A New Bound for Sullivan's Theorem for Simply Connected Domains

Martin Bridgeman, Richard D. Canary, and Andrew Yarmola

Department of Mathematics, Boston College   Department of Mathematics, University of Michigan

Introduction

For a simply connected domain Ω in $\mathbb{C} \subset \partial\mathbb{H}^3$, let $\mathrm{Dome}(\Omega)$ denote the boundary of the convex hull of $\partial\mathbb{H}^3 \setminus \Omega$. The surface $\mathrm{Dome}(\Omega)$ is hyperbolic in its intrinsic metric ([5]). The following result of Sullivan is of interest.

Sullivan's Theorem

Theorem. *There exists a universal constant $K > 1$ such that for any a proper simply connected domain Ω in $\mathbb{C}$ and the group Γ of Möbius transformations preserving Ω, there is a Γ-equivariant K-quasiconformal homeomorphism $f_\Omega : \Omega \to \mathrm{Dome}(\Omega)$ extending continuously to the identity map on the common boundary of Ω and $\mathrm{Dome}(\Omega)$.*

Let K_{eq} denote the best such K. Epstein, Marden and Markovic [3] have shown that $2.1 \leq K_{eq} \leq 13.88$. If the condition for equivariance is dropped, Bishop showed that the optimal K is less than 7.88 [1]. Expanding of previous techniques, we have been able to show that $K_{eq} \leq 7.12$.

Pleated Surfaces

Definition. Let $\mathcal{G}(\mathbb{H}^2)$ we the space of unoriented geodesics in $\mathbb{H}^2$. A measured lamination (Λ, μ) is a pair where μ is a Borel measure on $\mathcal{G}(\mathbb{H}^2)$, $\Lambda = \mathrm{supp}(\mu)$, and no two distinct geodesics in Λ intersect. We refer to elements of Λ as *leafs* and components of $\mathbb{H}^2 \setminus \cup\Lambda$ as *flats*.

Fix a real $c \geq 0$. If Λ is finite, then we may choose orientations for the leafs of (Λ, μ) and define the earthquake map $E_{c\mu} : \mathbb{H}^2 \to \mathbb{H}^2$ by taking translations along each leaf $l \in \Lambda$ by amount $c\mu(\{l\})$. Under $E_{c\mu}$, (Λ, μ) is carried to a new measured lamination (Λ^*, μ^*). We also define the pleating map $P_{c\mu} : \mathbb{H}^2 \to \mathbb{H}^3$ by fixing an embedding of a flat in $\mathbb{H}^3$ and mapping all other flats by bending along every $l \in \Lambda$ by the amount $c\mu(\{l\})$. These maps can be defined for general (Λ, μ) and are continuous except for possibly at elements of Λ, see [2, 3].

