Convex hulls in hyperbolic 3-space and generalized orthospectral identities

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A dissertation

submitted to the Faculty of

the Department of Mathematics

in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

Boston College
Morrissey College of Arts and Sciences
Graduate School

May 2016

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Abstract

We begin this dissertation by studying the relationship between the Poincaré metric of a simply connected domain $\Omega \subset \mathbb{C}$ and the geometry of $\mathrm{Dome}(\Omega)$, the boundary of the convex hull of its complement. Sullivan showed that there is a universal constant K_{eq} such that one may find a conformally natural K_{eq}-quasiconformal map from Ω to $\mathrm{Dome}(\Omega)$ which extends to the identity on $\partial\Omega$. Explicit upper and lower bounds on K_{eq} have been obtained by Epstein, Marden, Markovic and Bishop. We improve upon these upper bounds by showing that one may choose $K_{eq} \leq 7.1695$. As part of this work, we provide stronger criteria for embeddedness of pleated planes. In addition, for Kleinian groups Γ where $N = \mathbb{H}^3/\Gamma$ has incompressible boundary, we give improved bounds for the average bending on the convex core of N and the Lipschitz constant for the homotopy inverse of the nearest point retraction.

In the second part of this dissertation, we prove an extension of Basmajian's identity to n-Hitchin representations of compact bordered surfaces. For 3-Hitchin representations, we provide a geometric interpretation of this identity analogous to Basmajian's original result. As part of our proof, we demonstrate that for a closed surface, the Lebesgue measure on the Frenet curve of an n-Hitchin representation is zero on the limit set of any incompressible subsurface. This generalizes a classical result in hyperbolic geometry. In our final chapter, we prove the Bridgeman-Kahn identity for all finite volume hyperbolic n-manifolds with totally geodesic boundary. As part of this work, we correct a commonly referenced expression of the volume form on the unit tangent bundle of $\mathbb{H}^n$ in terms of the geodesic end point parametrization.

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Acknowledgements

I would first and foremost like to thank my advisor Martin Bridgeman for his amazing guidance, support, and patience during my graduate studies. I am particularly grateful to him for introducing me to the mathematical community at large. I thank the committee members, Ian Biringer, Martin Bridgeman, and Rob Meyerhoff, for generously giving their time to review my work. I thank Nicholas Vlamis, my academic brother and collaborator, for many wonderful and productive conversations over the past several years. I would also like to extend my gratitude to Rob Meyerhoff, Richard Canary, Greg McShane, and David Gabai for their extensive support and mentorship. I am deeply grateful to the students, postdocs, and professors of the Boston College Mathematics Department for creating such an amazing academic community.

I wish to thank my wonderful parents – Elena and Valeriy, my grandfather – Georgiy, my sister – Tatiana, and my amazing girlfriend – Liana, for their love, encouragement, care, and support over all these years.

CHAPTER 1

Introduction

1. Convex Hulls, Sullivan's Theorem and Lipschitz Bounds

In Chapter 3 of this thesis, we will consider the relationship between the Poincaré metric on a simply connected hyperbolic domain $\Omega \subset \hat{\mathbb{C}} = \partial \mathbb{H}^3$ and the geometry of the boundary of the $\mathbb{H}^3$-convex hull of its complement, denoted by $\mathrm{Dome}(\Omega)$. Sullivan [**Sul81**] (see also Epstein-Marden [**EM87**]) showed that there exists a universal constant $K_{eq} > 0$ such that there is a conformally natural K_{eq}-quasiconformal map $f : \Omega \to \mathrm{Dome}(\Omega)$ which extends to the identity on $\partial\Omega$. Epstein, Marden and Markovic provided bounds for the value of K_{eq}.

Theorem 1.1. (Epstein-Marden-Markovic [**EMM04, EMM06**]) *There exists $K_{eq} \leq 13.88$ such that if $\Omega \subset \hat{\mathbb{C}}$ is a simply connected hyperbolic domain, then there is a conformally natural K_{eq}-quasiconformal map $f : \Omega \to \mathrm{Dome}(\Omega)$ which extends continuously to the identity on $\partial\Omega \subset \hat{\mathbb{C}}$. Moreover, one may not choose $K_{eq} \leq 2.1$.*

Recall that f is said to be *conformally natural* if for all conformal automorphism A of $\hat{\mathbb{C}}$ which preserve Ω, one has $\bar{A} \circ f = f \circ A$, where $\bar{A}$ is the extension of A to an isometry of $\mathbb{H}^3$. In particular, this result is of interest in the setting of Kleinian groups. If $\Gamma \leq \mathrm{Isom}^+(\mathbb{H}^3)$ is a Kleinian group such that $N = \mathbb{H}^3/\Gamma$ has non-empty incompressible boundary, then Sullivan's Theorem provides a universal bound on the Teichmüller distance between the hyperbolic structure on the convex core of N and its conformal structure at infinity. The setting where N has compressible boundary has been extensively studied in [**BC03, BC13**].

If one does not require that the quasiconformal map $f : \Omega \to \mathrm{Dome}(\Omega)$ to be conformally natural, Bishop [**Bis04**] obtained a better uniform bound on the quasiconformality constant. However, Epstein and Markovic [**EM05**] showed that even in this setting one cannot uniformly bound the quasiconformality constant above by 2.

Theorem 1.2. (Bishop [**Bis04**]) *There exists $K' \leq 7.88$ such that if $\Omega \subset \hat{\mathbb{C}}$ is a simply connected hyperbolic domain, then there is a K'-quasiconformal map $f : \Omega \to \mathrm{Dome}(\Omega)$ which extends continuously to the identity on $\partial\Omega \subset \hat{\mathbb{C}}$.*

In joint work with Bridgeman and Canary [**BCY16**], we further improve the upper bound.

THEOREM 1.3. *There exists* $K_{eq} \leq 7.1695$ *such that if* $\Omega \subset \hat{\mathbb{C}}$ *is a simply connected hyperbolic domain, then there is a conformally natural* K_{eq}*-quasiconformal map* $f : \Omega \to \mathrm{Dome}(\Omega)$ *which extends continuously to the identity on* $\partial\Omega \subset \hat{\mathbb{C}}$.

Chapter 3 is organized around the key techniques and results that culminate in the above Theorem. We begin by realizing $\mathrm{Dome}(\Omega)$ as the image of a pleated plane $P_\mu : \mathbb{H}^2 \to \mathbb{H}^3$ whose bending is encoded by a measured lamination μ. Given $L > 0$, we define the L-roundness $||\mu||_L$ of μ to be the least upper bound on the total bending of $P_\mu(\alpha)$ where α is an open geodesic segment in $\mathbb{H}^2$ of length L. This generalizes the notion of roundness introduced by Epstein-Marden-Markovic [**EMM04**]. Our first bound improves earlier work of Bridgeman [**Bri98, Bri03**] on the roundness of embedded pleated planes. Below is an extended version of what appears in our published work [**BCY16**].

THEOREM 1.4. *If* $L \in (0, 2\sinh^{-1}(2)]$, μ *is a measured lamination on* $\mathbb{H}^2$, *and* P_μ *is an embedding, then* $\|\mu\|_L \leq F(L)$ *where*

$$F(L) = \begin{cases} 2\cos^{-1}(-\sinh(L/2)) & \textit{for } L \in [0, 2\sinh^{-1}(1)] \\ 3\pi - 2\,\cos^{-1}\left(\left(\sqrt{\cosh(L)} - 1\right)/2\right) & \textit{for } L \in (2\sinh^{-1}(1), 2\sinh^{-1}(2)] \end{cases}$$

Next, we generalize the work of Epstein-Marden-Markovic [**EMM04**, Theorem 4.2, part 2] and an unpublished result of Epstein and Jerrard [**EJ**], to give criteria on L-roundness which guarantee that P_μ is an embedding.

THEOREM 1.5. *There exists a computable monotonic function* $G : (0, \infty) \to (0, \pi)$ *such that if* μ *is a measured lamination on* $\mathbb{H}^2$ *with* $||\mu||_L < G(L)$, *then* P_μ *is a quasi-isometric embedding. Moreover,* P_μ *extends continuously to* $\hat{P}_\mu : \mathbb{H}^2 \cup \mathbb{S}^1 \to \mathbb{H}^3 \cup \hat{\mathbb{C}}$ *with* $\hat{P}_\mu(\mathbb{S}^1)$ *a quasi-circle.*

With these bounds in place, we adapt the techniques of Epstein, Marden and Markovic [**EMM04, EMM06**] for using complex earthquakes and angle scaling to approximate distances in universal Teichüller space. A computational approximation of an associated Riemann mapping completes the proof of Theorem 1.3.

The extended version of Theorem 1.4 allows us to improve bounds by Bridgeman [**Bri03**] on average bending and the Lipschitz constant for the homotopy inverse of the retraction map.

THEOREM 1.6. *Let Γ be a Kleinian group with the components of $\Omega(\Gamma)$ simply connected and let $N = \mathbb{H}^3/\Gamma$. There exist universal constants K_0, K_1 with $K_0 \leq 2.494$ and $K_1 \leq 3.101$ such that*

(i) if μ_Γ is the bending lamination of $\partial C(N)$, then

$$\ell_{\partial C(N)}(\mu_\Gamma) \leq K_0\, \pi^2\, |\chi\,(\partial C(N))|$$

(ii) for any closed geodesic α on $\partial C(N)$,

$$B_\Gamma(\alpha) = \frac{i(\alpha, \mu_\gamma)}{\ell(\alpha)} \leq K_1$$

where $B_\Gamma(\alpha)$ is called the average bending of α.

(iii) there exists a $(1+K_1)$-Lipschitz map $s : \partial C(N) \to \Omega(\Gamma)/\Gamma$ that is a homotopy inverse to the retract map $r : \Omega(\Gamma)/\Gamma \to \partial C(N)$.

2. Basmajian's Identity for Hitchin Representations

Spectral identities for hyperbolic manifolds express a constant quantity as a summation over the lengths of some class of curves. The first such identity was introduced by McShane in 1991 [**McS91**]. It was extended by Mirzakhani and used to give recursive formulas for Weil-Petersson volumes of moduli space [**Mir07b**] and to count simple closed geodesics on surfaces [**Mir08**]. This thesis will focus on the Basmajian and Bridgeman-Kahn identities.

Let Σ be a connected oriented compact surface with nonempty boundary whose double has genus at least 2. Given a finite area hyperbolic metric σ on Σ such that $\partial\Sigma$ is totally geodesic, Basmajian defined an *orthogeodesic* in (Σ, σ) to be an *oriented* proper geodesic arc perpendicular to $\partial\Sigma$ at both endpoints; denote the collection of all such arcs by $\mathcal{O}(\Sigma, \sigma)$. The *orthospectrum* $|\mathcal{O}(\Sigma, \sigma)|$ is the multiset of the lengths of orthogeodesics counted with multiplicity. *Basmajian's identity* [**Bas93**] states:

$$\ell_\sigma(\partial\Sigma) = \sum_{\ell \in |\mathcal{O}(\Sigma,\sigma)|} 2 \log \coth\left(\frac{\ell}{2}\right),$$

where $\ell_\sigma(\partial\Sigma)$ denotes the length of $\partial\Sigma$ measured in σ.

In Chapter 4, we formulate an extension of this identity to the setting of Hitchin representations using Labourie's notion of associated cross ratios [**Lab08**]. Hitchin representations are the connected component in the character variety $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(n,\mathbb{R})) /\!\!/ \mathrm{PSL}(n,\mathbb{R})$ containing the Veronese embedding of Fuchsian representations.

A Hitchin representation $\rho : \pi_1(\Sigma) \to \mathrm{PSL}(n,\mathbb{R})$ gives rise to a notion of length given by

$$\ell_\rho(\gamma) = \log \left| \frac{\lambda_{\max}(\rho(\gamma))}{\lambda_{\min}(\rho(\gamma))} \right| ,$$

where $\lambda_{\max}(\rho(\gamma))$ and $\lambda_{\min}(\rho(\gamma))$ are the eigenvalues of maximum and minimum absolute value of $\rho(\gamma)$, respectively. This definition make sense as $\rho(\gamma)$ is has all real eigenvalues with multiplicity one [**Lab06**].

Let Σ have m boundary components and choose $\mathcal{A} = \{\alpha_1, \ldots, \alpha_m\}$ to be a collection of primitive peripheral elements of $\pi_1(\Sigma)$ representing distinct boundary components oriented such that the surface is to the left. We call such a collection a *positive peripheral marking.* Set $H_i = \langle \alpha_i \rangle$, then we define the *orthoset* to be the following disjoint union of cosets:

$$\mathcal{O}(\Sigma, \mathcal{A}) = \left(\bigsqcup_{1 \le i \neq j \le m} H_i \backslash \pi_1(\Sigma) / H_j \right) \sqcup \left(\bigsqcup_{1 \le i \le m} (H_i \backslash \pi_1(\Sigma)/H_i) \smallsetminus H_i e H_i \right) ,$$

where $e \in \pi_1(\Sigma)$ is the identity. The orthoset serves as an algebraic replacement for $\mathcal{O}(\Sigma, \sigma)$ and there is clear bijection between $\mathcal{O}(\Sigma, \mathcal{A})$ and $\mathcal{O}(\Sigma, \sigma)$ in the hyperbolic setting.

A *cross ratio* on the boundary at infinity $\partial_\infty(\Sigma)$ of $\pi_1(\Sigma)$ is a Hölder function defined on

$$\partial_\infty(\Sigma)^{4*} = \{(x, y, z, t) \in \partial_\infty(\Sigma)^4 \colon x \neq t \text{ and } y \neq z\}$$

and invariant under the diagonal action of $\pi_1(\Sigma)$. In addition, it must satisfy several symmetry conditions (see Section 5 of Chapter 4). In [**Lab08**], Labourie showed how to associate a cross ratio B_ρ to a Hitchin representation ρ of a closed surface. For compact surfaces, this was done by Labourie and McShane [**LM09**] using a doubling construction. Define the function $G_\rho : \mathcal{O}(\Sigma, \mathcal{A}) \to \mathbb{R}$ by

$$G_\rho(H_i g H_j) = \log B_\rho(\alpha_i^+, g \cdot \alpha_j^+, \alpha_i^-, g \cdot \alpha_j^-) ,$$

where $\alpha^+, \alpha^- \in \partial_\infty(\Sigma)$ denote the attracting and repelling fixed points of $\alpha \in \pi_1(S)$, respectively. In joint work with Nicholas Vlamis [**VY15**], we prove :

Theorem 2.1. *Let Σ be a compact connected surface with $m > 0$ boundary components whose double has genus at least 2. Let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_m\}$ be a positive peripheral marking. If ρ is a Hitchin representation of $\pi_1(\Sigma)$, then*

$$\ell_\rho(\partial\Sigma) \;=\; \sum_{x \in \mathcal{O}(\Sigma,\mathcal{A})} G_\rho(x)\,,$$

where $\ell_\rho(\partial\Sigma) = \sum_{i=1}^m \ell_\rho(\alpha_i)$. Furthermore, if ρ is Fuchsian, this is Basmajian's identity.

In order to prove this result, we need to understand the measure of $\partial_\infty(\Sigma)$ in the limit set of its double. In [**Lab06**], Labourie defines a Hölder map $\xi_\rho : \partial_\infty(S) \to \mathbb{PR}^n$ for an n-Hitchin representation ρ of a closed surface S, which we call the *limit curve* associated to ρ. The image of this curve is a $C^{1+\alpha}$ submanifold and thus determines a measure class μ_ρ on $\partial_\infty(S)$ via the pullback of the Lebesgue measure. With respect to this measure we prove :

Theorem 2.2. *Let S be a closed surface and $\Sigma \subset S$ an incompressible subsurface. Let ρ be a Hitchin representation of S and ξ_ρ the associated limit curve. If μ_ρ is the pullback of the Lebesgue measure on the image of ξ_ρ, then $\mu_\rho(\partial_\infty(\Sigma)) = 0$.*

This result generalizes a classical fact about the measure of the limit set of a subsurface for closed hyperbolic surfaces (see [**Nic89**, Theorem 2.4.4]).

In Section 8 of Chapter 4, we give a geometric picture and motivation for our definitions and techniques by considering the case of 3-Hitchin representations. These correspond to convex real projective structures on surfaces as seen in the work of Choi and Goldman [**Gol90, CG93**]. We also show that our formulation recovers Basmajian's identity for Fuchsian representations. In Section 10, we demonstrate the relationship between Theorem 2.1 and Labourie-McShane's extension [**LM09**] of the McShane-Mirzakhani identity [**McS91, McS98, Mir07a**] to the setting of Hitchin representations. Both identities calculate the length of the boundary by giving a countable full-measure decomposition. We explain how these decompositions are related in both the classic hyperbolic setting and that of Hitchin representations.

3. Bridgeman-Kahn Identity for Finite Volume Hyperbolic Manifolds

In Chapter 5, we provide an extension of the Bridgeman-Kahn orthospectral identity to the setting of all finite volume hyperbolic manifolds with totally geodesic boundary. As part of our work, we provide the following correction to [**Nic89**, Theorem 8.1.1], where the statement is off by a factor of 2^{n-2}.

THEOREM 3.1. *Let $\Omega = dV d\omega$ be the standard volume form in $\mathrm{T}_1\mathbb{H}^{n+1}$. Then, with the following coordinates arising from the upper half space and conformal ball models for the geodesic endpoint parametrization*

$$\mathrm{T}_1\mathbb{H}^{n+1} \cong \{(\mathbf{x}, \mathbf{y}, t) \in \hat{\mathbb{R}}^n \times \hat{\mathbb{R}}^n \times \mathbb{R} : \mathbf{x} \neq \mathbf{y}\}$$

$$\mathrm{T}_1\mathbb{H}^{n+1} \cong \{(\mathbf{p}, \mathbf{q}, t) \in \mathbb{S}^n \times \mathbb{S}^n \times \mathbb{R} : \mathbf{p} \neq \mathbf{q}\},$$

we have

$$d\Omega = \frac{2^n d\mathbf{x}\, d\mathbf{y}\, dt}{|\mathbf{x} - \mathbf{y}|^{2n}} = \frac{2^n d\omega(\mathbf{p})\, d\omega(\mathbf{p})\, dt}{|\mathbf{p} - \mathbf{q}|^{2n}}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ respectively.

Let M be a finite volume hyperbolic n-manifold with non-empty totally geodesic boundary. As before, the collection of *oriented* geodesic arcs perpendicular to ∂M at both ends is called the *orthoset* of M, denoted $\mathcal{O}_M$. The *Bridgeman-Kahn identity* states

THEOREM 3.2. (Bridgeman-Kahn [**BK10**]) *Let M be a compact hyperbolic n-manifolds with totally geodesic boundary, then*

$$\mathrm{Vol}(M) = \sum_{l \in \mathcal{O}_M} F_n(l)$$

where $F_n : \mathbb{R}_+ \to \mathbb{R}_+$ is a decreasing function expressed as an integral of an elementary function and satisfies

(i) there exists $D_n > 0$, depending only on n, such that

$$F_n(l) \leq \frac{D_n}{(e^l - 1)^{n-2}}$$

(ii) let $H(m)$ denote the m^{th} harmonic number and $\Gamma(m) = (m-1)!$, then

$$\lim_{l \to 0} l^{n-2} F_n(l) = \frac{\pi^{\frac{n-2}{2}} H(n-2)\, \Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}$$

(iii)

$$\lim_{l \to \infty} \frac{e^{(n-1)l}}{l} F_n(l) = \frac{2^{n-1} \pi^{\frac{n-2}{2}} \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)^2}$$

Utilizing their identity, Bridgeman and Kahn were able to provide lower bounds on volume.

THEOREM 3.3. (Bridgeman-Kahn [**BK10**]) *There exists $C_n > 0$, depending only on n, such that if M is a compact hyperbolic n-manifolds with totally geodesic boundary, then*

$$\mathrm{Vol}(M) \geq C_n \,\mathrm{Area}(\partial M)^{\frac{n-1}{n-2}}.$$

REMARK 3.1. It is important to note that we have taken the liberty to correct the asymptotics in Theorem 3.2 to agree with our Theorem 3.1 as the authors of [**BK10**] referenced the volume form from [**Nic89**].

The proof of the identity relies on a clever decomposition of the unit tangent bundle. Calagrai [**Cal10**] produced a similar identity for the orthospectrum using a rather different decomposition. Recently, Masai and McShane [**MM13**], using a countable equidecomposability argument, demonstrated that the Bridgeman-Kahn and Calagari identities are one and the same. Additionally, they show that[1]

$$F_3(l) = \frac{2\pi(l+1)}{e^{2l}-1}.$$

Note that the asymptotics agree with that of Theorem 3.2.

In the case of surfaces, Bridgeman extended his identity to surfaces with cusped boundary. A boundary cusp of a surface looks like a vertex of an ideal hyperbolic polygon.

THEOREM 3.4. (Bridgeman [**Bri11**]) *Let S be a finite area hyperbolic surface with totally geodesic boundary and m boundary cusps, then*

$$\mathrm{Area}(S) = \frac{\pi}{3}m + \sum_{l \in |\mathcal{O}_S|} \frac{2}{\pi}\,\mathcal{L}\left(\mathrm{sech}^2\,\frac{l}{2}\right)$$

where $\mathcal{L}(z) = \frac{1}{2}\log|z|\log(1-z) + \sum_{k=1}^{\infty}\frac{z^k}{k^2}$ is the Rogers dilogarithm.

By applying this identity to simple hyperbolic surfaces with cusped boundary, Bridgeman was able to recover classical functional equations for the Rogers dilogarithm and provide infinite families of new ones.

In joint work with Nicholas Vlamis, we have extended the Bridgeman-Kahn identity to all finite volume hyperbolic n-manifolds with totally geodesic boundary.

[1]In [**MM13**], the authors compute the integral formula of [**Cal10**], which already takes into account dividing by $\mathrm{Vol}(\mathbb{S}^2)$.

THEOREM 3.5. *For $n \geq 3$ and M a finite volume hyperbolic n-manifold with totally geodesic boundary, let $\mathfrak{C}$ to be the set of ∂-cusps of M and $|\mathcal{O}(M)|$ the orthospectrum. For every $\mathfrak{c} \in \mathfrak{C}$, let $B_{\mathfrak{c}}$ be the maximal horoball in M and $d_{\mathfrak{c}}$ the Euclidean distance along $\partial B_{\mathfrak{c}}$ between the two boundary components of $\mathfrak{c}$. Then*

$$\mathrm{Vol}(M) = \sum_{\ell \in |\mathcal{O}(M)|} F_n(\ell) + \frac{H(n-2)\,\Gamma\left(\frac{n-2}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \sum_{\mathfrak{c} \in \mathfrak{C}} \frac{\mathrm{Vol}(B_{\mathfrak{c}})}{d_{\mathfrak{c}}^{n-1}}$$

where $\Gamma(m) = (m-1)!$ and $H(m)$ is the m^{th} harmonic number.

REMARK 3.2. Observe that by (ii) of Theorem 3.2, one has

$$\lim_{l \to 0} l^{n-2}\, F_n(l) = \frac{\pi^{\frac{n-2}{2}}\, H(n-2)\,\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\,\Gamma\left(\frac{n+1}{2}\right)} = \frac{\mathrm{Vol}(\mathbb{S}^n)}{2\pi}\,\frac{H(n-2)\,\Gamma\left(\frac{n-2}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}.$$

Since both of these quantities compute volumes of tangent vectors, it is possible that there might be a direct relationship using some kind of geometric rescaling argument. Unfortunately, our proof of Theorem 3.5 does not provide such an insight.

CHAPTER 2

Background

1. Hyperbolic Space

1.1. Models. Hyperbolic n-space, denoted by $\mathbb{H}^n$, is the unique complete, simply connected, Riemannian n-manifold of constant sectional curvature -1. Throughout, $d_{\mathbb{H}^n}$ will denote the hyperbolic metric, ds will be the length element, and dV will be the volume element of $\mathbb{H}^n$. We will also need to consider the standard compactification of $\mathbb{H}^n$ via the *boundary at infinity* $\partial_\infty \mathbb{H}^n$. We will think of $\partial_\infty \mathbb{H}^n$ as the visual sphere at infinity from any point of $\mathbb{H}^n$ or as the space of endpoints of geodesic rays. A good reference for all of the following details is [**Rat13**].

The *conformal ball model*, is given by

$$\mathbb{H}^n \cong \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < 1\} = \mathbb{B}^n, \quad \partial_\infty \mathbb{H}^n \cong \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\} = \mathbb{S}^{n-1},$$

$$ds = \frac{2|d\mathbf{x}|}{1-|\mathbf{x}|^2}, \quad \text{and} \quad dV = \frac{2^n d\mathbf{x}}{(1-|x|^2)^n}.$$

Here, complete geodesics are realized as circular arcs perpendicular to $\mathbb{S}^{n-1}$ and a hyperbolic hyperplane is the intersection of $\mathbb{B}^n$ with an $(n-1)$-sphere perpendicular to $\mathbb{S}^{n-1}$.

In the *upper half space model*, one has

$$\mathbb{H}^n \cong \{\mathbf{x} \in \mathbb{R}^n : x_n > 0\} = \mathbb{U}^n, \quad \partial_\infty \mathbb{H}^n = \{\mathbf{x} \in \mathbb{R}^n : x_n = 0\} \cup \{\infty\} = \hat{\mathbb{R}}^{n-1}$$

$$ds = \frac{|d\mathbf{x}|}{x_n}, \quad \text{and} \quad dV = \frac{d\mathbf{x}}{(x_n)^n}.$$

Similarly, complete geodesics are circular arcs or lines perpendicular to $\mathbb{R}^{n-1}$ and a hyperbolic hyperplane is the intersections of $\mathbb{U}^n$ with an $(n-1)$-sphere or a Euclidean hyperplane perpendicular to $\mathbb{R}^{n-1}$.

A *half space* is the closure of a connected component of $\mathbb{H}^n$ cut by a hyperplane. A *horoball* is a Euclidean ball tangent to $\partial_\infty \mathbb{H}^n$ and contained in $\mathbb{H}^n$ in either of these models. In the upper half space model, a horoball can also be realized as $\{\mathbf{x} \in \mathbb{R}^n : x_n > a > 0\}$. The boundary of a horoball is called a *horosphere* and is Euclidean in the induced path metric from $\mathbb{H}^n$.

The space of unoriented geodesics of $\mathbb{H}^n$ will be denoted by

$$\mathscr{G}(\mathbb{H}^n) = (\partial_\infty \mathbb{H}^n \times \partial_\infty \mathbb{H}^n \smallsetminus \Delta)/(\mathbb{Z}_2)$$

and the space of hyperplanes by $\mathscr{P}(\mathbb{H}^n)$.

We will also make mention of the *hyperboloid model*. Let $\langle \mathbf{x}, \mathbf{y} \rangle_q = x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1}$ be the Lorentzian inner product on $\mathbb{R}^{n+1}$, then

$$\mathbb{H}^n \cong \mathbb{P}\{\mathbf{x} \in \mathbb{R}^{n+1} : \langle \mathbf{x}, \mathbf{x} \rangle_q = -1\}, \quad \partial_\infty \mathbb{H}^n = \mathbb{P}\{\mathbf{x} \in \mathbb{R}^{n+1} : \langle \mathbf{x}, \mathbf{x} \rangle_q = 0\}, \text{ and}$$

$$\mathscr{P}(\mathbb{H}^n) \cong \mathbb{P}\{\mathbf{x} \in \mathbb{R}^{n+1} : \langle \mathbf{x}, \mathbf{x} \rangle_q = 1\}$$

where the induced metric from $\langle \cdot, \cdot \rangle_q$ is the hyperbolic metric on $\mathbb{H}^n$ and a pseudo-Riemannian metric on $\mathscr{P}(\mathbb{H}^n)$.

1.2. Isometries. The group of orientation preserving isometries $\operatorname{Isom}^+(\mathbb{H}^n)$ acts transitively on $\mathbb{H}^n$ and can be realized in several different ways. Let $M(\hat{\mathbb{R}}^n)$ denote the group of transformations of $\hat{\mathbb{R}}^n$ is generated by reflections in hyperplanes and inversions in spheres. Elements of $M(\hat{\mathbb{R}}^n)$ are called *Möbius transformations*. The isometry groups $\operatorname{Isom}^+(\mathbb{U}^n)$ and $\operatorname{Isom}^+(\mathbb{B}^n)$ are precisely the orientation preserving elements of $M(\hat{\mathbb{R}}^n)$ that preserve $\mathbb{U}^n$ and $\mathbb{B}^n$, respectively. Note that a Möbius transformation of $\mathbb{H}^{n-1}$ or $\mathbb{S}^{n-1}$ extends to Möbius transformation that preserves $\mathbb{U}^n$ and $\mathbb{B}^n$, respectively (see [**Rat13**] for details).

Additionally, we have $\operatorname{Isom}^+(\mathbb{H}^n) \cong \operatorname{SO}^+(n,1)$ acting on the hyperboloid model. Here $\operatorname{SO}^+(n,1)$ is the identity component in $\operatorname{SO}(n,1)$. In fact, one may realize $\mathbb{H}^n$ as the homogeneous space $\operatorname{SO}^+(n,1)/\operatorname{SO}(n)$. The metric and volume forms arise in this setting by projecting the Killing form and the Haar measure. For a detailed reference on this perspective, see [**FLJ12**].

For $n = 2$ and $n = 3$ we can identify $\operatorname{Isom}^+(\mathbb{U}^2) \cong \operatorname{PSL}(2,\mathbb{R})$ and $\operatorname{Isom}^+(\mathbb{U}^3) \cong \operatorname{PSL}(2,\mathbb{C})$ acting on $z \in \mathbb{U}^2$ or $z \in \hat{\mathbb{C}} = \partial_\infty \mathbb{U}^3$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \mapsto \frac{az+b}{cz+d}$$

Elements of $\operatorname{Isom}^+(\mathbb{H}^n)$ are classified into three different types. We say $g \in \operatorname{Isom}^+(\mathbb{H}^n)$ is

(i) *elliptic* if g fixes a point of $\mathbb{H}^n$.

(ii) *parabolic* if g fixes no point of $\mathbb{H}^n$ and unique point of $\partial_\infty \mathbb{H}^n$.

(iii) *hyperbolic* or *loxodromic* if g fixes no point of $\mathbb{H}^n$ and exactly two points of $\partial_\infty \mathbb{H}^n$.

2. Teichmüller Space and Quasiconformal Maps

Let S be a closed smooth surface of genus $g \geq 2$. A *(marked) hyperbolic structure* on S is a diffeomorphism $f : S \to X$ where $X = \mathbb{H}^2/\Gamma$ for a discrete torsion free subgroup Γ of $\mathrm{Isom}^+(\mathbb{H}^2) \cong \mathrm{PSL}(2, \mathbb{R})$. For $\alpha \in \pi_1(S)$, this allows us to define $\ell_X(\alpha)$ as the length of the unique geodesic representative in the free homotopy class $f_*(\alpha)$ on X. We say that $(X, f) \sim (Y, g)$ whenever $g \circ f^{-1}$ is isotopic to an isometry between X and Y. Define the *Teichmüller space* of S as

$$\mathrm{Teich}(S) = \{(X, f) \mid f : S \to X \text{ is a hyperbolic structure}\}/\sim .$$

This space non-empty for $g \geq 2$ and is homeomorphic to $\mathbb{R}^{6g-6}$. Notice that every hyperbolic structure (X, f) gives rise to a holonomy representation $f_* : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$. In fact, let $AH(\pi_1(S), \mathrm{PSL}(2, \mathbb{R}))$ denote the space of conjugacy classes of discrete, faithful representations ρ with $\mathbb{H}^2/\rho(\pi_1(S))$ compact, then the holonomy map defines a natural homeomorphism

$$\mathrm{Teich}(S) \cup \mathrm{Teich}(\overline{S}) \cong AH(\pi_1(S), \mathrm{PSL}(2, \mathbb{R}))$$

where $\overline{S}$ is S with the reversed orientation.

A metric on $\mathrm{Teich}(S)$ can be defined by measuring how far $g \circ f^{-1}$ is from being isotopic to an isometry. A homeomorphism h form a plane domain $\Omega \subset \mathbb{C}$ onto $f(\Omega)$ is said to be K-*quasiconformal*, if h has locally integrable, distributional derivatives $h_z, h_{\bar{z}}$ and

$$\frac{1 + |h_{\bar{z}}/h_z|}{1 - |h_{\bar{z}}/h_z|} \leq K \text{ almost everywhere on } \Omega$$

The quasiconformal constant $K(h)$ is the smallest such K. Note, that h is 1-quasiconformal if and only is it is conformal. Since X is locally a plane domain, we can define

$$d_{\mathrm{Teich}}((X, f), (Y, g)) = \inf\{K(h) \mid h \text{ is isotopic to } g \circ f^{-1}\}.$$

Let $k : \mathbb{D} \to \mathbb{D}$ be a quasiconformal map with $k(1) = 1, k(i) = i$, and $k(-1) = -1$, then k extends to *quasisymmetric* map on $qs(k) : \mathbb{S}^1 \to \mathbb{S}^1$. A map $h : \mathbb{S}^1 \to \mathbb{S}^1$ that fixes three points is said to be *quasisymmetric* if there exists $M > 0$ such that

$$\frac{1}{M} \leq \frac{|h\left(e^{i(x+t)}\right) - h\left(e^{ix}\right)|}{|h\left(e^{ix}\right) - h\left(e^{i(x-t)}\right)|} \leq M \text{ for all real } x, t \neq 0 \bmod 2\pi.$$

A powerful result of Ahlfors [**Ahl66**] shows that any quasisymmetric map extends to to a quasiconformal map of the unit disk. Further, if $\Gamma \leq \mathrm{Isom}^+(\mathbb{D})$ is discrete and $h\gamma h^{-1}$ is a restriction of a Möbius transformation for all $\gamma \in \Gamma$ (this is called the automorphy condition), then the quasiconformal extension of h also satisfies the automorphy condition. For details, see [**GL99**]. With this in mind, we define the *universal Teichmüller space* as

$$\mathcal{U} = \{h : \mathbb{S}^1 \to \mathbb{S}^1 \mid h \text{ is quasisymmetric}\}/\,\mathrm{Isom}^+(\mathbb{D}).$$

Observe that if we fix a point $(X, f) \in \mathrm{Teich}(S)$, then we get an embedding $\mathrm{Teich}(S) \to \mathcal{U}$ given by $(Y, g) \mapsto qs([g \circ f^{-1}])$, where $[g \circ f^{-1}]$ the quasiconformal map isotopic to $g \circ f^{-1}$ which attains the minimal quasiconformal constant.

We would also like to mention that one can define a non-symmetric metric on $\mathrm{Teich}(S)$, called the *Thurston metric*, by considering the minimal Lipschitz constant in the isotopy class of $g \circ f^{-1}$. Recall that a map $h : X \to Y$ between two metric spaces is K-bi-Lipschitz if

$$\frac{1}{K}\, d_X(x, y) \leq d_Y(h(x), h(y)) \leq K\ d_x(x, y) \quad \text{ for all } x, y \in X$$

and K-Lipschitz if only the right hand inequality holds.

Lastly, recall that a *quasi-isometric embedding* is a map $h : X \to Y$ such that there exist constants A, K with

$$\frac{1}{K}\, d_X(x, y) - A \leq d_Y(h(x), h(y)) \leq K\ d_x(x, y) + A \quad \text{ for all } x, y \in X.$$

3. Kleinian Groups and Convex Hulls

Let $\Gamma \leq \mathrm{Isom}^+(\mathbb{H}^n)$ be a discrete torsion free subgroup. Define the *limit set* of Γ to be $\Lambda_\Gamma = \overline{\Gamma\, x} \cap \partial_\infty \mathbb{H}^n$ for any $x \in \mathbb{H}^n$. This definition is independent of the choice of x. We say that Γ is a *Kleinian group* (or a *Fuchsian group* for $n = 2$) if Λ_Γ contains at least 3 points. The set $\Omega(\Gamma) = \partial_\infty \mathbb{H}^n \smallsetminus \Lambda_\Gamma$ is called the *domain of discontinuity* of Γ. It can be equivalently defined as the largest open subset in $\partial_\infty \mathbb{H}^n$ where Γ acts properly discontinuously.

The *convex hull* $\mathrm{CH}(X)$ of a closed set $X \subset \partial_\infty \mathbb{H}^n$ is smallest convex subset of $\mathbb{H}^n$ such that $\overline{\mathrm{CH}(X)} \cap \partial_\infty \mathbb{H}^n = X$. We require that X contain more than two points. For a Kleinian group Γ, the convex hull of Γ is $\mathrm{CH}(\Lambda_\Gamma)$ and the convex hull of $\mathbb{H}^n/\Gamma$ is $\mathrm{CH}(\Lambda_\Gamma)/\Gamma$, which is the smallest π_1-injective convex submanifold.

One defines the *nearest point retraction* $r : \overline{\mathbb{H}}^n \to \mathrm{CH}(X)$ as follows. For $x \in \overline{\mathbb{H}}^n \setminus \overline{\mathrm{CH}(X)}$ let $B_t(x)$ denote the 1-parameter family of hyperbolic balls or horoballs centered at x with $B_{t_0}(x) \subset B_{t_1}(x)$ for all $t_0 < t_1$. Then, for $x \in \overline{\mathrm{CH}(X)}$, we define $r(x) = x$ and for all other x, $r(x)$ is the first (unique) intersection point of $\overline{\mathrm{CH}(X)}$ with $B_t(x)$. See [**EM87**] for a proof that this is a well defined continuous distance decreasing map. For a Kleinian group Γ, this map projects to $r : \mathbb{H}^n/\Gamma \to \mathrm{CH}(\Lambda_\Gamma)/\Gamma$.

We focus our attention to the case where $n = 3$ and $\partial_\infty \mathbb{H}^3$ is identified with $\hat{\mathbb{C}}$. A *hyperbolic domain* Ω in $\hat{\mathbb{C}}$ is a connected open set such that $\hat{\mathbb{C}} \setminus \Omega$ is at least 3 points. In particular, a connected component of $\Omega(\Gamma)$ for a Kleinian group Γ is a hyperbolic domain. Let $X = \hat{\mathbb{C}} \setminus \Omega$. Epstein and Marden [**EM87**] show that if X is not contained in a circle, then $\mathrm{CH}(X)$ has non empty interior and a well defined boundary, denoted $\mathrm{Dome}(\Omega) = \partial \mathrm{CH}(X)$. If X lies in a circle, then $\mathrm{CH}(X)$ lies in a hyperbolic plane and is bounded by a countable collection of complete geodesics. In this setting, $\mathrm{Dome}(\Omega)$ is defined as the double $\mathrm{CH}(X)$ along those geodesics.

