Q CURVATURE ON A CLASS OF MANIFOLDS WITH DIMENSION AT LEAST 5

FENGBO HANG AND PAUL C. YANG

ABSTRACT. For a smooth compact Riemannian manifold with positive Yamabe invariant, positive Q curvature and dimension at least 5, we prove the existence of a conformal metric with constant Q curvature. Our approach is based on the study of extremal problem for a new functional involving the Paneitz operator.

1. INTRODUCTION

Recall the definition of the fourth order Paneitz operator and its associated Q curvature [B, P]: when (M, g) is a smooth compact n dimensional Riemannian manifold with $n \neq 5$, the Q curvature is given by

$$Q = -\frac{1}{2(n-1)}\Delta R - \frac{2}{(n-2)^2} |Rc|^2 + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} R^2 \quad (1.1)$$

= $-\Delta J - 2|A|^2 + \frac{n}{2}J^2.$

Here R is the scalar curvature, Rc is the Ricci tensor and

$$J = \frac{R}{2(n-1)}, \quad A = \frac{1}{n-2} \left(Rc - Jg \right).$$
(1.2)

The Paneitz operator is given by

$$P\varphi$$

$$= \Delta^{2}\varphi + \frac{4}{n-2}\operatorname{div}\left(Rc\left(\nabla\varphi, e_{i}\right)e_{i}\right) - \frac{n^{2}-4n+8}{2\left(n-1\right)\left(n-2\right)}\operatorname{div}\left(R\nabla\varphi\right) + \frac{n-4}{2}Q\varphi$$

$$= \Delta^{2}\varphi + \operatorname{div}\left(4A\left(\nabla\varphi, e_{i}\right)e_{i} - \left(n-2\right)J\nabla\varphi\right) + \frac{n-4}{2}Q\varphi.$$

$$(1.3)$$

Here e_1, \dots, e_n is a local orthonormal frame with respect to g. Under conformal change of the metric, the operator satisfies

$$P_{\rho^{\frac{4}{n-4}}g}\varphi = \rho^{-\frac{n+4}{n-4}}P_g\left(\rho\varphi\right). \tag{1.4}$$

This is similar to the conformal Laplacian operator, which appears naturally when considering transformation law of the scalar curvature under conformal change of metric ([LP]). As a consequence we have

$$P_{\rho^{\frac{4}{n-4}}g}\varphi\cdot\psi d\mu_{\rho^{\frac{4}{n-4}}g} = P_g\left(\rho\varphi\right)\cdot\rho\psi d\mu_g.$$
(1.5)

Here μ_g is the measure associated with metric g.

The research of Yang is supported by NSF grant 1104536.

In dimension four, the Paneitz operator is given by

$$P\varphi = \Delta^2 \varphi + 2\operatorname{div}\left(\operatorname{Rc}\left(\nabla\varphi, e_i\right)e_i\right) - \frac{2}{3}\operatorname{div}\left(\operatorname{R}\nabla\varphi\right),\tag{1.6}$$

and its conformal covariance property takes the following form

$$P_{e^{2w}g}\varphi = e^{-4w}P_g\varphi. \tag{1.7}$$

Following the basic work [CGY] in dimension four on the fourth order Q curvature equation, there has been several studies on this equation in dimension three by [HY1, XY, YZ], and in dimensions greater than four by [DHL, DM, HeR1, HeR2, HuR, QR1, QR2].

While it is important to determine conditions under which the Paneitz operator is positive, we discover that it is sufficient for our purpose in this article to determine when its Green's function is positive. This is a property that is conformally invariant: observe that by (1.4),

$$\ker P_g = 0 \Leftrightarrow \ker P_{\rho^{\frac{4}{n-4}}g} = 0, \tag{1.8}$$

and under this assumption, the Green's functions G_P satisfy the transformation law

$$G_{P,\rho^{\frac{4}{n-4}}g}(p,q) = \rho(p)^{-1} \rho(q)^{-1} G_{P,g}(p,q).$$
(1.9)

In analogy with the preliminary study of the classical Yamabe problem ([LP]), the first question would be whether one can find a conformal invariant condition for the existence of a conformal metric with positive Q curvature. In the case Yamabe invariant Y(g) > 0, the existence of a conformal metric with positive Q curvature is equivalent to the requirements that ker P = 0 and the Green's function $G_P > 0$ ([HY4]).