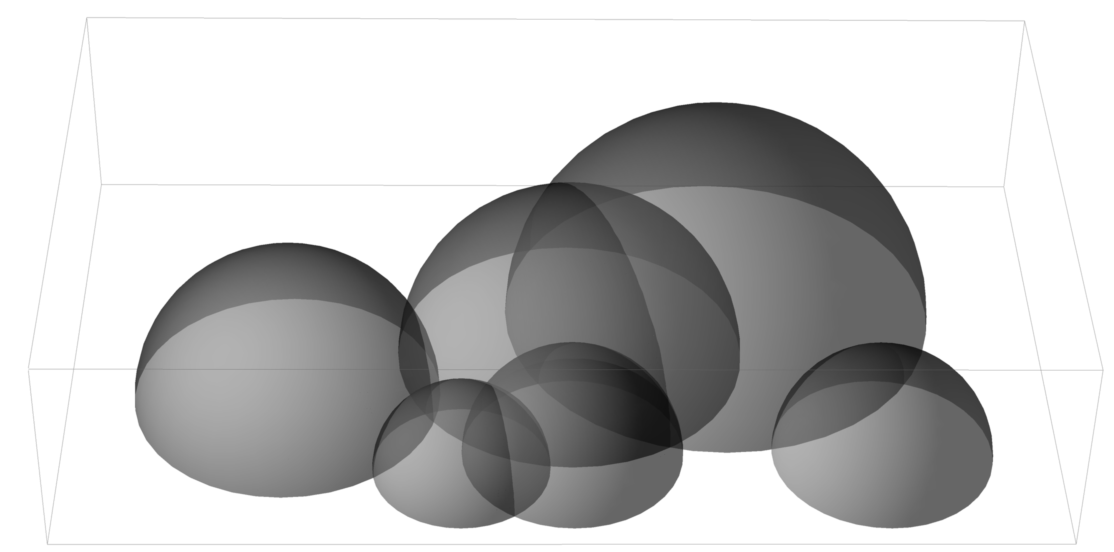

Figure 1: An example of $\mathrm{Dome}(\Omega)$ in the upper halfspace model, where Ω is the domain below all the hemispheres. Notice that this is also a pleated plane for some finite measured lamination.

For a given $z = x + iy \in \mathbb{C}$, we define the complex earthquake map $\mathbb{C}E_z : \mathbb{H}^2 \to \mathbb{H}^3$ by $\mathbb{C}E_z = P_{y\mu^*} \circ E_{x\mu}$. From this, we obtain a continuous family of pleated surface parametrized by z. Further, $\mathbb{C}E_z$ extends to a holomorphic motion of $\partial\mathbb{H}^2$ in $\partial\mathbb{H}^3$ [3].

In [3], Epstein, Marden and Markovic show given $\mathrm{Dome}(\Omega)$, one my find a measured lamination such that $\mathrm{im}\,\mathbb{C}E_i = \mathrm{Dome}(\Omega)$. Further, they show that this measured lamination, and the corresponding families of pleated surfaces, can be effectively approximated by finite measured laminations.

Bending and Roundedness

For an arc C transverse to a measured lamination (Λ, μ), we define

$$|\mu|(C) = \mu(\{l \in \Lambda \mid l \cap C \neq \emptyset\}).$$

The L-norm of (Λ, μ) is

$$\|\mu\|_L = \sup_C |\mu|(C),$$

where the supremum is taken over all arcs transverse to Λ of length L. We have the following new result about the L-norm of μ given that P_μ is embedded.

Theorem. *Let $B : [0, 2\sinh -1(1)] \to [\pi, 2\pi]$ be given by $B(L) = 2\cos^{-1}(-\sinh(L/2))$. If P_μ is embedded, then $\|\mu\|_L \leq B(L)$.*

There is also a result in another direction, which gives a bound on $\|\mu\|_L$ that guarantees that P_μ is embedded. From an unpublished manuscript of Epstein and Jerrard we have:

Theorem. *There exists a well defined monotonic function $G : (0, \infty) \to (0, \pi)$ such that if $\|\mu\|_L \leq G(L)$ then P_μ is embedded.*

Following Epstein, Marden and Markovic in [2], we use these results to find a region $\mathfrak{T}_0^L \subset \mathbb{C}$ such that $\mathbb{C}E_z$ if embedded for all $z \in \mathfrak{T}_0^L$ and (Λ, μ) with $\|\mu\|_L = 1$. For $L \in [0, 2\sinh -1(1)]$ and $x \in \mathbb{R}$, we define the functions

- $F_1(x, L) = \min(Le^{|x|/2}, \sinh^{-1}(e^{|x|}\sinh(L))$
- $F_2(x, L) = \max(Le^{-|x|/2}, \sinh^{-1}(e^{-|x|}\sinh(L))$
- $Q(x, L) = \max\left(\frac{G(L)}{\lceil F_1(x,L)/L \rceil}, G(F_2(x, L))\right)$.

Then the region $\mathfrak{T}_0^L$ is given by

$$\mathfrak{T}_0^L = \{x + iy \mid |y| < Q(x, L)\}.$$

We also define the region

$$\mathfrak{T}^L = \{x + iy \mid y > -Q(x, L)\}.$$

Main Results

Let $\mathcal{T}$ be the universal Teichmüller space defined as the space of all quasisymmetric homeomorphism of $\mathbb{S}^1$ modulo the action of the group of Möbius transformations by left composition. A straightforward generalization of the results and techniques of Epstein, Marden and Markovic used in [3] give us the following theorems.

