Notes for a talk on cohomology of compact Lie groups

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BASED ON THE PAPER OF MARK REEDER *"On the cohomology of compact Lie groups,"* L'ENSEIGNEMENT MATH., T. 41, (1995), PAGES 181-200 AND NOTES OF YANG ZHANG

1 Introduction

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and T a maximal torus of G with Lie algebra $\mathfrak{t}$. Let $W = N_G(T)/T$ be the Weyl group of T in G. Recall that W acts on $\mathfrak{t}$ through the Ad-representation. W is generated by reflections across kernels of roots of $\mathfrak{t}$ in $\mathfrak{g} \otimes \mathbb{C}$ or if you like the positive real roots.

The main result of these notes is that $H(G/T)$ vanishes in odd degrees. We will, in fact, provide a ring isomorphism $H(G/T)$ to a purely algebraic structure.

2 Background/Review

Let $\langle\ ,\ \rangle$ be the Ad-invariant inner product on $\mathfrak{g}$ (average all inner products on $\mathfrak{g}$ or take the negative of the Killing Form). We then have an orthogonal decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{t}$. For $X, Y, Z \in \mathfrak{g}$, the inner product satisfies $\langle [X,Y], X\rangle + \langle Y, [X,Z]\rangle = 0$. Note that $\mathrm{Ad}(T)$ has no nonzero invariant vectors in $\mathfrak{m}$ and no nonzero element of $\mathfrak{m}$ has zero bracket with all of $\mathfrak{t}$ (by the maximally of $\mathfrak{t}$ as an abelian subalgebra).

An element $H_0 \in \mathfrak{t}$ is called *regular* or *generic*, if the powers of $\exp H_0$ are dense in T. Note that $H_0 \in t$ is regular iff its $\mathrm{Ad}(G)$-centralizer is precisely $\mathrm{Ad}(T)$. For the remainder of this text, we choose some particular generic element $H_0 \in \mathfrak{t}$

Let $\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_\nu$ be an orthogonal decomposition given by the real irreducible representations of T, which are 2 dimensional. For $H \in t$, the eigenvalues of $\mathrm{Ad}(\exp H)$ on $\mathfrak{m}_i$ are $\{\exp(\pm\sqrt{-1}\,\alpha_i(H)\}$, where $\alpha_i \in \mathfrak{t}^*$. We let the set of positive roots $\Delta^+ = \{\alpha_1, \ldots, \alpha_\nu\}$ be the set of roots that take positive values on our generic element H_0. Note that since W acts faithfully on $\mathfrak{t}$, its image in $\mathrm{GL}(\mathfrak{t})$ is generated by reflections about the kernels of elements in Δ^+.

Since the $\mathfrak{m}_i$ are preserved by $\mathrm{ad}(\mathfrak{t})$, we can choose an orthonormal basis $\{X_i, X_{i+v}\}$ for $\mathfrak{m}_i$ such that the matrix for $\mathrm{ad}(H)\,|_{\mathfrak{m}_i}$ with $H \in \mathfrak{t}$ is

$$\begin{bmatrix} 0 & \alpha_i(H) \\ -\alpha_i(H) & 0 \end{bmatrix}.$$

By the ad-invariance of the inner product,

$$\langle H, [X_i, X_j] \rangle = -\langle [X_i, H], X_j \rangle = \langle [H, X_i], X_j \rangle = -\alpha_i(H) \langle X_{i+v}, X_j \rangle$$

for $1 \leq i \leq v$, $1 \leq j \leq 2v$. Above, the right hand side can be nonzero only if $j = i + v$. Thus, if $j \neq i \pm v$, then $[X_i, X_j] \in \mathfrak{m}$.

For $1 \leq i \leq v$, we let $H_i = [X_i, X_{i+v}]$, which is $\mathrm{Ad}(T)$-invariant so $H_i \in \mathfrak{m}$ and $\mathrm{ad}(H_i)\,\mathfrak{m}_i \subset \mathfrak{m}_i$. The span of X_i, X_{i+v}, H_i is a Lie subalgebra of $\mathfrak{g}$ that is actually isomorphic to $\mathfrak{su}(2)$.