Points on $\mathrm{Dome}(\Omega)$ can be connected by rectifiable paths along $\mathrm{Dome}(\Omega)$ and so it inherits a path metric from $\mathbb{H}^3$. Thurston [**Thu91**] showed that this path metric is, in fact, a complete hyperbolic metric. Further, he demonstrates that the covering map $\mathbb{H}^2 \to \mathrm{Dome}(\Omega)$ as a very specific structure that we now describe.

A *geodesic lamination* on $\mathbb{H}^2$ is a closed subset $\lambda \subset \mathscr{G}(\mathbb{H}^2)$ which does not contain any intersecting geodesics. It can be realized on $\mathbb{H}^2$ as a closed set foliated by complete geodesics and therefore the elements of λ are called *leaves*. A *measured lamination* μ on $\mathbb{H}^2$ is a non-negative countably additive measure μ on $\mathscr{G}(\mathbb{H}^2)$ supported on a geodesic lamination. A geodesic arc α in $\mathbb{H}^2$ is said to be *transverse to* μ, if it is transverse to every geodesic in $\mathrm{supp}(\mu)$. Whenever α is transverse to μ, we define

$$i(\mu, \alpha) = \mu\left(\{\gamma \in \mathscr{G}(\mathbb{H}^2) \mid \gamma \cap \alpha \neq \emptyset\}\right).$$

If α is not transverse to μ, then it is contained in a geodesic of $\mathrm{supp}(\mu)$ and we let $i(\mu, \alpha) = 0$.

Given a measured lamination μ on $\mathbb{H}^2$, we may construct a *pleated plane* $P_\mu : \mathbb{H}^2 \to \mathbb{H}^3$, well-defined up to post-composition with elements of $\mathrm{Isom}^+(\mathbb{H}^3)$. P_μ is an isometry on the components of $\mathbb{H}^2 \setminus \mathrm{supp}(\mu)$, which are called *flats*. If μ is a finite-leaved lamination, then

P_μ is simply obtained by bending, consistently rightward, by the angle $\mu(\{l\})$ along each leaf l of μ. Since any measured lamination is a limit of finite-leaved laminations, one may define P_μ in general by taking limits (see [**EM87**, Theorem 3.11.9]).

LEMMA 3.1. [**EM87**] *If Ω is a hyperbolic domain, there is a lamination μ on $\mathbb{H}^2$ such that P_μ is a locally isometric covering map with image* $\mathrm{Dome}(\Omega)$.

CHAPTER 3

Convex Hulls, Sullivan's Theorem and Lipschitz bounds

1. Pleated Planes and L-roundness

1.1. Pleated Planes. For any point $x \in \mathrm{Dome}(\Omega)$, a *support plane* P at x is a totally geodesic plane through x which is disjoint from the interior of the convex hull of $\hat{\mathbb{C}} \setminus \Omega$. At least one support plane exists at every point $x \in \mathrm{Dome}(\Omega)$ and $\mathrm{Dome}(\Omega) \cap P$ is either a geodesic line with endpoints in $\partial\Omega$, called a *bending line*, or a *flat*, which is the convex hull of a subset of ∂P containing at least 3 points. The boundary geodesics of a flat will also be called bending lines. Support planes come with a preferred normal direction pointing *away* from $\mathrm{CH}(\hat{\mathbb{C}} \setminus \Omega)$. The closure of the complement of $\mathbb{H}^3 \setminus P$ that lies in this direction is called the associated *half space*, denoted H_P. A detailed discussion and proofs on these facts can be found in [**EM87**].

For a curve $\alpha : (a, b) \to \mathrm{Dome}(\Omega)$, it is natural to consider the space of support planes at each point $\alpha(t)$. A theorem of Kulkarni and Pinkall [**KP94**] asserts that the space of support planes to $\mathrm{Dome}(\Omega)$ is an $\mathbb{R}$-*tree* in the induced path metric from $\mathscr{P}(\mathbb{H}^3)$ whenever Ω is a simply connected hyperbolic domain. Recall that an $\mathbb{R}$-*tree* is a simply connected, geodesic metric space such that for any two points there is a unique embedded arc connecting them. Therefore, dual to any rectifiable path $\alpha : (a, b) \to \mathrm{Dome}(\Omega)$, there is a continuous path $P_t : (c, d) \to \mathcal{P}(\mathbb{H}^3)$ and a map $p : (c, d) \to (a, b)$ such that P_t is a support plane at $\alpha(p(t))$. It also follows that we can define terminal support planes on the ends of α by $P_a = \lim_{t\to c^+} P_t$ and $P_b = \lim_{t\to d^-} P_t$.

Epstein and Marden further show that for every point $x \in \mathrm{Dome}(\Omega)$, there is a neighborhood $W \subset \mathbb{H}^3$ of x such that if l_1, l_2 are bending lines that meet W, then any support plane that meets l_1 intersects all support planes that meet l_2 [**EM87**, Lemma 1.8.3]. The transverse intersection of two support planes P, Q is called a *ridge line*. Notice that if two support planes P, Q intersect, they either do so at a ridge line or $P = Q$. If $P = Q$ and the interiors of H_P, H_Q are not equal, then $\hat{\mathbb{C}} \setminus \Omega$ is contained in a the circle ∂P.

The exterior angle, denoted $\angle_{ext}(P, Q)$, between two intersecting or tangent support planes is the angle between their normal vectors at any point of intersection or tangency. We define the interior angle by $\angle_{int}(P, Q) = \pi - \angle_{ext}(P, Q)$.

Let μ be the measured lamination on $\text{Dome}(\Omega)$ such that $P_\mu : \mathbb{H}^2 \to \text{Dome}(\Omega)$ is the pleated plane. By a *transverse geodesic* arc $\alpha : (a, b) \to \text{Dome}(\Omega)$, we will mean arc such that $P_\mu^{-1}(\alpha)$ is a geodesic arc in $\mathbb{H}^2$ and transverse to $\text{supp}(\mu)$. We say the terminal support planes P_a, P_b form a *roof* over α if the interiors of the associated half spaces H_t intersects H_a for all t. Roofs play an important role in approximating the bending along α.

LEMMA 1.1. (Lemmas 4.1 and 4.2 [**BC03**]) *Let μ be the measured lamination on* $\text{Dome}(\Omega)$*. If $\alpha : (a, b) \to \text{Dome}(\Omega)$ is a transverse geodesic arc such that the terminal support planes P_a, P_b form a roof over α then $i(\alpha, \mu) \leq \angle_{ext}(P, Q) = \pi - \angle_{int}(P, Q)$.*

LEMMA 1.2. *Let Ω be a simply connected hyperbolic domain and $\alpha : (a, b) \to \text{Dome}(\Omega)$ a transverse geodesic arc. If the interiors of the terminal half spaces H_a, H_b intersect, then P_a and P_b form a roof over α.*

PROOF. Intuitively, this is a consequence of the fact that support planes can't form "loops" when Ω is simply connected. Recall that the space of support planes to $\text{Dome}(\Omega)$ is an $\mathbb{R}$-tree. Since α is geodesic, the of support planes P_t to α must be embedded, and therefore the unique path between P_a and P_b. As the interiors of H_a, H_b intersect, either $P_a = P_b$ or P_a, P_b intersect at a ridge line ℓ_r. In the former case, it follows that $P_t = P_a = P_b$ is constant and therefore $H_t = H_a$ for all t.

In the later case, consider the path β which goes from $\alpha(a)$ to ℓ_r along P_a and from ℓ_r to $\alpha(b)$ along P_b. We can project β to $r(\beta) \subset \text{Dome}(\Omega)$. Since P_t is the unique path connecting P_a to P_b, it follows that the path of support planes along $r(\beta)$ must fun over all of P_t. By construction, every support plane to $r(\beta)$ must contain the ridge line ℓ_r. Thus, the interiors of H_t and H_a intersect for all t and P_a, P_b is a roof over α. $\square$

1.2. *L*-roundness. For a measured lamination μ on $\mathbb{H}^2$, Epstein, Marden and Markovic [**EMM04**] defined the *roundness* of μ to be $||\mu|| = \sup i(\mu, \alpha)$ where the supremum is taken over all open unit length geodesic arcs in $\mathbb{H}^2$. The roundness bounds the total bending of P_μ on any segment of length 1 and is closely related to average bending, which was introduced earlier by Bridgeman [**Bri98**].

In our work, we consider the *L-roundness* of a measured lamination for any $L > 0$

$$||\mu||_L = \sup i(\alpha, \mu)$$

where now the supremum is taken over all open geodesic arcs of length L in $\mathbb{H}^2$. We note that the supremum over open geodesic arcs of length L, is the same as that over *half* open geodesic arcs of length L.

In [**Bri03**], Bridgeman obtained an upper bound on the L-roundness of an embedded pleated plane.

THEOREM 1.1. (Bridgeman [**Bri03**]) *There exists a strictly increasing homeomorphism $F : [0, 2\sinh^{-1}(1)] \to [\pi, 2\pi]$ such that if μ is a measured lamination on $\mathbb{H}^2$ and P_μ is an embedding, then $||\mu||_L \leq F(L)$ for all $L \leq 2\sinh^{-1}(1)$. In particular,*

$$||\mu|| \leq F(1) = 2\pi - 2\sin^{-1}\left(\frac{1}{\cosh(1)}\right) \approx 4.8731.$$

Epstein, Marden and Markovic [**EMM04**] provided a criterion guaranteeing that a pleated plane is a bi-Lipschitz embedding.

THEOREM 1.2. (Epstein-Marden-Markovic [**EMM04**, Theorem 4.2, part 2]) *If μ is a measured lamination on $\mathbb{H}^2$ such that $||\mu|| \leq c_2 = 0.73$, then P_μ is a bi-Lipschitz embedding which extends to an embedding $\hat{P}_\mu : \mathbb{H}^2 \cup \mathbb{S}^1 \to \mathbb{H}^3 \cup \hat{\mathbb{C}}$ such that $\hat{P}_\mu(\mathbb{S}^1)$ is a quasi-circle.*

In [**EMM06**], Epstein, Marden and Markovic comment "unpublished work by David Epstein and Dick Jerrard should prove that $c_2 > .948$, though detailed proofs have not yet been written". David Epstein kindly provided their notes. In Section 4 of this Chapter, we prove a generalization of their result using the approach outlined in their notes.

2. An Upper Bound on L-roundness for Embedded Pleated Planes

In this section, we adapt the techniques of [**Bri03**] to obtain an improved bound on the L-roundness of an embedded pleated plane. As it appears here, Theorem 1.4 is an extended version of our work in [**BCY16**, Theorem 3.1] and will appear in a separate manuscript.

THEOREM 1.4. *If $L \in (0, 2\sinh^{-1}(2)]$, μ is a measured lamination on $\mathbb{H}^2$, and P_μ is an embedding, then $\|\mu\|_L \leq F(L)$ where*

$$F(L) = \begin{cases} 2\cos^{-1}(-\sinh(L/2)) & \text{for } L \in [0, 2\sinh^{-1}(1)] \\ 3\pi - 2\cos^{-1}\left(\left(\sqrt{\cosh(L)} - 1\right)/2\right) & \text{for } L \in (2\sinh^{-1}(1), 2\sinh^{-1}(2)] \end{cases}$$

The proof relies on a careful analysis of minimal lengths of arcs joining a sequence of 3 or 4 pleated planes. We present these arguments as Lemmas 2.1, 2.2, 2.4.

LEMMA 2.1. *Let P_0, P_1, P_2 be planes in $\mathbb{H}^3$ with boundary circles $C_i \in \partial_\infty \mathbb{H}^3$. Assume that $C_0 \cap C_2 = \{a\}$, $a \notin C_1$, and the minor angles $\angle_m(C_0, C_1) = \angle_m(C_1, C_2) = \theta < \pi/2$. If $\alpha : [0,1] \to \mathbb{H}^3$ is a rectifiable path with $\alpha(0) \in P_0, \alpha(1) \in P_2$ and $\alpha(t_1) \in P_1$ for some $t_1 \in (0,1)$. Then,*

$$\ell(\alpha) \geq 2\sinh^{-1}(\cos\theta).$$

PROOF. Since $a \notin C_1$, there is a plane $T \subset \mathbb{H}^3$ perpendicular to all P_i. Let $\lambda_i = T \cap P_i$. Take $\overline{\alpha}$ to be the nearest point projection of α onto T. Since nearest point projections shrink distances, $\ell(\alpha) \geq \ell(\overline{\alpha})$. In addition, as T is perpendicular to P_i, we have $\overline{\alpha}(0) \in \lambda_0, \overline{\alpha}(1) \in \lambda_2$ and $\overline{\alpha}(t_1) \in \lambda_1$. We can identify T with the Poincare disk and conjugate λ_i as in Figure 1.

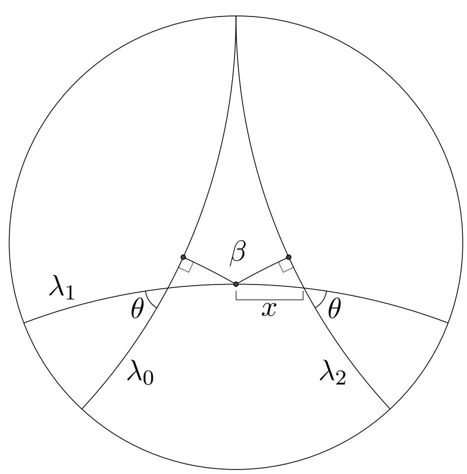

FIGURE 1. Configuration of $\lambda_i \subset T$ in the Poincare disk model for Lemma 2.1

By symmetry, the shortest curve connecting λ_0 to λ_2 via λ_1 is the symmetric piecewise geodesic β depicted in Figure 1. Let x be the sub-arc of λ_1 between $\lambda_1 \cap \lambda_2$ and $\lambda_1 \cap \beta$. Then, one may apply hyperbolic trigonometry formulae [**Bea95**, Theorem 7.9.1] and [**Bea95**, Theorem 7.11.2] to obtain

$$\sinh(x)\tan\theta = 1 \quad \text{and} \quad \sinh(\ell(\beta)/2) = \sinh(x)\sin\theta.$$

Therefore,

$$\ell(\alpha) \geq \ell(\beta) \geq 2\sinh^{-1}(\cos\theta).$$

$\square$

LEMMA 2.2. *Let P_0, P_1, P_2, P_3 be planes in $\mathbb{H}^3$ with boundary circles $C_i \in \partial_\infty \mathbb{H}^3$. Assume*

(i) $P_0 \cap P_2 = P_1 \cap P_3 = P_0 \cap P_3 = \emptyset$

(ii) $C_0 \cap C_3 = \{a\}$ *and* $C_1 \cap C_2 = \{b\}$

(iii) $a \notin C_1 \cup C_2$ *and* $b \notin C_0 \cup C_3$.

(iv) let η_i be normal directions to C_i such that η_0, η_3 point away from each other and η_1, η_2 point toward each other, then $\angle(\eta_0, \eta_1) = \angle(\eta_2, \eta_3) = \theta < \pi/2$.

If $\alpha : [0,1] \to \mathbb{H}^3$ is a rectifiable path with $\alpha(0) \in P_0, \alpha(1) \in P_3$, $\alpha(t_1) \in P_1$, and $\alpha(t_2) \in P_2$ for some $t_1, t_2 \in (0,1)$ with $t_1 < t_2$. Then,

$$\ell(\alpha) \geq \cosh^{-1}\left((2\cos\theta + 1)^2\right).$$

PROOF. Let ρ_i denote the reflection across P_i and $\rho_{i,j} = \rho_i \circ \rho_j$. Since α is supported by the planes P_i, we may look at pieces of α under a series of reflections. In particular, consider the curve

$$\beta = \alpha[0, t_1] \cup \rho_1(\alpha[t_1, t_2]) \cup \rho_{1,2}(\alpha[t_2, 1]).$$

Notice that β is a curve from P_0 to $\rho_{1,2}(P_3)$ and $\ell(\alpha) = \ell(\beta)$. Our goal is now to find a lower bound for $\ell(\beta)$ in terms of θ.

By construction, β is longer than the geodesic from P_0 to $\rho_{1,2}(P_3)$. Notice that this geodesic intersects P_1 and $\rho_1(P_2)$, so after reflecting some pieces, it satisfies the assumptions of the Lemma. Let T be the hyperplane going through the Euclidean centers of C_0 and $\rho_{1,2}(C_3)$. Since the geodesic between P_0 and $\rho_{1,2}(P_3)$ is unique, it must lie in T. Refer to Figure 2 for the generic configuration.

We need to say a few words about the validity of Figure 2 for our computations. Conjugating, we can map the points $a \to 0$ and $b \to \infty$. It follows from (ii) and (iii) that C_1, C_2 are parallel lines and C_0, C_3 are circles in the plane. Assumptions (i) and (iv) also guarantee

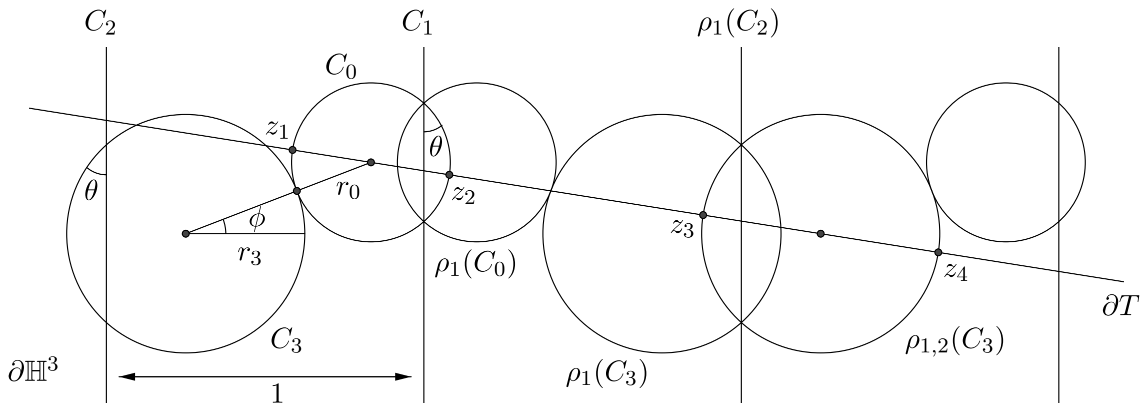

FIGURE 2. Boundaries of the planes P_i in Lemma 2.2 and their reflections in the upper half space model.

that, maybe after flipping, $0 \leq \phi \leq \pi/2$. It is straightforward to check that assumption (iv) on a choice of normal directions guarantees that θ is correctly labeled in Figure 2.

Identify T with $\mathbb{U}^2$ so that the center of C_0 corresponds to 0. We compute the distance between the two disjoint geodesics $\lambda = T \cap P_0$ and $\gamma = T \cap \rho_{1,2}(P_3)$. Let $z_1 < z_2 < z_3 < z_4$, $z_i \in \mathbb{R} \subset \partial T$ be the points $\partial\lambda \cup \partial\gamma$. We can use the standard cross ration to compute

$$(z_1, z_3; z_2, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} = \coth^2\left(\frac{1}{2} d_{\mathbb{H}}(\lambda, \gamma)\right) > 0$$

$$d_{\mathbb{H}}(\lambda, \gamma) = \log\left(\frac{\sqrt{(z_1, z_3; z_2, z_4)} + 1}{\sqrt{(z_1, z_3; z_2, z_4)} - 1}\right).$$

Let r_0, r_1, ϕ, θ be as in Figure 2 and normalize the diagram as shown. By directly constructing a diagram from our parameters, one checks that a configuration satisfies out assumptions if and only if

$$0 \leq \theta < \pi/2 \quad \text{and} \quad 0 \leq \phi \leq \pi/2$$

$$0 \leq r_i + r_i \cos\theta \leq 1 \text{ for } i = 0, 1$$

$$1 = (r_0 + r_1)(\cos\theta + \cos\phi)$$

To evaluate the cross ratio, let $z_1 = -r_0, z_2 = r_0, z_3 = c - r_0$, and $z_4 = c + r_0$, where c is the distance between the Euclidean centers of C_0 and $\rho_{1,2}(C_3)$. Computing, we have

$$c^2 = (r_0 + r_1)^2 \sin^2\phi + (2 - (r_0 + r_1)\cos\phi)^2 = 4 - 4(r_0 + r_1)\cos\phi + (r_0 + r_1)^2.$$

The cross ratio of these point is then

$$x = (-r_0, c - r_1; r_0, c + r_1) = \frac{(r_0 - r_1)^2 - c^2}{(r_0 + r_1)^2 - c^2} = 1 + \frac{r_0 r_1}{1 - (r_0 + r_1)\cos\phi} = 1 + \frac{r_0 r_1}{(r_0 + r_1)\cos\theta}.$$

Therefore,

$$\ell(\alpha) \geq d_{\mathbb{H}}(P_0, \rho_{1,2}(P_3)) \geq \inf_{r_0, r_1, \phi} \log\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) = \inf_{r_0, r_1, \phi} \log\left(1 + \frac{2}{\sqrt{x}-1}\right).$$

Since $\log(1 + 2/(\sqrt{x} - 1))$ is a decreasing function of x, our goal is to maximize x over all allowable configurations with fixed $0 \leq \theta < \pi/2$. Our parameter conditions imply

$$0 \leq r_i + r_i \cos\theta \leq 1, \quad \frac{1}{1 + \cos\theta} \leq (r_0 + r_1), \quad \text{and} \quad (r_0 + r_1) \leq \frac{1}{\cos\theta}.$$

Since $0 \leq \theta \leq \pi/2$, it is easy to see that this region is a triangle in the (r_0, r_1)-plane bounded by $r_i = 1/(1 + \cos\theta)$ and $(r_0 + r_1) = 1/(1 + \cos(\theta))$, see Figure 3.

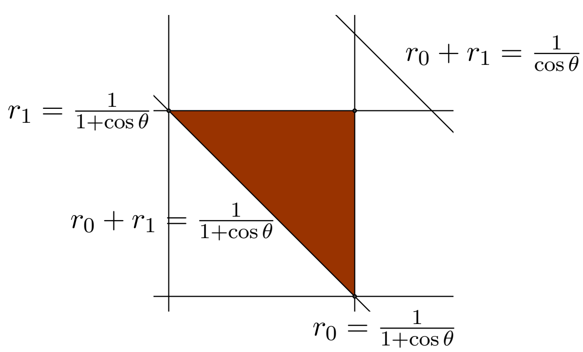

FIGURE 3. Constraints for maximizing $x = 1 + \frac{r_0 r_1}{(r_0+r_1)\cos\theta}$ in Lemma 2.2.

We also have

$$\frac{\partial x}{\partial r_i} = \frac{r_j^2}{(r_i + r_j)^2} > 0 \text{ for } r_i, r_j > 0 \text{ where } \{i, j\} = \{0, 1\},$$

so the maximum value of x is attained on the boundary of our triangle. On the edges corresponding to $r_i = 1/(1 + \cos\theta)$, we get a maximum when $r_0 = r_1 = 1/(1 + \cos\theta)$. For the edge corresponding to $(r_0 + r_1) = 1/(1 + \cos(\theta))$, we have a maximum at $r_0 = r_1 = 1/(2 + 2\cos\theta)$. Of these two points, x has the largest value at the former, so

$$\sup_{r_0, r_1, \phi} x = x\,|_{r_i = 1/(1+\cos\theta)} = 1 + \frac{1}{2(1 + \cos\theta)\cos\theta}$$

Lastly, note that using $\cosh(z) = (e^z + e^{-z})/2$, we have

$$\cosh\left(\log\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)\right) = \frac{1}{2}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1} + \frac{\sqrt{x}-1}{\sqrt{x}+1}\right) = \frac{x+1}{x-1}$$

Our desired results follows,

$$\ell(\alpha) \geq \inf_{r_0,r_1,\phi} \log\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) = \inf_{r_0,r_1,\phi} \cosh^{-1}\left(\frac{x+1}{x-1}\right) = \cosh^{-1}\left((2\cos\theta+1)^2\right).$$

□

COROLLARY 2.3. *The shortest rectifiable path $\alpha(t) \subset \mathbb{H}^3$ connecting four mutually tangent hyperplanes in $\mathbb{H}^3$ has length $2\sinh^{-1}(2)$ and is attained when the planes support four of the faces of a standard ideal octahedron.*

PROOF. If $\theta = 0$, then the geodesic we have find in Lemma 2.2 has length

$$\cosh^{-1}\left((2\cos(0)+1)^2\right) = \cosh^{-1}(9) = 2\sinh^{-1}(2).$$

The critical values of r_0, r_1 were $r_i = 1/(1+\cos\theta) = 1/2$, so $1 = (r_0+r_1)(\cos\theta+\cos\phi) = 1+\cos\phi$ and $\phi = \pi/2$. This configuration and the other four planes supporting a standard ideal octahedron are shown in Figure 4. □

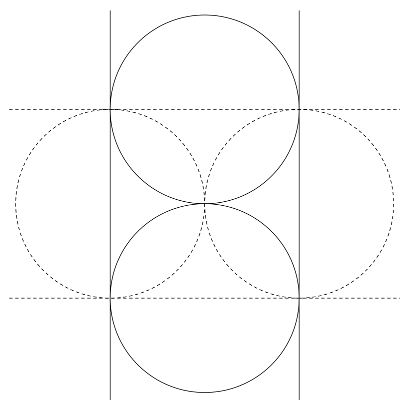

FIGURE 4. The supporting planes of a standard ideal octahedron in Cor 2.3.

Next, we prove a slight generalization of Lemma 2.2 where we replace the tangency of P_0 and P_3 for another condition.

LEMMA 2.4. *Let P_0, P_1, P_2, P_3 be planes in $\mathbb{H}^3$ with boundary circles $C_i \in \partial_\infty\mathbb{H}^3$. Assume*

(i) $P_0 \cap P_2 = P_1 \cap P_3 = \emptyset$

(ii) $C_1 \cap C_2 = \{b\}$ *and* $b \notin C_0 \cup C_3$.

(iii) let $P_\star$ be the unique plane between P_1 and P_2 tangent to P_3, then $\partial P_\star \cap C_0 \neq \emptyset$

(iv) let η_i be normal directions to C_i such that η_0, η_3 point away from each other and η_1, η_2 point toward each other, then $\angle(\eta_0,\eta_1) = \angle(\eta_2,\eta_3) = \theta < \pi/2$.

If $\alpha : [0,1] \to \mathbb{H}^3$ *is a rectifiable path with* $\alpha(0) \in P_0, \alpha(1) \in P_3$*,* $\alpha(t_1) \in P_1$*, and* $\alpha(t_2) \in P_2$ *for some* $t_1, t_2 \in (0,1)$ *with* $t_1 < t_2$*. Then,*

$$\ell(\alpha) \geq \cosh^{-1}\left((2\cos\theta + 1)^2\right).$$

PROOF. We will reduce to the case of Lemma 2.2 as follows. We can conjugate $b \to \infty$ and build as similar diagram with C_3 "below" C_0 as before, except they may no longer be tangent. Condition (iii) implies that there is some "slide" of P_0 along $P_\star$ to a plane P_0' that is tangent to P_3, see Figure 5.

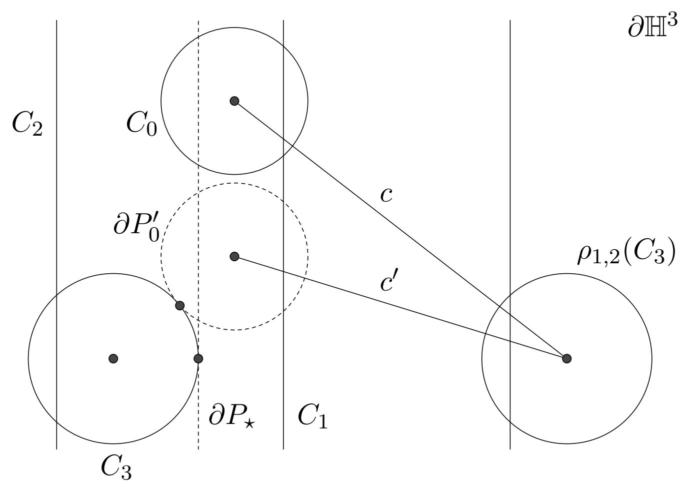

FIGURE 5. The "slide" move of P_0 to P_0' in Lemma 2.4. Notice that the Euclidean length $c \geq c'$.

Notice that the "slide" operation does not change radii of the circles in our configuration. In the proof of Lemma 2.2, the cross ratio was given as

$$x = \frac{(r_0 - r_1)^2 - c^2}{(r_0 + r_1)^2 - c^2}$$

This function is decreasing in c, so if we replace c with the shorter c' as in Figure 5. This gives a larger value of x and, therefore, a shorter geodesic. Thus, we replace P_0 with P_0' and apply Lemma 2.2. $\square$

PROOF OF THEOREM 1.4. Fix $L \in (0, 2\sinh^{-1}(2)]$. If we fix $\|\mu\|_L$, then for every $\epsilon > 0$, we can find a geodesic arc $\alpha : (0,1) \to P_\mu$ with $\ell(\alpha) = L$ such that $\|\mu\|_L - \epsilon < i(\alpha, \mu) \leq \|\mu\|_L$. Let $\{P_t\}$ for $t \in [0,1]$ denote the path of support planes to α and $p : [0,1] \to [0,1]$ be such that P_t is a support plane at $\alpha(p(t))$. Here, we take $P_0 = \lim_{t \to 0^+} P_t$ and $P_1 = \lim_{t \to 1^-} P_t$. We will divide our argument into cases via bounds on $\|\mu\|_L$.

Case $\|\mu\|_L \leq \pi$. This is the trivial case as $0 \leq L$ implies $\|\mu\|_L \leq \pi = F(0) \leq F(L)$.

Case $\pi < \|\mu\|_L \leq 2\pi$. Fix $\epsilon > 0$ small enough and α of length L such that

$$\pi < \|\mu\|_L - \epsilon < i(\alpha, \mu) \leq \|\mu\|_L \leq 2\pi.$$

Let $2\theta = 2\pi - \|\mu\|_L + \epsilon < \pi$, then by assumption $2\pi - 2\theta < i(\alpha,\mu)$. As the interior angle between P_0 and P_t decreases continuously, it follows from the roof property (Lemma 1.1) that there must be a t_1 such that $\angle_{int}(P_0, P_{t_1}) = \theta$ as $i(\alpha,\mu) > \pi - \theta$. Similarly, there must be at t_2 such that $\angle_{int}(P_{t_1}, P_{t_2}) = \theta$ as $i(\alpha, \mu) > 2\pi - 2\theta$. Notice that $P_0 \cap P_{t_2} = \emptyset$, as otherwise either they form a roof over α by Lemma 1.2 and $i(\alpha,\mu) \leq \pi$, a contraction.

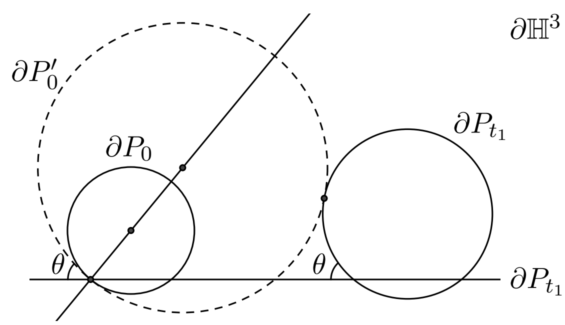

FIGURE 6. The "grow" move of P_0 to P_0' in **Case** $\pi < \|\mu\|_L \leq 2\pi$ of Theorem 1.4.

Since $2\theta < \pi$, our planes P_0, P_{t_1}, P_{t_2} *almost* satisfy the conditions of Lemma 2.1. By mapping P_{t_1} to a vertical plane in the upper half space model for $\mathbb{H}^3$, we easy see that we can "grow" P_0 to a plane P_0' that is tangent to P_{t_2} while keeping the interior angle with P_{t_1} equal to θ, see Figure 6. The plane P_0' is not a support plane, but a sub-arc of $\alpha[p(0), p(t_2)]$ joins it to P_{t_2}. Therefore, the shortest curve between P_0' and P_{t_2} with a point on P_{t_1} is shorter than α. We apply Lemma 2.1 to P_0', P_{t_1}, P_{t_3} and see

$$L \geq 2\sinh^{-1}(\cos\theta) \quad \Longrightarrow \quad \cos^{-1}(\sinh(L/2)) \leq \theta.$$

$$\|\mu\|_L = 2\pi - 2\theta + \epsilon \leq 2\pi - 2\cos^{-1}(\sinh(L/2)) + \epsilon = 2\cos^{-1}(-\sinh(L/2)) + \epsilon.$$

Since $\epsilon > 0$ can be taken arbitrarily small and $F(L)$ is an increasing function, $\|\mu\|_L \leq F(L)$.

Case $2\pi < \|\mu\|_L \leq 3\pi$. Fix $\epsilon > 0$ small enough and α of length L such that

$$2\pi < \|\mu\|_L - \epsilon < i(\alpha, \mu) \leq \|\mu\|_L \leq 3\pi.$$

Let $2\theta = 3\pi - \|\mu\|_L + \epsilon < \pi$, then by assumption $3\pi - 2\theta < i(\alpha, \mu)$. As before, since the interior angle decreases and we cannot violate the roof property, there exists t_1 such that $\angle_{int}(P_0, P_{t_1}) = \theta$, the smallest t_2 such that $\angle_{int}(P_{t_1}, P_{t_2}) = 0$, and t_3 such that $\angle_{int}(P_{t_2}, P_{t_3}) = \theta$.

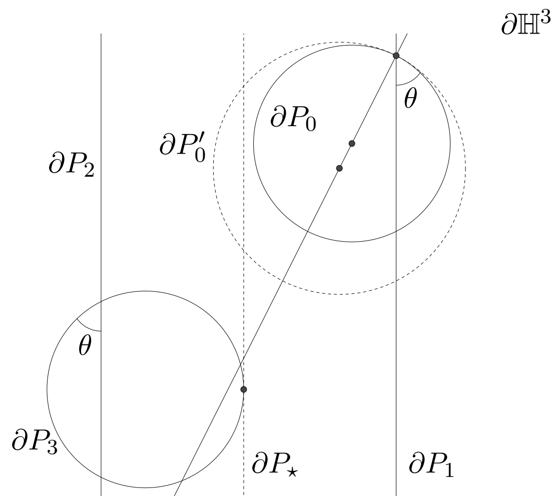

FIGURE 7. The "grow" move of P_0 to P_0' in **Case** $2\pi < \|\mu\|_L \leq 3\pi$ of Theorem 1.4.

We want to modify our set of planes slightly to satisfy the assumptions of Lemma 2.4. We see that $P_0 \cap P_{t_2} = P_{t_1} \cap P_{t_3} = P_0 \cap P_{t_3} = \emptyset$ by Lemma 1.2 as any roofs over subarcs of α would decrease its bending. Let $P_\star$ be the unique plane between P_{t_1} and P_{t_2} that is tangent to P_{t_3}. If $P_\star \cap P_0 = \emptyset$, we can then "grow" P_0 to P_0' so that $P_0' \cap P_\star \neq \emptyset$, see Figure 7. As before, P_0' is joined to P_{t_3} by a sub-arc of $\alpha[p(0), p(t_3)]$. As $\theta < \pi/2$, all the assumptions of Lemma 2.4 are satisfied, so we have

$$L \geq \cosh^{-1}\left((2\cos\theta + 1)^2\right) \quad \Longrightarrow \quad \cos^{-1}\left(\left(\sqrt{\cosh(L)} - 1\right)/2\right) \leq \theta.$$

$$\|\mu\|_L = 3\pi - 2\theta + \epsilon \leq 3\pi - 2\cos^{-1}\left(\left(\sqrt{\cosh(L)} - 1\right)/2\right) + \epsilon.$$

Since $\epsilon > 0$ can be taken arbitrarily small, $\|\mu\|_L \leq F(L)$.

Case $\|\mu\|_L > 3\pi$**.** We can choose α of length L such that $i(\alpha, \mu) > 3\pi$. As before, we find the smallest t_1, t_2, t_3 (in that order) such that $\angle_{int}(P_0, P_{t_1}) = \angle_{int}(P_{t_1}, P_{t_2}) = \angle_{int}(P_{t_2}, P_{t_3}) = 0$. Notice that P_{t_3} is *not* the terminal support plane for α, as $i(\alpha, \mu) > 3\pi$. After a possible "grow" move, this configuration corresponds to the case of Lemma 2.4 with $\theta = 0$. This, however, implies

$$L > \cosh^{-1}\left((2\cos(0)+1)^2\right) = \cosh^{-1}(9) = 2\sinh^{-1}(2)$$

which contradicts the fact that we fixed $L \in (0, 2\sinh^{-1}(2)]$. $\square$

3. Improved Bounds on Average Bending and Lipschitz Constants

We take an aside from bounds on Sullivan's Theorem to improve the Lipschitz and average bending bounds of [**Bri03**, Theorem 1.2]. One may revisit this section at a later time.

THEOREM 1.6. *Let Γ be a Kleinian group with the components of $\Omega(\Gamma)$ simply connected and let $N = \mathbb{H}^3/\Gamma$. There exist universal constants K_0, K_1 with $K_0 \leq 2.494$ and $K_1 \leq 3.101$ such that*

(i) if μ_Γ is the bending lamination of $\partial C(N)$, then

$$\ell_{\partial C(N)}(\mu_\Gamma) \leq K_0\,\pi^2\,|\chi\,(\partial C(N))|$$

(ii) for any closed geodesic α on $\partial C(N)$,

$$B_\Gamma(\alpha) = \frac{i(\alpha, \mu_\gamma)}{\ell(\alpha)} \leq K_1$$

where $B_\Gamma(\alpha)$ is called the average bending of α.

(iii) there exists a $(1+K_1)$-Lipschitz map $s : \partial C(N) \to \Omega(\Gamma)/\Gamma$ that is a homotopy inverse to the retract map $r : \Omega(\Gamma)/\Gamma \to \partial C(N)$.

PROOF. Our result is a direct generalization of [**Bri03**] by using our function $F(L)$ from Theorem 1.4. We provide an outline of the proof.

