\mu_t}\) be a lift. Then (ii) follows, as

\[
B_\Gamma(\alpha) = B_\Gamma(\tilde{\alpha}) \leq K_1.
\]

The statement of (iii) can be derived from (ii). Let \(K_s\) be the minimal Lipschitz constant of \(s : \partial C(N) \to \Omega(\Gamma) / \Gamma\). Then, Thurston characterized

\[
K_s = \sup \left\{ \frac{\ell(s, \alpha)}{\ell(\alpha)} \mid \alpha \text{ is a simple closed curve on } \partial C(N) \right\}
\]

and McMullen’s showed that \(\ell(s, \alpha) \leq \ell(\alpha) + i(\alpha, \mu_t)\) (see [Thu98, Theorem 8.5] and [McM98, Theorem 3.1]). Combining these two facts gives \(K_s \leq 1 + B_\Gamma(\tilde{\alpha}) \leq 1 + K_1\), so (iii) holds.

For (i), we use a computation from [Bri03, Section 5] to bound \(\ell(\mu_t)\) by integrating along the unit tangent bundle of \(\partial C(N)\). Fix \(L \in (0, 2 \sinh^{-1}(2)]\) and for \(v \in T_1(\partial C(N))\), let \(\alpha_v : (0, L) \to \partial C(N)\) be the unit speed geodesic in the direction \(v\). Then, Bridgeman and Canary [BC05] show

\[
\ell(\mu_t) = \frac{1}{4L} \int_{T_1(\partial C(N))} i(\alpha_v, \mu_t) d\Omega
\]

By taking a maximal lamination \(\tilde{\mu} \supset \mu_t\), one can integrate our bound \(F(L) \geq i(\alpha_v, \mu_t)\) over the set of ideal triangles \(\partial C(N) \setminus \tilde{\mu}\). In [Bri03, Section 5], Bridgeman works out this
integral and shows that
\[
\frac{\ell(\mu_\Gamma)}{\pi^2 |\chi(\partial C(N))|} \leq \frac{3}{\pi^2 L} \int_{(x,y) \in U} \frac{dx \, dy}{y^2} \int_0^{\cos^{-1}\left(\frac{D(x,y)}{\tanh(L)}\right)} F\left(L - \tanh^{-1}\left(\frac{D(x,y)}{\cos \theta}\right)\right) d\theta = K_0
\]
where \(U\) is the ideal triangle
\[
U = \{(x, y) \mid -1 \leq x \leq 1, y \geq \sqrt{1 - x^2}\}
\]
and
\[
D(x, y) = \frac{x^2 + y^2 - 1}{\sqrt{(x^2 + y^2 - 1)^2 + 4y^2}}
\]
computes the length of the unique perpendicular from \((x, y)\) to the “bottom” edge of \(U\). We compute this integral with using numerical approximation in Mathematica. We choose \(L = \sinh^{-1}(89/10) < 2\sinh^{-1}(2)\) and find the upper bound
\[
\frac{\ell(\mu_\Gamma)}{\pi^2 |\chi(\partial C(N))|} \leq K_0 \leq 2.494.
\]
\(\square\)

References


Department of Mathematics, Boston College, 140 Commonwealth Ave., Chestnut Hill, MA 02467

*E-mail address*: andrew.yarmola@bc.edu

*URL*: https://www2.bc.edu/andrew-v-yarmola