3 Invariant Theory

Let $\mathscr{P} = \bigoplus_{p=0}^{\infty} \mathscr{P}^p$ be the symmetric algebra on $\mathfrak{t}^*$ (i.e. $\mathscr{P}^p = (\mathfrak{t}^*)^{\otimes p}/\sim$ where $\lambda_1 \otimes \ldots \otimes \lambda_p \sim \lambda_{\sigma(1)} \otimes \ldots \otimes \lambda_{\sigma(p)}$ for $\sigma \in S_p$). One can think of $\mathscr{P}$ as polynomials over $\mathbb{R}$ where the monomials are products of functionals on $\mathfrak{t}$. The adjoint action of W on $\mathfrak{t}$ induces an action/representations of W on $\mathscr{P}$ by degree-preserving algebra automorphisms (for $\lambda \in \mathfrak{t}^*$ and $w \in W$, the action is $\lambda \mapsto \lambda \circ \mathrm{Ad}(w^{-1})$). We will be interested in the W-invariant polynomials $\mathscr{P}^W$.

Example 1 For $U(n)$, $\mathscr{P}^W$ is generated by elementary symmetric polynomials. For $U(n)$, $\mathfrak{t}$ is the set of diagonal complex matrices with $a_j\sqrt{-1}$ on the diagonal and W acts as S_n $\mathfrak{t}$ on by permuting a_j.

Theorem 2 *(Chevalley) The ring $\mathscr{P}^W$ has algebraically independent homogeneous generators $F_1, \ldots, F_l$ with $\mathscr{P}^W = \mathbb{R}[F_1, \ldots, F_l]$, where $l = \dim \mathfrak{t}$. (Recall: algebraically independent means that the homomophism $\mathbb{R}[X_1, \ldots, X_l] \to \mathbb{R}[F_1, \ldots, F_l]$ given by $X_i \mapsto F_i$ is an isomorphism)*

The generators are numbered such that $\deg F_1, \leq \ldots, \deg F_l$. We will call the numbers $m_i = \deg F_i - 1$ the *exponents* of W acting on $\mathfrak{t}$. It is known that $m_1 + \ldots + m_l = v$ and $(1 + m_1) \ldots (1 + m_l) = |W|$.

Example 3 For $SU(n)$, $\{m_i\}$ is $\{1, \ldots, n-1\}$ and for G_2 they are $\{1, 5\}$. Note that for $SU(n)$ you loose the generator in degree 1, which you had for $U(n)$, because of linear dependence. For G_2, the Lie algebra of T is that of $SU(3)$ but the action of W is extended by an inversion.

Let $\mathscr{D}$ be the ring of constant coefficient differential operators on $\mathscr{P}$. We can think of $\mathscr{D}$ as the symmetric algebra $S(\mathfrak{t})$, where $H \in \mathfrak{t}$ corresponds to the function on $\mathfrak{t}^*$ given by evaluation at H (e.g. $H \cdot (\lambda_1\lambda_2) = \lambda_1(H)\lambda_2 + \lambda_2(H)\lambda_1$ or the directional derivative for the vector H). We have that W acts naturally on $\mathscr{D}$ (by it's action on $S(\mathfrak{t})$) and we define the "harmonic polynomials" in $\mathscr{P}$ to be those annihilated by the W-invariant differential operators

$$\mathscr{H} = \{f \in \mathscr{P} : \mathscr{D}^W f = 0\}.$$

One can think of $\mathscr{H}$ as the solution to a set of differential equations.

Let $\mathscr{H}^p = \mathscr{H} \cap \mathscr{P}^p$, then $\mathscr{H} = \oplus_p \mathscr{H}^p$ since a differential operator is W invariant if and only if each homogeneous component in W invariant (think of about the action of W on $S(\mathfrak{t})$). Note that the action of W on $\mathscr{P}$ preserves $\mathscr{H}$ (for $g \in W$, $p \in \mathscr{P}$, $D \in \mathscr{D}$, we have that $D(g \cdot p) = (g^{-1} \cdot D)(p)$).

Proposition 4 *If $\mathscr{J}$ is the ideal generated by the elements of $\mathscr{P}^W$ of positive degree, then $\mathscr{P} = \mathscr{H} \oplus \mathscr{J}$ and multiplication is a linear isomorphism $\mathscr{H} \otimes \mathscr{P}^W \xrightarrow{\sim} \mathscr{P}$.*