Let δ be a geodesic arc on P_{μ_Γ} and fix $L \in (0, 2\sinh^{-1}]$. Set $\lceil x \rceil$ to be the least integer $\geq x$. By subdividing δ into arcs or length $\leq L$, we see

$$B_\Gamma(\delta) \leq \frac{\|\mu_\Gamma\|_L}{\ell(\delta)}\left\lceil\frac{\ell(\delta)}{L}\right\rceil \leq \frac{\|\mu_\Gamma\|_L}{\ell(\delta)}\left(\frac{\ell(\delta)}{L}+1\right) = \frac{\|\mu_\Gamma\|_L}{L}\left(1+\frac{L}{\ell(\delta)}\right) \leq \frac{F(L)}{L}\left(1+\frac{L}{\ell(\delta)}\right)$$

For an infinite length geodesic β on P_{μ_Γ} and a point $x \in \beta$, let β_x^t denote the sub-arc centered at x of length $2t$. One can define average bending for β as

$$B_\Gamma(\beta_x) = \limsup_{t\to\infty} B_\Gamma(\beta_x^t).$$

In [**Bri98**], Bridgeman shows that this notion is well defined and independent of x. In particular, by taking $\ell(\delta) \to \infty$ in the bound on $B_\Gamma(\delta)$, we see that for any infinite length geodesic β on P_{μ_Γ},

$$B_\Gamma(\beta) \leq \frac{F(L)}{L} \text{ for all } L \in (0, 2\sinh^{-1}(2)].$$

Let

$$K_1 = \min\left[\left(3\pi - 2\cos^{-1}\left(\frac{\sqrt{\cosh(L)}}{2} - \frac{1}{2}\right)\right)/L\right] \text{ over } L \in (2\operatorname{arcsinh}(1), 2\sinh^{-1}(2)]$$

Then, $B_\Gamma(\beta) \leq K_1 \leq 3.101$, where the minimum is attained at $L \approx 2.74104$.

For a closed geodesic α on $\partial C(N)$, let $\widetilde{\alpha} \subset P_{\mu_\Gamma}$ be a lift. Then (ii) follows, as

$$B_\Gamma(\alpha) = B_\Gamma(\widetilde{\alpha}) \leq K_1.$$

The statement of (iii) can be derived from (ii). Let K_s be the minimal Lipschitz constant of $s : \partial C(N) \to \Omega(\Gamma)/\Gamma$. Then, Thurston characterized

$$K_s = \sup\left\{ \frac{\ell(s_*\alpha)}{\ell(\alpha)} \middle| \alpha \text{ is a simple closed curve on } \partial C(N)\right\}$$

and McMullen's showed that $\ell(s_*\alpha) \leq \ell(\alpha) + i(\alpha, \mu_\Gamma)$ (see [**Thu98**, Theorem 8.5] and [**McM98**, Theorem 3.1]). Combining these two facts gives $K_s \leq 1 + B_\Gamma(\alpha) \leq 1 + K_1$, so (iii) holds.

For (i), we use a computation from [**Bri03**, Section 5] to bound $\ell(\mu_\Gamma)$ by integrating along the unit tangent bundle of $\partial C(N)$. Fix $L \in (0, 2\sinh^{-1}(2)]$ and for $v \in T_1(\partial C(N))$, let $\alpha_v : (0, L) \to \partial C(N)$ be the unit speed geodesic in the direction v. Then, Bridgeman and Canary [**BC05**] show

$$\ell(\mu_\Gamma) = \frac{1}{4L}\int_{T_1(\partial C(N))} i(\alpha_v, \mu_\Gamma)\, d\Omega$$

By taking a maximal lamination $\widetilde{\mu} \supset \mu_\Gamma$, one can integrate our bound $F(L) \geq i(\alpha_v, \mu_\Gamma)$ over the set of ideal triangles $\partial C(N) \setminus \widetilde{\mu}$. In [**Bri03**, Section 5], Bridgeman works out this integral and shows that

$$\frac{\ell(\mu_\Gamma)}{\pi^2\left|\chi\left(\partial C(N)\right)\right|} \leq \frac{3}{\pi^2 L}\int_{(x,y)\in U} \frac{dx\, dy}{y^2}\int_0^{\cos^{-1}\left(\frac{D(x,y)}{\tanh(L)}\right)} F\left(L - \tanh^{-1}\left(\frac{D(x,y)}{\cos\theta}\right)\right) d\theta = K_{eq}$$

where U is the ideal triangle

$$U = \{(x, y) \mid -1 \leq x \leq 1,\, y \geq \sqrt{1 - x^2}\} \text{ and}$$

$$D(x,y) = \frac{x^2+y^2-1}{\sqrt{(x^2+y^2-1)^2+4y^2}}$$

computes the length of the unique perpendicular from (x, y) to the "bottom" edge of U. We compute this integral with using numerical approximation in Mathematica. We choose $L = \sinh^{-1}(89/10) < 2\sinh^{-1}(2)$ and find the upper bound

$$\frac{\ell(\mu_\Gamma)}{\pi^2\,|\chi\,(\partial C(N))|} \leq K_{eq} \leq 2.494.$$

□

4. A New Criterion for Embeddedness of Pleated Planes

In this section, we provide a new criterion which guarantees the embeddedness of a pleated plane. This section is a revised version of what appears in [**BCY16**]. Our results generalize earlier work of Epstein-Marden-Markovic [**EMM04**] referenced here as Theorem 1.2 and an unpublished work of Epstein-Jerrard [**EJ**].

THEOREM 4.1. *There exists a computable increasing function $G : (0,\infty) \to (0,\pi)$, such that if μ is a measured lamination on $\mathbb{H}^2$ and*

$$||\mu||_L < G(L),$$

then P_μ is a bi-Lipschitz embedding and extends continuously to a map $\hat{P}_\mu : \mathbb{H}^2 \cup \mathbb{S}^1 \to \mathbb{H}^3 \cup \hat{\mathbb{C}}$. Further, $\hat{P}_\mu(\mathbb{S}^1)$ is a quasi-circle.

Since $G(1) \approx 0.948$, we recover this result claimed by Epstein and Jerrard as a special case.

COROLLARY 4.1. (Epstein-Jerrard [**EJ**]) *If μ is a measured lamination on $\mathbb{H}^2$ such that*

$$||\mu||_1 < .948$$

then P_μ is a bi-Lipschitz embedding and extends continuously to a map $\hat{P}_\mu : \mathbb{H}^2 \cup \mathbb{S}^1 \to \mathbb{H}^3 \cup \hat{\mathbb{C}}$. Further, $\hat{P}_\mu(\mathbb{S}^1)$ is a quasi-circle.

The derivation begins by finding an embedding criterion for piecewise geodesics. This portion of the proof follows Epstein and Jerrard's outline quite closely. Such a criterion is easily translated into a criterion for the embeddedness of pleated planes associated to laminations with finitely many leaves. We proceed to show that, in the finite-leaved lamination case, the pleated planes are in fact quasi-isometric embeddings with uniform bounds on the quasi-isometry constants. The general case is handled by approximating a pleated plane by pleated planes associated to finite-leaved laminations.

Remark 4.2. As in [**EMM04**, Theorem 4.2] one can consider a horocycle H in $\mathbb{H}^2$ and a consecutive sequence of points on H hyperbolic distance L apart. Connecting these points in sequence, one obtains an embedded piecewise geodesic γ in $\mathbb{H}^3$. Let $P_\mu(\mathbb{H}^2)$ be the pleated plane in $\mathbb{H}^3$ obtained by extending each flat in γ to a flat in $\mathbb{H}^3$. One may check that

$$||\mu||_L = 2\sin^{-1}\left(\tanh\left(\frac{L}{2}\right)\right).$$

This is the conjectured optimal bound. Since $2\sin^{-1}(\tanh(1/2)) \approx .96076$, Theorem 4.1 is nearly optimal when $L = 1$. In Figure 8, we observe that our $G(L)$ is close to optimal for all $L \in [0, 2\sinh^{-1}(1)]$.

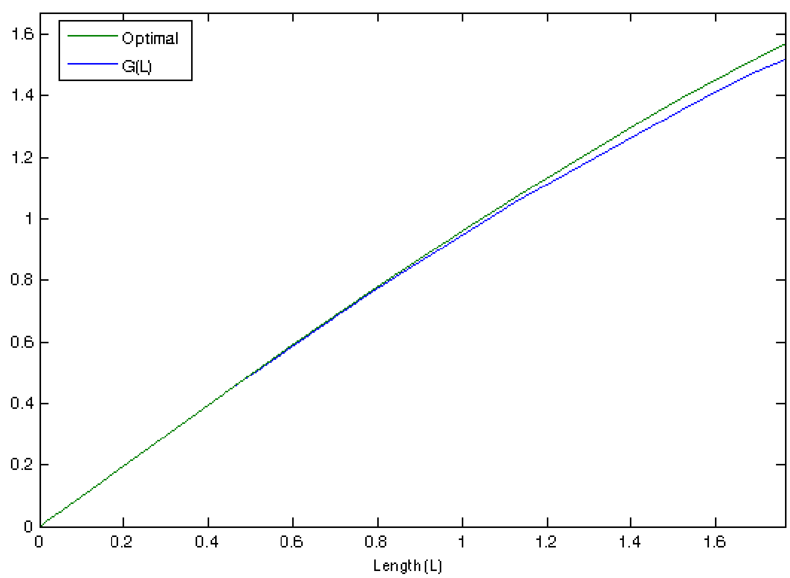

Figure 8. $G(L)$ and the conjectured optimal bound $2\sin^{-1}(\tanh(L/2))$ on $[0, 2\sinh^{-1}(1)]$

4.1. Piecewise Geodesics. Let J be an interval in $\mathbb{R}$ containing 0. A continuous map $\gamma : J \to \mathbb{H}^3$ will be called a *piecewise geodesic* if there exists a discrete set $\{t_i\} \subset \mathrm{int}(J)$, parameterized by an interval in $\mathbb{Z}$ (possibly infinite), such that, for all i, $t_i < t_{i+1}$ and $\gamma\mid_{(t_i,t_{i+1})}$ is a *unit* speed geodesic. We call t_i (or $\gamma(t_i)$) a *bending point* of γ. The *bending* *angle* ϕ_i at t_i is the angle between $\gamma([t_{i-1}, t_i])$ and $\gamma([t_i, t_{i+1}))$. We will further assume that $\phi_i > 0$ for all i. By analogy with the definition of L-roundness, we define $||\gamma||_L$ to be the supremum of the total bending angle in any open subsegment of γ of length $L > 0$.

For $t \neq t_i$ for any i, let $\theta(t) \in [0, \pi]$ be the angle between the ray from $\gamma(0)$ to $\gamma(t)$ and the tangent vector $\gamma'(t)$. For $i = 1, \ldots, n$, define $\theta^{\pm}(t_i) \in [0, \pi]$ to be the angle between the ray from $\gamma(0)$ to $\gamma(t)$ and $\lim_{t \to t_i^+} \gamma'(t)$ or $\lim_{t \to t_i^-} \gamma'(t)$, respectively. We set $\theta^{\pm}(t) = \theta(t)$ for $t \neq t_i$. Notice that $\theta(t)$ smooth and non-increasing on (t_i, t_{i+1}) for all i and that

$$|\theta^+(t_i) - \theta^-(t_i)| \leq \phi_i \text{ for all } i. \tag{4.1}$$

If $t \neq t_i$ for any i, Epstein-Marden-Markovic [**EMM04**, Lemma 4.4] show that for

$$r_\gamma(t) = d_{\mathbb{H}^3}(\gamma(0), \gamma(t))$$

one has

$$r'_\gamma(t) = \cos(\theta(t)) \quad \text{and} \quad \theta'(t) = -\frac{\sin(\theta(t))}{\tanh(r_\gamma(t))} \leq -\sin(\theta(t)). \tag{4.2}$$

4.2. The Hill Function of Epstein and Jerrard. A key tool in Epstein and Jerrard's work is the *hill function* $h(x)$, where

$$h : \mathbb{R} \to (0, \pi) \text{ is given by } h(x) = \cos^{-1}(\tanh(x)). \tag{4.3}$$

The hill function is convex, decreasing, and a homeomorphism, with the key features

$$h'(x) = -\operatorname{sech}(x) = -\sin(h(x)) \quad \text{and} \quad h(0) = \frac{\pi}{2}. \tag{4.4}$$

For fixed $L > 0$, we consider solutions for x to the equation

$$h'(x) = \frac{h(x) - h(x - L)}{L}.$$

Geometrically, this corresponds to finding a point on the graph of h such that the tangent line at $(x, h(x))$ intersects the graph at the point $(x - L, h(x - L))$, see Figure 9. We will show that there is a unique solution $x = c(L)$ and that $c(L) \in (0, L)$.

Given $x \in \mathbb{R} \setminus \{0\}$, the tangent line at $(x, h(x))$ to the graph of h intersects it in two distinct points $(x, h(x))$ and $(f(x), h(f(x))$. Letting $f(0) = 0$, we see that function f is continuously differentiable and odd. Define $A(x) = x - f(x)$ and note that it is also continuously differentiable and odd. We argue that A is strictly increasing. Since A is odd, it suffices to work over $[0,\infty)$. Suppose that $0 \leq x_1 < x_2$, and that T_i is the tangent line to h at x_i. As h is convex on $[0,\infty)$, $T_1 \cap T_2 = (x_0, y_0)$ lies below the graph of h and $x_1 < x_0 < x_2$. For $x \leq x_0$, it follows that T_2 lies below T_1 and $f(x_2) < f(x_1) \leq f(0) = 0$. We conclude that f is decreasing and $A(x) = x - f(x)$ is increasing with $A(x) > x$ for all $x \in (0,\infty)$. The function c is the inverse of A, so c is also continuously differentiable and strictly increasing. Since $A(x) > x$ for $x > 0$, $c(L) \in (0, L)$.

Define

$$\Theta(L) = h(c(L)) \quad \text{and} \quad G(L) = h(c(L) - L) - h(c(L)) = -Lh'(c(L)).$$

We observe that G is monotonic. Define $B(x) = h(f(x)) - h(x)$ to be the height difference between intersection points of the tangent line at $(x, h(x))$ with the graph of h. As h and f are both strictly decreasing continuous functions, B is strictly increasing and continuous. By definition, $G(L) = B(c(L))$, so G is also strictly increasing and continuous.

The following fact is the key estimate in the proof of Theorem 4.1.

Lemma 4.3. *If $\gamma : [0,\infty) \to \mathbb{H}^3$ is piecewise geodesic with a first bending point, $L > 0$, and*

$$||\gamma||_L \leq G(L),$$

then for all $t > 0$

$$\theta^+(t) \leq \Theta(L) + G(L) = h(c(L) - L) < \pi.$$

Proof. Our argument will proceed by contradiction. Fix $L > 0$, $c = c(L)$, $G = G(L)$, $\Theta = \Theta(L)$ and choose our indexing so that $t_1 > 0$ is the first bending point of γ. Suppose there exits T_0 with $\theta^+(T_0) > \Theta + G$. Define

$$T = \inf\{t \in [0,\infty) \mid \theta^+(t) > \Theta + G > 0\}$$

and note that $T > 0$ as $\theta^+(t) = 0$ on $[0, t_1)$. In addition, T is a bending point of γ as $\theta^+(t)$ is continuous and non-increasing on $[t_i, t_{i+1})$ for all i. It follows that T is the first bending point with $\theta^+(T) > \Theta + G$. Also, since $\theta^-(t) = \theta^+(t)$ on (t_i, t_{i+1}), we have

$$0 < \theta^\pm(t) \leq \Theta + G < \pi \text{ for all } t \in (t_1, T).$$

Using equation (4.2), we see that

$$\theta'(t) < -\sin(\theta(t)) \text{ for } t \in (t_i, t_{i+1}) \subset (t_1, T).$$

In particular, θ is decreasing on those intervals. For the remainder of the argument, we only consider $(t_i, t_{i+1}) \subset (t_1, T)$.

Define

$$s_0 = \sup\{s \in (0, T] \mid \theta^-(s) \leq \Theta\}.$$

By continuity, $\theta^-(s_0) \leq \Theta$. Observe that $s_0 < T$, as otherwise $s_0 = T$ and we obtain a contraction by

$$||\gamma||_L \leq G \quad \Longrightarrow \quad |\theta^+(T) - \theta^-(s_0)| \leq G \quad \Longrightarrow \quad \theta^+(T) \leq \theta^-(s_0) + G \leq \Theta + G.$$

Further, we must have $s_0 = t_i$ for some i, as otherwise the fact that θ^- is continuous and decreasing on all intervals (t_i, t_{i+1}) would contradict the choice of s_0.

If $T - s_0 < L$, then $[s_0, T]$ is contained in an open interval of length L and we again obtain a contradiction by

$$||\gamma||_L \leq G \quad \Longrightarrow \quad \theta^+(T) \leq \theta^-(s_0) + G \leq \Theta + G.$$

Thus, we may assume that $T - s_0 \geq L$ and $\theta^-(t) > \Theta$ on $(s_0, s_0 + L]$ by our choice of s_0. In addition, note that $\theta^+(t) \in [\Theta, \Theta + G]$ for $t \in [s_0, s_0 + L)$, as otherwise the decreasing nature of $\theta^\pm$ on (t_i, t_{i+1}) contradicts the definition of s_0 or T. We now proceed to obtain a contradiction and complete the proof. Our trick will be to use the hill function h to keep track of the drops in $\theta(t)$ over (t_i, t_{i+1}) and the jumps at t_i.

To have a visual picture for our construction, we define maps $P^\pm : (t_1, T) \to \mathbb{R}^2$ which are continuous away from $\{t_i\}$ and whose images lie on the graph of h. Since h is a homeomorphism onto $(0, \pi)$ and $0 < \theta^\pm(t) < \pi$ for all $t \in (t_1, T)$, we can find a unique $g^\pm(t) \in \mathbb{R}$, such that

$$h(g^\pm(t)) = \theta^\pm(t) \text{ for } t \in (t_1, T).$$

We then define

$$P^\pm(t) = (P_1^\pm(t), P_2^\pm(t)) = (g^\pm(t), h(g^\pm(t))) = (g^\pm(t), \theta^\pm(t)).$$

Since the functions P^+ and P^- agree away from the bending points, we denote the common functions by $P(t)$, $g(t)$, and $\theta(t)$ on the intervals (t_i, t_{i+1}).

Notice that as one moves along the geodesic ray γ, the functions $\theta^{\pm}(t)$ decrease on each interval (t_i, t_{i+1}) and have vertical jumps equal to $\psi_i = \theta^+(t_i) - \theta^-(t_i)$ at each t_i. By equation 4.1, we have

$$|\psi_i| = |\theta^+(t_i) - \theta^-(t_i)| \leq \phi_i.$$

Correspondingly, the points $P^{\pm}(t)$ slide rightward and downward along the graph of h for $t \in (t_i, t_{i+1})$ and jump vertically, either upward or downward, by ψ_i at t_i, see Figure 9.

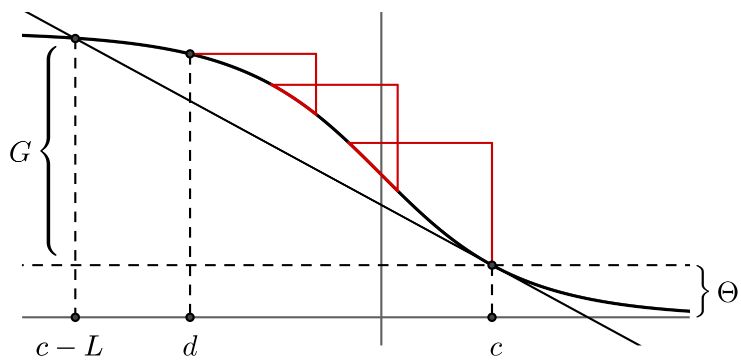

FIGURE 9. Jumps and slides of $P^{\pm}(t)$ on the graph of h

Under our hypotheses, a careful analysis of this picture will lead to the contradiction that $\theta^-(s_0 + L) \leq \Theta$. The key observation in the proof is that

$$h'(g(t))g'(t) = \theta'(t) < -\sin(\theta(t)) = -\sin(h(g(t))) = h'(g(t))$$

where the last equality follows from equation (4.4). Since $h'(g(t)) < 0$, we conclude that $g'(t) > 1$ for all $t \in (t_i, t_{i+1})$ and therefore, by the Mean Value Theorem, for all i we have

$$g^-(t_{i+1}) - g^+(t_i) > t_{i+1} - t_i. \tag{4.5}$$

Let $\{s_0 = t_j, t_{j+1}, \ldots, t_{j+m}\}$ be the bending points in the interval $[s_0, s_0 + L)$. For convenience, we redefine $t_{j+m+1} = s_0 + L$. Since $||\gamma||_L \leq G$, the total vertical jump in the region $[s_0, s_0 + L)$ is at most G, that is

$$\sum_{i=j}^{j+m} |\theta^+(t_i) - \theta^-(t_i)| \leq G.$$

Let

$$d = \min\{g^+(t) \mid t \in [s_0, s_0 + L)\}.$$

Notice that $d \in [c-L, c]$ since $\theta^+(t) \in [\Theta, \Theta+G]$ for all $t \in [s_0, s_0+L)$ by our choice of s_0 and T. We now break the proof into two cases on the values of d.

Case $d \in [-c, c]$**.** If $d \in [-c, c]$ then $g^+([s_0, s_0+L)) \subseteq [-c, c]$. Since $\theta^-(t) > \Theta$ on $(s_0, s_0+L]$, we have $g^-((s_0, s_0+L]) \subseteq [-c, c]$. Notice that since $h'(x) = -\sin(h(x))$ and h is decreasing, then for $x \in [-c, c]$,

$$h'(x) \leq h'(c) = -\frac{G}{L}.$$

Therefore, applying equation (4.5) and the Mean Value Theorem, we see that

$$\theta^-(t_{i+1}) - \theta^+(t_i) \leq h'(c)(g^-(t_{i+1}) - g^+(t_i)) = -\frac{G}{L}(g^-(t_{i+1}) - g^+(t_i)) < -\frac{G}{L}(t_{i+1} - t_i)$$

for all $i = j, \ldots, j+m$. Thus,

$$\begin{aligned} \theta^-(s_0+L) - \theta^-(s_0) &= \left(\sum_{i=j}^{j+m} \theta^+(t_i) - \theta^-(t_i)\right) + \left(\sum_{i=j}^{j+m} \theta^-(t_{i+1}) - \theta^+(t_i)\right) \\ &< \left(\sum_{i=j}^{j+m} |\theta^+(t_i) - \theta^-(t_i)|\right) - \left(\sum_{i=1}^{j+m} \frac{G}{L}(t_{i+1} - t_i)\right) \\ &\leq G - \frac{G}{L}\sum_{i=1}^{j+m}(t_{i+1} - t_i) = 0. \end{aligned}$$

Since $\theta^-(s_0) \leq \Theta$, this implies that $\theta^-(s_0+L) \leq \Theta$, which contradicts the choice of s_0.

Case II: $d \in [c-L, -c)$**.** If $d \in [c-L, -c)$, then for all $t \in [s_0, s_0+L)$, we have

$$|h'(g(t))| \geq |h'(d)|$$

Another application of the Mean Value Theorem gives

$$\theta^+(t_i) - \theta^-(t_{i+1}) \geq |h'(d)|(g^-(t_{i+1}) - g^+(t_i)) > |h'(d)|(t_{i+1} - t_i) \tag{4.6}$$

for all $i = j, \ldots, j+m$. Whenever $g^+(t_i) < g^-(t_i)$ for $i = j, \ldots, j+m$, we also obtain

$$\theta^+(t_i) - \theta^-(t_i) \geq |h'(d)|(g^-(t_i) - g^+(t_i)) > 0 \tag{4.7}$$

Notice that as g^+ is increasing on $[t_i, t_{i+1})$ for all i, there exists a largest $k \in \{j, \ldots, j+m\}$ with $g^+(t_k) = d$. By (4.6), we obtain

$$\sum_{i=j}^{k-1} \theta^+(t_i) - \theta^-(t_{i+1}) > |h'(d)|(t_k - s_0).$$

Since $\theta^+(t_k) = h(d)$ and $\theta^-(t_j) = \theta^-(s_0) \leq \Theta$,

$$\sum_{i=j}^{k} \theta^+(t_i) - \theta^-(t_i) > h(d) - \Theta + |h'(d)|(t_k - s_0).$$

Therefore, as the total jump on the interval $[s_0, s_0 + L)$ is at most G,

$$\sum_{i=k+1}^{j+m} \theta^+(t_i) - \theta^-(t_i) \leq G - (h(d) - \Theta) - |h'(d)|(t_k - s_0) = h(c-L) - h(d) - |h'(d)|(t_k - s_0).$$

Since $g^+(t_k) = d$ and $s_0 + L = t_{j+m+1}$,

$$g^-(s_0 + L) = d + \left(\sum_{i=k}^{j+m} g^-(t_{i+1}) - g^+(t_i) \right) - \left(\sum_{i=k+1}^{j+m} g^-(t_i) - g^+(t_i) \right).$$

Let $I = \{i \mid k+1 \leq i \leq j+m \text{ and } g^-(t_i) - g^+(t_i) > 0\}$. After dropping any terms in the right hand sum with indices not in I, we can applying inequalities (4.5) and (4.7) to obtain

$$\begin{aligned} g^-(s_0 + L) &> d + \left(\sum_{i=k}^{j+m} t_{i+1} - t_i \right) - \frac{1}{|h'(d)|} \left(\sum_{i \in I} \theta^+(t_i) - \theta^-(t_i) \right) \\ &> d + (s_0 + L - t_k) - \frac{1}{|h'(d)|} \big(h(c-L) - h(d) - |h'(d)|(t_k - s_0) \big) \\ &= d + L - \left(\frac{h(c-L) - h(d)}{|h'(d)|} \right). \end{aligned}$$

Since h' is negative and decreasing on the interval $[c - L, d]$, we can take the tangent line at d and observe that

$$h(c-L) \leq h(d) + h'(d)(c - L - d)$$

which implies that

$$\frac{h(c-L) - h(d)}{|h'(d)|} \leq d - c + L.$$

Therefore,

$$g^-(s_0 + L) > d + L + (d - c + L) = c$$

Since $\Theta = h(c)$, this implies that $\theta^-(s_0 + L) \leq \Theta$ contradicting the definition of s_0. This final contradiction completes the proof. $\square$

As a nearly immediate corollary, we obtain an embeddedness criterion for piecewise geodesics.

COROLLARY 4.4. *If $\gamma : [0, \infty) \to \mathbb{H}^3$ is a piecewise geodesic with a first bending point, and $||\gamma||_L \leq G(L)$ for some $L > 0$, then γ is an embedding.*

PROOF. If the corollary fails, then there exist $0 \leq a < b$ such that $\gamma(a) = \gamma(b)$. Let $\beta : [0, \infty) \to \mathbb{H}^3$ be given by $\beta(t) = \gamma(t+a)$. Then $||\beta||_L \leq G(L)$ and since a, b are separated by a finite number of bending points, β has a first bending point. By definition, there exists $t_i \in (0, b)$ such that β is geodesic on $[t_i, b]$. However, this implies that $\theta^+(t) = \pi$ on (t_i, b), a contradiction to Lemma 4.3 above. □

For a finite-leaved measured lamination μ on $\mathbb{H}^2$ and any geodesic ray $\alpha : [0, \infty) \to \mathbb{H}^2$, the curve $\gamma = P_\mu \circ \alpha$ is a piecewise geodesic and $||\gamma||_L \leq ||\mu||_L$ by definition. Since any two points in $\mathbb{H}^2$ can be joined by a geodesic ray, we immediately obtain an embeddedness criterion for pleated planes.

COROLLARY 4.5. *If μ is a finite-leaved measured lamination on $\mathbb{H}^2$ and $||\mu||_L \leq G(L)$ for some $L > 0$, then $P_\mu : \mathbb{H}^2 \to \mathbb{H}^3$ is an embedding.*

4.3. Uniformly Bi-Lipschitz Embeddings. We next prove that if $\gamma : \mathbb{R} \to \mathbb{H}^3$ is a piecewise geodesic and $||\gamma||_L < G(L)$, then γ is uniformly bi-Lipschitz. We note that since γ is 1-Lipschitz by definition, we only have to prove a lower bound K for the bi-Lipschitz constant depending only on L and $\|\mu\|_L$. This will immediately imply that if μ is a finite-leaved lamination on $\mathbb{H}^2$ and $||\mu_L|| < G(L)$, then P_μ is a K-bi-Lipschitz embedding.

PROPOSITION 4.6. *If $\gamma : \mathbb{R} \to \mathbb{H}^3$ is a piecewise geodesic such that*

$$||\gamma||_L < G(L),$$

then γ is K-bi-Lipschitz where K depends only on L and $||\gamma||_L$.

PROOF. We first set our notation. We may assume, without loss of generality, that 0 is not a bending point of γ. Let $t_0 = 0$ and index the bending points in $(0, \infty)$ by an interval of positive integers beginning with 1 and the bending points in $(-\infty, 0)$ by an interval of negative integers ending with -1. As before, ϕ_i will be the bending angle of γ at t_i.

The following lemma will allow us to reduce to the planar setting.

LEMMA 4.7. *There exists an embedded piecewise geodesic* $\alpha : \mathbb{R} \to \mathbb{H}^2$ *with the same bending points as* γ *such that*

(i) if the bending angle of α *at a bending point* t_i *is given by* ϕ_i', *then* $\phi_i' \leq \phi_i$,

(ii) $d(\alpha(0), \alpha(t)) = d(\gamma(0), \gamma(t))$ *for all* t, *and*

(iii) there exists a continuous non-decreasing $\Psi : \mathbb{R} \to (-\pi, \pi)$ *such that for* $t > 0$, $\Psi(t)$ *is the angle between* $\alpha([0, t_1])$ *and the geodesic joining* $\alpha(0)$ *to* $\alpha(t)$ *and for* $t < 0$, $\Psi(t)$ *is the angle between* $\alpha([-t_1, 0])$ *and the geodesic joining* $\alpha(0)$ *to* $\alpha(t)$.

PROOF. Let f_i be the geodesic arc from $\gamma(0)$ to $\gamma(t_i)$ and let T_i be the possibly degenerate hyperbolic triangle with vertices $\gamma(0)$, $\gamma(t_i)$, and $\gamma(t_{i+1})$ and edges f_i, $\gamma([t_i, t_{i+1}])$ and f_{i+1}. We construct α by first placing an isometric copy of T_0 in $\mathbb{H}^2$, so that f_1 is counterclockwise from f_0.We then iteratively place a copy of T_i counterclockwise from a copy of T_{i-1} along the image of f_i for all positive t_i. We then place a copy of T_{-1} in $\mathbb{H}^2$ so that the image of f_{-2} is clockwise from f_{-1}, T_{-1} and T_0 meet at the image of $\gamma(0)$, and the images of f_1 and f_{-1} lie in a geodesic. We then iteratively place a copy of T_{-i-1} clockwise from a copy of T_{i-1} along the image of f_i for all negative t_{-i}. See Figure 10.

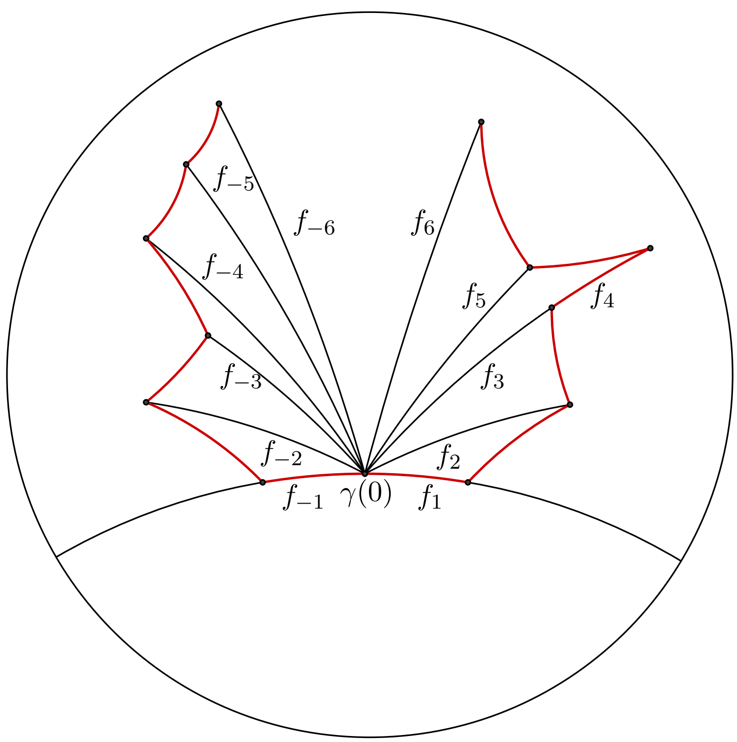

FIGURE 10. The construction of α, shown in red, for Lemma 4.7

Let $\alpha : \mathbb{R} \to \mathbb{H}^2$ be the piecewise geodesic traced out by the images of the pieces of γ. By construction, α has the same bending points as γ and, moreover, since $d(\alpha(0), \alpha(t))$ is realized in the isometric copy of T_i for $t \in [t_i, t_{i+1}]$, it is immediate that $d(\alpha(0), \alpha(t)) = d(\gamma(0), \gamma(t))$ for all t.

We next check that the bending angle ϕ_i' of α at t_i is at most ϕ_i. Note that the possibilities for gluing T_i to T_{i-1} in $\mathbb{H}^3$ are given by the one-parameter family of triangles obtained by rotating T_i about f_i. Consider the vectors $v_i^- = \gamma_-'(t_i)$ and $v_i^+ = \gamma_+'(t_i)$ at $\gamma(t_i)$. Then the exterior angle ϕ_i is the distance between v_i^- and v_i^+ in the unit tangent sphere at $\gamma(t_i)$. The edge f_i defines an axis r_i in this unit sphere and our one-parameter family corresponds to rotating v_i^+ around r_i. It is straightforward to see that that distance between v_i^- and v_i^+ is minimized when v_i^+, v_i^- and r_i are coplanar and the interiors of T_i and T_{i-1} are disjoint. See Figure 11. We conclude that, $\phi_i' \le \phi_i$. It follows that

$$||\alpha||_L \le ||\gamma||_L < G(L),$$

so Corollary 4.4 implies that α is an embedding.

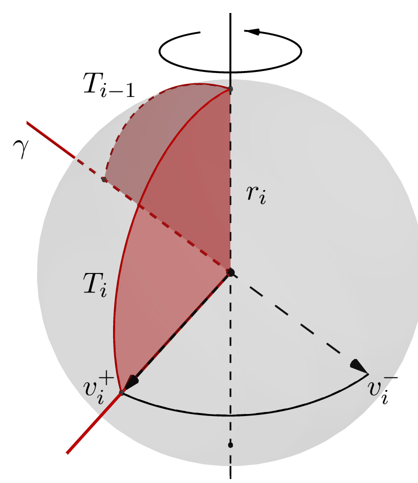

FIGURE 11. Configuration of triangles T_i, T_{i_1}, vectors $v_i^{\pm}$, and the axis r_i for the piecewise geodesic γ in Lemma 4.7

We can now define a continuous non-decreasing function $\Psi : \mathbb{R} \to \mathbb{R}$ with $\Psi(0) = 0$ and the property that for $t > 0$, $\Psi(t)$ mod 2π is the angle between $\alpha([0, t_1])$ and the geodesic joining $\alpha(0)$ to $\alpha(t)$ and for $t < 0$, $\Psi(t)$ mod 2π is the angle between $\alpha([-t_1, 0])$ and the geodesic joining $\alpha(0)$ to $\alpha(t)$.

To conclude (iii), we show that $\Psi(t) < \pi$ for all $t > 0$. If this fails, then γ intersects the geodesic g_0 containing $\alpha([0, t_1])$. Suppose that $\alpha(b) \in g_0$ for some $b > 0$. Then, consider the piecewise geodesic $\hat{\alpha}$ given by $\hat{\alpha}(t) = \alpha(b-t)$ for $t \in [0,b]$ and unit speed along g_0 in the direction of $\alpha(b)$ for $t > b$. Notice that $\hat{\alpha}$ is not an embedding. However,

$$||\hat{\alpha}||_L \leq ||\alpha||_L < G(L),$$

so Corollary 4.4 would imply otherwise. Therefore, $\hat{\alpha}$ cannot exist and $\Psi(t) < \pi$. One similarly argues that $\Psi(t) > -\pi$ for all $t < 0$, completing the proof of (iii). □

Our next goal, before we can address the bi-Lipschitz constants of α and γ, is to show that α is proper. Since Ψ is monotone and bounded we may define

$$\Psi_{+\infty} = \lim_{t\to\infty} \Psi(t) \quad \text{and} \quad \Psi_{-\infty} = \lim_{t\to-\infty} \Psi(t).$$

Lemma 4.8. *The piecewise geodesic $\alpha : \mathbb{R} \to \mathbb{H}^2$ constructed in Lemma 4.7 is proper.*

Proof. The basic idea is that by monotonicity of Ψ, $\alpha([0,\infty))$ can only accumulate on the geodesic ray ρ_+ emanating from $\alpha(0)$ and making angle Ψ_+ with $\alpha([0,t_1])$. If it has an accumulation point $q \in \mathbb{H}^2$, then there must be infinitely many segments of α running nearly parallel to ρ_+ and $q \in \rho_+$. However, Lemma 4.3 tell us that no segment of α can be "pointing" nearly straight back to $\alpha(0)$. In particular, the total length of these segments which are "pointing" towards $\alpha(0)$ is finite. This will allow us to arrive at a contradiction.

Fix $L > 0$ and assume that α is not proper on the ray $\alpha|_{[0,\infty)}$. Recall that if t is not a bending point, then $\theta(t)$ is the angle between $\alpha'(t)$ and the geodesic segment joining $\alpha(0)$ to $\alpha(t)$. Lemma 4.3 implies that for all $t > 0$

$$\theta^+(t) \leq \Theta_0 = \Theta(L) + G(L) < \pi.$$

Since $\alpha|_{[0,\infty)}$ is not proper, there is an accumulation point q of $\alpha|_{[0,\infty)}$ on the ray ρ_+ emanating from $\alpha(0)$ which makes an angle $\Phi_{+\infty}$ with $\alpha([0,t_1])$.