The basic question of interest is to find constant Q curvature metric in a conformal class, in the same spirit as Yamabe problem. The main aim of the present article is to prove the following

Theorem 1.1. Let (M, g) be a smooth compact n dimensional Riemannian manifold with $n \ge 5$, Y(g) > 0, $Q \ge 0$ and not identically zero, then ker P = 0, the Green's function of P is positive and there exists a conformal metric \tilde{g} with $\tilde{Q} = 1$.

The fundamental difficulty of the lack of maximum principle in this fourth order equation has recently been overcome by the work in [GM]. Following this development, similar results in dimension 3 were proved in [HY3, HY4]. Dimension 4 case does not suffer from this difficulty and was treated in many articles like [CY, DM, FR] and so on. For locally conformally flat manifold with positive Yamabe invariant and Poincare exponent less than $\frac{n-4}{2}$ (see [SY]), Theorem 1.1 was proved in [QR2] by apriori estimates and connecting the equation to Yamabe equation through a path of integral equations. Under the slightly more stringent conditions R > 0 and Q > 0, Theorem 1.1 as well as the positivity of mass of the 4th order Paneitz operator was proved in [GM] through the study of a non-local flow. Here we will derive Theorem 1.1 by maximizing a functional (see (1.16) and (2.2)) involving the Paneitz operator (see Theorem 1.3 for more details). For $u, v \in C^{\infty}(M)$, we denote the quadratic form associated with P as

$$E(u,v)$$

$$= \int_{M} Pu \cdot v d\mu$$

$$= \int_{M} \left(\Delta u \Delta v - \frac{4}{n-2} Rc \left(\nabla u, \nabla v \right) + \frac{n^2 - 4n + 8}{2 \left(n - 1 \right) \left(n - 2 \right)} R \nabla u \cdot \nabla v + \frac{n - 4}{2} Quv \right) d\mu$$

$$= \int_{M} \left(\Delta u \Delta v - 4A \left(\nabla u, \nabla v \right) + \left(n - 2 \right) J \nabla u \cdot \nabla v + \frac{n - 4}{2} Quv \right) d\mu,$$
(1.10)

and

$$E(u) = E(u, u).$$
 (1.11)

By the integration by parts formula in (1.10) we know E(u, v) also makes sense for $u, v \in H^2(M)$.

To find the metric \tilde{g} in Theorem 1.1, we write $\tilde{g} = \rho^{\frac{4}{n-4}}g$, then the equation $\tilde{Q} = 1$ becomes

$$P_{g}\rho = \frac{n-4}{2}\rho^{\frac{n+4}{n-4}}, \quad \rho \in C^{\infty}(M), \rho > 0.$$
(1.12)

Let

$$Y_4(g) = \inf_{u \in H^2(M) \setminus \{0\}} \frac{E(u)}{\|u\|_{L^{\frac{2n}{n-4}}}^2},$$
(1.13)

then $Y_4\left(\tau^{\frac{4}{n-4}}g\right) = Y_4(g)$ for any positive smooth function τ . Hence $Y_4(g)$ is a conformal invariant. If (M,g) is not locally conformally flat and $n \geq 8$, or (M,g) is locally conformally flat with Y(g) > 0, ker P = 0 and the Green's function of $P, G_P > 0$, or n = 5, 6, 7 with Y(g) > 0, ker P = 0 and $G_P > 0$, one can show $Y_4(g)$ is achieved (see [ER, R, GM]), but in general it is difficult to know whether the minimizer is positive. Under the additional assumption $Y_4(g) > 0$ and $G_P > 0$, it was observed in [R] that the minimizer can not change sign. Combine with the positivity criterion of Green's function in [HY4], we arrive at

Theorem 1.2. Let (M, g) be a smooth compact n dimensional Riemannian manifold with $n \ge 5$, Y(g) > 0, $Y_4(g) > 0$, $Q \ge 0$ and not identically zero, then

- (1) $Y_4(g) \leq Y_4(S^n)$, equality holds if and only if (M,g) is conformal diffeomorphic to the standard sphere.
- (2) $Y_4(g)$ is always achieved. Any minimizer must be smooth and can not change sign. In particular we can find a constant Q curvature metric in the conformal class.
- (3) If (M,g) is not conformal diffeomorphic to the standard sphere, then the set of all minimizers u for $Y_4(g)$, after normalizing with $||u||_{L^{\frac{2n}{n-4}}} = 1$, is compact in C^{∞} topology.