The former gives us that $\mathscr{P}/\mathscr{J}$ is isomorphic to $\mathscr{H}$ as W modules (Note: they are in fact isomorphic to the regular representation of W). The isomorphism $\mathscr{H} \otimes \mathscr{P}^W \simeq \mathscr{P}$ implies

$$\sum_{p \geq 0} \dim \mathscr{H}^p t^p = \prod_{i=1}^{l} (1 + t + t^2 + \ldots + t^{m_i}) \text{ (where } l = \dim \mathfrak{t})$$

which shows that $\dim \mathscr{H}^v = 1$ and $\mathscr{H}^p = 0$ for $p > v$. This formula is deduced from

$$\sum_p \dim \mathscr{P}^p t^p = \left(\sum_p \dim \mathscr{H}^p t^p\right)\left(\sum_p \dim(\mathscr{P}^W \cap \mathscr{P}^p) t^p\right),$$

$$\sum_p \dim \mathscr{P}^p t^p = (1 + t + t^2 + \ldots)^l = \frac{1}{(1-t)^l}, \text{ and}$$

$$\sum_p \dim(\mathscr{P}^W \cap \mathscr{P}^p) t^p = \prod_{i=1}^{l} \frac{1}{(1 - t^{m_i+1})}.$$

The primordial harmonic polynomial is $\Pi = \prod_{\alpha \in \Delta^+} \alpha \in \mathscr{H}^v$. For $U(n)$ this is the Vandermonde determinant $\prod_{i<j}(x_i - x_j)$, which is transformed by the sign character

via the action of S_n. In general, W acts like the sign character on the span of Π, where the sign character $\varepsilon : W \to \{\pm 1\}$ gives the parity of the number of reflections for each $g \in W$. Any other polynomial whose span is preserved by the action of the sign character vanishes on all root hyperplanes and so is divisible by Π. Thus Π generates $\mathscr{H}^v$ as $\dim \mathscr{H}^v = 1$.

We may now state the theorem we will discuss at the end of this talk

Theorem 5 *(Borel) There is a degree-doubling W-equivariant ring isomorphism*

$$c : \mathscr{P}/\mathscr{J} \to H(G/T).$$

Consequently, $\mathscr{H}_{(2)} \simeq H(G/T)$, where the subscript indicated degree doubling.

4 Invariant Differential Forms

Let G act transitively on a manifold M (think $M = G/T$). If τ_g is the diffeomorphism given by $g \in G$, then a differential p-form $\omega \in \Omega^p(M)$ is G-invariant if $\tau_g^* \omega = \omega$ for all $g \in G$. Since G acts transitively, such a form is determined by its value at one point on M.

Lemma 6 *Every de Rham cohomology class of M is represented by a G-invariant form and the complex of G-invariant forms is preserved by the exterior derivative.*

Definition 7 We define $\Lambda^p \mathfrak{n}^*$ as the set of all skew-symmetric multilinear maps $\omega : \mathfrak{n} \times \ldots \times \mathfrak{n} \to \mathbb{R}$ where the domain has p terms.

Proposition 8 *The complex $\{(\Lambda^p \mathfrak{n}^*)^K, \delta\}$ computes $H^*(M)$, where K is the stabilizer of a point $o \in M$, $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{n}$ with $\mathfrak{r}$ the Lie algebra of K, and δ is defined below.*

Proof Identify $M = G/K$ and note that $T_o(M)$ is naturally identified with $\mathfrak{n}$. Thus, an invariant form $\tilde{\omega}$ is determined by a skew-symmetrc multilinear map

$$\omega = \tilde{\omega}_o : \mathfrak{n} \times \ldots \times \mathfrak{n} \to \mathbb{R},$$

that is $\omega \in \Lambda^p \mathfrak{n}^*$. The invariance of $\tilde{\omega}$ under K implies that ω is $\mathrm{Ad}(K)$ invariant. Conversely, any element $\omega \in (\Lambda^p \mathfrak{n}^*)^K$ determines a G invariant form $\tilde{\omega}$ by

$$\tilde{\omega}_{g \cdot o}((d\tau_g)X_1, \ldots, (d\tau_g)X_p) = \omega(X_1, \ldots, X_p),$$

for $X_1, \ldots, X_p \in \mathfrak{n} \simeq T_o(M)$ and $g \in G$. Thus, we may identify the G-invariant p-forms with $(\Lambda^p \mathfrak{n}^*)^K$. The exterior derivative then becomes $\delta : (\Lambda^p \mathfrak{n}^*)^K \to (\Lambda^{p+1} \mathfrak{n}^*)^K$ given by

$$\delta\omega(X_0, \ldots, X_p) = \frac{1}{p+1} \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j]_{\mathfrak{n}}, X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p).$$

Where $[X_i, X_j]_{\mathfrak{n}}$ is the projection of $[X_i, X_j]$ on $\mathfrak{n}$ along $\mathfrak{r}$ and $\hat{}$ means the term is omitted. By the Lemma, the complex $\{(\Lambda^p \mathfrak{n}^*)^K, \delta\}$ computes $H^*(M)$.