Working in the disk model, we let $\alpha(0) = 0$ and $\alpha([0,t_1])$ lie on the positive real axis. For small $\epsilon > 0$, we consider the region B_ϵ given in hyperbolic polar coordinates (r,φ) at 0 by

$$B_\epsilon = [r(q)-\epsilon, r(q)+\epsilon] \times [\varphi(q)-\epsilon, \varphi(q)] \subset \mathbb{D}^2.$$

A standard computation in these coordinates shows that the hyperbolic metric on B_ϵ given by $ds^2 = dr^2 + \sinh^2(r)d\varphi^2$. On B_ϵ, we also consider the taxicab metric given by

$d_T((r_1,\varphi_1),(r_2,\varphi_2)) = |r_1 - r_2| + |\varphi_1 - \varphi_2|$. It follows that d_T and $d_{\mathbb{H}}$ are bi-Lipshitz equivalent on B_ϵ. We will show that $\alpha([0,\infty)) \cap B_\epsilon$ has finite length in the taxi cab metric.

Let $J = \alpha^{-1}(B_\epsilon)$ and note that J is a countable collection of disjoint arcs with $\alpha(J) = \alpha([0,\infty)) \cap B_\epsilon$. Since Ψ is monotonic, the φ coordinate of α is also monotonic, and therefore, the total length of $\alpha(J)$ in the φ direction is bounded above by ϵ. In addition, since α accumulates on q, the signed length of $\alpha(J)$ in the r direction is bounded above by 2ϵ.

We will now use the fact that $\theta^+(t) \le \Theta_0$, to show that the length in the negative r-direction is bounded. Let $(\Psi(t), r_\alpha(t))$ parametrize α in polar coordinates over some $t \in [a,b) \subset J$ away from bending points. By the law of sines on the triangle with vertices $\alpha(0), \alpha(a), \alpha(t)$ for $t \in (a,b)$ and since α is unit speed, we have

$$\frac{\sin(\Psi(t) - \Psi(a))}{\sinh(t-a)} = \frac{\sin(\theta(t))}{\sinh(r_\alpha(a))}.$$

Taking the limit as $t \to a$, we obtain

$$\Psi'(a) = \frac{\sin(\theta(a))}{\sinh(r_\alpha(a))}.$$

Equation (4.2) then gives

$$\frac{dr_\alpha}{d\Psi}(\Psi(t))\,\Psi'(t) = r_\alpha'(t) = \cos(\theta(t)) \quad \Longrightarrow \quad \frac{dr_\alpha}{d\Psi}(\Psi(a)) = \cot(\theta(a))\sinh(r_\gamma(a)).$$

Since $0 < \theta(a) \le \Theta_0 < \pi$ and $r_\gamma(a) \le r(q) + \epsilon$, we have $\frac{dr_\alpha}{d\Psi}(\Psi(a)) \ge \cot(\Theta_0)\sinh(r(q)+\epsilon)$. Integrating over Ψ, we see that the total length of of $\alpha(J)$ in the negative r-direction is bounded by $\epsilon\,|\cot(\Theta_0)|\sinh(r(q)+\epsilon)$. Therefore, using the 2ϵ bound on the signed length, the total length of $\alpha(J)$ in the r-direction is bounded by $2\,\epsilon\,(1 + |\cot(\Theta_0)|\sinh(r(q)+\epsilon))$.

It follows that $\alpha(J)$ has finite length in the taxicab metric on B_ϵ. We can therefore choose $\bar{t} \in J$, so that $\alpha(J \cap [\bar{t},\infty))$ has length less than $\epsilon/4$ in the taxicab metric and $d_T(\alpha(\bar{t}), q) < \epsilon/4$. Therefore, $\alpha(J \cap [\bar{t},\infty)) \subset B_{\epsilon/2}(q)$ and $\overline{B_{\epsilon/2}(q))} \subset B_\epsilon$, where $B_{\epsilon/2}(q)$ is the neighborhood of radius $\epsilon/2$ of q in the taxicab metric on B_ϵ. This implies that $[\bar{t},\infty) \subset J$ and therefore $\alpha([\bar{t},\infty))$ has finite hyperbolic because d_T and $d_{\mathbb{H}}$ are bi-Lipshitz equivalent on B_ϵ. However, α is unit speed, so this is a contraction. Therefore, α must be proper. □

Returning to the proof of Proposition 4.6, we note that it suffices to show that there exists K, depending only on L and $\|\mu\|_L$, such that for all $t \in \mathbb{R}$,

$$r_\gamma(t) = d(\gamma(0), \gamma(t)) = d(\alpha(0), \alpha(t)) \ge K|t|.$$

Indeed, if $\gamma : \mathbb{R} \to \mathbb{H}^3$ is any piecewise geodesic with $||\gamma||_L < G(L)$ and $a < b$, then we can consider the new piecewise geodesic $\gamma_a : \mathbb{R} \to \mathbb{H}^3$ given by $\gamma_a(t) = \gamma(t+a)$. Then by construction $||\gamma_a||_L = ||\gamma||_L \leq G(L)$ and

$$r_{\gamma_a}(t) = d(\gamma_a(0), \gamma_a(t)) \geq K|t| \quad \Longrightarrow \quad d(\gamma(a), \gamma(b)) = r_{\gamma_a}(b-a) \geq K|b-a|.$$

Since γ is 1-Lipschitz by definition, it would follow that γ is a K-bi-Lipschitz embedding.

As we have show that α is proper and Ψ is monotone, α has two unique limit points ξ^- and ξ^+ in $\mathbb{S}^1$ which are endpoints of the geodesic rays from $\alpha(0)$ that make angles $\Psi_{-\infty}$ and $\Psi_{+\infty}$ with $\alpha([t_{-1}, t_1])$. Since α is embedded,

$$\Psi_{+\infty} - \Psi_{-\infty} \leq \pi.$$

In fact, this inequality is strict by the following argument. Let $B = (G(L) - ||\alpha||_L)/2$ and construct a new piecewise geodesic $\alpha_1 : \mathbb{R} \to \mathbb{H}^2$ which has a bend of angle B at 0. Then, by definition,

$$||\alpha_1||_L \leq ||\alpha||_L + B = ||\alpha||_L + (G(L) - ||\alpha||_L)/2 < G(L)$$

and by Corollary 4.4, α_1 is an embedding. Therefore,

$$\Psi_{+\infty} - \Psi_{-\infty} \leq \pi - B < \pi.$$

Let g be the geodesic joining ξ^- to ξ^+. By the above inequality, the visual distance between ξ^+ and ξ^-, as viewed from $\alpha(0)$ is at least B. It follows that there exists C, depending only on B, so that $d(\alpha(0), g)) \leq C$. In fact, one may apply [**Bea95**, Theorem 7.9.1] to choose

$$C = \cosh^{-1}\left(\frac{1}{\sin(B/2)}\right).$$

Notice that, by applying the above argument to $\alpha_t(s) = \alpha(s+t)$, we see that the visual distance between ξ^+ and ξ^- is at least B as viewed from $\alpha(t)$ for any $t \in \mathbb{R}$. Therefore, $\alpha(t)$ lies within C of g for any $t \in \mathbb{R}$.

We now claim that there exists $K > 0$ such that if $p : \mathbb{H}^2 \to g$ is the orthogonal projection, then $p \circ \alpha$ is a 1-Lipschitz and K-bi-Lipschitz orientation-preserving embedding. The fact that $p \circ \alpha$ is 1-Lipschitz follows immediately since both p and α are 1-Lipschitz. Let h_0 be the oriented orthogonal geodesic through $\alpha(0)$ toward g and let ν_0 be the angle between h_0

and the oriented geodesic segment $\alpha([t_{-1},t_1])$. Since $\Psi_{+\infty}-\Psi_{-\infty}\le \pi - B$, one has that

$$\frac{B}{2}\le \nu_0 \le \pi-\frac{B}{2}$$

and, therefore, restriction of $p\circ\alpha$ to $[t_{-1},t_1]$ is an orientation-preserving embedding. Let η_0 be a unit tangent vector at $\alpha(0)$ perpendicular to h_0. Then, since ρ is an projection,

$$||p'(\alpha(0))(\eta_0)|| = \frac{1}{\cosh(d(\alpha(0),g))}\ge \frac{1}{\cosh(C)} = \sin(B/2)$$

Since $B/2\le\nu_0\le\pi-B/2$, the projection of $\alpha'(0)$ onto η_0 has lengths at least $\sin(B/2)$, so

$$||(p\circ\alpha)'(0)||\ge \frac{\sin(B/2)}{\cosh(C)} = \sin^2(B/2) = \frac{1}{K}.$$

By reparameterizing $\alpha_t(s)=\alpha(s+t)$, we conclude that away from bending points, $p\circ\alpha$ is an orientation-preserving local homeomorphism and $||(p\circ\alpha)'(t)||\ge\frac{1}{K}$. Therefore, for all t,

$$d(p(\gamma(0)),p(\gamma(t))\ge\frac{t}{K}.$$

Lastly, since p is 1-Lipschitz (in fact, it decreases lengths),

$$r_\gamma(t) = d(\alpha(0),\alpha(t))\ge d(p(\gamma(0)),p(\gamma(t)))\ge\frac{t}{K}$$

Our previous remarks show that this is enough to guarantee that γ is K-bi-Lipschitz. □

As an immediate corollary, we obtain a version of Theorem 4.1 for finite-leaved laminations.

Corollary 4.9. *If μ is a finite-leaved measured lamination on $\mathbb{H}^2$ such that*

$$||\mu||_L < G(L),$$

then P_μ is a K-bi-Lipschitz embedding, where K depends only on L and $||\mu||_L$.

Proof of Theorem 4.1. Suppose that μ is a measured lamination on $\mathbb{H}^2$ with $||\mu||_L < G(L)$. By [**EMM06**, Lemma 4.6], there exists a sequence $\{\mu_n\}$ of finite-leaved measured laminations which converges to μ such that $||\mu_n||_L = ||\mu||_L$ for all n. Corollary 4.9 implies that each P_{μ_n} is a K-bi-Lipschitz embedding where K depends only on L and $||\mu||_L$. The maps $\{P_{\mu_n}\}$ converges uniformly on compact sets to P_μ (see [**EM87**, Theorem III.3.11.9]), so P_μ is also a K-bi-Lipschitz embedding. It follows that P_μ extends continuously to $\hat{P}_\mu : \mathbb{H}^2\cup\mathbb{S}^1_\infty\to\mathbb{H}^3\cup\mathbb{S}^2_\infty$ and $\hat{P}_\mu(\mathbb{S}^1)$ is a quasi-circle. □

5. Complex Earthquakes

In this section, we use Theorem 4.1 to give improved bounds in results of Epstein-Marden-Markovic which will lead to the improved bound obtained in Theorem 1.3. We first obtain new bounds guaranteeing that complex earthquakes extend to homeomorphisms at infinity, see Corollaries 5.1 and 5.2. Once we have done so, we obtain a generalization of [**EMM04**, Theorem 4.14] which produces a family of conformally natural quasiconformal maps associated to complex earthquakes with the same support μ which satisfy the bounds obtained in Corollary 5.1 or Corollary 5.2. Finally, we give a version of [**EMM06**, Theorem 4.3] which gives rise to a family of quasiregular maps associated to all complex earthquakes with positive bending along μ. Recall that a map $g = h \circ f$ is *quasiregular* if f is a quasiconformal homeomorphism and h is locally injective and holomorphic on the image of f.

The goal of building these families will be to construct a holomorphic map $\mathcal{F}$ from the largest possible domain in $\mathbb{C}$ into the universal Teichmüller space such that the image contains the quasisymmetric map associated of $r : \Omega \to \mathrm{Dome}(\Omega)$ and the identity map. The quasiconformal constant for the retraction map corresponds to the distance between these two points in universal Teichmüller space. The larger we can make the domain, the better the Poincare metric on the domain approximates Teichmüller distance.

If μ is a measured lamination on $\mathbb{H}^2$, we let $E_\mu : \mathbb{H}^2 \to \mathbb{H}^2$ to be the *earthquake map* defined by fixing a component of the complement of μ and left-shearing all other components by an amount given by the measure on μ. An earthquake map is continuous except on leaves of μ with discrete measure and extends to a homeomorphism of $\mathbb{S}^1$. In particular, any measured lamination λ on $\mathbb{H}^2$ is mapped to a well-defined measured lamination on $\mathbb{H}^2$, which we denote $E_\mu(\lambda)$.

Given a measured lamination μ on $\mathbb{H}^2$ and $z = x + iy \in \mathbb{C}$, we define the *complex earthquake*

$$\mathbb{C}E_z = P_{yE_{x\mu}} \circ E_{x\mu} : \mathbb{H}^2 \to \mathbb{H}^3$$

to be the composition of earthquaking along $x\mu$ and then bending along the lamination $yE_{x\mu}(\mu)$. The sign of y determines the direction of the bending. By linearity,

$$||yE_{x\mu}(\mu)||_L = |y| \; ||E_{x\mu}(\mu)||_L.$$

See Epstein-Marden [**EM87**, Chapter 3] or Epstein-Marden-Markovic [**EMM04**, Section 3] for a detailed discussion of complex earthquakes.

The following estimate allows one to bound $||E_{x\mu}(\mu)||_L$.

THEOREM 5.1. (Epstein-Marden-Markovic [**EMM04**, Theorem 4.12]) *Let ℓ_1 and ℓ_2 be distinct leaves of a measured lamination μ on $\mathbb{H}^2$. Suppose that α is a closed geodesic segment with endpoints on ℓ_1 and ℓ_2 and let $x = i(\alpha, \mu)$. Let ℓ_1' and ℓ_2' be the images of ℓ_1 and ℓ_2 under the earthquake E_μ. Then*

$$\sinh(d(\ell_1', \ell_2')) \leq e^x \sinh(d(\ell_1, \ell_2)) \quad \text{and} \qquad d(\ell_1', \ell_2') \leq e^{x/2} d(\ell_1, \ell_2).$$

Furthermore,

$$\sinh(d(\ell_1, \ell_2)) \leq e^x \sinh(d(\ell_1', \ell_2')) \quad \text{and} \qquad d(\ell_1, \ell_2) \leq e^{x/2} d(\ell_1', \ell_2').$$

Motivated by this result, Epstein, Marden, and Markovic define the function

$$f(L, x) = \min\left(Le^{|x|/2}, \sinh^{-1}(e^{|x|}\sinh(L))\right). \tag{5.1}$$

Corollary 4.13 in [**EMM04**] to Theorem 5.1 generalizes to give:

COROLLARY 5.1. *If μ is a measured lamination on $\mathbb{H}^2$, $z = x + iy \in \mathbb{C}$, and $L > 0$, then*

$$||E_{x\mu}(\mu)||_L \leq \left\lceil \frac{f(L, x)}{L} \right\rceil ||\mu||_L.$$

Furthermore, if

$$|y| < \frac{G(L)}{\left\lceil \frac{f(L,x)}{L} \right\rceil ||\mu||_L},$$

then $\mathbb{C}E_z$ extends to an embedding of $\mathbb{S}^1$ into $\hat{\mathbb{C}}$.

We similarly define

$$g(L, x) = \max\left(Le^{-|x|/2}, \sinh^{-1}(e^{-|x|}\sinh(L))\right) \tag{5.2}$$

and combine Theorem 5.1 and Theorem 4.1 to obtain :

COROLLARY 5.2. *If μ is a measured lamination on $\mathbb{H}^2$, $z = x + iy \in \mathbb{C}$, and $L > 0$, then*

$$||E_{x\mu}(\mu)||_{g(L,x)} \leq ||\mu||_L.$$

Furthermore, if

$$|y| < \frac{G(g(L,x))}{||\mu||_L},$$

then $P_{yE_{x\mu}}$ is a bi-Lipschitz embedding and $\mathbb{C}E_z$ extends to an embedding of $\mathbb{S}^1$ into $\hat{\mathbb{C}}$.

Note, we will show later that if $2\tanh(L) > L$ then $g(L,x) = Le^{-|x|/2}$. See Lemma 7.1.

Proofs. The proofs of Corollaries 5.1 and 5.2 both follow the same outline as the proof of [**EMM04**, Corollary 4.13].

Let μ be a measured lamination on $\mathbb{H}^2$, $z = x + iy \in \mathbb{C}$, and fix $L > 0$. Suppose that $A > 0$ and that α is an open geodesic arc in $\mathbb{H}^2$ of length A which is transverse to $E_{x\mu}(\mu)$. Theorem 5.1 guarantees that one can choose an open geodesic arc β in $\mathbb{H}^2$ of total length at most $f(A,x)$ which intersects exactly the leaves of ℓ of μ for which $E_{x\mu}(\ell)$ intersects α. By construction,

$$i(\alpha, E_{x\mu}(\mu)) = i(\beta,\mu) \leq ||\mu||_{f(A,x)},$$

and therefore,

$$||E_{x\mu}(\mu)||_A \leq ||\mu||_{f(A,x)}. \tag{5.3}$$

For the proof of Corollary 5.2, inequality 5.3 immediately implies that

$$||E_{x\mu}(\mu)||_{g(L,x)} \leq ||\mu||_{f(g(L,x),x)} = ||\mu||_L.$$

Thus, by linearity,

$$|y| < \frac{G(g(L,x))}{||\mu||_L} \quad \Longrightarrow \quad ||y\ E_{x\mu}||_{g(L,x)} < G(g(L,x)).$$

Theorem 4.1 then implies that $P_{yE_{x\mu}}$ is a bi-Lipschitz embedding which extends to an embedding of $\mathbb{S}^1$ into $\hat{\mathbb{C}}$. Since $E_{x\mu}$ extends to a homeomorphism of $\mathbb{S}^1$, it follows that $\mathbb{C}E_z$ extends to an embedding of $\mathbb{S}^1$ into $\hat{\mathbb{C}}$. This completes the proof of Corollary 5.2.

We now turn to the proof of Corollary 5.1. If we subdivide a half open geodesic arc in $\mathbb{H}^2$ of length $f(L,x)$ into $\lceil f(L,x)/L \rceil$ half open geodesic arcs of length less than or equal to L, then (5.3) implies

$$||E_{x\mu}(\mu)||_L \leq ||\mu||_{f(L,x)} \leq \left\lceil \frac{f(L,x)}{L} \right\rceil ||\mu||_L.$$

Therefore, linearity again gives

$$|y| < \frac{G(L)}{\left\lceil \frac{f(L,x)}{L} \right\rceil ||\mu||_L} \quad \Longrightarrow \quad ||y \; E_{x\mu}(\mu)||_L < G(L).$$

and we may again use Theorem 4.1 to complete the proof of Corollary 5.1. □

For all $L > 0$, define

$$Q(L,x) = \max\left(\frac{G(L)}{\left\lceil \frac{f(L,x)}{L} \right\rceil}, G(g(L,x)) \right) \tag{5.4}$$

and

$$\mathcal{T}_0^L = \operatorname{int}(\{x + iy \mid |y| < Q(L,x)\}\,. \tag{5.5}$$

The following theorem is a direct generalization of Theorem 4.14 in Epstein-Marden-Markovic [**EMM04**]. In its proof, we simply replace their use of Corollary 4.13 in [**EMM04**] with our Corollaries 5.1 and 5.2.

THEOREM 5.2. *Suppose that $L > 0$ and μ is a measured lamination on $\mathbb{H}^2$ such that $||\mu||_L = 1$. Then, for $t \in \mathcal{T}_0^L$,*

(i) $\mathbb{C}E_t$ extends to an embedding $\phi_t : \mathbb{S}^1 \to \hat{\mathbb{C}}$ which bounds a region Ω_t.

(ii) There is a quasiconformal map $\Phi_t : \mathbb{D}^2 \to \Omega_t$ with domain the unit disk and quasiconformal dilatation K_t bounded by

$$K_t \le \frac{1 + |h(z)|}{1 - |h(z)|}$$

where $h : \mathcal{T}_0^L \to \mathbb{D}^2$ is a Riemann map with $h(0) = 0$. Moreover, the map $\Phi_t \cup \phi_t : \mathbb{D}^2 \cup \mathbb{S}^1 \to \hat{\mathbb{C}}$ is continuous.

(iii) If G is a group of Möbius transformations preserving μ, then Φ_t can be chosen so that there is a homomorphism $\rho_t : G \to G_t$ where G_t is also a group of Möbius transformations and

$$\Phi_t \circ g = \rho_t(g) \circ \Phi_t$$

for all $g \in G$.

In order to extend the family of quasisymmetric maps that arise from Φ_t to a larger domain, Epstein, Marden and Markovic introduce the theory of complex angle scaling maps and use them to produce a quasiregular family $\Psi_t \circ \Phi_{t_0}$ which agree on $\mathbb{S}$ with Φ_t for $t \in \mathcal{T}^L$. Given Theorems 4.1 and 5.2, their proof of this extension follows immediately:

THEOREM 5.3. ([**EMM06**, Theorem 4.13]) *Suppose that $L > 0$, μ is a measured lamination on $\mathbb{H}^2$ with $||\mu||_L = 1$, $v_0 > 0$ and $t_0 = iv_0 \in \mathcal{T}_0^L$. If $t \in \mathcal{T}_0^L$, let Ω_t be the the image of $\mathbb{D}^2$ under the map Φ_t given by Theorem 5.2. Then there exists a continuous map $\Psi : \mathbb{U}^2 \times \Omega_{t_0} \to \hat{\mathbb{C}}$, such that*

(i) $\Psi_{t_0} = id$.

(ii) For each $z \in \Omega_{t_0}$, $\Psi(t, z)$ depends holomorphically on t.

(iii) For each $t \in \mathcal{T}_0^L \cap \mathbb{U}^2$, Ψ_t can be continuously extended to $\partial\Omega_{t_0}$ such that

$$\Psi_t \circ \Phi_{t_0}|_{\mathbb{S}^1} = \Phi_t|_{\mathbb{S}^1} .$$

In particular $\Psi_0 : \partial\Omega_{t_0} \to \mathbb{S}^1$ and $\Phi_{t_0} : \mathbb{S}^1 \to \partial\Omega_0$ are inverse homeomorphisms.

(iv) If $t \in \mathcal{T}_0^L \cap \mathbb{U}^2$, then Ψ_t is injective and $\Psi_t(\Omega_0) = \Phi_t(\mathbb{D}^2) = \Omega_t$.

(v) If $t = u + iv \in \mathbb{U}^2$, then Ψ_t is locally injective K_t-quasiregular mapping where

$$K_t = \frac{1 + |\kappa(t)|}{1 - |\kappa(t)|}, \qquad |\kappa(t)| = \frac{\sqrt{u^2 + (v - v_0)^2}}{\sqrt{u^2 + (v + v_0)^2}}$$

(vi) If G is a group of Möbius transformations preserving Ω_0, then there is a homomorphism $\rho_t : G \to G_t$ where G_t is also a group of Möbius transformations, such that

$$\Psi_t \circ g = \rho_t(g) \circ \Psi_t$$

for all $g \in G$.

6. Quasiconfomal Bounds

By combining their version of Theorem 5.2 and 5.3, Epstein, Marden and Markovic [**EMM06**] produce a family of quasiregular mappings indexed by

$$\mathcal{S}^L = \text{int}\left\{x + iy \in \mathbb{C} \mid y > -\frac{0.73}{f(1, x)}\right\}$$

such that if $|\text{Im}(t)| < \frac{0.73}{f(1,x)}$, then Φ_t is quasiconformal. We consider the enlarged region

$$\mathcal{T}^L = \mathcal{T}_0^L \cup \mathbb{U}^2 = \text{int}\left\{x + iy \in \mathbb{C} \mid y > -Q(x, L)\right\}.$$

One can now readily adapt the techniques of the proof of Epstein-Marden-Markovic [**EMM06**, Theorem 6.11] to establish:

THEOREM 6.1. *If Ω is a simply connected hyperbolic domain in $\hat{\mathbb{C}}$ and $L > 0$, then there is a conformally natural K-quasiconformal map $f : \Omega \to \mathrm{Dome}(\Omega)$ which extends to the identity on $\partial\Omega \subset \hat{\mathbb{C}}$ such that*

$$\log(K) \le d_{\mathcal{T}^L}(ic_1(L), 0)$$

where $d_{\mathcal{T}^L}$ is the Poincaré metric on the domain $\mathcal{T}^L$ and $c_1(L) = 2\cos^{-1}\left(-\sinh\left(\frac{L}{2}\right)\right)$.

We offer a brief sketch of the proof in order to indicate where our new bounds, as given in Theorems 1.4, 5.2 and 5.3, are used in the argument.

We recall that universal Teichmüller space $\mathcal{U}$ is the space of quasisymmetric homeomorphisms of the unit ciricle $\mathbb{S}^1$, modulo the action of Möbius transformations by post-composition (see, for example, Ahlfors [**Ahl66**, Chapter VI]). The Teichmüller metric on the space $\mathcal{U}$ is defined by

$$d_{\mathcal{U}}(f, g) = \log \inf K(\hat{f}^{-1} \circ \hat{g})$$

where the infimum is over all quasiconformal extensions $\hat{f}$ and $\hat{g}$ of f and g to maps from the unit disk to itself and $K(\hat{f}^{-1} \circ \hat{g})$ is the quasiconformal dilatation of $\hat{f}^{-1} \circ \hat{g}$. If Γ is a group of conformal automorphisms of $\mathbb{D}^2$, we define $\mathcal{U}(\Gamma) \subseteq \mathcal{U}$ to be the quasisymmetric homeomorphisms which conjugate the action of Γ to the action of an isomorphic group of conformal automorphisms. The Teichmüller metric on $\mathcal{U}(\Gamma)$ is defined similarly by considering extensions which conjugate Γ to a group of conformal automorphisms.

Let $g : \mathbb{D}^2 \to \hat{\mathbb{C}}$ be a locally injective quasiregular map, i.e. $g = h \circ f$ where f is a quasiconformal homeomorphism and h is locally injective and holomorphic on the image of f. We may define a complex structure C_g on $\mathbb{D}^2$ by pulling back the complex structure on $\hat{\mathbb{C}}$ via g. The identity map defines a quasiconformal homeomorphism $\hat{g} : \mathbb{D}^2 \to C_g$. We then uniformize C_g by a conformal map $R : C_g \to \mathbb{D}^2$ and consider the quasiconformal map $R \circ \hat{g} : \mathbb{D}^2 \to \mathbb{D}^2$. This map extends to the boundary to give a quasisymmetric map $qs(g) : \mathbb{S}^1 \to \mathbb{S}^1$.

Fix $L \in (0, 2\sinh^{-1}(1)]$ and choose μ so that $\mathrm{Dome}(\Omega) = P_{c\mu}(\mathbb{D}^2)$ where $||\mu||_L = 1$ and $c > 0$. We use Theorem 5.2 to define a map

$$\mathcal{F} : \mathcal{T}_0^L \to \mathcal{U}(\Gamma) \quad \text{by} \quad \mathcal{F}(t) = qs(\Phi_t) = (R_t \circ \Phi_t)|_{\mathbb{S}^1}$$

where Γ is the group of conformal automorphisms of $\mathbb{H}^2$ preserving μ and $R_t : \Omega_t \to \mathbb{D}^2$ is a uniformization of Ω_t.

Similarly, we may use Theorem 5.3, with some choice of $t_0 = iv_0 \in \mathcal{T}_0^L$, to define a map

$$\mathcal{G} : \mathbb{U} \to \mathcal{U}(\Gamma) \quad \text{by } \mathcal{G}(t) = qs(\Psi_t \circ \Phi_{t_0}).$$

If t lies in the intersection of the domains of $\mathcal{F}$ and $\mathcal{G}$, then even though Φ_t and $\Psi_t \circ \Phi_{t_0}$ need not agree on $\mathbb{D}^2$, Theorem 5.3 implies that they have the same boundary values and quasi-disk image Ω_t. Therefore F and G agree on the overlap $\mathcal{T}_0^L \cap \mathbb{U}$ of their domains. We may combine the functions to obtain a well-defined function

$$\bar{\mathcal{F}} : \mathcal{T}^L \to \mathcal{U}(\Gamma).$$

Theorem [**EMM06**, Theorem 6.5 and Proposition 6.9] further shows that $\bar{\mathcal{F}}$ is holomorphic.

The Kobayashi metric on a complex manifold M is defined to be the largest metric on M with the property that for any holomorphic map $f : \mathbb{D}^2 \to M$, f is 1-Lipschitz with respect the hyperbolic metric on $\mathbb{D}^2$. Therefore, holomorphic maps between complex manifolds are 1-Lipschitz with respect to their Kobayashi metrics. The Kobayashi metric on $\mathcal{U}$ and $\mathcal{U}(\Gamma)$ turns out to be equivalent to the Teichmüller metric we describer earlier (see [**GL99**, Chapter 7]). Moreover, the Poincaré metric on any simply connected domain, in particular $\mathcal{T}^L$, agrees with its Kobayashi metric. It follows then that for any $t \in \mathcal{T}^L$,

$$d_{\mathcal{U}(\Gamma)}(\bar{\mathcal{F}}(t), \bar{\mathcal{F}}(0)) \leq d_{\mathcal{T}^L}(t, 0).$$

Since we normalized $\mathrm{Dome}(\Omega) = P_{c\mu}(\mathbb{D}^2)$ and $||\mu||_L = 1$, Theorem 1.4, our upper bound on L-roundedness, implies that

$$c \leq c_1(L) = 2\cos^{-1}\left(-\sinh\left(\frac{L}{2}\right)\right).$$

Therefore,

$$d_{\mathcal{U}(\Gamma)}(\bar{F}(ic), \bar{F}(0)) \leq d_{\mathcal{T}^L}(ic, 0) \leq d_{\mathcal{T}^L}(ic_1(L), 0).$$

Since $\mathbb{C}E_{ic} = P_{c\mu}$ and $\Omega = \Omega_{ic}$ is simply connected, the map $g_{ic} = \Psi_{ic} \circ \Phi_{t_0}$ is a conformally natural quasiconformal mapping with image Ω. Moreover, $P_{c\mu} \circ g_{ic}^{-1} : \Omega \to \text{Dome}(\Omega)$ extends to the identity on $\partial\Omega = \partial\text{Dome}(\Omega)$. For more details, see the discussion in the proofs of [**EMM06**, Theorem 6.11] or [**BC13**, Theorem 1.1].

We have that $\bar{\mathcal{F}}(ic) = qs(g_{ic}) = (R \circ g_{ic})|_{\mathbb{S}^1}$ where $R : \Omega \to \mathbb{D}^2$ is a uniformization map. Therefore,

$$d_{\mathcal{U}(\Gamma)}(\bar{\mathcal{F}}(ic), \bar{\mathcal{F}}(0)) = d_{\mathcal{U}(\Gamma)}(\bar{\mathcal{F}}(ic), Id) = \log \inf K(h)$$

where the infimum is taken over all quasiconformal maps from $\mathbb{D}^2$ to $\mathbb{D}^2$ extending $(R \circ g_{ic})|_{\mathbb{S}^1}$ and conjugating Γ to a group of conformal automorphisms. By basic compactness results for families of quasiconformal maps, this infimal quasiconformal dilatation is achieved by a quasiconformal map $h : \mathbb{D}^2 \to \mathbb{D}^2$. If $f : \Omega \to \mathbb{D}^2$ is given by $f = h^{-1} \circ R$, then

$$K(f) = K(h) = d_{\mathcal{U}(\Gamma)}(\bar{F}(ic), \bar{F}(0)) \leq d_{\mathcal{T}^L}(ic_1(L), 0).$$

Since h and $R \circ g_{ic}$ are quasiconformal maps with the same extension to $\partial\mathbb{H}^2$, they are boundedly homotopic (see, e.g., [**EMM06**, Lemma 5.10]). Therefore, f is boundedly homotopic to g_{ic}^{-1} and $P_{c\mu} \circ f : \Omega \to \text{Dome}(\Omega)$ is boundedly homotopic to $P_{c\mu} \circ g_{ic}^{-1}$. Since $P_{c\mu} \circ g_{ic}^{-1}$ extends to the identity on $\partial\Omega$, it follows that $P_{c\mu} \circ f$ also extends to the identity on $\partial\Omega$. Therefore, $P_{c\mu} \circ f : \Omega \to \text{Dome}(\Omega)$ is the desired conformally natural K-quasiconformal map which extends to the identity on $\partial\Omega$ such that

$$\log(K) \leq d_{\mathcal{T}^L}(ic_1(L), 0).$$

This completes the sketch of the proof of Theorem 6.1.

Remark: Epstein, Marden and Markovic showed that if Ω is simply connected, then a quasiconformal map between Ω and $\text{Dome}(\Omega)$ extends to the identity on $\partial\Omega$ if and only if it is boundedly homotopic to the nearest point retraction from Ω to $\text{Dome}(\Omega)$. See [**EMM06**, Theorem 5.9].

7. Derivation of Numeric Bound

In order to complete the proof of Theorem 1.3, it suffices to show that one can choose $L \in (0, 2\sinh^{-1}(1)$ such that

$$d_{\mathcal{T}^L}(ic_1(L), 0) < 7.1695.$$

Motivated by computer calculations for various values of L, we choose $L = 1.48$.

We will construct an inscribed polygonal approximation $\mathcal{T}^L_{pol}$ for the region $\mathcal{T}^L$ and then use an implementation of the Schwarz-Christoffel formula to approximate the Poincare distance. The approximation is constructed using MATLAB's Symbolic Math Toolbox and variable

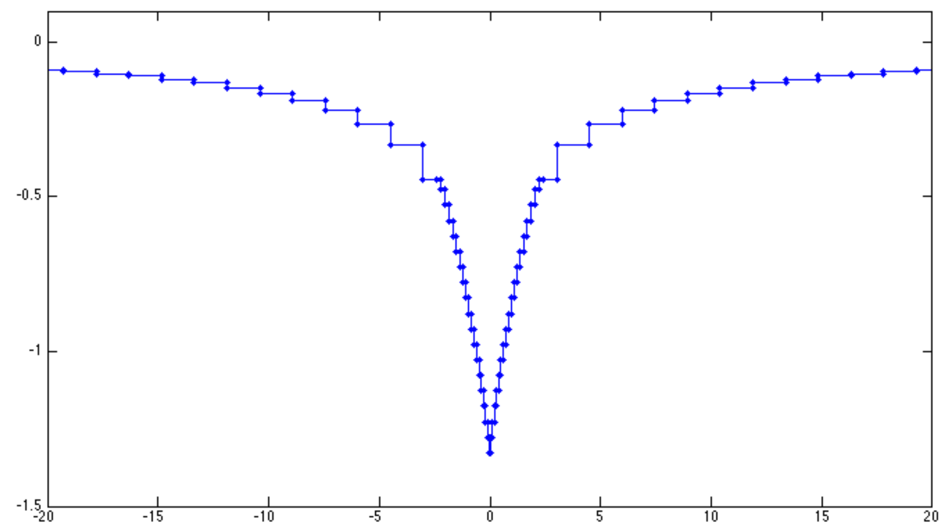

FIGURE 12. Polygonal approximation $\mathcal{T}^L_{pol}$ of $\mathcal{T}^L$

precision arithmetic. Variable precision arithmetic allows us to compute vertex positions to arbitrary precision. In particular, we can deduce sign changes to find intervals containing intersection points.

Our polygonal region will be bounded by a step function $Step(x) \leq Q(L, x)$, as seen in Figure 12. Let us recall that

$$Q(L, x) = \max\left(\frac{G(L)}{\left\lceil \frac{f(L,x)}{L} \right\rceil}, G(g(L, x))\right).$$

To construct $Step(x)$, we find all intervals where $\frac{G(L)}{\lceil f(L,x)/L \rceil}$ and $G(g(L, x))$ intersect in a desired range of $x \in [-a, a]$. For values where $\frac{G(L)}{\lceil f(L,x)/L \rceil}$ dominates, we bound $Q(L, x)$ by truncated decimal expansions (i.e. lower bounds) of values of $\frac{G(L)}{\lceil f(L,x)/L \rceil}$, which we compute using variable precision arithmetic.

For parts dominated by $G(g(L, x))$, we simplify our computation by using the following Lemma.

LEMMA 7.1. *Let $L_0 > 0$ be the unique positive solution to $2\tanh(L) = L$. If $L < L_0 \approx 1.91501$, then $2\tanh(L) > L$ and $g(L,x) = Le^{-|x|/2}$.*

PROOF. Recall that

$$g(L,x) = \max\left(Le^{-|x|/2}, \sinh^{-1}(e^{-|x|}\sinh L)\right).$$

Let $L < L_0$ and consider the function $j(x) = e^x \sinh(Le^{-x/2})$. It has a critical point precisely when

$$2\tanh(Le^{-x/2}) = Le^{-x/2}.$$

Since $L < L_0$, we have $Le^{-x/2} < L_0$ when $x \geq 0$, so j has no critical points in the interval $[0,\infty)$. Since $j'(0) = \sinh L - \frac{L}{2}\cosh L > 0$, j is increasing on the interval $[0,\infty)$. Therefore,

$$j(x) = e^x \sinh(Le^{-x/2}) \geq \sinh(L) = j(0)$$

for all $x \geq 0$, so

$$Le^{-x/2} \geq \sinh^{-1}(e^{-x}\sinh(L))$$

for all $x \geq 0$. Thus, $g(L,x) = Le^{-|x|/2}$ for all x. $\square$

From our initial analysis of the hill function h, we know that $G(t)$ is an increasing function on $t \in [0,\infty)$. It follows that $G(g(L,x))$ is a decreasing function for $x \in [0,\infty)$. Therefore, we can approximate $G(g(L,x))$ by a step function from below.

To compute the values of $G(g(L,x))$, recall that $G(t) = h(c(t)-t) - h(c(t))$. The function $c(t)$ can be computed to arbitrary precision from the equation

$$t\,h'(c(t)) = h(c(t)) - h(c(t)-t).$$

In particular, variable precision arithmetic can give us truncated decimal expansions of values of $G(g(L,x))$. We sample at a collection of points to obtain a step function where $G(g(L,x))$ dominates.

These computations give $Step(x) \leq Q(L,x)$ on some interval $[-a,a]$. Outside of that interval, we set $Step(x) = 0$. The graph of $-Step(x)$ gives bounds of our region $\mathcal{T}^L_{pol}$, which is properly in $\mathcal{T}^L$. Proper containment implies that the inclusion map $\mathcal{T}^L_{pol} \to \mathcal{T}^L$ is 1-Lipschitz in the Poincare metric.

```
L=1.48
G(L) = 1.327185362837166
HPL(0) = 0.000007509959438 + 0.009347547230674i
HPL(B) =  0.000009420062234 + 0.067016970686742i
H(L) = 1.969831901361628
K(L) = 7.169471208698489
```

FIGURE 13. Output of our program for computing a bound on the quasi-conformal constant in Sullivan's Theorem

Using the Schwarz-Christoffel mapping toolbox developed by Toby Driscoll [**Dri**], the images of the points 0 and $2\cos^{-1}\left(-\sinh\left(\frac{L}{2}\right)\right)i$ are computed under a Riemann mapping for $\mathcal{T}^L_{pol}$ to the upper half plane. Computing the hyperbolic distance between the images provides the result. The Schwarz-Christoffel mapping toolbox provides precision and error estimates. The error bounds are on the order of 10^{-5}.