In general it is not known whether $Y(g) > 0, Q \ge 0$ and not identically zero would imply $Y_4(g) > 0$. To get around this difficulty when proving Theorem 1.1 we note that by [HY4, Proposition 1.1] if $Y(g) > 0, Q \ge 0$ and not identically zero

- 3

then ker P = 0, and the Green's function of P, $G_P > 0$. Hence we can define an integral operator (the inverse of P) as

$$G_{P}f(p) = \int_{M} G_{P}(p,q) f(q) d\mu(q).$$
 (1.14)

If we denote $f = \rho^{\frac{n+4}{n-4}}$, then equation (1.12) becomes

$$G_P f = \frac{2}{n-4} f^{\frac{n-4}{n+4}}, \quad f \in C^{\infty}(M), f > 0.$$
 (1.15)

Let

$$\Theta_{4}(g) = \sup_{\substack{f \in L^{\frac{2n}{n+4}}(M) \setminus \{0\}}} \frac{\int_{M} G_{P} f \cdot f d\mu}{\|f\|_{L^{\frac{2n}{n+4}}}^{2}}$$
(1.16)
$$= \sup_{\substack{f \in L^{\frac{2n}{n+4}}(M) \setminus \{0\}}} \frac{\int_{M \times M} G_{P}(p,q) f(p) f(q) d\mu(p) d\mu(q)}{\|f\|_{L^{\frac{2n}{n+4}}}^{2}}.$$

It follows from the classical Hardy-Littlewood-Sobolev inequality ([S]) that $\Theta_4(g)$ is always finite. Moreover it follows from (1.9) that for positive smooth function ρ , $\Theta_4\left(\rho^{\frac{4}{n-4}}g\right) = \Theta_4(g)$ i.e. $\Theta_4(g)$ is a conformal invariant. If $\Theta_4(g)$ is achieved by a maximizer f, using the fact $G_P > 0$, we easily deduce that f can not change sign. $\Theta_4(g)$ has a nice invariant description (see Lemma 2.1):

$$\Theta_4(g) = \frac{2}{n-4} \sup\left\{ \frac{\int_M \widetilde{Q} d\widetilde{\mu}}{\left\|\widetilde{Q}\right\|_{L^{\frac{2n}{n+4}}(M,d\widetilde{\mu})}^2} : \widetilde{g} \in [g] \right\}$$
(1.17)

Here [g] denotes the conformal class of g i.e.

$$[g] = \left\{ \rho^2 g : \rho \in C^{\infty}(M), \rho > 0 \right\}.$$
 (1.18)

Theorem 1.3. Assume (M,g) is a smooth compact n dimensional Riemannian manifold with $n \ge 5$, Y(g) > 0, $Q \ge 0$ and not identically zero, then

- (1) $\Theta_4(g) \ge \Theta_4(S^n)$, here S^n has the standard metric. $\Theta_4(g) = \Theta_4(S^n)$ if and only if (M,g) is conformal diffeomorphic to the standard sphere.
- (2) $\Theta_4(g)$ is always achieved. Any maximizer f must be smooth and can not change sign. Say f > 0, then after scaling we have $G_P f = \frac{2}{n-4} f^{\frac{n-4}{n+4}}$ i.e. $Q_{f^{\frac{4}{n+4}}g} = 1$.
- (3) If (M, g) is not conformal diffeomorphic to the standard sphere, then the set of all maximizers f for $\Theta_4(g)$, after normalizing with $||f||_{L^{\frac{2n}{n+4}}} = 1$, is compact in C^{∞} topology.

It is worthwhile to note the similarity of Theorem 1.2 and 1.3 to classical Yamabe problem ([LP]) and the integral equation considered in [HWY1, HWY2]. Indeed, the formulation of our approach follows that of [HWY2]. A similar functional for the conformal Laplacian operator, Θ_2 (see (5.19)) is also considered in [DZ]. In section 2 below we will first give other expressions for $\Theta_4(g)$ and discuss its relation with $Y_4(g)$, then we will derive an almost sharp Sobolev inequality related to extremal problem of $\Theta_4(g)$ and find the asymptotic expansion formula for the Green's function of Paneitz operator. In section 3 we will apply the concentration compactness principle to deduce a criterion for the existence of maximizers of $\Theta_4(g)$. In section 4 we will show maximizers always exist and they are smooth. In particular Theorem 1.3 follows. At last in section 5 we will prove Theorem 1.2. Moreover we will show the approach to Theorem 1.3 gives another way to find constant scalar curvature metrics in a conformal class.