□

Example 9 Define $\omega(X, Y, Z) = \langle X, [Y, Z] \rangle$ then $[\omega] \neq 0 \in H^3(G)$. In particular, S^n is not a Lie group for $n > 3$.

5 Cohomology of Flag Manifolds

We will use Morse Theory to show that the odd dimensional cohomology of G/T vanishes. We can further use this approach to decompose the flag manifold G/T into cells. This is called the *Bruhat Decomposition*. This process will be the generalization of decomposing the $S^2 = SU(2)/T$ into a 0-cell and a 2-cell.

We will find a Morse function f on G/T. For a smooth manifold M, a morse function $f : M \to \mathbb{R}$ is a smooth function with non-singular Hessian $H_x f$ at each critical point x. The function we find will be the analogue of the dot product of vectors on a 2-sphere with the vector pointing to the north pole. The span of the gradient flow lines emanating from a critical point will provide us with a cell decomposition. For the sphere the flow lines from the south pole give us the 2-cell and the north pole, which has no flow lines emanating, gives us the 0-cell.

If f is a Morse function and x is critical point, let $\lambda(x)$ be the number of negative eigenvalues of $H_x f$. Then the Morse polynomial is $\mathcal{M}_t(f) = \sum t^{\lambda(x)}$ over the critical points x of f.

Theorem 10 *For a morse function $f : M \to \mathbb{R}$, we have that $\mathcal{M}_t(f) \geq \sum_i \dim H^i(M) t^i$. Moreover, if the morse polynomial has no consecutive exponents, equality holds.*

To construct a Morse function on G/T, we take the regular element $H_0 \in \mathfrak{t}$ that we chose for the positive roots. Recall that the $\mathrm{Ad}(G)$ centralizer of H_0 is exactly $\mathrm{Ad}(T)$, so we may view $G/T \subset \mathfrak{g}$ as the $\mathrm{Ad}(G)$ orbit of H_0 (analogous to $S^2 \subset \mathbb{R}^3$). We define $f : G/T \to \mathbb{R}$ by

$$f(gT) = \langle \mathrm{Ad}(g)H_0, H_0 \rangle.$$

For $X \in \mathfrak{g}$, we can compute the vector field

$$\bar{X}f(gT) = \frac{d}{ds} f(\exp(sX)gT) \mid_{s=0} = \langle \mathrm{Ad}(g)H_0, [H_0, X] \rangle,$$

where the last equality is given by ad invariance of the inner product. Since the centralizer of H_0 in $\mathfrak{g}$ is exactly $\mathfrak{t}$ as H_0 is regular, it follows that the image of $\mathrm{ad}(H_0)$ is $\mathfrak{m}$. So gT is a critical point of f if and only if $\langle \mathrm{Ad}(g)H_0, \mathfrak{m} \rangle = 0$. Therefore, $\mathrm{Ad}(g)H_0 \in \mathfrak{t}$ by the orthogonal decomposition of $\mathfrak{g}$. It follows that $\mathrm{Ad}(g)H_0 = \mathrm{Ad}(w)H_0$ for some $w \in W$ and that wT, for $w \in W$, are precisely the critical points of f.

Let $X_1, \dots, X_{2\nu}$ be the orthonormal basis for $\mathfrak{m}$ we discussed earlier. Note that the differential of $\pi : G \to G/T$ maps $\mathrm{Ad}(w)\,\mathfrak{m} = \mathfrak{m}$ isomorphically onto $T_{wT}(G/T)$, so we may use our basis to compute the Hessian at each point wT. If h_{ij} is the ij entry in $H_{wT}f$, then using our identities for the inner product

$$h_{ij}(wT) = \bar{X}_i\bar{X}_j f(wT) = \langle [X_i, \mathrm{Ad}(w)H_0], [H_0, X_j] \rangle = -\alpha_i(\mathrm{Ad}(w)H_0)\alpha_j(H_0)\langle X_{i\pm\nu}, X_{j\pm\nu} \rangle.$$

Note that it follows that $h_{ij} = 0$ for $i \neq j$ and $h_{ii}(wT) = -\alpha_i(\mathrm{Ad}(w)H_0)\alpha_i(H_0)$. Since H_0 is regular, then so is $\mathrm{Ad}(w)H_0$ and therefore $h_{ii}(w) \neq 0$ and $H_{wT}f$ is non singular. Thus, as $\dim \mathfrak{m} = 2\nu$, the index $\lambda(wT)$ is twice the number $m(w)$ of positive roots α such that $H \mapsto \alpha(\mathrm{Ad}(w)H)$ (i.e. $w^{-1} \cdot \alpha$) is again a positive root.