We found that the optimal bound is given when L is approximately 1.48. Using $L = 1.48$, the point

$$B = c_1(L)i = 2\cos^{-1}\left(-\sinh\left(\frac{L}{2}\right)\right)i \approx 5.027888826784i$$

and

$$e^{d_{\mathcal{T}^L}(ic_1(L),0)} \approx 7.16947.$$

A truncated version of the output (Figure 13) provides the values of $G(L)$, $HPL(0)$, and $HPL(B)$, where $HPL : \mathcal{T}^L_{pol} \to \mathbb{H}^2$ is the Riemann mapping. We also have the computed values $H(L) = d_{\mathbb{H}^2}(HPL(0), HPL(B))$ and $K(L) = exp(d_{\mathcal{T}^L}(ic_1(L), 0)$.

CHAPTER 4

Basmajian's Identity for Hitchin Representations

1. Hitchin Representations

Let Σ be a connected compact oriented surface possibly with boundary and with negative Euler characteristic. A homomorphism $\rho : \pi_1(\Sigma) \to \mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$ is said to be *Fuchsian* if it is faithful with discrete image Γ such that $\mathrm{CH}(\Gamma)/\Gamma$ is compact (i.e. convex cocompact). Let $\iota\colon \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R})$ be a preferred representative arising from the unique irreducible representation of $\mathrm{SL}(2,\mathbb{R})$ into $\mathrm{SL}(n,\mathbb{R})$. An *$n$-Fuchsian* homomorphism is defined to be a homomorphism ρ that factors as $\rho = \iota \circ \rho_0$, where ρ_0 is Fuchsian.

Following the definition in [**LM09**], a *Hitchin* homomorphism from $\pi_1(\Sigma) \to \mathrm{PSL}(n,\mathbb{R})$ is one that may be deformed into an n-Fuchsian homomorphism such that the image of each boundary component stays purely loxodromic at each stage of the deformation. An element of $\mathrm{PSL}(n,\mathbb{R})$ is *purely loxodromic* if it has all real eigenvalues with multiplicity 1.

For the rest of the Chapter, we will let ρ denote the conjugacy class of a Hitchin homomorphism and refer to this class as a Hitchin representation.

2. Doubling a Hitchin Represenation

In this section, we will recall relevant details from the construction of Labourie and McShane on doubling of Hitchin representations. See [**LM09**, §9] for a complete discussion.

Let Σ be a connected compact oriented surface with boundary whose double $\widehat{\Sigma}$ has genus at least 2 and let ρ be an n-Hitchin representation of $\pi_1(\Sigma)$. There are two injections $\iota_0, \iota_1\colon \Sigma \to \widehat{\Sigma}$ and an involution $\iota : \widehat{\Sigma} \to \widehat{\Sigma}$ fixing all points on $\partial\Sigma$ such that $\iota \circ \iota_0 = \iota_1$. Fix a point $v \in \partial\Sigma$ and a primitive element $\partial_v \in \pi_1(\widehat{\Sigma}, v)$ corresponding to the boundary component of v. For $\gamma \in \pi_1(\widehat{\Sigma}, v)$, define $\bar{\gamma} = \iota_*(\gamma)$. One can choose $R : \pi_1(\Sigma, v) \to \mathrm{PSL}(n,\mathbb{R})$ in the conjugacy class of ρ with $R(\partial_v)$ a diagonal matrix with decreasing entries.

Such a representative is called a *good representative*. Define

$$J_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & & 1 \end{pmatrix},$$

then Corollary 9.2.2.4 of [**LM09**] constructs a Hitchin representation $\widehat{\rho}$ of $\pi_1(\widehat{\Sigma})$ whose restriction to $\pi_1(\Sigma)$ is ρ; furthermore, for any good representative R of ρ there exists $\widehat{R} : \pi_1(\widehat{\Sigma}, v) \to \mathrm{PSL}(n, \mathbb{R})$ in the conjugacy class of $\widehat{\rho}$ with

$$\widehat{R}(\bar{\gamma}) = J_n \cdot \widehat{R}(\gamma) \cdot J_n$$

for all $\gamma \in \pi_1(\widehat{\Sigma}, v)$. We refer to such a $\widehat{\rho}$ as the *Hitchin double of* ρ and we will refer to $\widehat{R}$, as constructed from R, as a *good representative of* $\widehat{\rho}$.

From this construction and [**Lab06**, Theorem 1.5], it follows that for a Hitchin representation ρ, the image $\rho(\gamma)$ of any nontrivial element of $\pi_1(\Sigma)$ is purely loxodromic. In particular, associated to a Hitchin representation ρ there is a length function ℓ_ρ defined by

$$\ell_\rho(\gamma) := \log \left| \frac{\lambda_{\max}(\rho(\gamma))}{\lambda_{\min}(\rho(\gamma))} \right|, \tag{2.1}$$

where $\lambda_{\max}(\rho(\gamma))$ and $\lambda_{\min}(\rho(\gamma))$ are the eigenvalues of maximum and minimum absolute value of $\rho(\gamma)$, respectively. Note that for a 2-Hitchin representation (i.e. a Fuchsian representation) this length function agrees with the hyperbolic length.

3. The Boundary at Infinity

Let Σ be a connected compact oriented surface with negative Euler characteristic and choose a finite area hyperbolic metric σ such that if $\partial\Sigma \neq \emptyset$, then $\partial\Sigma$ is totally geodesic. We can then identify the universal cover $\widetilde{\Sigma}$ of Σ with $\mathbb{H}^2$ if $\partial\Sigma = \emptyset$ or with a convex subset of $\mathbb{H}^2$ cut out by disjoint geodesics in the case that $\partial\Sigma \neq \emptyset$.

One defines the *boundary at infinity* $\partial_\infty(\Sigma)$ of $\pi_1(\Sigma)$ to be $\overline{\widetilde{\Sigma}} \cap \partial\overline{\mathbb{H}}^2$. With this definition, it makes sense to talk about Hölder functions on $\partial_\infty(\Sigma)$. Recall that a map $f : X \to Y$ between metric spaces is α-Hölder for $0 < \alpha \leq 1$, if there exists $C > 0$ such that,

$$d_Y(f(x), f(y)) \leq C\, d_X(x,y)^\alpha \quad \text{for all } x, y \in X$$

Clearly, Hölder functions are closed under composition, though the constant may change. For any two hyperbolic metrics σ_1, σ_2 on Σ, there exists a unique $\pi_1(\Sigma)$-equivariant quasisymmetric map $\partial_\infty(\Sigma, \sigma_1) \to \partial_\infty(\Sigma, \sigma_2)$ (see [**Ahl66**, IV.A]). This map is a Hölder homeomorphism (see [**GH02**, Lemma 1]) and therefore a Hölder map on $\partial_\infty(\Sigma)$ will remain so if we choose a different metric. Our definition of $\partial_\infty(\Sigma)$ topologically coincides with the Gromov boundary of a hyperbolic group (see [**BH13**, III.H.3]), however the Hölder structure is additional.

Note that if Σ is closed, then $\partial_\infty(\Sigma) \cong \mathbb{S}^1$. If Σ has boundary and a double of at least genus 2, then $\partial_\infty(\Sigma)$ is a Cantor set. Further, $\partial_\infty(\Sigma)$ is identified as a subset of $\mathbb{S}^1_\infty = \partial\overline{\mathbb{H}}^2$ and therefore admits a natural cyclic ordering from the orientation of Σ. For convention, we will view the ordering as counterclockwise.

We will use the notation $(x, y) \subset \partial_\infty(\Sigma)$ to denote the open set consisting of points z such that the tuple (x, z, y) is positively oriented. Note that $(y, x) \cap (x, y) = \emptyset$.

We say that a quadruple (x, y, z, t) is *cyclically ordered* if either (x, y, z), (y, z, t) and (z, t, x) are all positively or negatively oriented.

4. The Frenet Curve

Let $\mathscr{F}$ be the complete flag variety for $\mathbb{R}^n$, i.e. the space of all maximal sequences $V_1 \subset V_2 \ldots V_{n-1}$ of proper linear subspaces of $\mathbb{R}^n$. Consider a curve $\Xi\colon \mathbb{S}^1 \to \mathscr{F}$ with $\Xi = (\xi_1, \xi_2, \ldots, \xi_{n-1})$. We say that Ξ is a *Frenet curve* if

- for all sets of pairwise distinct points $(x_1, \ldots, x_l)$ in $\mathbb{S}^1$ and positive integers $d_1 + \cdots + d_l = d \leq n$,
$$\bigoplus_{i=1}^{l} \xi_{d_i}(x_i) = \mathbb{R}^d\,.$$
- for all x in $\mathbb{S}^1$ and positive integers $d_1 + \cdots + d_l = d \leq n$,
$$\lim_{\substack{(y_1,\ldots,y_l)\to x,\\ y_i \text{ all distinct}}} \left(\bigoplus_{i=1}^{i=l} \xi_{d_i}(y_i) \right) = \xi_d(x)\,.$$

We call $\xi = \xi_1$ the *limit curve* and $\theta = \xi_{n-1}$ the *osculating hyperplane.* The second property above guarantees that the image of ξ is a C^1-submanifold of $\mathbb{PR}^n$.

It turns out that given a Hitchin representation of a closed surface, one can construct an associated Frenet curve. As a set of points, this curve is the closure of the attracting fixed points of $\rho(\gamma)$ for all $\gamma \in \pi_1(S)$.

THEOREM 4.1. [**Lab06**, Theorem 1.4] *Let ρ be an n-Hitchin representation of the fundamental group of a closed connected oriented surface S of genus at least 2. Then there exists a ρ-equivariant Hölder Frenet curve on $\partial_\infty(S)$.*

The metric on $\mathscr{F}$ arrises from a choice of inner product on $\mathbb{R}^n$ and the associated embedding $\mathscr{F} \to \prod_{i=1}^{n-1} \mathbb{PR}^n$. In particular, we may use the usual spherical angle metric on $\operatorname{im} \xi_1$. Since ξ_2 is Hölder, we have the immediate Corollary.

COROLLARY 4.1. *If $\xi : \partial_\infty(S) \to \mathscr{F}$ is the Frenet curve associated to an n-Hitchin representation, then* $\operatorname{im}(\xi)$ *is a $C^{1+\alpha}$ submanifold of $\mathbb{PR}^n$.*

For a closed surface, let ξ_ρ and θ_ρ be the limit curve and osculating hyperplane associated to a Hitchin representation ρ, respectively. For a connected compact surface Σ with boundary and a Hitchin representation ρ, we define ξ_ρ to be the restriction of $\xi_{\widehat{\rho}}$ to $\pi_1(\Sigma)$, where $\widehat{\rho}$ is the Hitchin double of ρ.

5. Cross Ratios

In this section, following [**Lab08**], we construct the Hölder cross ratio on $\partial_\infty(\Sigma)$ associated to a Hitchin representation. Let

$$\partial_\infty(\Sigma)^{4*} = \{(x, y, z, t) \in \partial_\infty(\Sigma) \,|\, x \neq t \text{ and } y \neq z\}\,.$$

A *cross ratio* on $\partial_\infty(\Sigma)$ is a $\pi_1(\Sigma)$-invariant Hölder function $B\colon \partial_\infty(\Sigma)^{4*} \to \mathbb{R}$ satisfying:

$$B(x, y, z, t) = 0 \iff x = y \text{ or } z = t\,, \tag{5.1}$$

$$B(x, y, z, t) = 1 \iff x = z \text{ or } y = t\,, \tag{5.2}$$

$$B(x, y, z, t) = B(x, y, w, t)B(w, y, z, t)\,, \tag{5.3}$$

$$B(x, y, z, t) = B(x, y, z, w)B(x, w, z, t)\,. \tag{5.4}$$

In addition, the above conditions imply the following symmetries:

$$B(x,y,z,t) = B(z,t,x,y)\,, \tag{5.5}$$

$$B(x,y,z,t) = B(z,y,x,t)^{-1}\,, \tag{5.6}$$

$$B(x,y,z,t) = B(x,t,z,y)^{-1}\,. \tag{5.7}$$

The *period* of a nontrivial element γ of $\pi_1(\Sigma)$ with respect to B is

$$\ell_B(\gamma) := \log|B(\gamma^+, x, \gamma^-, \gamma x)| = \log|B(\gamma^-, \gamma x, \gamma^+, x)|\,,$$

where γ^+ (rest., γ^-) is the attracting (rest., repelling) fixed point of γ on $\partial_\infty(\Sigma)$ and x is any element of $\partial_\infty(\Sigma) \smallsetminus \{\gamma^+, \gamma^-\}$. This definition is independent of the choice of x.

A cross ratio B is said to be ordered, if in addition B satisfies

$$B(x,z,t,y) > 1\,, \tag{5.8}$$

$$B(x,y,z,t) < 0\,, \tag{5.9}$$

whenever the quadruple (x,y,z,t) is cyclically ordered.

This definition of the cross ratio is motivated by the classical cross ratio $B_{\mathbb{P}}$ on $\mathbb{RP}^1$ defined in an affine patch as

$$B_{\mathbb{P}}(x,y,z,t) = \frac{(x-y)(z-t)}{(x-t)(z-y)}\,. \tag{5.10}$$

Before we associate a cross ratio to a Hitchin representation, consider the following construction. If $L \subset \mathbb{RP}^n$ is a projective line, let $\mathbb{RP}^{n\,*}_V = \{Z \in \mathbb{RP}^{n*} : V \not\subset Z\}$ and let $\eta_V : \mathbb{RP}^{n\,*}_V \to \mathbb{RP}^n$ be given by $\eta_V(w) = w \cap V$. For points $p, q \in \mathbb{RP}^n$ with $V = p \oplus q$ and $r, s \in \mathbb{RP}^{n\,*}_V$, define

$$\mathfrak{B}(r,p,s,q) := B_V\left(\eta_V(r), p, \eta_V(s), q\right)\,,$$

where B_V is the classical cross ratio on V. Note that $\mathfrak{B}$ is a smooth function on its domain.

Let ρ be a Hitchin representation for Σ, a connected compact oriented surface with double of genus at least 2. We can then define B_ρ, *the cross ratio associated to* ρ, for a quadruple $(x,y,z,t) \in \partial_\infty(\Sigma)^{4*}$ by

$$B_\rho(x,y,z,t) := \mathfrak{B}\left(\theta_\rho(x), \xi_\rho(y), \theta_\rho(z), \xi_\rho(t)\right)\,. \tag{5.11}$$

By [**Lab06**, Theorem 1.4] and [**LM09**, Theorem 9.1], B_ρ is an ordered cross ratio. Furthermore,

$$\ell_{B_\rho}(\gamma) = \ell_\rho(\gamma)$$

for any nontrivial element γ of $\pi_1(\Sigma)$.

Remark. We should note that the cross ratio associated to a Hitchin representation ρ as defined here is referred to as B_{ρ^*} in [**Lab08**] and [**LM09**], where $\rho^*(\gamma) = \rho(\gamma^{-1})^t$. The cross ratio used in [**Lab08**] and [**LM09**] has $B_\rho(x, y, z, t) = B_{\rho^*}(y, x, t, z)$. Both cross ratios have all the same properties, as shown in [**Lab08**]. The choice to use this definition is a cosmetic one for the case of $\mathbb{RP}^2$-surfaces considered below.

6. Lebesgue Measure on the Frenet Curve

Let S be a closed surface and $\Sigma \subset S$ an incompressible connected subsurface. A complete hyperbolic structure on S gives an identification of $\partial_\infty(S)$ with $\mathbb{S}^1_\infty = \partial\mathbb{H}^2$. It is a classical result that under this identification $\partial_\infty(\Sigma)$ is measure 0 with respect to the Lebesgue measure on $\mathbb{S}^1_\infty$ (for instance, see [**Nic89**, Theorem 2.4.4]). The goal of this section is to show that this holds true with respect to the Lebesgue measure on the limit curve associated to a Hitchin representation.

For the entirety of this section, if ρ is a Hitchin representation of a surface with boundary, we will use R to denote a good representative. Further, we will assume that $\xi_\rho = \xi_R$.

LEMMA 6.1. *Let $\widehat{\rho}$ be the Hitchin double of $\rho : \pi_1(\Sigma) \to \mathrm{PSL}(n, \mathbb{R})$, then J_n preserves the limit curve $\xi_{\widehat{\rho}} \subset \mathbb{PR}^n$ associated to $\widehat{\rho}$.*

PROOF. Let $\xi = \xi_{\widehat{\rho}} = \xi_{\widehat{R}}$. Since the attracting fixed points of $\widehat{R}$ are dense in ξ, we will first show that J_n preserves the set of attracting fixed points.

Let $\gamma \in \pi_1(\widehat{\Sigma})$, then by equivariance, $\xi(\gamma^+)$ is the attracting fixed point of $\widehat{R}(\gamma)$. It follows that $J_n \cdot \xi(\gamma^+)$ is fixed by $J_n \cdot \widehat{R}(\gamma) \cdot J_n = \widehat{R}(\overline{\gamma})$. Recall that $\overline{\gamma}$ is the image of γ under the induced map of the canonical involution of $\widehat{\Sigma}$. Choose $x \notin \widehat{R}(\overline{\gamma})^\perp$ such that $y = J_n \cdot x \notin \widehat{R}(\gamma)^\perp$. Here, $\widehat{R}(\gamma)^\perp$ is the hyperplane spanned by the eigenvectors associated to the eigenvalues of non-maximal absolute value. We then have that

$$\lim_{k\to\infty} \left(\widehat{R}(\bar{\gamma})^k \cdot x \right) = \xi(\bar{\gamma}^+)$$

and also

$$\begin{aligned}\lim_{k\to\infty}\left(\widehat{R}(\bar\gamma)^k\cdot x\right) &= \lim_{k\to\infty}\left(J_n\cdot\widehat{R}(\gamma)^k\cdot J_n\cdot x\right)\\ &= J_n\cdot\left(\lim_{k\to\infty}\widehat{R}(\gamma)^k\cdot y\right)\\ &= J_n\cdot\xi(\gamma^+)\,.\end{aligned}$$

In particular, $\xi(\bar\gamma^+) = J_n\cdot\xi(\gamma^+)\in\xi$ is the attracting fixed point of $\widehat{R}(\bar\gamma)$. Now choose $z\in\xi$, then there exists a sequence $\{\gamma_j\}$ in $\pi_1(\widehat{\Sigma})$ such that

$$\lim_{j\to\infty}\xi(\gamma_j^+) = z\,.$$

Hence,

$$\lim_{j\to\infty}\left(J_n\cdot\xi(\gamma_j^+)\right) = J_n\cdot z$$

and as ξ is closed, we have $J_n\cdot z\in\xi$. Therefore J_n preserves ξ. $\square$

Definition 6.2. A finite positive measure μ on $\partial_\infty(S)$ is *quasi-invariant* if, for every $g\in\pi_1(S)$, the pushforward measure $g_*(\mu)$ is absolutely continuous with respect to μ. In addition, if the Radon-Nikodym derivative is Hölder, we say μ is *Hölder quasi-invariant.*

Let $\xi_\rho\colon\partial_\infty(S)\to\mathbb{PR}^n$ be the limit curve associated to an n-Hitchin representation ρ of $\pi_1(S)$. By Corollary 4.1, the image of ξ_ρ is a $C^{1+\alpha}$ submanifold, so we let $\eta_\rho\colon\mathbb{S}^1\to\operatorname{im}(\xi)$ be a C^1-parameterization with Hölder derivatives. W further assume that η_ρ is constant speed $\|\eta_\rho'\| = c_\rho$ (recall that $\mathbb{PR}^n$ carries the standard spherical metric). Let λ be the Lebesgue measure on $\mathbb{S}^1$ and define $\mu_\rho = (\xi_\rho^{-1}\circ\eta_\rho)_*\lambda$.

Lemma 6.3. *The measure μ_ρ is Hölder quasi-invariant.*

Proof. Fix $\gamma\in\pi_1(S)$ and let $A\subset\partial_\infty(S)$ be measurable. By definition,

$$\gamma_*\mu_\rho(A) = \mu_\rho(\gamma^{-1}A) = \lambda\left(\eta_\rho^{-1}\circ\xi\left(\gamma^{-1}A\right)\right) = \lambda\left(\eta_\rho^{-1}\circ\rho(\gamma^{-1})\circ\xi\left(A\right)\right).$$

Let $s_\gamma(t) = \eta_\rho^{-1}\circ\rho(\gamma^{-1})\circ\eta_\rho(t)$ then,

$$\gamma_*\mu_\rho(A) = \lambda(s_\gamma(\eta_\rho^{-1}\circ\xi(A))) = \int_{\eta_\rho^{-1}\circ\xi(A)} s_\gamma'\,d\lambda = \int_A s_\gamma'\circ\eta_\rho^{-1}\circ\xi\,d\mu_\rho. \tag{6.1}$$

Since η is constant speed and $\rho(\gamma^{-1})$ preserves $\operatorname{im}(\xi)$,

$$s_\gamma'(t) = \|D_{\rho(\gamma^{-1})}(\eta_\rho(t))\cdot\eta_\rho'(t)\|/c_\rho.$$

Because $D_{\rho(\gamma^{-1})}$ is continuously differentiable on $\mathrm{T}(\mathbb{PR}^n)$ (and therefore Hölder) and $\eta_\rho'(t)$ is Hölder by construction, it follows that $s_\gamma'(t)$ is as well. Additionally, since η_ρ is constant speed, the quantity

$$\sup_{p,q\in\mathbb{S}^2} \frac{d_{\mathbb{S}^1}(p,q)}{d_{\mathbb{PR}^n}(\eta_\rho(p),\eta_\rho(q))} < +\infty.$$

It follows that $s_\gamma' \circ \eta_\rho^{-1} \circ \xi$ is Hölder and therefore μ is a Hölder quasi-invariant measure with respect to the action of $\pi_1(S)$ on $\partial_\infty(S)$ by (6.1). □

This Lemma along an argument of Anosov, as cited by Ledrappier [**Led94**, Section e], tells us that that μ_ρ is $\pi_1(S)$-*ergodic*. In particular, if $A \subset \partial_\infty(S)$ is a $\pi_1(S)$-invariant set, then A has either null or full measure. We now apply this $\pi_1(S)$-ergodicity to obtain:

LEMMA 6.4. *For a compact bordered surface Σ with a double $\widehat{\Sigma}$ of genus at least 2, fix ρ a Hitchin representation of $\pi_1(\Sigma)$ and its Hitchin double $\hat{\rho}$. Then, viewing $\partial_\infty(\Sigma) \subset \partial_\infty(\widehat{\Sigma})$,*

$$\mu_{\hat{\rho}}(\partial_\infty(\Sigma)) = 0.$$

PROOF. Fix a basepoint in Σ and consider $\partial_\infty(\Sigma) \subset \partial_\infty(\widehat{\Sigma})$ via the natural inclusion $\pi_1(\Sigma) \to \pi_1(\widehat{\Sigma})$. Let $\xi = \xi_{\widehat{\rho}} = \xi_{\widehat{R}}$. Define

$$U = \bigcup_{g\in\pi_1(\widehat{\Sigma})} g \cdot \partial_\infty(\Sigma)\,.$$

As μ_ρ is ergodic, either $\mu_\rho(U) = 0$ or U has full measure. Let ι be the involution on $\partial_\infty(\widehat{\Sigma})$ defined by $\xi^{-1} \circ J_n \circ \xi$. Then,

$$U' = \iota(U)$$

is another $\pi_1(S)$-invariant set implying it either has null or full measure. Moreover, since $J_n|_{\mathrm{im}(\xi_\rho)}$ is C^1,

$$\mu_\rho(U) = 0 \iff \mu_\rho(U') = 0\,.$$

Notice that both $\mu_\rho(U)$ and $\mu_\rho(U')$ cannot be full measure, as $U \cap U'$ consists of the attracting and repelling fixed points of primitive peripheral elements and must be countable. Therefore, $0 = \mu_\rho(U) \geq \mu_{\hat{\rho}}(\partial_\infty(\Sigma))$ □

This measure property for the Hitchin double will be enough to prove the general case where $\Sigma \subset S$ is an incompressible surface. For this, make use of the Hausdorff measure in $\mathbb{R}^{n-1}$.

LEMMA 6.5 (Theorem 3.2.3 [**Fed69**]). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz function for $m \leq n$. If A is an λ^m (Lebesgue) measurable set, then*

$$\int_A J_m(f(x))\, d\lambda^m x = \int_{\mathbb{R}^n} N(f \mid A, y)\, d\mathcal{H}^m y$$

where $\mathcal{H}^m$ is the m-dimensional Hausdorff measure, $N(f \mid A, y) = \#\{x \in A \mid f(x) = y\}$, and $J_m(f(x)) = \sqrt{\det(Df^t \cdot Df)}(x)$.

THEOREM 2.2. *Let S be a closed surface and $\Sigma \subset S$ an incompressible subsurface. Let ρ be a Hitchin representation of S and ξ_ρ the associated limit curve. If μ_ρ is the pullback of the Lebesgue measure on the image of ξ_ρ, then $\mu_\rho(\partial_\infty(\Sigma)) = 0$.*

PROOF. By fixing a basepoint on Σ, there are natural inclusions $i : \pi_1(\Sigma) \to \pi_1(S)$ and $\hat{\imath} : \pi_1(\Sigma) \to \pi_1(\widehat{\Sigma})$ and the induced inclusions $i_*, \hat{\imath}_*$ on the boundaries at infinity. Let R be a representative of ρ such that $R \circ i$ is a good representative for $\rho|_{\pi_1(\Sigma)}$ and build $\hat{R}$ by doubling $R \circ i$.

Let $\xi = \xi_R$ and $\hat{\xi} = \xi_{\widehat{R}}$ be the limit curves associated to ρ and $\hat{\rho}$, respectively. For $\gamma \in \pi_1(\Sigma)$,

$$\xi \circ i_*(\gamma^+) = R(i(\gamma))^+ = \widehat{R}(\hat{\imath}(\gamma))^+ = \hat{\xi} \circ \hat{\imath}_*(\gamma^+)$$

as $R(i(\gamma)) = \widehat{R}(\hat{\imath}(\gamma))$. By the density of attracting fixed points in $\partial_\infty(\Sigma)$ we see that $\xi \circ i_* = \hat{\xi} \circ \hat{\imath}_*$. In particular, they have the same image $\Lambda_\Sigma = \xi \circ i_*(\partial_\infty(\Sigma)) = \hat{\xi} \circ \hat{\imath}_*(\partial_\infty(\Sigma))$.

Fix some affine chart $\mathbb{R}^{n-1}$ of $\mathbb{PR}$ containing Λ_Σ (and by convexity $\mathrm{im}(\xi)$ and $\mathrm{im}(\hat{\xi})$ as well). Let $\eta, \hat{\eta} : \partial_\infty(S) \to \mathbb{R}^{n-1}$ be the two $C^{1+\alpha}$ constant speed parametrization for $\mathrm{im}(\xi), \mathrm{im}(\hat{\xi})$ of constant speed $c_\rho, c_{\hat{\rho}}$, respectively. We apply Theorem 6.5 and Lemma 6.4. By construction, $J_{n-1}(\eta) = c_\rho$ and $J_{n-1}(\eta) = c_{\hat{\rho}}$ and therefore

$$\mu_\rho(i_*(\partial_\infty(\Sigma))) = \int_{\eta_\rho^{-1}(\Lambda_\Sigma)} c_\rho \, d\lambda = \mathcal{H}^1(\Lambda_\Sigma)$$

$$0 = \mu_{\hat{\rho}}(\hat{\imath}_*(\partial_\infty(\Sigma))) = \int_{\eta_{\hat{\rho}}^{-1}(\Lambda_\Sigma)} c_{\hat{\rho}} \, d\lambda = \mathcal{H}^1(\Lambda_\Sigma)$$

It follows that $\mu_\rho(i_*(\partial_\infty(\Sigma))) = 0$, as desired. $\square$

REMARK 6.6. Notice that we have show that the Hausdorff dimensions of Λ_Σ is ≤ 1. A questions of interest would be to understand the variation this quantity under deformations of ρ as one leaves the Fuchsian locus.

7. Orthogeodesics and Double Cosets

Let Σ be connected compact orientable surface with genus g and $m > 0$ boundary components such that the double of Σ has genus at least 2. Fix a finite volume hyperbolic metric σ on Σ such that $\partial\Sigma$ is totally geodesic. In particular, we can fix an identification of the universal cover U of Σ with a convex subset of $\mathbb{H}^2$ cutout by geodesics. This also gives an identification of $\pi_1(\Sigma)$ with a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^2)$.

An *orthogeodesic* in (Σ, σ) is an oriented properly embedded arc perpendicular to $\partial\Sigma$ at both endpoints. Denote the collection of orthogeodesics as $\mathcal{O}(\Sigma, \sigma)$. The *orthospectrum* is the multiset containing the lengths of orthogeodesics with multiplicity and is denoted by $|\mathcal{O}(\Sigma, \sigma)|$. Observe that every element of $|\mathcal{O}(\Sigma, \sigma)|$ appears at least twice as orthogeodesics are oriented. Also note that $\mathcal{O}(\Sigma, \sigma)$ is countable as the orthogeodesics correspond to a subset of the oriented closed geodesics in the double of (Σ, σ). Let $\ell_\sigma(\partial\Sigma)$ be the length of $\partial\Sigma$ in (Σ, σ), then recall Basmajian's identity [**Bas93**]

$$\ell_\sigma(\partial\Sigma) = \sum_{\ell \in |\mathcal{O}(\Sigma,\sigma)|} 2 \log \coth\left(\frac{\ell}{2}\right).$$

In order to extend this identity to the setting of Hitchin representations, we first need to replace the geometric object $\mathcal{O}(\Sigma, \sigma)$ with an algebraic object; this is the goal of this section.

Let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_m\} \subset \pi_1(\Sigma)$ be a collection of primitive elements representing the m components of $\partial\Sigma$ in $\pi_1(\Sigma)$ oriented such that the surface is to the left. We will call such a set $\mathcal{A}$ a *positive peripheral marking.* Set $H_i = \langle \alpha_i \rangle$ and, treating $\pi_1(\Sigma)$ as a subgroup of $\mathrm{PSL}(2, \mathbb{R})$, let $\widetilde{\alpha}_i \subset \mathbb{H}^2$ be the lift of α_i such that $H_i = \mathrm{Stab}(\widetilde{\alpha}_i)$.

Fix $1 \leq i, j \leq n$ (possibly $i = j$), then for $g \in \pi_1(\Sigma)$ (or $g \in \pi_1(\Sigma) \smallsetminus H_i$ if $i = j$) define the arc $\widetilde{\alpha}_{i,j}(g)$ to be the minimal length arc oriented from $\widetilde{\alpha}_i$ to $g \cdot \widetilde{\alpha}_j$. Now $\widetilde{\alpha}_{i,j}(g)$ descends to an orthogeodesic $\alpha_{i,j}(g)$ on (Σ, σ). For $i \neq j$, we denote the set of double cosets

$$\mathcal{O}_{i,j}(\Sigma, \mathcal{A}) = H_i \backslash \pi_1(\Sigma) / H_j = \{H_i g H_j : g \in \pi_1(\Sigma)\}$$

and for $i = j$, define

$$\mathcal{O}_{i,i}(\Sigma, \mathcal{A}) = (H_i \backslash \pi_1(\Sigma) / H_i) \smallsetminus \{H_i e H_i\},$$

where $e \in \pi_1(\Sigma)$ is the identity. We will denote an element of $\mathcal{O}_{i,j}(\Sigma, \mathcal{A})$ corresponding to $H_i g H_j$ as $[g]_{i,j}$. Associated to the pair $(\Sigma, \mathcal{A})$ we define the *orthoset* to be the collection of all such cosets

$$\mathcal{O}(\Sigma, \mathcal{A}) = \bigsqcup_{1 \le i,j \le n} \mathcal{O}_{i,j}(\Sigma, \mathcal{A}) .$$

From the definitions, it is clear that the map

$$\Phi : \mathcal{O}(\Sigma, \mathcal{A}) \to \mathcal{O}(\Sigma, \sigma)$$

given by

$$\Phi([g]_{i,j}) = \alpha_{i,j}(g)$$

is well-defined.

Proposition 7.1. *The map Φ is a bijection.*

Proof. We first show it is injective. Suppose $\Phi([g]_{i,j}) = \Phi([g']_{i',j'})$. First note that $i' = i$ and $j = j'$ since the arcs must be oriented from α_i to α_j. Now $\widetilde{\alpha}_{i,j}(g)$ and $\widetilde{\alpha}_{i,j}(g')$ must differ by an element of $\pi_1(\Sigma)$. Since both these arcs start on $\widetilde{\alpha}_i$ it is clear that there exists $h_i \in H_i$ such that

$$\widetilde{\alpha}_{i,j}(g) = h_i \cdot \widetilde{\alpha}_{i,j}(g') .$$

In particular, we must have that $g \cdot \widetilde{\alpha}_j = (h_i g') \cdot \widetilde{\alpha}_j$ implying

$$(g')^{-1} h_i^{-1} g \in H_j .$$

Set $h_j = (g')^{-1} h_i^{-1} g \in H_j$, then

$$g = h_i g' h_j \in H_i g' H_j ,$$

so that $[g]_{i,j} = [g']_{i,j}$ and Φ is injective.

To see that Φ is surjective, take an orthogeodesic $\beta \in \mathcal{O}(\Sigma, \sigma)$ from α_i to α_j. Choose a lift $\widetilde{\beta}$ of β such that $\widetilde{\beta}$ starts on $\widetilde{\alpha}_i$. But $\widetilde{\beta}$ must also end on some lift of α_j which we can write as $g \cdot \widetilde{\alpha}_j$, so that $\Phi([g]_{i,j}) = \beta$ and Φ is surjective. Notice that if $i = j$, then $g \notin H_i e H_i$ as β is a non-trivial orthogeodesic. □

We will see how to rewrite Basmajian's identity in terms of the orthoset as a corollary of generalizing the identity to real projective structures.

Remark 7.2. (1) In his paper, Basmajian [**Bas93**] uses the fact that an orthogeodesic can be obtained from $g \in \pi_1(\Sigma)$; in our notation, he constructs $\alpha_{i,i}(g)$ for a fixed i.

(2) Despite using the language and setting of surfaces, the above discussion holds just as well for connected compact hyperbolic n-manifolds with totally geodesic boundary.

8. Real Projective Structures ($n = 3$)

A convex real projective surface, or convex $\mathbb{RP}^2$-surface, is a quotient Ω/Γ where $\Omega \subset \mathbb{RP}^2$ is a convex domain in the complement of some $\mathbb{RP}^1$ and $\Gamma < \mathrm{PGL}(3, \mathbb{R})$ is a discrete group acting properly on Ω. A convex $\mathbb{RP}^2$-structure on a surface S is a diffeomorphism $f : S \to \Omega/\Gamma$. The work of Goldman [**Gol90**] tells us that the conjugacy class of the holonomy coming from a convex $\mathbb{RP}^2$-structure on a surface S is a Hitchin representation $\pi_1(S) \to \mathrm{SL}(3, \mathbb{R})$. In fact, for closed surfaces this identification is a bijection by the work of Choi-Goldman [**CG93**].

In this section we give a generalization of Basmajian's identity to convex $\mathbb{RP}^2$-surfaces and by extension to 3-Hitchin representations. This result is an immediate corollary of Theorem 2.1, however, the proof here is geometric in nature and will closely follow Basmajian's original proof in [**Bas93**]. Further, it motivates the general case.

8.1. Hilbert metric. Let $F = \Omega/\Gamma$ be a convex $\mathbb{RP}^2$-surface, then F carries a natural Finsler metric called the Hilbert metric, which we now describe.

Let $x, y \in \Omega$ and define $L \subset \mathbb{RP}^2$ to be the projective line connecting x and y. L intersects $\partial\Omega$ in two points p, q such that p, x, y, q is cyclically ordered on L. Choose any affine patch containing these four points, then the Hilbert distance between x and y is

$$h(x, y) := \log B_{\mathbb{P}}(p, y, q, x)\,,$$

where $B_{\mathbb{P}}$ is the projective cross-ratio defined in (5.10). The geodesics in the Hilbert geometry correspond to the intersection of projective lines with Ω. As the cross-ratio is invariant under projective transformations we see that the Hilbert metric descends to a metric on F.

Let ρ be the holonomy associated to a convex $\mathbb{RP}^2$-structure on a surface S, then for a primitive element $g \in \pi_1(\Sigma)$, the length $\ell_\rho(g)$ (see (3.3)) agrees with the translation length of the geodesic representative of $\rho(g)$ in the Hilbert metric.

Note that when Ω is a conic, it is projectively equivalent to a disk. In this case, F is hyperbolic and $h = 2d_{\mathbb{H}}$ where $d_{\mathbb{H}}$ denotes the hyperbolic metric. For more details on Hilbert geometry see [**BK12**].

8.2. Basmajian's identity. Let F be a connected compact orientable convex $\mathbb{RP}^2$-surface with non-empty totally geodesic boundary whose double is at least genus 2. Using the doubling construction described in §2 for Hitchin representations, let $\widehat{F} = \Omega/\Gamma$ be the double of F. Then $\widehat{F}$ is a closed convex $\mathbb{RP}^2$-surface. Note that there is also a doubling construction in [**Gol90**] inherent to convex $\mathbb{RP}^2$-surfaces, which is essentially a more geometric version of the Hitchin doubling that we have already discussed.