The authors would like to thank Gursky and Malchiodi for making their work available.

2. Some preparations

2.1. The conformal invariants $Y_4(g)$, $Y_4^+(g)$ and $\Theta_4(g)$. Throughout this subsection we will assume (M, g) is a smooth compact n dimensional Riemannian manifold with $n \geq 5$. Recall

$$Y_4(g) = \inf_{u \in H^2(M) \setminus \{0\}} \frac{E(u)}{\|u\|_{L^{\frac{2n}{n-4}}}^2} = \inf_{u \in C^{\infty}(M) \setminus \{0\}} \frac{\int_M Pu \cdot ud\mu}{\|u\|_{L^{\frac{2n}{n-4}}}^2}.$$
 (2.1)

If in addition Y(g) > 0, $Q \ge 0$ and not identically zero, then

$$\Theta_{4}(g) = \sup_{f \in L^{\frac{2n}{n+4}}(M) \setminus \{0\}} \frac{\int_{M} G_{P} f \cdot f d\mu}{\|f\|_{L^{\frac{2n}{n+4}}}^{2}}$$

$$= \sup_{u \in W^{4,\frac{2n}{n+4}}(M) \setminus \{0\}} \frac{\int_{M} Pu \cdot u d\mu}{\|Pu\|_{L^{\frac{2n}{n+4}}}^{2}}.$$
(2.2)

The second equality in (2.2) is very useful for us later on because the expression is local. It will facilitate our calculations in estimating $\Theta_4(g)$. $\Theta_4(g)$ also has an invariant description.

Lemma 2.1. If $n \ge 5$, Y(g) > 0, $Q \ge 0$ and not identically zero, then

$$\Theta_4(g) = \frac{2}{n-4} \sup\left\{ \frac{\int_M \widetilde{Q} d\widetilde{\mu}}{\left\|\widetilde{Q}\right\|_{L^{\frac{2n}{n+4}}(M,d\widetilde{\mu})}^2} : \widetilde{g} \in [g] \right\}.$$
(2.3)

Here [g] is the conformal class of Riemannian metrics associated with g.

Proof. Note that

$$\frac{2}{n-4} \sup \left\{ \frac{\int_{M} \widetilde{Q} d\widetilde{\mu}}{\left\|\widetilde{Q}\right\|_{L^{\frac{2n}{n+4}}(M,d\widetilde{\mu})}^{2}} : \widetilde{g} \in [g] \right\}$$
$$= \sup \left\{ \frac{\int_{M} Pu \cdot u d\mu}{\left\|Pu\right\|_{L^{\frac{2n}{n+4}}}^{2}} : u \in C^{\infty}(M), u > 0 \right\}$$
$$\leq \Theta_{4}(g).$$

On the other hand, by the positivity of G_P we have

$$\begin{split} &\Theta_{4}\left(g\right)\\ = &\sup\left\{\frac{\int_{M}G_{P}f\cdot fd\mu}{\left\|f\right\|_{L^{\frac{2n}{n+4}}}^{2}}:f\in L^{\frac{2n}{n+4}}\left(M\right)\setminus\left\{0\right\},f\geq 0\right\}\\ &=&\sup\left\{\frac{\int_{M}G_{P}f\cdot fd\mu}{\left\|f\right\|_{L^{\frac{2n}{n+4}}}^{2}}:f\in C^{\infty}\left(M\right)\setminus\left\{0\right\},f\geq 0\right\}\\ &=&\sup\left\{\frac{\int_{M}Pu\cdot ud\mu}{\left\|Pu\right\|_{L^{\frac{2n}{n+4}}}^{2}}:u\in C^{\infty}\left(M\right)\setminus\left\{0\right\},Pu\geq 0\right\}\\ &\leq&\sup\left\{\frac{\int_{M}Pu\cdot ud\mu}{\left\|Pu\right\|_{L^{\frac{2n}{n+4}}}^{2}}:u\in C^{\infty}\left(M\right),u>0\right\}\\ &=&\frac{2}{n-4}\sup\left\{\frac{\int_{M}\widetilde{Q}d\widetilde{\mu}}{\left\|\widetilde{Q}\right\|_{L^{\frac{2n}{n+4}}\left(M,d\widetilde{\mu}\right)}^{2}}:\widetilde{g}\in\left[g\right]\right\}. \end{split}$$

In between we have used the fact for smooth function $u, Pu \ge 0$ and u not identically zero implies u > 0.