The Morse polynomial of f is then $M_t(f) = \sum_{w \in W} t^{2m(w)}$. Since all the exponents of $M_f(t)$ are odd, $M_t(f) = \sum_i H^i(M)t^i$ and it follows that $H^i(M) = 0$ for i odd. In particular, $\sum_i \dim H^{2i}(G/T) = |W|$.

The *Schubert cell* X_w in the Bruhat Decomposition is the cell spanned by the flow lines of the gradient of f emanating from wT. The dimension of this cell is then the number of positive eigenvalues of the $H_{wT}f$, or, equivalently, twice the number of positive roots that become negative under $w^{-1} \cdot \alpha$.

Note that W acts on G/T by $w \cdot gT = gw^{-1}T$, which gives us an action of W on $H(G/T)$. Since $H(G/T)$ vanishes in odd degrees, the Lefschetz number associated to w is equal to the trace of its action on $H(G/T)$. If $w \neq 1$, then it has not fixed points so the Lefshetz number is zero. If $w = 1$, then the Lefshetz number is simply the Euler characteristic, which is $|W|$. Hence, the action is that of the regular representation, so $H(G/T) \simeq \mathbb{R}[W]$ as W-modules.

We can now give the proof of our final result, which we restate here.

Theorem 11 *(Borel) There is a degree-doubling W-equivariant ring isomorphism*

$$c : \mathscr{P}/\mathscr{J} \to H(G/T).$$

Consequently, $\mathscr{H}_{(2)} \simeq H(G/T)$, where the subscript indicated degree doubling.

Proof The idea is to describe $H(G/T)$ in terms of G-invariant differential forms. For each $\lambda \in \mathfrak{t}^*$, we extend λ to all of $\mathfrak{g}$ by making it zero on $\mathfrak{m}$ and define an $\mathrm{Ad}(T)$-invariant 2-form on $\mathfrak{m}$ by

$$\omega_\lambda(X,Y) = \lambda([X,Y]).$$

We can identify ω_λ with an honest G-invariant differential form $\tilde{\omega}_\lambda$ as before. The action of W on G-invariant forms is given by its action on G/T. One can compute that $w \cdot \omega_\lambda = \omega_{w\cdot\lambda}$. Further, the Jacobi identity implies that $\delta\omega_\lambda(X,Y,Z) = \frac{1}{3}([[X,Z]_\mathfrak{m}, Y] - [[X,Y]_\mathfrak{m}, Z] - [[Y,Z]_\mathfrak{m}, X]) = 0$. We let $c(\lambda) = [\tilde{\omega}_\lambda] \in H^2(G/T)$ and extend it to degree-doubling map

$$c : \mathscr{P} \to H(G/T)$$

which preserves the W-action on both sides. Since $H(G/T)$ is the regular representation of W, its W-invariants are 1-dimensional and can therefore only occur in $H^0(G/T)$. Since c is W-equivariant, it follows that the kernel of c contains the ideal $\mathscr{J}$. The rest of the proof deals with showing that $\mathscr{J}$ is exactly the kernel of c.

To prove that $\ker c = \mathscr{J}$, it suffices to show that c is injective on $\mathscr{H}$ as $\mathscr{P} = \mathscr{H} \oplus \mathscr{J}$. This is done by induction starting at the highest degree of 2ν and descending down. For degree 2ν it suffices to show that $c(\Pi)$, where Π is the primordial harmonic polynomial, is non zero in $H^{2\nu}(G/T)$.