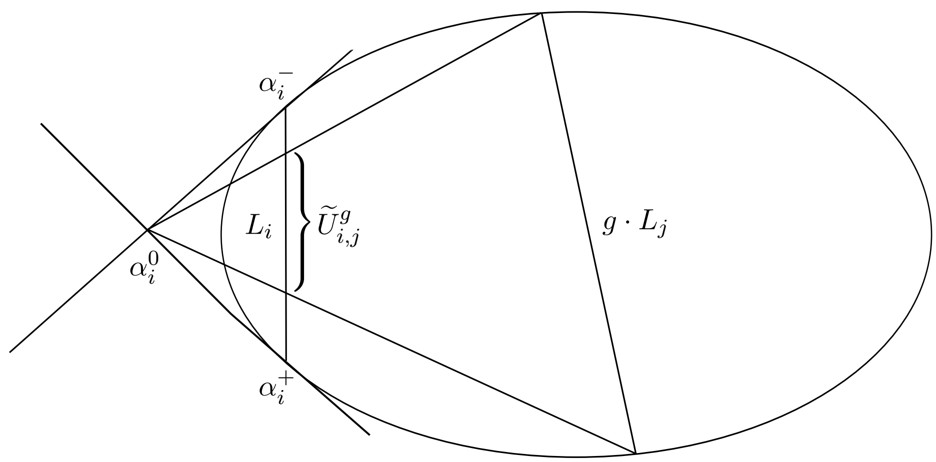

FIGURE 1. Orthogonal projection of $g \cdot L_j$ onto L_i whose image we defined as $\widetilde{U}_{i,j}^g$.

Choose a positive peripheral marking $\mathcal{A} = \{\alpha_i\}_{i=1}^m$. Let $\widetilde{F} \subset \Omega$ be the universal cover of F and let L_i be the geodesic in Ω stabilized by $\alpha_i \in \Gamma$. In projective geometry, orthogonal projection to L_i is defined as follows: As Ω is strictly convex [**Gol90**], for $x \in \partial\Omega$ let $\theta(x)$ denote the line tangent to $\partial\Omega$ at x. Set $\alpha_i^0 = \theta(\alpha_i^+) \cap \theta(\alpha_i^-)$, then the projection to L_i is defined to be $\eta_i : \overline{\Omega} \to L_i$ where $\eta_i(y)$ is the intersection of the line connecting α_i^0 and y and the line L_i. For $[g]_{i,j} \in \mathcal{O}(F, \mathcal{A})$, we let

$$\widetilde{U}_{i,j}^g = \eta_i(g \cdot L_j)$$

be the orthogonal projection of $g \cdot L_j$ onto L_i. This is shown in Figure 1.

LEMMA 8.1. *Let $\pi : \Omega \to \widehat{F}$ be the universal covering map, then $\pi|\widetilde{U}_{i,j}^g$ is injective.*

PROOF. Suppose that $\pi|\widetilde{U}_{i,j}^g$ were not injective, then $(\alpha_i \cdot U_{i,j}^g) \cap U_{i,j}^g \neq 0$. This can only happen if $(\alpha_i g) \cdot L_j$ and $g \cdot L_j$ intersect in Ω, which is impossible as the boundary is totally geodesic. $\square$

By Lemma 8.1, we may define $U_{i,j}^g = \pi(\widetilde{U}_{i,j}^g)$.

LEMMA 8.2. *If $[g]_{i,j}, [h]_{r,s} \in \mathcal{O}(F, \mathcal{A})$ are distinct elements, then $U^g_{i,j} \cap U^h_{r,s} = \emptyset$.*

PROOF. If $U^g_{i,j}$ intersects $U^h_{r,s}$, then $i = r$ and by fixing lifts, one has $g \cdot L_j \cap h \cdot L_s \neq \emptyset$, which would mean that $\partial\Sigma$ is not totally geodesic. $\square$

We define $G_F \colon \mathcal{O}(F, \mathcal{A}) \to \mathbb{R}^+$ by

$$G_F([g]_{i,j}) = \log B_{\mathbb{P}}(\alpha_i^+, \eta_i(g \cdot \alpha_j^+), \alpha_i^-, \eta_i(g \cdot \alpha_j^-))$$

for $[g]_{i,j} \in \mathcal{O}(F, \mathcal{A})$. Let ρ be a 3-Hitchin representation realizing F, then by a standard fact in projective geometry about cross-ratios of four lines

$$G_F([g]_{i,j}) = \log B_\rho(\alpha_i^+, g \cdot \alpha_j^+, \alpha_i^-, g \cdot \alpha_j^-)\,,$$

which agrees with our function in Theorem 2.1. We can then write Basmajian's identity:

PROPOSITION 8.3 (Basmajian's identity for $\mathbb{RP}^2$-surfaces). *Let F be a connected compact orientable convex $\mathbb{RP}^2$-surface with non-empty totally geodesic boundary whose double has genus at least 2. Let $\mathcal{A} = \{\alpha_1, \dots, \alpha_m\}$ be a positive peripheral marking. Then,*

$$\ell_F(\partial F) = \sum_{x \in \mathcal{O}(F,\mathcal{A})} G_F(x)$$

where ℓ_F measures length in the Hilbert metric on F and $\ell_F(\partial F) = \sum_{i=1}^n \ell_F(\alpha_i)$. Furthermore, if F is hyperbolic, then this is Basmajian's identity.

PROOF. Abusing notation, we will use α_i to denote both the element in $\pi_1(F)$ and its geodesic representative in F. From above, we have $U^g_{i,j}$ is an interval embedded in α_i and by construction

$$\ell(U^g_{i,j}) = \log B_{\mathbb{P}}(\alpha_i^+, \eta_i(g \cdot \alpha_j^+), \alpha_i^-, \eta_i(g \cdot \alpha_j^-))\,.$$

For a fixed i, the complement of

$$\bigcup_{[g]_{i,j} \in \mathscr{O}(F,\mathcal{A})} U^g_{i,j}$$

in α_i is the projection of $\partial_\infty(F)$, or $\pi(\eta_i(\partial_\infty(F)))$, which has measure zero by Lemma 6.4. This gives the identity as stated.

We now show that this is Basmajian's identity in the case that Σ is hyperbolic. In this case, we may draw a standard picture with Ω being the unit disk in an affine patch as in Figure 2. The line connecting $g \cdot L_j$ and L_i is a lift of the orthogeodesic corresponding to the

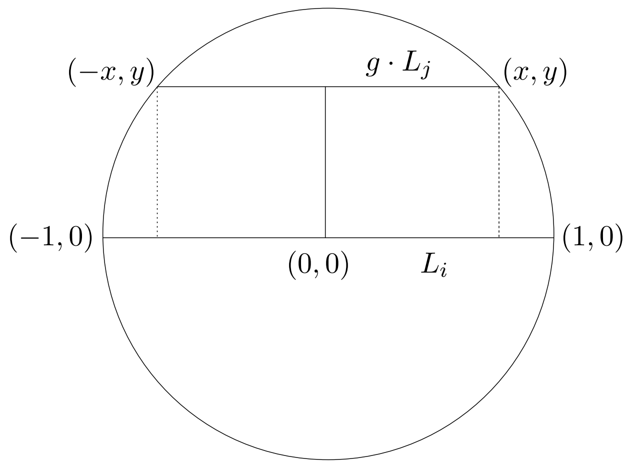

FIGURE 2. A standard diagram for the orthogonal projection of $g \cdot L_j$ onto L_i in the hyperbolic case.

element $[g]_{i,j}$ (this can be seen by considering the corresponding geodesics in the Poincaré disk model). We have

$$\ell = \log B_{\mathbb{P}}((0, -1), (0, y), (0, 1), (0, 0)) = \log \left(\frac{1 + y}{1 - y} \right)$$

is the length of this orthogeodesic in the Hilbert metric and let

$$L = \log B_{\mathbb{P}}((-1, 0), (x, 0), (1, 0), (-x, 0)) = 2 \log \left(\frac{1 + x}{1 - x} \right)$$

be the length of the projection of $g \cdot L_j$ onto L_i. From this we see that

$$x = \tanh \frac{L}{4} \text{ and } y = \tanh \frac{\ell}{2}.$$

From $x^2 + y^2 = 1$ we see that

$$1 = \tanh^2 \left(\frac{L}{4} \right) + \tanh^2 \left(\frac{\ell}{2} \right) \quad \Longrightarrow \quad \tanh^2 \left(\frac{L}{4} \right) = \operatorname{sech}^2 \left(\frac{\ell}{2} \right).$$

Now using the fact that

$$\operatorname{arctanh}(z) = \frac{1}{2} \log \left(\frac{1 + z}{1 - z} \right)$$

we have

$$L = 4 \operatorname{arctanh} \left(\operatorname{sech} \left(\frac{\ell}{2} \right) \right) = 2 \log \left(\frac{1 + \operatorname{sech} \left(\frac{\ell}{2} \right)}{1 - \operatorname{sech} \left(\frac{\ell}{2} \right)} \right) = 4 \log \coth \left(\frac{\ell}{4} \right).$$

Recalling that the Hilbert metric is twice the hyperbolic metric, we recover

$$\ell_h(\partial F) = \sum_{\ell_h \in |\mathcal{O}(F)|} 2 \log \coth\left(\frac{\ell_h}{2}\right),$$

where $\ell_h(\gamma)$ measures length of γ in the hyperbolic metric on F, which is as desired. $\square$

9. Basmajian's Identity

We saw in the case of convex $\mathbb{RP}^2$-structures (or 3-Hitchin representations) on a bordered surface that Basmajian's identity is derived by computing the lengths of orthogonal projections in the universal cover. In the n-Hitchin case, we no longer have the same picture of a universal cover (for $n > 3$), but the idea is roughly the same. In fact, in terms of cross ratios, we will be using the same function on the orthoset as the summand.

Let Σ be a compact surface with $m > 0$ boundary components whose double has genus at least 2. Choose a positive peripheral marking $\mathcal{A} = \{\alpha_1, \cdots, \alpha_m\}$, then for an ordered cross ratio B on $\partial_\infty(\Sigma)$ we define the function $G_B \colon \mathcal{O}(\Sigma, \mathcal{A}) \to \mathbb{R}_+$ by

$$G_B\left([g]_{i,j}\right) := \log B\left(\alpha_i^+, g \cdot \alpha_j^+, \alpha_i^-, g \cdot \alpha_j^-\right).$$

For a Hitchin represenation ρ and the associated cross ratio B_ρ we set

$$G_\rho = G_{B_\rho}.$$

We think of $G_\rho([g]_{i,j})$ as measuring the length of the projection of the line connecting $g \cdot \alpha_j^+$ and $g \cdot \alpha_j^-$ to the line connecting α_i^+ and α_i^-.

Theorem 2.1. *Let Σ be a compact connected surface with $m > 0$ boundary components whose double has genus at least 2. Let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_m\}$ be a positive peripheral marking. If ρ is a Hitchin representation of $\pi_1(\Sigma)$, then*

$$\ell_\rho(\partial\Sigma) \;=\; \sum_{x \in \mathcal{O}(\Sigma, \mathcal{A})} G_\rho(x),$$

where $\ell_\rho(\partial\Sigma) = \sum_{i=1}^m \ell_\rho(\alpha_i)$. Furthermore, if ρ is Fuchsian, this is Basmajian's identity.

Remark. Theorem 2.1 holds for surfaces Σ with $m > 0$ boundary components and p cusps if one *assumes* that Hitchin representations of $\widehat{\Sigma}$, the double, have associated Frenet curves. In particular, our result relies of the existence of a Frenet curve for closed surfaces.

It is currently not known if all Hitchin representations with parabolic holonomy around punctures admit Frenet curves.

Proof. We use the framework from [**LM09**, Theorem 4.1.2.1]. Let us focus our attention on a single boundary component. Let $\alpha = \alpha_1$. Fix a finite area hyperbolic structure on Σ so that $\partial\Sigma$ is totally geodesic. Identify Σ with U/Γ for a convex set $U \subset \mathbb{H}^2$ whose boundary in $\mathbb{H}^2$ is a disjoint union of geodesics. With this identification, $\partial_\infty(\Sigma) \cong \partial_\infty U = \overline{U} \cap \mathbb{S}^1_\infty$ and $\pi_1(\Sigma) \cong \Gamma$. Moreover, $\mathbb{S}^1_\infty \setminus \partial_\infty U$ is a union of *disjoint* intervals of the form $\tilde{I}_\beta = (\beta^-, \beta^+)$ for primitive peripheral elements $\beta \in \Gamma$ which have Σ on their left. By construction, $\beta = g\alpha_j g^{-1}$ for some a_j in the positive peripheral marking $\mathcal{A}$ and $g \in \Gamma$.

Observe that $\left(g\alpha_j g^{-1}\right)^\pm = \alpha_k^\pm$ if and only if $g\alpha_j g^{-1} = \alpha_k$ because $g\alpha_j g^{-1}$ and α_k are primitive. In particular, we must have that $j = k$ and $g \in H_j = \langle \alpha_j \rangle$. We therefore conclude that $\tilde{I}_{g_1\alpha_j g_1^{-1}} = \tilde{I}_{g_2\alpha_k g_2^{-1}}$ if and only if $j = k$ and $g_2^{-1}g_1 \in H_j$, giving us the bijection

$$\left\{\text{Components } \tilde{I}_\beta \text{ of } \mathbb{S}^1_\infty \setminus \partial_\infty U\right\} \Longleftrightarrow \bigsqcup_{1\le j\le n} \pi_1(\Sigma)/H_j\,.$$

Let $B = B_\rho$ be the cross ratio associated to ρ and fix some $\zeta \in (\alpha^+, \alpha^-) \subset \partial_\infty(\Sigma)$ in order to define the continuous function $F_B\colon (\alpha^+, \alpha^-) \to \mathbb{R}$ by

$$F_B(x) = \log B(\alpha^+, x, \alpha^-, \zeta). \tag{9.1}$$

Note that $B(\alpha^+, x, \alpha^-, \zeta)$ is positive by (5.8) and (5.7).

Lemma 9.1. *F_B is a homeomorphism onto its image. Further, if Σ is closed, then F_B is surjective.*

Proof. This follows from the proof of [**LM09**, Theorem 4.1.2.1] bur we include an argument here for completeness. First injectivity: if $B(\alpha_+, x, \alpha, \zeta) = B(\alpha_+, x', \alpha, \zeta)$, then $B(\alpha^+, x, \alpha^-, x') = 1$ by (5.3); hence, $x = x'$ by (5.2). Furthermore, the inequality (5.8) implies F_B preserves the ordering and therefore it is a homeomorphism onto its image. Lastly, note that as $x \to \alpha^\pm$ we have $F_B(x) \to \mp\infty$ by (5.1) and (5.7) and that (α^+, α^-) is connected if Σ is closed. $\square$

Since F_B is increasing, we see that the set $\mathbb{R} \smallsetminus F_B(\partial_\infty U)$ is a union of disjoint intervals $\hat{I}_\beta = (F_B(\beta^-), F_B(\beta^+))$. Further,

$$\begin{aligned} F_B(\alpha \cdot x) &= \log B(\alpha^+, \alpha \cdot x, \alpha^-, \zeta) \\ &= \log \frac{B(\alpha^+, x, \alpha^-, \zeta)}{B(\alpha^+, x, \alpha^-, \alpha \cdot x)} \\ &= F_B(x) - \ell_\rho(\alpha) \end{aligned}$$

by (5.4) and (5.7). Now, set $\mathbb{T} = \mathbb{R}/\ell_\rho(\alpha)\mathbb{Z}$ and define $\pi : \mathbb{R} \to \mathbb{T}$ to be the projection. From above, we have that $\hat{I}_{\alpha\beta\alpha^{-1}} \cap \hat{I}_\beta = \emptyset$ and

$$\hat{I}_{\alpha\beta\alpha^{-1}} = (F_B(\alpha \cdot \beta^+), F_B(\alpha \cdot \beta^-)) = \hat{I}_\beta - \ell_\rho(\alpha)$$

so $\pi|_{\hat{I}_\beta}$ is injective. Define $I_\beta = \pi(\hat{I}_\beta)$ and observe that

$$\{\text{Components } I_\beta \text{ of } \mathbb{T} \smallsetminus \pi(F_B(\partial_\infty U))\} \Longleftrightarrow \left(\bigsqcup_{1 \le j \le m} H_1 \backslash \pi_1(\Sigma)/H_j \right) \smallsetminus \{H_1 e H_1\},$$

where we remove $H_1 e H_1$ as it corresponds to the interval $\tilde{I}_\alpha$, which is outside (α^+, α^-). Using our notation from §7, the right hand side is simply $\bigsqcup_{1 \le j \le m} \mathcal{O}_{1,j}(\Sigma, \mathcal{A})$.

For each I_β, there is a j and an element $[g]_{1,j} \in \mathcal{O}_{1,j}(\Sigma, \mathcal{A})$, where $\beta = g\alpha_j g^{-1}$. With this representative, we see that if λ is the Lebesgue measure on $\mathbb{R}$, then

$$\begin{aligned} \lambda(I_\beta) &= F_B(\beta^+) - F_B(\beta^-) \\ &= \log \frac{B(\alpha^+, \beta^+, \alpha^-, \zeta)}{B(\alpha^+, \beta^-, \alpha^-, \zeta)} \\ &= \log \big(B(\alpha^+, \beta^+, \alpha^-, \zeta) \cdot B(\alpha^+, \zeta, \alpha^-, \beta^-) \big) \text{ (by (5.7))} \\ &= \log B(\alpha^+, \beta^+, \alpha^-, \beta^-) \text{ (by (5.4))} \\ &= \log B(\alpha^+, g \cdot \alpha_j^+, \alpha^-, g \cdot \alpha_j^-) \\ &= G_\rho([g]_{1,j}). \end{aligned}$$

It follows that

$$\ell_\rho(\alpha) = \lambda(\mathbb{T}) = \lambda(\pi(F_B(\partial_\infty U))) + \sum_{1 \le j \le m} \sum_{x \in \mathcal{O}_{1,j}(\Sigma, \mathcal{A})} G_\rho(x).$$

Lemma 6.4 tells us that $\lambda(\pi(F_B(\partial_\infty U))) = 0$ giving the identity for a single boundary component. By doing the same for the other boundary components and summing, we have arrived at

$$\ell_\rho(\partial\Sigma) = \sum_{x\in\mathcal{O}(\Sigma,\mathcal{A})} G_\rho(x)\,.$$

We finish by noting that the proof of Proposition 8.3 implies that if ρ is Fuchsian then we recover Basmajian's original identity. $\square$

Remark. In Theorem 2.1, G_ρ is defined using the Frenet curve associated to the doubled representation $\hat{\rho} : \pi_1(\hat{\Sigma}) \to \mathrm{PSL}(n,\mathbb{R})$. However, if we are given Σ as a subsurface of a closed surface S and a Hitchin representation $\rho' : \pi_1(S) \to \mathrm{PSL}(n,\mathbb{R})$, then we may use the cross ratio associated to ρ' restricted to $\pi_1(\Sigma)^{4*}$, which agrees with that of $\hat{\rho}$ as seen in the proof of Theorem 2.2.

10. Relations to the McShane-Mirzakhani Identity

In this section, we discuss the relation between our identity and Labourie-Mcshane's generalization of the McShane-Mirzakhani identity. We will first consider the hyperbolic surface case and then generalize to Hitchin representations.

There are three spectral identities on hyperbolic surfaces with nonempty totally geodesic boundary (the McShane-Mirzkani [**Mir07a**], Basmajian [**Bas93**], and Bridgeman [**Bri11**] identities) that originally appeared to be using completely different ideas, but were put into a unified framework by S.P. Tan by viewing them as different decompositions of the geodesic flow. This viewpoint is outlined in the survey [**BT16**]. These ideas led to the Luo-Tan identity for closed surfaces [**LT14**]. This is the viewpoint we take in this section.

We note that finding relationships between the identities listed has been of recent interest. Connections between Basmajian and Bridgeman's identities were explored in [**BT14**] and [**Vla15**]. Also, in a sense, the identity of Luo-Tan for closed surfaces gives connections between Bridgeman's identity and that of McShane-Mirzakhani.

The McShane-Mirzakhani identity gives the length of a boundary component as sum over a collection of pairs of pants in the surface. As the geometry of a pair of pants is dictated by the lengths of its boundary components, the summands depend on the lengths of simple closed geodesics in the surface. In order to prove this identity, one has to give a decomposition of the boundary into intervals. As this is the same idea for Basmajian's identity, the goal of this section is to relate the Basmajian decomposition of the boundary to that of the McShane-Mirzakhani decomposition.

10.1. McShane-Mirzakhani Decomposition. Let F be a compact hyperbolic surface with nonempty totally geodesic boundary. Fix α to be a component of ∂F. For a point $x \in \alpha$, let $\beta_x(t)$ be geodesic obtained by flowing the unit vector v_x normal to α at x for time t. Define $t_x \in \mathbb{R}_+$ to be either

- the first value of t such that there exists $t_0 \in [0, t)$ with $\beta_x(t) = \beta_x(t_0)$, i.e. t_x is the first time the geodesic obtained by flowing v_x hits itself, or
- if the arc obtained from this flow is simple and returns to the boundary, then we let t_x to be the time it takes to return to ∂F, i.e. $\beta_x(t_x) \in \partial F$, or
- if the arc is simple and infinite in length, let $t_x = \infty$.

Note that the set of boundary points with $t_x = \infty$ is measure zero as the limit set projects to a set of measure zero on α in the natural Lebesgue measure class. For those $x \in \alpha$ with $t_x < \infty$, define the geodesic arc $\delta_x = \beta_x([0, t_x])$. The arc δ_x defines a pair of pants P_x as follows (there are two cases):

(i) If δ_x is simple and finite, let α' be the components of ∂F containing $n_t(t_x)$ (possibly $\alpha' = \alpha$) and define P_x to be the neighborhood of $\delta_x \cup \alpha \cup \alpha'$ with totally geodesic boundary.

(ii) If δ_x is not simple, then define P_x to be the neighborhood of $\delta_x \cup \alpha$ with totally geodesic boundary. This case is shown in Figure 3.

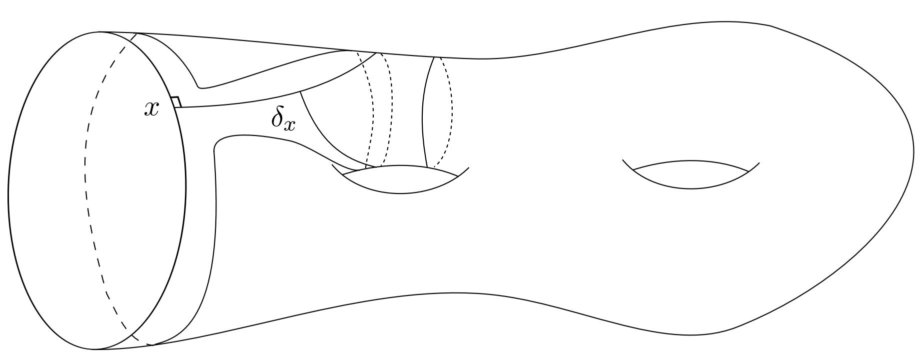

FIGURE 3. An example of P_x with δ_x non-simple.

The McShane-Mirzakhani decomposition of the boundary is as follows. Let $\mathcal{P}_\alpha(F)$ be the set of embedded pairs of pants $P \subset F$ with geodesic boundary and with α as a boundary component. For $P \in \mathcal{P}_\alpha(F)$ set

$$V_P = \{x \in \alpha \colon P_x = P\},$$

We then have that V_P is a disjoint union of two intervals unless P contains two components of ∂F, in which case V_P is a single interval. Further, $V_P \cap V_{P'} = \emptyset$ for $P \neq P'$ yielding

$$\ell(\alpha) = \ell\left(\bigcup_{P \in \mathcal{P}_\alpha(F)} V_P\right) = \sum_{P \in \mathcal{P}_\alpha(F)} \ell(V_P)\,,$$

see [**Mir07a**] or [**BT16**] for details. The McShane-Mirzakhani identity is derived from computing $\ell(V_P)$ for $P \in \mathcal{P}_\alpha(F)$.

In the case that F has a single boundary component, this identity becomes

$$\ell(\alpha) = \sum_{P \in \mathcal{P}_\alpha(F)} \log\left(\frac{e^{\frac{\ell(\partial P)}{2}} + e^{\ell(\partial F)}}{e^{\frac{\ell(\partial P)}{2}} + 1}\right).$$

10.2. Comparing Decompositions. In §8, we saw how to decompose the boundary for Basmajian's identity using orthogonal projection in the universal cover; let us give the same decomposition from a slightly different perspective that better matches the discussion on the McShane-Mirzakhani decomposition.

For $x \in \partial F$, let β_x be the oriented geodesic obtained by flowing the vector normal to ∂F based at x as before. β_x will have finite length and terminate in ∂F for almost every $x \in \partial F$ as the limit set projects to a set of measure zero on ∂F. For every orthogeodesic $\beta \in \mathcal{O}(F)$ we define

$$U_\beta = \{x \in \partial F \colon \beta_x \text{ is properly isotopic to } \beta\}.$$

As no two orthogeodesics are properly isotopic, we see that $U_\beta \cap U_{\beta'} = \emptyset$ and as almost every β_x is properly isotopic to some orthogeodesic we again arrive at Basmajian's identity

$$\ell(\partial F) = \ell\left(\bigcup_{\beta \in \mathcal{O}(F)} U_\beta\right) = \sum_{\beta \in \mathcal{O}(F)} \ell(U_\beta) = \sum_{\beta \in \mathcal{O}(F)} 2 \log\coth\frac{\ell(\beta)}{2}\,.$$

As the McShane-Mirzakhani identity calculates the length of a particular boundary component, for α a component of ∂F, let $\mathcal{O}_\alpha(F)$ be the collection of orthogeodesics emanating from α.

PROPOSITION 10.1. *Let F be a compact hyperbolic surface with nonempty totally geodesic boundary. For each $\beta \in \mathcal{O}_\alpha(F)$, there exists $P \in \mathcal{P}_\alpha(F)$ such that $U_\beta \subset V_P$.*

PROOF. There exists x such that $\beta = \beta_x$, so we set $P = P_x$. Given $y \in U_\beta$, we know that there is a proper isotopy taking β_y to β, which must also take δ_y to δ_x. Given the definition of P_y, we have that $P_y = P$. $\square$

For $P \in \mathcal{P}_\alpha(F)$, let

$$\mathcal{O}_P = \{\beta \in \mathcal{O}_\alpha(F) \colon U_\beta \subset V_P\}.$$

We then immediately have:

COROLLARY 10.2. *Let F be a compact hyperbolic surface with nonempty totally geodesic boundary. For $P \in \mathcal{P}(F)$*

$$\ell(V_P) = \sum_{\beta \in \mathcal{O}_P} 2 \log \coth \frac{\ell(\beta)}{2}.$$

10.3. Decompositions in the Hitchin Setting. In order to proceed, we need to translate the geometric language in the two decompositions to information about the fundamental groups of the surface. We have already seen how to do this in the context of Basmajian's identity using the orthoset in §7. Now let us do the same for the McShane-Mirzakhani identity following [**LM09**].

Let Σ be a compact connected oriented surface with nonempty boundary whose double has genus at least two. Fix a hyperbolic metric σ on Σ such that $\partial\Sigma$ is totally geodesic. As we have done before, let us identify the universal cover of Σ with a convex subset of $\mathbb{H}^2$ cut out by geodesics. Fix a positive peripheral marking $\mathcal{A} = \{\alpha_1, \ldots, \alpha_m\}$ for Σ and let $\alpha = \alpha_1$ be a fixed peripheral element. As in the previous section, we have the set $\mathcal{P}_\alpha(\Sigma, \sigma)$ consisting of embedded pairs of pants with totally geodesic boundary containing the component of $\partial\Sigma$ represented by α. We would like to replace these geometric objects with topological ones. In particular, we will translate V_P into a subset of S^1_∞ instead of a subset of α itself. In the geometric setting, this would be done via projection from α to $(\alpha^+, \alpha^-) \subset \mathbb{S}^1_\infty$.

Given $P \in \mathcal{P}_\alpha(\Sigma, \sigma)$ we can find a *good* pair $(\beta, \gamma) \in \pi_1(P)^2$ such that $\alpha\gamma\beta = e$ and β, γ oriented with P on the left. Let (β', γ') be another good pair, then we will say that

$(\beta,\gamma) \sim (\beta',\gamma')$ if for some n

$$\beta' = \alpha^n \beta \alpha^{-n}$$
$$\gamma' = \alpha^n \gamma \alpha^{-n} .$$

Up to his equivalence there only exist two such pairs: (β,γ) and $(\gamma,\gamma\beta\gamma^{-1})$. These pairs and equivalences depend only on the topology, so let us define $\mathcal{P}_\alpha(\Sigma)$ to be the set of isotopy classes of embedded pairs of pants in Σ containing α as a boundary component. Note that we have a natural bijection $\mathcal{P}_\alpha(\Sigma,\sigma) \to \mathcal{P}_\alpha(\Sigma)$ by sending P to its isotopy class $[P]$.

The pairs (β,γ) and $(\gamma,\gamma\beta\gamma^{-1})$ correspond to the two isotopy classes of embeddings of a fixed pair of pants P_0 into Σ with a choice of peripheral elements $\alpha_0,\beta_0,\gamma_0 \in \pi_1(P_0)$ with P_0 on the left, $\alpha_0\gamma_0\beta_0 = e$ and $\alpha_0 \mapsto \alpha$. This language is used in [**LM09**].

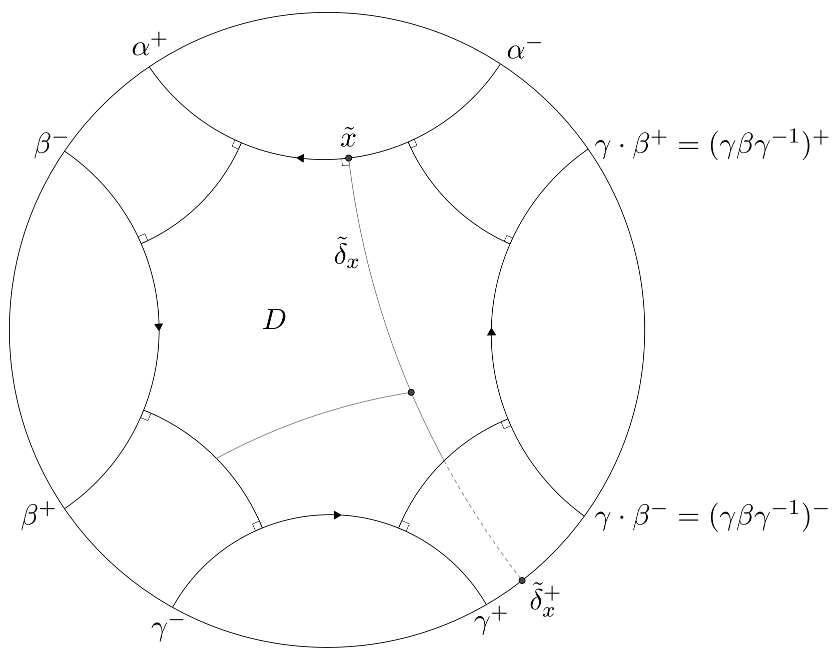

FIGURE 4. A fundamental domain D for P and the lift of δ_x. One can verify that $\alpha\gamma\beta = e$ and $\alpha^{-1} \cdot \beta^\pm = \gamma \cdot \beta^\pm$.

Let us fix $[P] \in \mathcal{P}_\alpha(\Sigma)$ with $P \in \mathcal{P}_\alpha(\Sigma,\sigma)$ and fix a good pair $(\beta,\gamma) \in \pi_1(P)^2 \subset \pi_1(\Sigma)^2$. We can draw a fundamental domain D for P as in Figure 4. Abusing notation and letting α also denote its geodesic representative in $\partial\Sigma$, let $x \in \alpha$ be such that $P_x = P$. Lift x to $\tilde{x} \in D$ on the geodesic $\mathfrak{G}(\alpha^-,\alpha^+) \subset \mathbb{H}^2$ and let $\tilde{\delta}_x$ be the lift of δ_x (as defined in the previous subsection) living in this fundamental domain.

Assuming β and γ are not peripheral, observe that δ_x determines P if and only if $\delta_x \subset P$ and has finite length, see Figure 3 for an example. In particular, this means that δ_x stays inside P and either self interests or hits α. This is equivalent to having $\tilde{\delta}_x^+$ in the set

$$\tilde{J}_P = (\beta^+, \gamma^-) \cup (\gamma^+, \gamma \cdot \beta^-) \subset \mathbb{S}^1_\infty$$

as shown in Figure 4. The orthogonal projection of J_P to the geodesic $\mathfrak{G}(\alpha^-, \alpha^+) \subset \mathbb{H}^2$ followed by the universal covering projection to $\partial\Sigma$ is injective and corresponds to V_P.

Now suppose only γ is peripheral, then δ_x determines P if and only if $\tilde{\delta}_x^+$ is in the interval $\tilde{J}_P = (\beta^+, \gamma \cdot \beta^-)$. We simply add in the interval (γ^-, γ^+) to allow for simple arcs δ_x that hit the boundary component γ for the scenario in the previous paragraph.

Similarly, if only β is peripheral, then δ_x determines P if and only if $\tilde{\delta}_x^+$ is some α translate of a point in the interval

$$\tilde{J}_P = (\alpha \cdot \gamma^+, \gamma^-) = (\beta^-, \gamma^-) \cup \alpha \cdot (\gamma^+, \gamma \cdot \beta^-)\,.$$

The technicality of translating by α arrises because be chose our lift $\tilde{x} \in D$ and want to write $\tilde{J}_P$ as one interval.

If both β and γ are peripheral, then $\Sigma = P$ and the interval is simply $J_P = (\beta^-, \gamma \cdot \beta^-)$. The same sequence of projections also gives V_P in these cases.

Let ρ be a Hitchin representation of Σ and let $B = B_\rho$ be the associated cross ratio. We define the *pants gap function* $H_\rho : \mathcal{P}_\alpha(\Sigma) \to \mathbb{R}$ as follows. Let $[P] \in \mathcal{P}_\alpha(\Sigma)$ and let $(\beta, \gamma) \in \pi_1(P)^2 \subset \pi_1(\Sigma)$ be a good pair. Define the auxiliary function $i_\partial : \pi_1(\Sigma) \to \{0,1\}$ by $i_\partial(\omega) = 1$ if ω is primitive peripheral and $i_\partial(\omega) = 0$ otherwise. Then

$$\begin{aligned} H_\rho([P]) = \log\left[B(\alpha^+, \gamma^-, \alpha^-, \beta^+) \cdot B(\alpha^+, \gamma \cdot \beta^-, \alpha^-, \gamma^+)\right] + \\ + i_\partial(\beta) \log B(\alpha^+, \beta^+, \alpha^-, \beta^-) + i_\partial(\gamma) \log B(\alpha^+, \gamma^+, \alpha^-, \gamma^-)\,. \end{aligned}$$

With this setup at hand, the McShane-Mirzakhani identity for Hitchin representations from [**LM09**] states

$$\ell_\rho(\alpha) = \sum_{[P] \in \mathcal{P}_\alpha(\Sigma)} H_\rho([P])\,.$$

If we let $\mathbb{T} = \mathbb{R}/\ell_\rho(\alpha)\mathbb{Z}$ and let J_P be the projection of $\tilde{J}_P$ under the composition of the projection $\pi : \mathbb{R} \to \mathbb{T}$ and the map F_B defined in (9.1), then the McShane-Mirzakhani identity is saying that the J_P are all disjoint and give a full measure decomposition of $\mathbb{T}$.

As in the proof of Theorem 2.1, for a primitive peripheral element β of $\pi_1(\Sigma)$, let $\tilde{I}_\beta = (\beta^-, \beta^+)$ and $I_\beta = \pi((F_B(\beta^-), F_B(\beta_+)))$. Using $\alpha = \alpha_1$ in some positive peripheral marking, let

$$\mathcal{O}_\alpha(\Sigma, \mathcal{A}) = \mathcal{O}_{1,j}(\Sigma, \mathcal{A}) .$$

We saw that I_β corresponds to an element $x \in \mathcal{O}_\alpha(\Sigma, \mathcal{A})$, so let us rename this interval I_x. As the sets J_P are all disjoint, it follows that for $x \in \mathcal{O}_\alpha(\Sigma, \mathcal{A})$, there is a unique $[P] \in \mathcal{P}_\alpha(\Sigma)$ such that $I_x \subset J_P$. This gives the analog of Proposition 10.1:

PROPOSITION 10.3. *Let Σ be a compact connected orientable surface with nonempty boundary whose double has genus at least 2. For each $x \in \mathcal{O}_\alpha(\Sigma, \mathcal{A})$ there exists a unique $[P] \in \mathcal{P}_\alpha(\Sigma)$ such that $I_x \subset J_P$.*

For $[P] \in \mathcal{P}_\alpha(\Sigma)$, let

$$\mathcal{O}_P(\Sigma, \mathcal{A}) = \{x \in \mathcal{O}_\alpha(\Sigma, \mathcal{A}) \colon I_x \subset J_P\} .$$

We then immediately have the analog of Corollary 10.2:

COROLLARY 10.4. *Let Σ be a compact connected orientable surface with nonempty boundary whose double has genus at least 2 and let $[\rho]$ a Hitchin representation of $\pi_1(\Sigma)$. For $[P] \in \mathcal{P}_\alpha(\Sigma)$*

$$H_\rho([P]) = \sum_{x \in \mathcal{O}_P(\Sigma, \mathcal{A})} G_\rho(x) .$$

CHAPTER 5

Bridgeman-Kahn Identity for Finite Volume Hyperbolic Manifolds

1. Finite Volume Hyperbolic Manifolds with Totally Geodesic Boundary

For us, a *hyperbolic n-manifold with totally geodesic boundary M* can be defined as an *orientable* manifold with boundary that admits an atlas of charts $\{\varphi_\alpha : U_\alpha \to D_\alpha\}$, where $D_\alpha \subset \mathbb{H}^3$ are closed halfspaces, $\varphi_\alpha(U_\alpha \cap \partial M) = \varphi_\alpha(U_\alpha) \cap \partial D_\alpha$, and the transition maps are restrictions of elements of $\mathrm{Isom}^+(\mathbb{H}^3)$. We will assume that all our manifolds are *complete*, in the sense that the developing map $\mathcal{D} : \widetilde{M} \to \mathbb{H}^3$ is a covering map onto the convex hull of some subset of $\partial_\infty \mathbb{H}^n$. If fact, when M has finite volume, it can be show that $\mathcal{D}$ is an isometry and $\mathcal{D}(\widetilde{M})$ is a countable intersection of closed half-spaces bounded by mutually disjoint hyperplanes. Further, if Γ is the image of the holonomy map, $M \cong \mathrm{CH}(\Lambda_\Gamma)/\Gamma$ (see, for example, [**Deb07**]). To understand the structure of ∂M, we have the following result of Kojima.

THEOREM 1.1. (Kojima [**Koj90**]) *If M is a complete finite volume hyperbolic n-manifold with totally geodesic boundary, then ∂M is a complete finite volume hyperbolic $(n-1)$-manifold.*

In particular, if $X \subset \partial M$ is a boundary component, then $\widetilde{X}$ is a hyperplane on $\partial \widetilde{M}$.