To better understand the relation between $Y_{4}\left(g\right)$ and $\Theta_{4}\left(g\right)$, we define

$$Y_{4}^{+}(g) = \inf \left\{ \frac{\int_{M} Pu \cdot ud\mu}{\left\| u \right\|_{L^{\frac{2n}{n-4}}}^{2}} : u \in C^{\infty}(M), u > 0 \right\}$$

$$= \frac{n-4}{2} \inf \left\{ \frac{\int_{M} \widetilde{Q}d\widetilde{\mu}}{\left(\widetilde{\mu}(M)\right)^{\frac{n-4}{n}}} : \widetilde{g} \in [g] \right\}.$$

$$(2.4)$$

Clearly we have

$$Y_4(g) \le Y_4^+(g)$$
. (2.5)

Lemma 2.2. If $n \ge 5$, Y(g) > 0, $Q \ge 0$ and not identically zero, then

$$Y_4^+(g)\,\Theta_4(g) \le 1.$$
 (2.6)

Moreover if $Y_{4}^{+}(g)$ is achieved, then $Y_{4}^{+}(g) \Theta_{4}(g) = 1$ and $\Theta_{4}(g)$ must be achieved too.

Proof. It is clear that $\Theta_4(g) > 0$. To prove the inequality we only need to deal with the case $Y_4^+(g) > 0$. Under this assumption for $u \in C^{\infty}(M)$, u > 0, we have $\int_M Pu \cdot ud\mu > 0$. By Holder's inequality we have

$$\frac{\left(\int_{M} Pu \cdot u d\mu\right)^{2}}{\left\|u\right\|_{L^{\frac{2n}{n-4}}}^{2} \left\|Pu\right\|_{L^{\frac{2n}{n+4}}}^{2} \le 1.$$

It follows that

$$Y_{4}^{+}\left(g\right)\frac{\int_{M}Pu\cdot ud\mu}{\left\|Pu\right\|_{L^{\frac{2n}{n+4}}}^{2}}\leq1.$$

7

By the proof of Lemma 2.1 we have

$$\Theta_{4}\left(g\right) = \sup\left\{\frac{\int_{M} Pv \cdot vd\mu}{\left\|Pv\right\|_{L^{\frac{2n}{n+4}}}^{2}} : v \in C^{\infty}\left(M\right), v > 0\right\},\$$

hence $Y_{4}^{+}\left(g\right)\Theta_{4}\left(g\right) \leq 1$. If $Y_{4}^{+}\left(g\right)$ is achieved, say at $u \in C^{\infty}\left(M\right), u > 0$, then

$$Pu = \kappa u^{\frac{n+4}{n-4}}$$

for some constant κ . Since $G_P > 0$, we see $\kappa > 0$. Hence

$$\Theta_{4}\left(g\right) \geq \frac{\int_{M} Pu \cdot ud\mu}{\left\|Pu\right\|_{L^{\frac{2n}{n+4}}}^{2}} = \frac{1}{\kappa} \left\|u\right\|_{L^{\frac{2n}{n+4}}}^{-\frac{8}{n-4}} = \frac{1}{Y_{4}^{+}\left(g\right)} \geq \Theta_{4}\left(g\right)$$

Hence all the inequalities are equalities. $\Theta_4(g) = \frac{1}{Y_4^+(g)}$ and is achieved at utoo.

Remark 2.1. Assume $Y_4^+(g) \Theta_4(g) = 1$. Later we will show $\Theta_4(g)$ is always achieved by positive smooth functions i.e.

$$\Theta_{4}\left(g\right) = \frac{\int_{M} G_{P} f \cdot f d\mu}{\left\|f\right\|_{L^{\frac{2n}{n+4}}}^{2}} = \frac{\int_{M} P v \cdot v d\mu}{\left\|Pv\right\|_{L^{\frac{2n}{n+4}}}^{2}},$$

here $f \in C^{\infty}(M)$, f > 0, $v = G_P f$. Hence $v \in C^{\infty}(M)$, v > 0 and $Pv = \kappa v^{\frac{n+4}{n-4}}$

for some constant κ . Using $G_P > 0$ we see $\kappa > 0$. On the other hand

$$\Theta_4(g) = \frac{\int_M Pv \cdot v d\mu}{\left\| Pv \right\|_{L^{\frac{2n}{n+4}}}^2} = \kappa^{-1} \left\| v \right\|_{L^{\frac{2n}{n-4}}}^{-\frac{8}{n-4}}.$$