For each root $\alpha_i \in \Delta^+$, we have element $X_i, X_{i+\nu}$ that form a basis for $\mathfrak{m}_i$ such that $[X_i, X_{i+\nu}] = H_i \operatorname{im} \mathfrak{t}$. Recall that $[X_i, X_j] \in \mathfrak{m}$ is $j \neq i+\nu$ where $1 \le i \le \nu$. For each i, write $\omega_i = \omega_{\alpha_i}$. Then by definition $c(\Pi) = [\tilde{\omega}_{\alpha_1} \wedge \ldots \wedge \tilde{\omega}_{\alpha_\nu}]$ and we can evaluate

$$\omega_1 \wedge \ldots \wedge \omega_\nu(X_1, X_{1+\nu}, \ldots, X_\nu, X_{2\nu}) =$$

$$= \frac{1}{(2\nu)!} \sum_{\sigma \in S_{2\nu}} sgn(\sigma)\omega_1(X_{\sigma(1)}, X_{\sigma(1+\nu)}) \cdots \omega_\nu(X_{\sigma(\nu)}, X_{\sigma(2\nu)}) =$$

$$= \frac{1}{(2\nu)!} \sum_{\sigma \in S_{2\nu}} sgn(\sigma)\alpha_1([X_{\sigma(1)}, X_{\sigma(1+\nu)}]) \cdots \alpha_\nu([X_{\sigma(\nu)}, X_{\sigma(2\nu)}])$$

Since $\alpha_i([X_{\sigma(i)}, X_{\sigma(i+\nu)}]) = 0$ unless $[X_\sigma i, X_{\sigma(i+\nu)}] \in \mathfrak{m}$, the term for σ is zero unless σ permutes the pairs $\{i, i+\nu\}$, and possibly switches the order of members. Note that $\sigma(\sigma)$ is minus one the number of switches, so it follows that

$$\omega_1 \wedge \ldots \wedge \omega_\nu(X_1, X_{1+\nu}, \ldots, X_\nu, X_{2\nu}) =$$

$$= \frac{2^\nu}{(2\nu)!} \sum_{\sigma \in S_\nu} \alpha_1([X_{\sigma(1)}, X_{\sigma(1)+\nu}]) \cdots \alpha_\nu([X_{\sigma(\nu)}, X_{\sigma(\nu)+\nu}]) =$$

$$= \frac{2^\nu}{(2\nu)!} \sum_{\sigma \in S_\nu} \alpha_1(H_{\sigma(1)}) \cdots \alpha_\nu(H_{\sigma(\nu)}) = \frac{2^\nu}{(2\nu)!} \partial_1 \cdots \partial_\nu \Pi$$

where ∂_i is the derivation of $\mathscr{P}$ extending $\lambda \mapsto \lambda(H_i)$. Since the pairing $\mathscr{D} \otimes \mathscr{P} \to \mathbb{R}$ given by $(D,f) \mapsto (Df)(0)$ is perfect, it follows that there is a degree ν differential operator that pairs non trivially with Π. Further, since an irreducible W-module can only pair non trivially with its dual, and the self-dual character ε ocurs with multiplicity one in $\mathscr{D}^\nu$, afforded by $\partial_1 \cdots \partial_\nu$, it follows that $\partial_1 \cdots \partial_\nu \Pi \neq 0$ and $c(\Pi) \neq 0$.

We may now inductively assume that $c : \mathscr{H}^k \to H^{2k}(G/T)$ is injective for some $k \leq \nu$. Let $V = \mathscr{H}^{k-1} \cap \ker c$. Note that V is preserved by W since c is W-equivariant. Since the sign character is absent from $\mathscr{H}^{k-1}$, there is a possible root α such that the reflection s_α along the associated hyperplane does not act like $-I$ on V. We can then decompose $V = V_+ \oplus V_-$ according to the eigenspaces of s_α. If $V \neq 0$, then $V_+ \neq 0$ so we may take some $f \in V_+$. Now $c(\alpha f) = c(\alpha)c(f) = 0$ and αf is in degree k, so $\alpha f \in \mathscr{J}$ by assumption. Let $h_1, \ldots, h_{|W|}$ be a basis for $\mathscr{H}$ with $h_1, \ldots, h_r$ s_α-skew and the rest s_α-invariant. By Chevalley's Theorem, we can write $\alpha f = \sum_i h_i \tau_i$, with τ_i W-invariant of positive degree. Since αf is s_α-skew by construction, the sum only goes up to r. For $i \leq r$, the polynomial h_i must vanish on $\ker \alpha$ and therefore $h_i = \alpha h_i'$ for some $h_i' \in \mathscr{P}$. Then it follows that $f = \sum_{i=1}^r h_i' \tau_i \in \mathscr{J}$ and f is harmonic. Thus, we must have that $f = 0$ and c is injective on $\mathscr{H}^{k-1}$. By induction, c is injective and since $H(G/T)$ vanishes in odd degree, the proof is complete.

□