1.1. Cusps. Let M be a complete finite volume hyperbolic n-manifold M and $\Gamma \leq \mathrm{Isom}^+(\mathbb{H}^n)$ the image of the holonomy map for M. A *cusps* $\mathfrak{c}$ of M is a Γ-orbit of the fixed point of some parabolic element $g \in \Gamma$. By the Margulis Lemma, $\mathfrak{c}$ admits an embedded horoball neighborhood $B_\mathfrak{c} \subset M$. Following [**Koj90**], $\mathfrak{c}$ arises in two different ways. We say $\mathfrak{c}$ is an *internal* cusp of M whenever $B_\mathfrak{c} \cong E_\mathfrak{c} \times [0, \infty)$ for some closed Euclidean $(n-1)$-manifold $E_\mathfrak{c}$. We call $\mathfrak{c}$ a *boundary cusp*, or ∂-cusps for short, whenever $B_\mathfrak{c} \cong E^\partial_\mathfrak{c} \times [0, \infty)$ for some compact Euclidean $(n-1)$-manifold $E^\partial_\mathfrak{c}$ with totally geodesic boundary. In the case of a ∂-cusp, the two components of $\partial E_{\mathfrak{c}}^{\partial}$ correspond to horoball neighborhoods of cusps of ∂M. In particular, ∂-cusps corresponds to some pairs of cusp of ∂M.

2. Volume Form on the Unit Tangent Bundle

2.1. Unit Tangent Bundle. The volume form Ω on $\mathrm{T}_1\mathbb{H}^n$ is invariant under the action of $\mathrm{Isom}^+(\mathbb{H}^n) \cong \mathrm{SO}^+(n,1)$, where the isomorphism is realized in the hyperboloid model. Here $\mathrm{SO}^+(n,1)$ is the identity component in $\mathrm{SO}(n,1)$. As homogeneous spaces, one may identify $\mathbb{H}^n \cong \mathrm{SO}^+(n,1)/\mathrm{SO}(n)$ and $\mathrm{T}_1\mathbb{H}^n \cong \mathrm{SO}^+(n,1)/\mathrm{SO}(n-1)$. The form Ω arrises by projecting the Haar measure from $\mathrm{SO}^+(n,1)$ to $\mathrm{T}_1\mathbb{H}^n$, which is unique up to scalar multiplication. Alternatively, we may also parametrize $\mathrm{T}_1\mathbb{H}^n \cong \mathbb{H}^n \times \mathbb{S}^{n-1}$ and consider the natural volume element $dV\ d\omega$. Since $\mathrm{SO}^+(n,1)$ acts on $\mathrm{T}_1\mathbb{B}^n$ by orientation preserving Möbius transformations, $dV\,d\omega$ is also invariant. With this in mind, we normalize to have

$$d\Omega = dV\,d\omega. \tag{2.1}$$

For a detailed reference on this perspective, see [**FLJ12**].

2.2. Stereographic Projection and Standard Volume Formulae. We will need a few facts about the standard volume element $d\omega$ on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. Instead of using spherical coordinates, we can parametrize $\mathbb{S}^n - \{e_{n+1}\}$ using *stereographic coordinates.* Define $\mathring{\pi}$: $\mathbb{R}^n \to \mathbb{R}^n$ by

$$\mathring{\pi}(\mathbf{x}) = \frac{2\mathbf{x}}{|\mathbf{x}|^2+1}.$$

Then *stereographic projection* $\pi : \mathbb{R}^n \to \mathbb{S}^n - \{\mathbf{e}_n\}$ is given by

$$\pi(\mathbf{x}) = \left(\mathring{\pi}(\mathbf{x}), \frac{|\mathbf{x}|^2-1}{|\mathbf{x}|^2+1}\right) = \left(\frac{2x_1}{|\mathbf{x}|^2+1}, \dots, \frac{2x_n}{|\mathbf{x}|^2+1}, \frac{|\mathbf{x}|^2-1}{|\mathbf{x}|^2+1}\right). \tag{2.2}$$

We will also make use of the *standard transformation* $\eta : \overline{\mathbb{U}^n} \to \overline{\mathbb{B}^n}$ given by $\eta = \sigma \circ r_n$ where σ is the reflection in the sphere $S(\mathbf{e}_n, \sqrt{2})$ and r_n is the reflection through the plane $\mathbf{e}_n = 0$. In coordinates, for $\mathbf{x} \in \overline{\mathbb{U}^n} - \{\infty\}$,

$$\eta(x_1,\dots,x_n) = \left(\frac{2x_1}{|\mathbf{x}|^2+2x_n+1}, \dots, \frac{2x_{n-1}}{|\mathbf{x}|^2+2x_n+1}, \frac{|\mathbf{x}|^2-1}{|\mathbf{x}|^2+2x_n+1}\right). \tag{2.3}$$

See [**Rat13**, §4.4] for details. Notice that $\pi = \eta\,|_{\mathbb{R}^n}$.

Next, we consider the volume form $\pi^*(d\omega)$ in stereographic coordinates.

LEMMA 2.1. *The pullback of ω from $\mathbb{S}^n$ to $\mathbb{R}^n$ via π has element*

$$\pi^*(d\omega) = \frac{2^n\, d\mathbf{x}}{(|\mathbf{x}|^2+1)^n}.$$

Unable to find a reference for this fact, we do the computation in the appendix (see 0.2).

Our volume form $d\omega$ is induced from $\mathbb{R}^n$ and therefore

$$\text{Vol}(\mathbb{S}^n) = \frac{2\,\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}, \tag{2.4}$$

$$\text{Vol}(\mathbb{S}^n)/\,\text{Vol}(\mathbb{S}^{n-1}) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \tag{2.5}$$

where $\Gamma(\cdot)$ is the Γ-function normalized to $\Gamma(n) = (n-1)!$ for $n \in \mathbb{Z}^+$.

Let $B_{\mathbb{H}^n}(R)$ denote a hyperbolic ball of radius R. Using spherical coordinates for $\mathbb{H}^n$, one can derive

$$\text{Vol}(B_{\mathbb{H}^n}(R)) = \text{Vol}(\mathbb{S}^n)\int_0^R \sinh^{n-1}(r)\; dr \tag{2.6}$$

see [**Rat13**, §3.4] for details. In the appendix, we provide a expression for this volume in terms of hypergeometric functions (see Proposition 0.3).

2.3. Möbius Transformations. For $\gamma \in M(\hat{\mathbb{R}}^n)$ and $\mathbf{x} \in \mathbb{R}^n$, $\gamma'(x)$ is a *conformal* matrix (i.e. a constant multiple of an orthogonal transformation). Thus, we can define $|\gamma'(x)| \in \mathbb{R}_+$ to be the unique number such that $\gamma'(\mathbf{x})/|\gamma'(\mathbf{x})|$ is orthogonal.

In this Chapter, $|\cdot|$ will *always* denote the standard Euclidean norm $|\mathbf{x}| = \sqrt{x_1^2+\ldots+x_n^2}$ for $\mathbf{x} \in \mathbb{R}^n$ and $\{\mathbf{e}_i\}_{i=1}^n$ will be the standard basis for $\mathbb{R}^n$. With this in mind, Nicholls provides the following useful formula for $\gamma \in M(\hat{\mathbb{R}}^n)$ and $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ [**Nic89**, (1.3.2)].

$$|\gamma(\mathbf{x}) - \gamma(\mathbf{y})| = |\gamma'(\mathbf{x})|^{1/2}\,|\gamma'(\mathbf{y})|^{1/2}\,|\mathbf{x}-\mathbf{y}| \tag{2.7}$$

We will make extensive use of equation (2.7). Additionally, we will need the following two constructions.

Let $\mathring{\sigma}(\mathbf{v}) = \mathbf{v}/|\mathbf{v}|^2$ be the inversion through the sphere of radius 1 around $\mathbf{0} \in \mathbb{R}^n$. A simple computation (see [**Rat13**, proof of Theorem 4.1.5]) shows that the Jacobian of $\mathring{\sigma}$ is

$$|\det \mathring{\sigma}'| = 1/|\mathbf{v}|^{2n} \tag{2.8}$$

Lastly, for $\mathbf{y} \in \mathbb{R}^{n-1} \subset \partial_\infty \mathbb{U}^n$, we construct a hyperbolic rotation $\gamma_{\mathbf{y}} : \overline{\mathbb{U}}^n \to \overline{\mathbb{U}}^n$ around $\mathbf{e}_n \in \mathbb{U}^n$ that takes $\mathbf{0}$ to $\mathbf{y}$. Observe that the stereographic projection π acts radially in the sense that the plane $\langle \mathbf{e}_n, \mathbf{y} \rangle$ equals $\langle \mathbf{e}_n, \pi(\mathbf{y}) \rangle$. Let $rot_{\mathbf{y}}$ be a rotation of $\mathbb{S}^{n-1}$ in the plane $\langle \mathbf{e}_n, \pi(\mathbf{y}) \rangle$ taking $-\mathbf{e}_n$ to $\pi(\mathbf{y})$ and define $\gamma_{\mathbf{y}} = \pi^{-1} \circ rot_{\mathbf{y}} \circ \pi$. From the definition, it is clear that $\gamma_{\mathbf{y}}$ acts on the line $\mathbb{R}\,\mathbf{y}$. A simple computation in this plane shows that

$$\gamma_{\mathbf{y}}\left(t\,\frac{\mathbf{y}}{|\mathbf{y}|}\right) = \frac{t + |\mathbf{y}|}{1 - |\mathbf{y}|\,t}\,\frac{\mathbf{y}}{|\mathbf{y}|} \qquad \text{for } t \in \mathbb{R}.$$

Since $\gamma_{\mathbf{y}}'$ is conformal, we can compute

$$|\gamma_{\mathbf{y}}'(\mathbf{0})| = \left.\frac{d}{dt}\right|_{t=0} \frac{t + |\mathbf{y}|}{1 - |\mathbf{y}|\,t} = 1 + |\mathbf{y}|^2 \tag{2.9}$$

2.4. Hypergeometric, Gamma, and Harmonic Number Functions. We will need to use a few special functions. Recall that the Γ function defined by $\Gamma(m) = (m-1)!$ for $m \in \mathbb{Z}^+$ satisfies the following doubling formula,

$$\Gamma(z)\Gamma\left(z + \tfrac{1}{2}\right) = 2^{1-2z}\,\sqrt{\pi}\,\Gamma(2z) \tag{2.10}$$

We will also use the m^{th} harmonic number $H(m)$ given as

$$H(m) = \sum_{k=1}^{m} \frac{1}{k} = \int_0^1 \frac{1 - w^m}{1 - w}\,dw. \tag{2.11}$$

Lastly, we will require a few facts about hypergeometric functions. Given $a, b, c \in \mathbb{C}$ with $c \notin \mathbb{Z}_- \cup \{0\}$ one defines the *hypergeometric function*

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\,\Gamma(b+k)}{\Gamma(c+k)}\,\frac{z^k}{k!} \quad \text{for } |z| < 1 \tag{2.12}$$

and by continuation elsewhere. Note that ${}_2F_1(a, b, c; z) = {}_2F_1(b, a, c; z)$. For a reference on hypergeometric functions see [**AAR99**].

THEOREM 2.1 (Euler 1769, [**AAR99**, Theorem 2.2.1] for proof). *If* $\Re(c) > \Re(b) > 0$, *then*

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 x^{b-1}\,(1-x)^{c-b-1}\,(1-zx)^{-a}\,dx \tag{2.13}$$

for $z \in \mathbb{C} - [1, \infty)$, $\arg(t) = \arg(1-t) = 0$ *and* $(1-zt)^{-a}$ *taking its principal value.*

Using this integral formula, one can prove the following transform due to Pfaff and Euler.

THEOREM 2.2 (Euler 1769, [**AAR99**, Theorem 2.2.5] for proof).

$$ {}_2F_1(a,b,c;z) = (1-z)^{c-a-b}\, {}_2F_1(c-a,c-b,c;z) \tag{2.14} $$

2.5. Geodesic Endpoint Parametrization. In this section, we will rewrite the natural volume element $d\Omega = dV\, d\omega$ on $\mathrm{T}_1\mathbb{H}^n$ in terms of the geodesic endpoint parametrization. A common reference for this formula can be found in [**Nic89**, Theorem 8.1.1], however, the formula is off by a scalar multiple. We correct this here. After this manuscript was complete, we did find a correct version in [**FLJ12**, Proposition III.6.2.6], however, our proofs are significantly different.

For this parametrization we fix a base point $O \in \mathbb{H}^n$. For convenience, we will choose the origin $\mathbf{0} \in \mathbb{R}^n$ in the conformal ball model and $\mathbf{e}_n \in \mathbb{U}^n$. In the *geodesic endpoint parametrization* a point $v \in \mathrm{T}_1\mathbb{H}^n$ is mapped to a triple $(\xi_-, \xi_+, t) \in \partial_\infty\mathbb{H}^n \times \partial_\infty\mathbb{H}^n \times \mathbb{R}$ where ξ_-, ξ_+ are the backwards and forwards endpoints of the geodesic defined by v, respectively. On this geodesic there is a closest point $p(\xi_-, \xi_+)$ to O, called the *reference point*. The value of t is the signed hyperbolic distance along this geodesic from $p(\xi_-, \xi_+)$ to the basepoint of v. This assignment is a bijection and we have

$$ \mathrm{T}_1\mathbb{H}^n \cong \{(\xi_-, \xi_+, t) \in \partial_\infty\mathbb{H}^n \times \partial_\infty\mathbb{H}^n \times \mathbb{R} : \xi_- \neq \xi_+\}. $$

For a a Möbius transformation γ of $\mathbb{H}^n$ and a point (ξ_-, ξ_+, t) we have that

$$ \gamma(\xi_-, \xi_+, t) = (\gamma(\xi_-), \gamma(\xi_+), t + s_\gamma(\xi_-, \xi_+)) \tag{2.15} $$

where $s_\gamma(\xi_-, \xi_+)$ is the signed distance between $p((\gamma(\xi_-), \gamma(\xi_+))$ and $\gamma(p(\xi_-, \xi_+))$ along the geodesic from $\gamma(\xi_-)$ to $\gamma(\xi_+)$.

The following proposition is the corrected version of [**Nic89**, Theorem 8.1.1].

THEOREM 3.1. *Let $\Omega = dV d\omega$ be the standard volume form in $\mathrm{T}_1\mathbb{H}^{n+1}$. Then, with the following coordinates arising from the upper half space and conformal ball models for the geodesic endpoint parametrization*

$$ \mathrm{T}_1\mathbb{H}^{n+1} \cong \{(\mathbf{x}, \mathbf{y}, t) \in \hat{\mathbb{R}}^n \times \hat{\mathbb{R}}^n \times \mathbb{R} : \mathbf{x} \neq \mathbf{y}\} $$

$$ \mathrm{T}_1\mathbb{H}^{n+1} \cong \{(\mathbf{p}, \mathbf{q}, t) \in \mathbb{S}^n \times \mathbb{S}^n \times \mathbb{R} : \mathbf{p} \neq \mathbf{q}\}, $$

we have

$$d\Omega = \frac{2^n d\mathbf{x}\, d\mathbf{y}\, dt}{|\mathbf{x}-\mathbf{y}|^{2n}} = \frac{2^n d\omega(\mathbf{p})\, d\omega(\mathbf{p})\, dt}{|\mathbf{p}-\mathbf{q}|^{2n}}$$

where $|\cdot|$ *is the Euclidean norm in* $\mathbb{R}^n$ *and* $\mathbb{R}^{n+1}$ *respectively.*

We first do a computation to show the second equality in Theorem 3.1.

LEMMA 2.2. *The standard transformation* $\eta : \overline{\mathbb{U}^n} \to \overline{\mathbb{B}^n}$ *induces the map* $\phi = \pi \times \pi \times id$ *between the two geodesic endpoint parametrization models, where* π *is the stereographic projection. In addition*

$$\phi^* \left(\frac{2^n d\omega(\mathbf{p})\, d\omega(\mathbf{p})\, dt}{|\mathbf{p}-\mathbf{q}|^{2n}} \right) = \frac{2^n d\mathbf{x}\, d\mathbf{y}\, dt}{|\mathbf{x}-\mathbf{y}|^{2n}}.$$

PROOF. Since $\eta(\mathbf{e_{n+1}}) = \mathbf{0}$ (our fixed base points) and η preserves the hyperbolic metric, the identity component of ϕ is clear. By constriction, $\eta\,|_{\mathbb{R}^n} = \pi$ and so Proposition 2.1 implies

$$\phi^* \left(\frac{2^n d\omega(\mathbf{p})\, d\omega(\mathbf{p})\, dt}{|\mathbf{p}-\mathbf{q}|^{2n}} \right) = \frac{2^n}{(|\mathbf{x}|^2+1)^n} \cdot \frac{2^n}{(|\mathbf{y}|^2+1)^n} \cdot \frac{2^n d\mathbf{x}\, d\mathbf{y}\, dt}{|\pi(\mathbf{x})-\pi(\mathbf{y})|^{2n}}. \tag{2.16}$$

Using the defining equation (2.2) for π, we get

$$\begin{aligned}
|\pi(x)-\pi(y)|^2 &= \left| \frac{2\mathbf{x}}{|\mathbf{x}|^2+1} - \frac{2\mathbf{y}}{|\mathbf{y}|^2+1} \right|^2 + \left(\frac{|\mathbf{x}|^2-1}{|\mathbf{x}|^2+1} - \frac{|\mathbf{y}|^2-1}{|\mathbf{y}|^2+1} \right)^2 \\
&= \frac{4|\mathbf{x}|^2}{(|\mathbf{x}|^2+1)^2} + \frac{4|\mathbf{y}|^2}{(|\mathbf{y}|^2+1)^2} - \frac{8\,\mathbf{x}\cdot\mathbf{y}}{(|\mathbf{x}|^2+1)(|\mathbf{y}|^2+1)} \\
&\quad + \frac{(|\mathbf{x}|^2-1)^2}{(|\mathbf{x}|^2+1)^2} + \frac{(|\mathbf{y}|^2-1)^2}{(|\mathbf{y}|^2+1)^2} - \frac{2(|\mathbf{x}|^2-1)(|\mathbf{y}|^2-1)}{(|\mathbf{x}|^2+1)(|\mathbf{y}|^2+1)}
\end{aligned}$$

Adding the terms vertically, we have

$$\begin{aligned}
|\pi(x)-\pi(y)|^2 &= 1+1-\frac{2(|\mathbf{x}|^2-1)(|\mathbf{y}|^2-1)+8\,\mathbf{x}\cdot\mathbf{y}}{(|\mathbf{x}|^2+1)(|\mathbf{y}|^2+1)} \\
&= \frac{2(|\mathbf{x}|^2+1)(|\mathbf{y}|^2+1)-2(|\mathbf{x}|^2-1)(|\mathbf{y}|^2-1)-8\,\mathbf{x}\cdot\mathbf{y}}{(|\mathbf{x}|^2+1)(|\mathbf{y}|^2+1)} \\
&= \frac{4|\mathbf{x}|^2+4|\mathbf{y}|^2-8\,\mathbf{x}\cdot\mathbf{y}}{(|\mathbf{x}|^2+1)(|\mathbf{y}|^2+1)} = \boxed{\frac{4|\mathbf{x}-\mathbf{y}|^2}{(|\mathbf{x}|^2+1)(|\mathbf{y}|^2+1)}}
\end{aligned}$$

Substituting into equation (2.16), we obtain the desired result

$$\phi^* \left(\frac{2^n d\omega(\mathbf{p})\, d\omega(\mathbf{p})\, dt}{|\mathbf{p}-\mathbf{q}|^{2n}} \right) = \frac{2^n d\mathbf{x}\, d\mathbf{y}\, dt}{|\mathbf{x}-\mathbf{y}|^{2n}}.$$

□

LEMMA 2.3. *There exists a constant $C \in \mathbb{R}$ such that*

$$C\, d\Omega = \frac{2^n d\mathbf{x}\, d\mathbf{y}\, dt}{|\mathbf{x}-\mathbf{y}|^{2n}} = \frac{2^n d\omega(\mathbf{p})\, d\omega(\mathbf{p})\, dt}{|\mathbf{p}-\mathbf{q}|^{2n}}$$

PROOF. By Lemma 2.2 we can work in the upper half space model. Recall from section 2.1 that the form Ω arrises from the Haar measure on $\mathrm{SO}^+(n,1)$, with $\mathrm{SO}^+(n,1)$ acting on $\mathrm{T}_1\mathbb{H}^n \cong \mathrm{SO}^+(n,1)/\,\mathrm{SO}(n-1)$ by orientation preserving Möbius transformations. Let γ be a Möbius transformation of $\mathbb{U}^{n+1}$, then

$$\begin{aligned}\gamma^*\left(\frac{2^n d\mathbf{x}\, d\mathbf{y}\, dt}{|\mathbf{x}-\mathbf{y}|^{2n}}\right) &= \frac{2^n d\gamma(\mathbf{x})\, d\gamma(\mathbf{y})\, d(t+s_\gamma(\mathbf{x},\mathbf{y}))}{|\gamma(\mathbf{x})-\gamma(\mathbf{y})|^{2n}} \\ &= \frac{2^n|\gamma'(\mathbf{x})|^n|\gamma'(\mathbf{y})|^n d\mathbf{x}\, d\mathbf{y}\, dt}{\left(|\gamma'(\mathbf{x})|^{1/2}|\gamma'(\mathbf{x})|^{1/2}|\mathbf{x}-\mathbf{y}|\right)^{2n}} = \frac{2^n d\mathbf{x}\, d\mathbf{y}\, dt}{|\mathbf{x}-\mathbf{y}|^{2n}}\end{aligned}$$

using equations (2.15) and (2.7) for Möbius transforamtions. By uniqueness of the Haar measure up to scalar multiple, our lemma follows. □

Define

$$B_h = \{\mathbf{z} \in \mathbb{H}^{n+1} \mid d_{\mathbb{H}}(\mathbf{z}, O) < \operatorname{arcsinh}(1)\}$$

In $\mathbb{U}^{n+1}$, B_h has Euclidean center $\sqrt{2}\,\mathbf{e}_{n+1}$ and radius 1. We will prove that

$$\int_{\mathrm{T}_1 B_h} d\Omega = \int_{\mathrm{T}_1 B_h} \frac{2^n d\mathbf{x}\, d\mathbf{y}\, dt}{|\mathbf{x}-\mathbf{y}|^{2n}}, \tag{2.17}$$

which implies $C = 1$ and Theorem 3.1 holds. We will make use of standard volume formulae (2.2) and hypergeometric functions (2.4.)

LEMMA 2.4.

$$\int_{\mathrm{T}_1 B_h} d\Omega = \mathrm{Vol}(\mathbb{S}^n)\,\mathrm{Vol}(\mathbb{S}^{n-1})\,\frac{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)}{2\,\Gamma\left(\frac{n+3}{2}\right)}\,{}_2F_1\left(\frac{1}{2}, \frac{n+1}{2}, \frac{n+3}{2}; -1\right). \tag{2.18}$$

PROOF. Using $d\Omega = dV\, d\omega$ on $\mathrm{T}_1 B_h \cong B_h \times \mathbb{S}^n$ and the volume formula (2.6),

$$\int_{\mathrm{T}_1 B_h} d\Omega = \mathrm{Vol}(\mathbb{S}^n)^2 \int_0^{\operatorname{arcsinh}(1)} \sinh^n(\rho)\, d\rho = \mathrm{Vol}(\mathbb{S}^n)^2 \int_0^1 \frac{t^{(n-1)/2}}{2\sqrt{1+t}}\, dt$$

where we made the substitution $\rho = \operatorname{arcsinh}(\sqrt{t})$ with $d\rho = dt/(2\sqrt{t}\,\sqrt{1+t})$.

It is straight forward to recognize this integral as a hypergeometric function with coefficients $a = 1/2,\ b = (n+1)/2,\ c = (n+3)/2$, and $z = -1$ (see 2.4). For future convenience, let

$$\begin{aligned}
I_\Omega &= \frac{\mathrm{Vol}(\mathbb{S}^n)}{\mathrm{Vol}(\mathbb{S}^{n-1})} \int_0^1 \frac{t^{(n-1)/2}}{2\sqrt{1+t}}\, dt = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \frac{\Gamma\left(b\right)\Gamma\left(c-b\right)}{2\,\Gamma\left(c\right)}\ {}_2F_1\left(a,b,c;z\right) \\
&= \frac{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(1\right)}{2\,\Gamma\left(\frac{n+3}{2}\right)}\ {}_2F_1\left(\frac{1}{2}, \frac{n+1}{2}, \frac{n+3}{2}; -1\right) \\
&= \boxed{\frac{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)}{2\,\Gamma\left(\frac{n+3}{2}\right)}\ {}_2F_1\left(\frac{1}{2}, \frac{n+1}{2}, \frac{n+3}{2}; -1\right)}
\end{aligned}$$

using equations (2.5) and (2.13). Note that $\int_{\mathrm{T}_1 B_h} d\Omega = \mathrm{Vol}(\mathbb{S}^n)\,\mathrm{Vol}(\mathbb{S}^{n-1})\ I_\Omega$. $\square$

For the sake of completeness, we write down the volume formula for a hyperbolic ball of arbitrary radius in the appendix (see 0.3).

LEMMA 2.5.

$$\int_{\mathrm{T}_1 B_h} \frac{2^n d\mathbf{x}\, d\mathbf{y}\, dt}{|\mathbf{x}-\mathbf{y}|^{2n}} = \mathrm{Vol}(\mathbb{S}^n)\,\mathrm{Vol}(\mathbb{S}^{n-1})\ \frac{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{n+3}{2}\right)}\ {}_2F_1\left(1, \frac{n}{2}+1, \frac{n+3}{2}; -1\right). \tag{2.19}$$

PROOF. Let $\mathscr{G}(\mathbf{x},\mathbf{y})$ be the complete oriented hyperbolic geodesic from $\mathbf{x}$ to $\mathbf{y}$ and define

$$L_{B_h}(\mathbf{x},\mathbf{y}) = \text{hyperbolic length of } B_h \cap \mathscr{G}(\mathbf{x},\mathbf{y}).$$

Note that L_{B_h} is invariant under any hyperbolic isometry fixing $O \in \mathbb{H}^{n+1}$. Then

$$\int_{\mathrm{T}_1 B_h} \frac{2^n d\mathbf{x}\, d\mathbf{y}\, dt}{|\mathbf{x}-\mathbf{y}|^{2n}} = \int_{\mathbb{R}^n} \int_{proj_{\mathbf{y}}(B_h)} \frac{2^n\, L_{B_h}(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{2n}}\, d\mathbf{x}\, d\mathbf{y}$$

where $proj_{\mathbf{y}}(B_h)$ is the geodesic visual projection of B_h from $\mathbf{y}$ onto $\partial_\infty \mathbb{H}^{n+1}$. By Fubini's Theorem, we can integrate $d\mathbf{x}$ and $d\mathbf{y}$ separately.

Let $\gamma_{\mathbf{y}}$ denote the hyperbolic rotation around $\mathbf{e}_{n+1}$ defined in Section 2.3 that takes $\mathbf{0}$ to $\mathbf{y}$. Recall that $|\gamma'_{\mathbf{y}}(\mathbf{0})| = 1 + |\mathbf{y}|^2$ by formula (2.9). Using the change of coordinates $\mathbf{x} = \gamma_{\mathbf{y}}(\mathbf{u})$

with $d\gamma_{\mathbf{y}}(\mathbf{u}) = |\gamma'_{\mathbf{y}}(\mathbf{u})|^n \, d\mathbf{u}$, equations (2.9) and (2.7), and Lemma 2.1, we obtain

$$\begin{aligned}
\int_{\mathrm{T}_1 B_h} \frac{2^n d\mathbf{x}\, d\mathbf{y}\, dt}{|\mathbf{x}-\mathbf{y}|^{2n}} &= \int_{\mathbb{R}^n} \int_{proj_{\mathbf{y}}(B_h)} \frac{2^n\, L_{B_h}(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{2n}}\, d\mathbf{x}\, d\mathbf{y} \\
&= \int_{\mathbb{R}^n} \int_{\gamma_{\mathbf{y}}^{-1}(proj_{\mathbf{y}}(B_h))} \gamma_{\mathbf{y}}^{*} \left(\frac{2^n\, L_{B_h}(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{2n}}\, d\mathbf{x} \right) d\mathbf{y} \\
&= \int_{\mathbb{R}^n} \int_{proj_{\mathbf{0}}(B_h)} \left(\frac{2^n\, L_{B_h}(\gamma_{\mathbf{y}}(\mathbf{u}), \gamma_{\mathbf{y}}(\mathbf{0}))}{|\gamma_{\mathbf{y}}(\mathbf{u}) - \gamma_{\mathbf{y}}(\mathbf{0})|^{2n}}\, d\gamma_{\mathbf{y}}(\mathbf{u}) \right) d\mathbf{y} \\
&= \int_{\mathbb{R}^n} \int_{proj_{\mathbf{0}}(B_h)} \left(\frac{2^n\, L_{B_h}(\mathbf{u},\mathbf{0})}{|\gamma'_{\mathbf{y}}(\mathbf{u})|^n |\gamma'_{\mathbf{y}}(\mathbf{0})|^n |\mathbf{u}|^{2n}} |\gamma'_{\mathbf{y}}(\mathbf{u})|^n \, d\mathbf{u} \right) d\mathbf{y} \\
&= \int_{\mathbb{R}^n} \frac{2^n\, d\mathbf{y}}{(1+|\mathbf{y}|^2)^n} \int_{proj_{\mathbf{0}}(B_h)} \frac{L_{B_h}(\mathbf{u},\mathbf{0})\, d\mathbf{u}}{|\mathbf{u}|^{2n}} \\
&= \mathrm{Vol}(\mathbb{S}^n) \int_{|\mathbf{u}|>1} \frac{L_{B_h}(\mathbf{u},\mathbf{0})\, d\mathbf{u}}{|\mathbf{u}|^{2n}}.
\end{aligned}$$

We used the fact that B_h has Euclidean center $\sqrt{2}\,\mathbf{e}_{n+1}$ and radius 1 to substitute $proj_{\mathbf{0}}(B_h) = \{\mathbf{u} \in \mathbb{R}^n \mid |\mathbf{u}| > 1\}$. For convenience, we set

$$I_\Psi = \frac{1}{\mathrm{Vol}(\mathbb{S}^n)\ \mathrm{Vol}(\mathbb{S}^{n-1})} \int_{\mathrm{T}_1 B_h} \frac{2^n d\mathbf{x}\, d\mathbf{y}\, dt}{|\mathbf{x}-\mathbf{y}|^{2n}} = \frac{1}{\mathrm{Vol}(\mathbb{S}^{n-1})} \int_{|\mathbf{u}|>1} \frac{L_{B_h}(\mathbf{u},\mathbf{0})\, d\mathbf{u}}{|\mathbf{u}|^{2n}}$$

Let $\mathring{\sigma}(\mathbf{v}) = \mathbf{v}/|\mathbf{v}|^2$ be the inversion through the sphere of radius 1 around $\mathbf{0}$ in $\mathbb{R}^n$. Note that $\mathring{\sigma}$ preserves B_h. By formula 2.8, the Jacobian is $|\det \mathring{\sigma}'| = 1/|\mathbf{v}|^{2n} = |\mathbf{u}|^{2n}$. Changing coordinates using $\mathbf{u} = \mathring{\sigma}(\mathbf{v})$ with $d\mathbf{u}/|\mathbf{u}|^{2n} = d\mathbf{v}$, we have

$$I_\Psi = \frac{1}{\mathrm{Vol}(\mathbb{S}^{n-1})} \int_{|\mathbf{v}|<1} L_{B_h}(\mathring{\sigma}(\mathbf{v}), \mathring{\sigma}(\infty))\, d\mathbf{v} = \frac{1}{\mathrm{Vol}(\mathbb{S}^{n-1})} \int_{|\mathbf{v}|<1} L_{B_h}(\mathbf{v},\infty)\, d\mathbf{v}.$$

By symmetry, the choice of B_h, and the formula for hyperbolic distance in $\mathbb{U}^2$, we have

$$L_{B_h}(\mathbf{v},\infty) = \log\left(\frac{\sqrt{2} + \sqrt{1-|\mathbf{v}|^2}}{\sqrt{2} - \sqrt{1-|\mathbf{v}|^2}} \right)$$

Let $d\mathbf{v} = \rho^{n-1} \sin^{n-2}(\theta_1) \ldots \sin(\theta_{n-2})\ d\rho\, d\theta_1 \ldots d\theta_{n-2}$ be the spherical change of coordinates for $\mathbf{v} \in \mathbb{R}^n$, then

$$
\begin{aligned}
I_\Psi &= \frac{1}{\mathrm{Vol}(\mathbb{S}^{n-1})}\int_{|\mathbf{v}|<1} L_{B_h}(\mathbf{v},\infty)\,d\mathbf{v} = \int_0^1 \rho^{n-1}\log\left(\frac{\sqrt{2}+\sqrt{1-\rho^2}}{\sqrt{2}-\sqrt{1-\rho^2}}\right)d\rho \\
&= \left[\frac{\rho^n}{n}\log\left(\frac{\sqrt{2}+\sqrt{1-\rho^2}}{\sqrt{2}-\sqrt{1-\rho^2}}\right)\right]_0^1 + \int_0^1 \frac{\rho^n}{n}\left(\frac{2\sqrt{2}\,\rho}{\sqrt{1-\rho^2}\,(1+\rho^2)}\right)d\rho \\
&= 0 + \frac{\sqrt{2}}{n}\int_0^1 \frac{2\,\rho^{n+1}}{\sqrt{1-\rho^2}\,(1+\rho^2)}\,d\rho = \frac{\sqrt{2}}{n}\int_0^1 \frac{t^{n/2}}{\sqrt{1-t}\,(1+t)}\,dt
\end{aligned}
$$

where we substitute $\rho=\sqrt{t}$ with $d\rho = dt/(2\sqrt{t})$. It is straight forward to recognize this integral as a hypergeometric function with coefficients $a=1,\ b=(n+2)/2,\ c=(n+3)/2$, and $z=-1$ (see 2.4). If follows that

$$
\begin{aligned}
I_\Psi &= \frac{\sqrt{2}}{n}\int_0^1 \frac{t^{n/2}}{\sqrt{1-t}\,(1+t)}\,dt = \frac{\sqrt{2}}{n}\frac{\Gamma(b)\,\Gamma(c-b)}{2\,\Gamma(c)}\ {}_2F_1(a,b,c;z) \\
&= \frac{\sqrt{2}\,\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{1}{2}\right)}{n\,\Gamma\left(\frac{n+3}{2}\right)}\ {}_2F_1\left(1,\frac{n}{2}+1,\frac{n+3}{2};-1\right) \\
&= \boxed{\frac{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{n+3}{2}\right)}\ {}_2F_1\left(1,\frac{n}{2}+1,\frac{n+3}{2};-1\right)}
\end{aligned}
$$

Note that $\int_{\mathrm{T}_1 B_h} 2^n d\mathbf{x}\,d\mathbf{y}\,dt/|\mathbf{x}-\mathbf{y}|^{2n} = \mathrm{Vol}(\mathbb{S}^n)\,\mathrm{Vol}(\mathbb{S}^{n-1})\ I_\Psi$. □

We can now combine all of our results to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. The work we have done in Lemmas 2.2, 2.3, 2.4, and 2.5 shows that we only need $I_\Omega = I_\Psi$. To do this, we use the symmetry of the a,b parameters and the hypergeometric transformation (2.14). With $a=1/2,\ b=(n+1)/2,\ c=(n+3)/2$, and $z=-1$, we have

$$
\begin{aligned}
{}_2F_1\left(\frac{1}{2},\frac{n+1}{2},\frac{n+3}{2};-1\right) &= (1-z)^{c-b-a}\ {}_2F_1(c-b,c-a,c;z) \\
&= \sqrt{2}\ {}_2F_1\left(1,\frac{n}{2}+1,\frac{n+3}{2};-1\right)
\end{aligned}
$$

Multiplying both sides by $\frac{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)}{2\,\Gamma\left(\frac{n+3}{2}\right)}$ gives $I_\Omega = I_\Psi$ as desired. □

3. Identity for Manifolds with Cusped Boundary

The Bridgman-Kahn identity for a compact hyperbolic n-manifold M with totally geodesic boundary can be expressed as

$$\mathrm{Vol}(\mathrm{T}_1 M) = \mathrm{Vol}(\mathbb{S}^{n-1}) \sum_{\ell \in |\mathcal{O}(M)|} F_n(\ell),$$

where $\mathcal{O}(M)$ is the set of all oriented orthogeodesics of M and $|\mathcal{O}(M)|$ is the orthospectrum. For each $v \in \mathrm{T}_1 M$, let $\exp_v : I_v \to M$ be the longest unit speed geodesic with $\exp_v'(0) = v$ and $I_v \subset \mathbb{R}$ and interval. Define ℓ_v to be the length of $\exp_v$. For each $\gamma \in \mathcal{O}(M)$, $\mathrm{Vol}(\mathbb{S}^{n-1})F_n(\ell_v)$ represents the volume of vectors

$$V_\gamma = \{v \in \mathrm{T}_1 M \mid \exp_v \text{ has finite length and } \exp_v \text{ is homotopic to } \gamma \text{ relative } \partial M\}.$$

A universal covering argument shows that F_n only depends on the length of γ. In this section, we extend this identity.

Theorem 3.5. *For $n \geq 3$ and M a finite volume hyperbolic n-manifold with totally geodesic boundary, let $\mathfrak{C}$ to be the set of ∂-cusps of M and $|\mathcal{O}(M)|$ the orthospectrum. For every $\mathfrak{c} \in \mathfrak{C}$, let $B_\mathfrak{c}$ be the maximal horoball in M and $d_\mathfrak{c}$ the Euclidean distance along $\partial B_\mathfrak{c}$ between the two boundary components of $\mathfrak{c}$. Then*

$$\mathrm{Vol}(M) = \sum_{\ell \in |\mathcal{O}(M)|} F_n(\ell) + \frac{H(n-2)\,\Gamma\left(\frac{n-2}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \sum_{\mathfrak{c} \in \mathfrak{C}} \frac{\mathrm{Vol}(B_\mathfrak{c})}{d_\mathfrak{c}^{n-1}}$$

where $\Gamma(m) = (m-1)!$ and $H(m)$ is the m^{th} harmonic number.