Theorem. *There is a holomorphic map $\mathfrak{G} : \mathfrak{T}^L \to \mathcal{T}$ such that if $P_{c\mu}$ if a pleated surface corresponding to $\mathrm{Dome}(\Omega)$ with $\|\mu\|_L = 1$, then the class $\mathfrak{G}(ic)$ recovers the map in Sullivan's Theorem.*

Main Theorem

Theorem. The optimal equivariant quasiconformal constant K_{eq} satisfies the inequality

$$\log(K_{eq}) \leq d_{\mathcal{T}}(\mathfrak{G}(iB(L)), 0) \leq d_{\mathfrak{T}^L}(iB(L), 0).$$

In particular, taking $L = 1.5$ we attain

$$K_{eq} \leq 7.12.$$

Computation

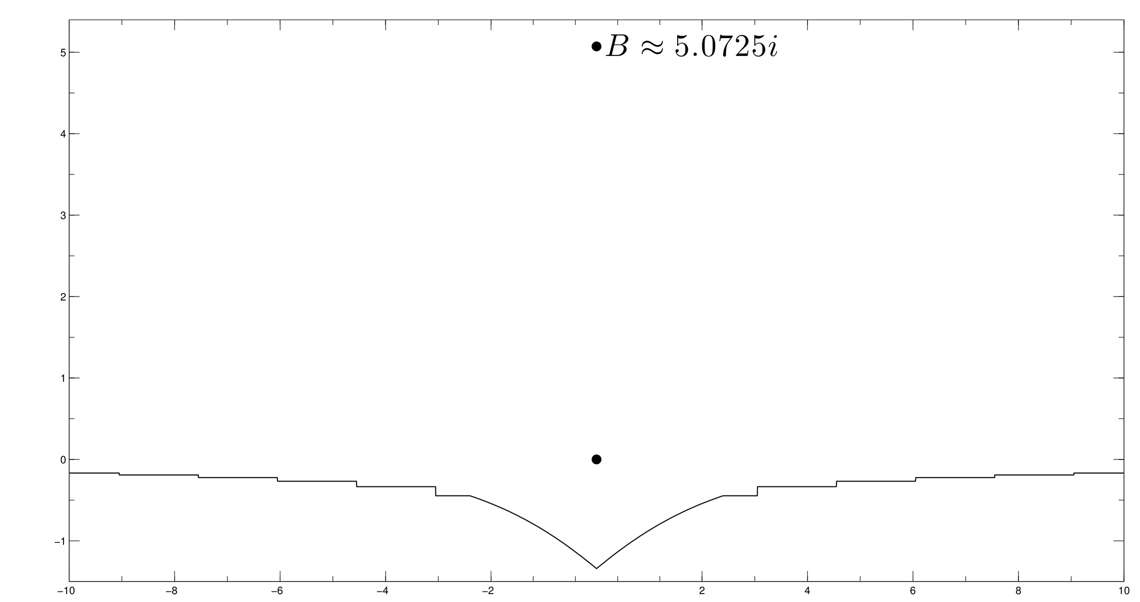

Figure 2: The graph of the function $-Q(x, L)$ for $L = 1.5$. The area above the graph is the region $\mathfrak{T}^L$. The hyperbolic distance between the two points 0 and $B = B(L)$ determines the upper bound for K_{eq}.

We used MATLAB to construct a polygonal approximation of infinite area for the region $\mathfrak{T}^L$. Using the Schwarz-Christoffel mapping toolbox by Toby Driscoll [4], we computed the images of the points 0 and B under a Riemman mapping of $\mathfrak{T}^L$ to the upper half plane. Computing the hyperbolic distance between the images provides the result.

References

[1] C.J. Bishop, "An explicit constant for Sullivan's convex hull theorem," in *In the tradition of Ahlfors and Bers III, Contemp. Math.* **355**(2007), Amer. Math. Soc., 41–69.

[2] D.B.A. Epstein, A. Marden and V. Markovic, "Quasiconformal homeomorphisms and the convex hull boundary," *Ann. of. Math.* **159**(2004), 305–336.

[3] D.B.A. Epstein, A. Marden and V. Markovic, "Complex earthquakes and deformations of the unit disk," *J. Diff. Geom* **73**(2006), 119–166.

[4] T. Driscoll, *A Matlab program for computing and visualizing conformal maps,* http://www.math.udel.edu/SC/

[5] W.P. Thurston, *The Geometry and Topology of 3-Manifolds*, Lecture Notes, Princeton University, 1979.

Acknowledgements

Richard D. Canary was partially supported by NSF grants DMS-0504791 and DMS-0554239.