The asymptotics of our coefficient are straightforward to analyze. In particular, one has

Proposition 3.1. *As $n \to \infty$,*

$$\frac{H(n-2)\,\Gamma\left(\frac{n-2}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \approx \sqrt{\frac{2}{\pi}}\left(\frac{\gamma}{\sqrt{n}} + \frac{\log(n)}{\sqrt{n}}\right) + O\left(\frac{1}{n^{3/2}}\right)$$

where γ is Euler's constant.

Proof. This observation follows directly of the well known asymptotic of $H(m)$ and $\Gamma(z)$. As $m, z \to \infty$,

$$H(m) \approx \gamma + \log(m) + \frac{1}{2m} + O\left(\frac{1}{m^2}\right),$$

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \approx z^{a-b}\left(1+\frac{(a-b)(a+b-1)}{2z}+O\left(\frac{1}{z^2}\right)\right)$$

where we take $z = n/2$, $a = -1$ and $b = -1/2$. $\square$

3.1. Decomposition of the Unit Tangent Bundle. For finite volume hyperbolic n-manifold M with totally geodesic boundary without ∂-cusps, the set $\bigcup_{\gamma \in \mathcal{O}(M)} V_\gamma$ is full measure in T_1M by ergodicity of the geodesic flow for the geometric double DM (see [**Nic89**, Theorem 8.3.7]). Indeed, ergodicity implies that for almost every vector $v \in \mathrm{T}_1M$, $\exp_v$ has finite length, and since the geometric structure on ∂M has no cusps, every such arc is homotopic to some orthogeodesic relative ∂M. To extend this construction to the case where ∂M has a geometric structure with cusps, we must consider the volume of vectors that exponentiate to finite arcs homotopic out a ∂-cusp of M relative ∂M. Notice, we do not worry about internal cusps of M as the set of vectors what wander off into an internal cusp has measure zero by ergodicity.

Fix a ∂-cusp $\mathfrak{c}$ of M and let

$$V_{\mathfrak{c}} = \{v \in \mathrm{T}_1M \mid \exp_v \text{ has finite length and } \exp_v \text{ is homotopic out } \mathfrak{c} \text{ relative } \partial M\}.$$

Then, immediately, we have

$$\mathrm{Vol}(\mathrm{T}_1M) = \sum_{\gamma \in \mathcal{O}(M)} \mathrm{Vol}(V_\gamma) + \sum_{\mathfrak{c} \in \mathfrak{C}} \mathrm{Vol}(V_{\mathfrak{c}}) \tag{3.1}$$

We now proceed to compute $\mathrm{Vol}(V_{\mathfrak{c}})$.

Let $v \in \mathrm{T}_1M$ be such that $\exp_v$ is of finite length and homotopic out $\mathfrak{c}$. Let X_- and X_+ be the backwards and forwards boundary components hit by $\exp_v$. They are precisely the components that meet every horoball neighborhood of $\mathfrak{c}$. As discussed in Section 1.1, any lift of a component of ∂M is a complete hyperplane in $\mathbb{H}^n$ that bounds $\widetilde{M}$. Thus, any lift $\widetilde{\exp}_v$ must terminate on two hyperplanes H_- and H_+ in $\mathbb{H}^n$ corresponding to X_- and X_+. Further, as we have remarked in Section 1.1, they are tangent. We fix a lift $\widetilde{\exp}_v$ and let $p = \overline{H}_- \cap \overline{H}_+$ be the unique point of tangency on $\partial_\infty \mathbb{H}^n$. Let $\Gamma_{\mathfrak{c}} \leq \pi_1(M)$ be the subgroup of elements fixing p. Recall that $\Gamma_{\mathfrak{c}}$ is a discrete group of parabolic transformations.

Let $B_{\mathfrak{c}}$ be the maximal horoball neighborhood of $\mathfrak{c}$ in M and let $d_{\mathfrak{c}}$ denote the Euclidean distance along $\partial B_{\mathfrak{c}}$ between X_- and X_+. Conjugating to take $p \mapsto \infty$, we can assume that every element $\gamma \in \Gamma_{\mathfrak{c}}$ acts on $\mathrm{span}\langle e_1, \ldots, e_{n-1}\rangle$ by $\gamma(x) = a_\gamma + A_\gamma x$, where A_γ is an orthogonal transformation, $a_\gamma \neq 0$, and $A_\gamma a_\gamma = a_\gamma$ [**Rat13**, Theorem 4.7.3]. We can further assume that in standard coordinates for $\mathbb{H}^n$

$$\partial \widetilde{B}_{\mathfrak{c}} = \{\mathbf{x} \in \mathbb{H}^n \mid x_n = 1\}$$

$$H_- = \{\mathbf{x} \in \mathbb{H}^n \mid x_1 = 0\}$$

$$H_+ = d_{\mathfrak{c}} e_1 + H_-.$$

In particular, this implies that $a_\gamma \cdot e_1 = 0$ and $A_\gamma e_1 = e_1$ for all $\gamma \in \Gamma$. Let

$$V = \{\mathbf{x} \in \mathbb{H}^n \mid 0 \leq x_1 \leq d_{\mathfrak{c}}\}$$

denote the region between H_- and H_+. We will also need to consider the subsets

$$U_- = \{\mathbf{x} \in \ \partial_\infty \mathbb{H}^n \mid x_1 < 0\}$$

$$U_+ = \{\mathbf{x} \in \partial_\infty \mathbb{H}^n \mid x_1 > d_{\mathfrak{c}}\}.$$

Note that $\Gamma_{\mathfrak{c}}$ naturally acts on $U_\pm$.

To compute $\mathrm{Vol}(V_{\mathfrak{c}})$, we must find the volume of all unit tangent vectors $v \in \mathrm{T}_1 V$ such that the complete geodesic $\exp_v$ has endpoints in U_- and U_+ up to the action of $\Gamma_{\mathfrak{c}}$. Let D be a fundamental domain for the action of $\Gamma_{\mathfrak{c}}$ on U_-, then

$$\mathrm{Vol}(V_{\mathfrak{c}}) = 2\,\mathrm{Vol}\{v \in \mathrm{T}_1 V \mid v \text{ is tangent to a complete geodesic going from } D \text{ to } U_+\}.$$

For points $\mathbf{x}, \mathbf{y} \in \partial_\infty \mathbb{H}^n$, let $\mathscr{G}(\mathbf{x}, \mathbf{y})$ be the complete hyperbolic geodesic connecting $\mathbf{x}$ and $\mathbf{y}$. Define

$$L(\mathbf{x}, \mathbf{y}) = \text{hyperbolic length of } V \cap \mathscr{G}(\mathbf{x}, \mathbf{y}).$$

Note that $L(\mathbf{x}, \mathbf{y}) = \ell_v$ for every vector tangent to $\mathscr{G}(\mathbf{x}, \mathbf{y}) \cap V$. See Figure 1.

From Theorem 3.1 it follows that

$$\mathrm{Vol}(V_{\mathfrak{c}}) = \int_{V_{\mathfrak{c}}} d\Omega = 2 \int_{\mathbf{y} \in U_+} \int_{\mathbf{x} \in D} \frac{2^{n-1}\, L(\mathbf{x}, \mathbf{y})\, d\mathbf{x}\, d\mathbf{y}}{|\mathbf{x} - \mathbf{y}|^{2n-2}} \tag{3.2}$$

where we integrate out the dt to get $L(\mathbf{x}, \mathbf{y})$.

To evaluate the quantity $L(\mathbf{x}, \mathbf{y})$, we will need a generalization of the following computation of Bridgeman and Dumas. See Figure 3.

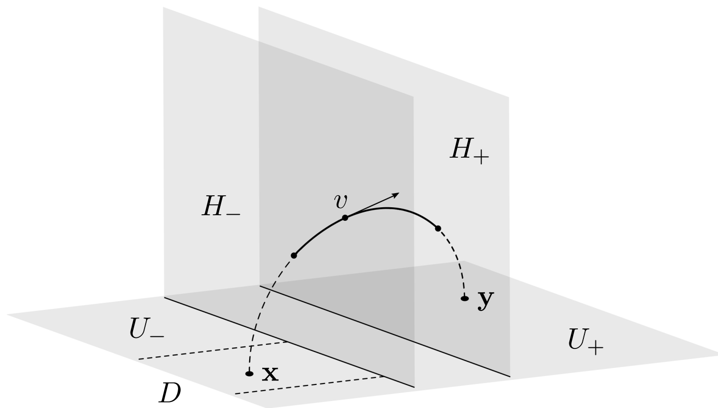

FIGURE 1. To compute $\mathrm{Vol}(V_{\mathfrak{c}})$, we must find the volume of all vectors $v \in V$ for which the corresponding complete geodesic emanates from D and terminates in U_+.

LEMMA 3.2. [**BD07**, Lemma 8] *For $n = 2$ and $d = 1$, for $x, y \in \mathbb{R} \subset \partial_\infty \mathbb{H}^2$, with $x < 0$ and $y > 1$,*

$$L(x, y) = \frac{1}{2} \log \left(\frac{y(x-1)}{x(y-1)} \right).$$

LEMMA 3.3. *As defined above, the function L only depends on the x_1, y_1 coordinates of $\mathbf{x}, \mathbf{y} \in \partial_\infty \mathbb{H}^n$ and on $d_{\mathfrak{c}}$. In particular,*

$$L(\mathbf{x}, \mathbf{y}) = \frac{1}{2} \log \left(\frac{y_1(x_1 - d_{\mathfrak{c}})}{x_1(y_1 - d_{\mathfrak{c}})} \right). \tag{3.3}$$

PROOF. Without loss of generality, we may fix $\mathbf{x} = (x_1, 0, \ldots, 0)$ by applying parabolic transformations that fix ∞ and preserve H_-, H_+. We will show that $L(\mathbf{x}, \mathbf{y})$ depends only on x_1, y_1 and $d_{\mathfrak{c}}$. Consider Figure 2 showing $\mathbf{x}, \mathbf{y}$ on $\partial_\infty \mathbb{H}^n$. Here, $\mathscr{G}(\mathbf{x}, \mathbf{y})$ is perpendicular to the page. There is a hyperbolic 2-plane $\mathbb{H}^2_{\mathbf{x},\mathbf{y}}$ in $\mathbb{H}^n$ whose boundary is the line through $\mathbf{x}, \mathbf{y}$. It follows that $L(\mathbf{x}, \mathbf{y})$ is the length of the arc on the geodesic $\mathscr{G}(-u, w+v)$ lying above the interval $(0, w)$ in $\mathbb{H}^2_{\mathbf{x},\mathbf{y}}$, where u, d, v are as in Figure 2. By construction, $w = \cos(\theta)\, d_{\mathfrak{c}}$, $u = \cos(\theta)\, |x_1|$ and $w + v = \cos(\theta)\, y_1$. Since multiplication by $\cos(\theta)$ is a hyperbolic isometry, $L(\mathbf{x}, \mathbf{y})$ we see that the length of the arc on the geodesic $\mathscr{G}(x_1, y_1)$ lying above the interval $(0, d_{\mathfrak{c}})$ in $\mathbb{H}^2$. See the diagram in Figure 3.

Rescaling further by $1/d_{\mathfrak{c}}$, we see by Lemma 3.2 that

$$L(\mathbf{x}, \mathbf{y}) = \frac{1}{2} \log \left(\frac{\frac{y_1}{d_{\mathfrak{c}}}(\frac{x_1}{d_{\mathfrak{c}}} - 1)}{\frac{x_1}{d_{\mathfrak{c}}}(\frac{y_1}{d_{\mathfrak{c}}} - 1)} \right) = \frac{1}{2} \log \left(\frac{y_1(x_1 - d_{\mathfrak{c}})}{x_1(y_1 - d_{\mathfrak{c}})} \right)$$

□

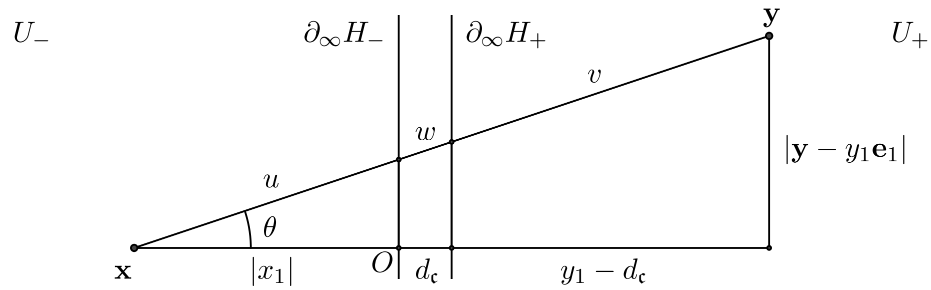

FIGURE 2. The diagram above shows the points $\mathbf{x}, \mathbf{y}$ on $\partial_\infty \mathbb{H}^n$ without ∞ in the $e_1, \ldots e_{n-1}$ coordinates. The point $O = (0, \ldots, 0)$ denotes the origin and horizontal is the e_1-axis.

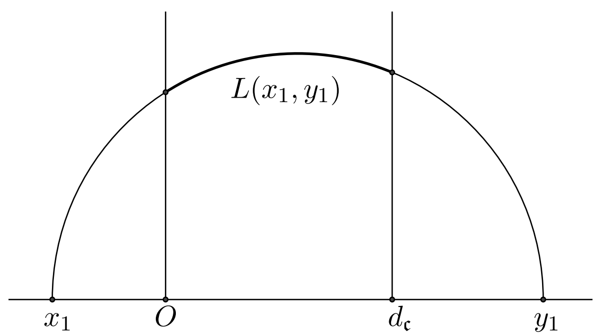

FIGURE 3. The diagram showing $L(x_1, y_1)$ in the plane $\mathbb{H}^2_{\mathbf{x},\mathbf{y}}$ for Lemma 3.3.

3.2. Integration. To set up the integration, we observe that $D = (-\infty, 0) \times D'$ where D' is a fundamental domain for the action of $\Gamma_\mathfrak{c}$ on $\partial_\infty H_- = \{\mathbf{x} \in \partial_\infty \mathbb{H}^n \mid x_1 = 0\}$. Also, $U_+ = (d_\mathfrak{c}, \infty) \times \mathbb{R}^{n-2}$, refer once again to Figure 1. Applying our observations to equation (3.2) and making the substituions $w_i = y_i - x_i$ for $i = 2, \ldots n-1$, we obtain

$$\begin{aligned}
\mathrm{Vol}(V_\mathfrak{c}) &= 2^{n-1} \int_{-\infty}^{0} \int_{d_\mathfrak{c}}^{\infty} \int_{D'} \int_{\mathbb{R}^{n-2}} \frac{\log\left(\frac{y_1(x_1 - d_\mathfrak{c})}{x_1(y_1 - d_\mathfrak{c})}\right) dy_2 \ldots dy_{n-1}\, dx_2 \ldots x_{n-1}\, dy_1\, dx_1}{\sqrt{(x_1 - y_1)^2 + \sum_{i=2}^{n-1}(x_i - y_i)^2}^{\,2n-2}} \\
&= 2^{n-1} \int_{-\infty}^{0} \int_{d_\mathfrak{c}}^{\infty} \int_{D'} \int_{\mathbb{R}^{n-2}} \frac{\log\left(\frac{y_1(x_1 - d_\mathfrak{c})}{x_1(y_1 - d_\mathfrak{c})}\right) dw_2 \ldots dw_n\, dx_2 \ldots x_n\, dy_1\, dx_1}{\sqrt{(x_1 - y_1)^2 + \sum_{i=2}^{n-1} w_i^2}^{\,2n-2}}.
\end{aligned} \tag{3.4}$$

To integrate out w_i for $i = 2, \ldots n$-1, one can show with induction on $k \geq 3$ and the substitution $w = A\tan(\theta)$ that

$$\int_{-\infty}^{\infty} \frac{dw}{\sqrt{w^2 + A^2}^{\,k}} = \frac{1}{A^{k-1}} \int_{-\pi/2}^{\pi/2} \cos^{k-2}(\theta) d\theta = \frac{\sqrt{\pi}\, \Gamma((k-1)/2)}{A^{k-1}\, \Gamma(k/2)}. \tag{3.5}$$

See Proposition 0.1 in the appendix for the proof.

For w_i with $i \geq 2$, we let $A = \sqrt{(x_1 - y_1)^2 + \sum_{j=i+1}^{n-1} w_j^2}$ and $k = 2n - i$. Applying equation (3.5) recursively for $i \geq 2$, we obtain

$$\begin{aligned}\mathrm{Vol}(V_{\mathfrak{c}}) &= \frac{2^{n-1}\pi^{(n-2)/2}\,\Gamma(n/2)}{\Gamma(n-1)} \int_{-\infty}^{0}\int_{d_{\mathfrak{c}}}^{\infty}\int_{D'} \frac{\log\left(\frac{y_1(x_1-d_{\mathfrak{c}})}{x_1(y_1-d_{\mathfrak{c}})}\right) dx_2 \dots x_n\, dy_1\, dx_1}{(y_1-x_1)^n} \\ &= \frac{2^{n-1}\pi^{(n-2)/2}\,\mathrm{Vol}(D')\,\Gamma(n/2)}{\Gamma(n-1)} \int_{-\infty}^{0}\int_{d_{\mathfrak{c}}}^{\infty} \frac{\log\left(\frac{y_1(x_1-d_{\mathfrak{c}})}{x_1(y_1-d_{\mathfrak{c}})}\right) dy_1\, dx_1}{(y_1-x_1)^n}\end{aligned}$$

Note that the Euclidean volume $\mathrm{Vol}(D')$ is finite by the following lemma.

Lemma 3.4. *In the given parametrization,*

$$\mathrm{Vol}(D') = \frac{(n-1)\,\mathrm{Vol}(B_{\mathfrak{c}})}{d_{\mathfrak{c}}}$$

Proof. By construction, $\Gamma_{\mathfrak{c}} \leq \pi_1(M)$ is the largest subgroup fixing ∞ and

$$B_{\mathfrak{c}} = \{x \in \mathbb{H}^n \mid x_n > 1\}/\Gamma_{\mathfrak{c}}.$$

Recall that $\gamma \in \Gamma_{\mathfrak{c}}$ acts on $\mathrm{span}\langle e_1, \dots, e_{n-1}\rangle$ by $\gamma(x) = a_\gamma + A_\gamma x$, where A_γ is an orthogonal transformation, $a_\gamma \neq 0$, and $A_\gamma a_\gamma = a_\gamma$ [**Rat13**, Theorem 4.7.3], and that $a_\gamma \cdot e_1 = 0$ and $A_\gamma e_1 = e_1$ for all $\gamma \in \Gamma$. In particular, the action of $\Gamma_{\mathfrak{c}}$ restricts to $\mathrm{span}\langle e_2, \dots, e_{n-1}\rangle$ with D' as a fundamental domain for this action. It follows that $[0, d_{\mathfrak{c}}] \times D' \times (0, \infty)$ is a fundamental domain for the action of $\Gamma_{\mathfrak{c}}$ on $\{x \in \mathbb{H}^n \mid x_n > 1\}$ and, by the volume form on $\mathbb{U}^n$, we have

$$\mathrm{Vol}(B_{\mathfrak{c}}) = d_{\mathfrak{c}}\,\mathrm{Vol}(D')/(n-1).$$

□

For the remaining integral, we turn to the following Lemma, which we will prove last.

Lemma 3.5. *For $n \geq 3$*

$$\int_{-\infty}^{0}\int_{d_{\mathfrak{c}}}^{\infty} \frac{\log\left(\frac{y(x-d_{\mathfrak{c}})}{x(y-d_{\mathfrak{c}})}\right) dy\, dx}{(y-x)^n} = \frac{2\,H(n-2)}{(n-1)(n-2)\,d_{\mathfrak{c}}^{n-2}}$$

It follows that

$$\mathrm{Vol}(V_{\mathfrak{c}}) = \frac{2^n\pi^{(n-2)/2}\;H(n-2)\,\Gamma(n/2)}{(n-2)\,\Gamma(n-1)}\frac{\mathrm{Vol}(B_{\mathfrak{c}})}{d_{\mathfrak{c}}^{n-1}}.$$

and

$$\frac{1}{\mathrm{Vol}(\mathbb{S}^{n-1})}\mathrm{Vol}(V_{\mathfrak{c}}) = \frac{2^{n-1}\;H(n-2)\,\Gamma(n/2)^2}{\pi\,(n-2)\,\Gamma(n-1)}\frac{\mathrm{Vol}(B_{\mathfrak{c}})}{d_{\mathfrak{c}}^{n-1}}.$$

By the duplication formula (2.10) for $\Gamma(z)$, one has

$$2^{1-(n-1)}\sqrt{\pi}\,\Gamma(n-1)=\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-1}{2}+\frac{1}{2}\right)=\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}\right).$$

Using this relation, we can simplify

$$\begin{aligned}\frac{1}{\mathrm{Vol}(\mathbb{S}^{n-1})}\mathrm{Vol}(V_{\mathfrak{c}}) &= \frac{2\,H(n-2)\,\Gamma(\frac{n}{2})}{\sqrt{\pi}\,(n-2)\,\Gamma(\frac{n-1}{2})}\frac{\mathrm{Vol}(B_{\mathfrak{c}})}{d_{\mathfrak{c}}^{n-1}}\\ &= \frac{2\,H(n-2)\,\left(\frac{n}{2}-1\right)\Gamma(\frac{n-2}{2})}{\sqrt{\pi}\,(n-2)\,\Gamma(\frac{n-1}{2})}\frac{\mathrm{Vol}(B_{\mathfrak{c}})}{d_{\mathfrak{c}}^{n-1}} \qquad (3.6)\\ &= \frac{H(n-2)\,\Gamma(\frac{n-2}{2})}{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}\frac{\mathrm{Vol}(B_{\mathfrak{c}})}{d_{\mathfrak{c}}^{n-1}}.\end{aligned}$$

Up to the proof of Lemma 3.5, our version of the Bridgeman-Kahn identity is complete by assembling our computations and the decomposition in equation (3.1).

$$\boxed{\mathrm{Vol}(M)=\sum_{\ell\in|\mathcal{O}(M)|}F_n(\ell)+\frac{H(n-2)\,\Gamma\left(\frac{n-2}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}\sum_{\mathfrak{c}\in\mathfrak{C}}\frac{\mathrm{Vol}(B_{\mathfrak{c}})}{d_{\mathfrak{c}}^{n-1}}}$$

Proof of Lemma 3.5. We first split up the integral into three pieces

$$I=\int_{-\infty}^{0}\int_{d_{\mathfrak{c}}}^{\infty}\frac{\log\left(\frac{y(x-d_{\mathfrak{c}})}{x(y-d_{\mathfrak{c}})}\right)dy\,dx}{(y-x)^n}=I_1-I_2-I_3$$

where

$$\begin{aligned}I_1&=\int_{-\infty}^{0}\int_{d_{\mathfrak{c}}}^{\infty}\frac{\log(d_{\mathfrak{c}}-x)\,dy\,dx}{(y-x)^n}\\ I_2&=\int_{-\infty}^{0}\int_{d_{\mathfrak{c}}}^{\infty}\frac{\log(-x/y)\,dy\,dx}{(y-x)^n}\\ I_3&=\int_{-\infty}^{0}\int_{d_{\mathfrak{c}}}^{\infty}\frac{\log(y-d_{\mathfrak{c}})\,dy\,dx}{(y-x)^n}\end{aligned}$$

We can easily compute I_1 to be

$$\begin{aligned}I_1&=\int_{-\infty}^{0}\int_{d_{\mathfrak{c}}}^{\infty}\frac{\log(d_{\mathfrak{c}}-x)\,dy\,dx}{(y-x)^n}\\ &=\frac{1}{n-1}\int_{-\infty}^{0}\frac{\log(d_{\mathfrak{c}}-x)\,dx}{(d_{\mathfrak{c}}-x)^{n-1}}\\ &=\frac{1}{n-1}\left[\frac{\log(d_{\mathfrak{c}}-x)}{(n-2)(d_{\mathfrak{c}}-x)^{n-2}}+\frac{1}{(n-2)^2(d_{\mathfrak{c}}-x)^{n-2}}\right]_{-\infty}^{0}\\ &=\boxed{\frac{(n-2)\log(d_{\mathfrak{c}})+1}{(n-1)(n-2)^2\,d_{\mathfrak{c}}^{n-2}}}\end{aligned}$$

For I_2, we first use the change of coordinates $z = x/y$ and $y = y$, where $dy\,dx = y dy\,dz$. With the proper change of limits of integration,

$$\begin{aligned} I_2 &= \int_{-\infty}^{0} \int_{d_{\mathfrak{c}}}^{\infty} \frac{\log(-x/y)\,dy\,dx}{(y-x)^n} \\ &= \int_{-\infty}^{0} \int_{d_{\mathfrak{c}}}^{\infty} \frac{\log(-z)\,dy\,dz}{y^{n-1}(1-z)^n} \\ &= \frac{1}{(n-2)\,d_{\mathfrak{c}}^{n-2}} \int_{-\infty}^{0} \frac{\log(-z)\,dz}{(1-z)^n} \end{aligned}$$

Next, we change coordinates to $w = 1/(1-z)$ with $dw = dz/(1-z)^2$, giving

$$\begin{aligned} I_2 &= \frac{1}{(n-2)\,d_{\mathfrak{c}}^{n-2}} \int_0^1 \log\left(\frac{1}{w} - 1\right) w^{n-2} dw \\ &= \boxed{\frac{-H(n-2)}{(n-1)(n-2)\,d_{\mathfrak{c}}^{n-2}}} \end{aligned}$$

by Lemma 3.6 below.

Lemma 3.6. *For $m \in \mathbb{Z}_{\geq 0}$,*

$$\int_0^1 \log\left(\frac{1}{w} - 1\right) w^m dw = \frac{-H(m)}{m+1}$$

Proof of Lemma 3.6. We begin by splitting the integral into two parts,

$$\begin{aligned} \int_0^1 \log\left(\frac{1}{w} - 1\right) w^m dw &= \int_0^1 \log(1-w)\,w^m - \log(w)\,w^m dw \\ &= \int_0^1 \frac{\log(1-w)}{-m-1} d\left(1 - w^{m+1}\right) - \int_0^1 \frac{\log(w)}{m+1} d\left(w^{m+1}\right) \end{aligned}$$

As $m \geq 0$, the two integrals inside are as follows

$$\begin{aligned} \int_0^1 \frac{\log(1-w)}{-m-1} d\left(1 - w^{m+1}\right) &= \left[\frac{\log(1-w)\left(1 - w^{m+1}\right)}{-m-1}\right]_0^1 - \frac{1}{m+1} \int_0^1 \frac{1 - w^{m+1}}{1-w} dw \\ &= 0 - \frac{H(m+1)}{m+1}. \end{aligned}$$

and

$$\begin{aligned} \int_0^1 \frac{\log(w)}{m+1} d\left(w^{m+1}\right) &= \left[\frac{\log(w)\,w^{m+1}}{m+1}\right]_0^1 - \frac{1}{m+1} \int_0^1 w^m dw \\ &= 0 - \frac{1}{(m+1)^2} \end{aligned}$$

Combining, we see that

$$\int_0^1 \log\left(\frac{1}{w}-1\right) w^m dw = \frac{1}{m+1}\left(-H(m+1)+\frac{1}{m+1}\right) = \frac{-H(m)}{m+1}$$

$\square$

Returning to the proof of Lemma 3.5, we compute I_3 using the change of coordinates $u = y/d_{\mathfrak{c}}$ and $z = x/y$, where $dy\,dx = u\,d_{\mathfrak{c}}^2\,du\,dz$.

$$\begin{aligned}
I_3 &= \int_{-\infty}^{0}\int_{d_{\mathfrak{c}}}^{\infty} \frac{\log(y-d_{\mathfrak{c}})\,dy\,dx}{(y-x)^n} \\
&= \int_{-\infty}^{0}\int_{1}^{\infty} \frac{\log(d_{\mathfrak{c}}(u-1))\,du\,dz}{u^{n-1}(1-z)^n\,d_{\mathfrak{c}}^{n-2}} \\
&= \frac{1}{d_{\mathfrak{c}}^{n-2}}\left(\int_{-\infty}^{0}\frac{\log(d_{\mathfrak{c}})\,dz}{(1-z)^n}\int_1^\infty\frac{du}{u^{n-1}} + \int_{-\infty}^{0}\frac{dz}{(1-z)^n}\int_1^\infty \frac{\log(u-1)\,du}{u^{n-1}}\right) \\
&= \frac{1}{d_{\mathfrak{c}}^{n-2}}\left(\frac{\log(d_{\mathfrak{c}})}{n-1}\frac{1}{n-2} + \frac{1}{n-1}\int_1^\infty \frac{\log(u-1)\,du}{u^{n-1}}\right).
\end{aligned}$$

We do one last change of coordinates to $w = 1/u$ with $dw = -du/u^2$ and apply Lemma 3.6 to obtain

$$\begin{aligned}
I_3 &= \frac{1}{(n-1)\,d_{\mathfrak{c}}^{n-2}}\left(\frac{\log(d_{\mathfrak{c}})}{n-2} + \int_0^1 \log\left(\frac{1}{w}-1\right) w^{n-3}dw\right) \\
&= \frac{1}{(n-1)\,d_{\mathfrak{c}}^{n-2}}\left(\frac{\log(d_{\mathfrak{c}})}{n-2} - \frac{H(n-3)}{n-2}\right) \\
&= \boxed{\frac{\log(d_{\mathfrak{c}})-H(n-3)}{(n-1)(n-2)\,d_{\mathfrak{c}}^{n-2}}}
\end{aligned}$$

Combining, we have our desired result.

$$\begin{aligned}
I &= I_1 - I_2 - I_3 \\
&= \frac{(n-2)\log(d_{\mathfrak{c}})+1}{(n-1)(n-2)^2\,d_{\mathfrak{c}}^{n-2}} + \frac{H(n-2)}{(n-1)(n-2)\,d_{\mathfrak{c}}^{n-2}} + \frac{H(n-3)-\log(d_{\mathfrak{c}})}{(n-1)(n-2)\,d_{\mathfrak{c}}^{n-2}} \\
&= \frac{1}{(n-1)(n-2)\,d_{\mathfrak{c}}^{n-2}}\left(\frac{1}{n-2}+H(n-2)+H(n-3)\right) \\
&= \frac{2\,H(n-2)}{(n-1)(n-2)\,d_{\mathfrak{c}}^{n-2}}.
\end{aligned}$$

$\square$

CHAPTER 6

Appendix

PROPOSITION 0.1. *For $k \geq 3$,*

$$\int_{-\pi/2}^{\pi/2} \cos^{k-2}(\theta) d\theta = \frac{\sqrt{\pi}\,\Gamma((k-1)/2)}{\Gamma(k/2)}.$$

PROOF. We proceed by induction on k. For $k = 3$, we have

$$\int_{-\pi/2}^{\pi/2} \cos(\theta) d\theta = 2 = \frac{\sqrt{\pi} \cdot 1}{\sqrt{\pi}/2} = \frac{\sqrt{\pi}\,\Gamma((3-1)/2)}{\Gamma(3/2)}.$$

We also need to compute for $k = 4$,

$$\int_{-\pi/2}^{\pi/2} \cos^2(\theta) d\theta = \left[\frac{\theta}{2} + \frac{\sin(\theta)\cos\theta}{2}\right]_{-\pi/2}^{\pi/2} = \frac{\pi}{2} = \frac{\sqrt{\pi}(\sqrt{\pi}/2)}{1} = \frac{\sqrt{\pi}\,\Gamma((4-1)/2)}{\Gamma(4/2)}.$$

Using the induction assumption for $k > 4$, we have

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \cos^{k-2}(\theta) d\theta &= \left[\frac{\cos^{k-3}(\theta)\sin(\theta)}{k-2}\right]_{-\pi/2}^{\pi/2} + \frac{k-3}{k-2}\int_{-\pi/2}^{\pi/2} \cos^{k-4}(\theta) d\theta \\ &= 0 + \frac{\sqrt{\pi}\,((k-3)/2)\,\Gamma((k-3)/2)}{((k-2)/2)\,\Gamma((k-2)/2)} = \frac{\sqrt{\pi}\,\Gamma((k-1)/2)}{\Gamma(k/2)}.\end{aligned}$$

□

PROPOSITION 0.2. *The pullback of ω from $\mathbb{S}^n$ to $\mathbb{R}^n$ via π has element*

$$\pi^*(d\omega) = \frac{2^n\, d\mathbf{x}}{(|\mathbf{x}|^2 + 1)^n}$$

PROOF. The induced Riemannian metric on the $\mathbb{S}^n$ from $\mathbb{R}^{n+1}$ in stereo graphic coordinates is given by

$$
\begin{aligned}
g_{\mathbb{S}^n} &= \left(dy_1^2 + \ldots + dy_{n+1}^2\right)\big|_{\mathbf{y}=\pi(\mathbf{x})} = \left(d\left(\frac{|\mathbf{x}|^2-1}{|\mathbf{x}|^2+1}\right)\right)^2 + \sum_{i=1}^n \left(d\left(\frac{2x_i}{|\mathbf{x}|^2+1}\right)\right)^2 \\
&= \left(\sum_{i=1}^n \frac{4x_i}{(|\mathbf{x}|^2+1)^2}\,dx_i\right)^2 + 4\sum_{i=1}^n\left(\frac{|\mathbf{x}|^2+1-2x_i^2}{(|\mathbf{x}|^2+1)^2}\,dx_i - \sum_{j=1,j\neq i}^n \frac{2\,x_i\,x_j}{(|\mathbf{x}|^2+1)^2}\,dx_j\right)^2 \\
&= \frac{4}{(|\mathbf{x}|^2+1)^2}\left(\left(\sum_{i=1}^n \frac{2x_i}{|\mathbf{x}|^2+1}\,dx_i\right)^2 + \sum_{i=1}^n\left(dx_i - \sum_{j=1}^n \frac{2\,x_i\,x_j}{|\mathbf{x}|^2+1}\,dx_j\right)^2\right)
\end{aligned}
$$

Let $\alpha = \sum_{i=1}^n x_i\,dx_i$, then

$$
\begin{aligned}
g_{\mathbb{S}^n} &= \frac{4}{(|\mathbf{x}|^2+1)^2}\left(\frac{4\,\alpha^2}{(|\mathbf{x}|^2+1)^2} + \sum_{i=1}^n\left(dx_i^2 + \frac{4\,x_i^2\,\alpha^2}{(|\mathbf{x}|^2+1)^2} - \frac{4x_i\,dx_i\,\alpha}{|\mathbf{x}|^2+1}\right)\right) \\
&= \frac{4}{(|\mathbf{x}|^2+1)^2}\left(\frac{4\,\alpha^2}{(|\mathbf{x}|^2+1)^2} + \frac{4\,|\mathbf{x}|^2\,\alpha^2}{(|\mathbf{x}|^2+1)^2} - \frac{4\,\alpha^2}{|\mathbf{x}|^2+1} + \sum_{i=1}^n dx_i^2\right) \\
&= \frac{4}{(|\mathbf{x}|^2+1)^2}\sum_{i=1}^n dx_i^2
\end{aligned}
$$

Since $g_{\mathbb{S}^n}$ is diagonal, it follows that

$$\pi^*\omega = \sqrt{\det g_{\mathbb{S}^n}}\,d\mathbf{x} = \frac{2^n\,d\mathbf{x}}{(|\mathbf{x}|^2+1)^n}$$

□

PROPOSITION 0.3. *Let $B_{\mathbb{H}^n}(R)$ denote a hyperbolic ball of radius R in $\mathbb{H}^n$, then*

$$
\begin{aligned}
\mathrm{Vol}\left(B_{\mathbb{H}^n}(R)\right) &= \mathrm{Vol}\left(\mathbb{S}^{n-1}\right)\frac{\sinh^n(R)}{n}\,{}_2F_1\left(\frac{1}{2},\frac{n}{2},\frac{n}{2}+1;-\sinh^2(R)\right) \\
&= \frac{\sinh^n(R)\,\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}\,{}_2F_1\left(\frac{1}{2},\frac{n}{2},\frac{n}{2}+1;-\sinh^2(R)\right)
\end{aligned}
$$

Note that there are several different transformations that can be applied to ${}_2F_1$ to get different version of this formula. We prefer this form because $\sinh(R)$ is the Euclidean radius of $B_{\mathbb{H}^n}(R)$ centered around $\mathbf{e}_n \in \mathbb{U}^n$.

Proof. The volume formula 2.6 gives

$$\mathrm{Vol}\left(B_{\mathbb{H}^n}(R)\right) = \mathrm{Vol}(\mathbb{S}^{n-1}) \int_0^R \sinh^{n-1}(\rho)\, d\rho = \mathrm{Vol}\left(\mathbb{S}^{n-1}\right) \int_0^1 \frac{\sinh^n(R)\, t^{(n-2)/2}}{2\sqrt{1+\sinh^2(R)\, t}}\, dt$$

where $\rho = \mathrm{arcsinh}(\sinh(R)\sqrt{t})$ with $d\rho = \sinh(R)\, dt / \left(2\sqrt{t}\sqrt{1+\sinh^2(R)\, t}\right)$.

It is straight forward to recognize this integral as a hypergeometric function with coefficients $a = 1/2,\ b = n/2,\ c = (n+2)/2$, and $z = -\sinh^2(R)$ (see 2.13). Therefore

$$\begin{aligned}
\mathrm{Vol}\left(B_{\mathbb{H}^n}(R)\right) &= \mathrm{Vol}\left(\mathbb{S}^{n-1}\right) \frac{\sinh^n(R)}{2} \frac{\Gamma(b)\,\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1(a,b,c;z) \\
&= \mathrm{Vol}\left(\mathbb{S}^{n-1}\right) \frac{\sinh^n(R)}{2} \frac{\Gamma\left(\frac{n}{2}\right)\Gamma(1)}{\Gamma\left(\frac{n}{2}+1\right)}\ {}_2F_1\left(\frac{1}{2}, \frac{n}{2}, \frac{n}{2}+1; -\sinh^2(R)\right) \\
&= \mathrm{Vol}\left(\mathbb{S}^{n-1}\right) \frac{\sinh^n(R)}{n}\ {}_2F_1\left(\frac{1}{2}, \frac{n}{2}, \frac{n}{2}+1; -\sinh^2(R)\right) \\
&= \frac{\sinh^n(R)\, \pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}\ {}_2F_1\left(\frac{1}{2}, \frac{n}{2}, \frac{n}{2}+1; -\sinh^2(R)\right)
\end{aligned}$$

$\square$

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