Hence

$$Y_{4}^{+}(g) = \kappa \left\| v \right\|_{L^{\frac{8}{n-4}}}^{\frac{8}{n-4}} = \frac{\int_{M} Pv \cdot v d\mu}{\left\| v \right\|_{L^{\frac{2n}{n-4}}}^{2}}$$

In another word, positive maximizers for $\Theta_4(g)$ are also minimizers for $Y_4^+(g)$.

2.2. The sphere S^n . On S^n $(n \ge 5)$ with standard metric we have

$$Q = \frac{n(n+2)(n-2)}{8}$$
(2.7)

and

$$Pu = \Delta^2 u - \frac{n^2 - 2n - 4}{2} \Delta u + \frac{n(n+2)(n-2)(n-4)}{16} u.$$
(2.8)

Let N be the north pole and $\pi_N: S^n \setminus \{N\} \to \mathbb{R}^n$ be the stereographic projection, use $x = \pi_N$ as the coordinate, then the Green's function of P with pole at N is given by

$$G_{P,N} = \frac{1}{n(n-2)(n-4)2^{n-3}\omega_n} \left(|x|^2 + 1\right)^{\frac{n-4}{2}}.$$
(2.9)

Here ω_n is the volume of the unit ball in \mathbb{R}^n i.e.

$$\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)},\tag{2.10}$$

 Γ is the Gamma function given by

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha - 1} dt \quad \text{for } \alpha > 0.$$
(2.11)

From [CLO, Li] we know

$$Y_{4}(S^{n}) = \inf_{u \in C_{c}^{\infty}(\mathbb{R}^{n}) \setminus \{0\}} \frac{\|\Delta u\|_{L^{2}(\mathbb{R}^{n})}^{2}}{\|u\|_{L^{2}(\mathbb{R}^{n})}^{2}}$$

$$= \frac{\|\Delta u_{1}\|_{L^{2}(\mathbb{R}^{n})}^{2}}{\|u_{1}\|_{L^{\frac{2n}{n-4}}(\mathbb{R}^{n})}^{2}}$$

$$= \frac{n(n+2)(n-2)(n-4)}{16} \frac{2^{\frac{4}{n}}\pi^{\frac{2(n+1)}{n}}}{\Gamma(\frac{n+1}{2})^{\frac{4}{n}}}$$

$$= Y_{4}^{+}(S^{n}).$$
(2.12)

Here

$$u_1(x) = \left(|x|^2 + 1\right)^{-\frac{n-4}{2}}.$$
(2.13)

For $\lambda > 0$, let

$$u_{\lambda}\left(x\right) = \lambda^{-\frac{n-4}{2}} u_{1}\left(\frac{x}{\lambda}\right) = \left(\frac{\lambda}{\left|x\right|^{2} + \lambda^{2}}\right)^{\frac{n-4}{2}},\qquad(2.14)$$

 then

$$\Delta^2 u_{\lambda} = n \left(n+2 \right) \left(n-2 \right) \left(n-4 \right) u_{\lambda}^{\frac{n+4}{n-4}}.$$
(2.15)

On the other hand it follows from [CLO, Li] that

$$\Theta_4\left(S^n\right) \tag{2.16}$$

$$= \frac{1}{2n(n-2)(n-4)\omega_{n}} \sup_{f \in L^{2}(\mathbb{R}^{n}) \setminus \{0\}} \frac{\int_{\mathbb{R}^{n} \times \mathbb{R}^{n}} \frac{f(x)f(y)}{|x-y|^{n-4}} dx dy}{\|f\|_{L^{\frac{2n}{n+4}}(\mathbb{R}^{n})}^{2}}$$

$$= \sup_{u \in C_{c}^{\infty}(\mathbb{R}^{n}) \setminus \{0\}} \frac{\int_{\mathbb{R}^{n}} (\Delta u)^{2} dx}{\|\Delta^{2} u\|_{L^{\frac{2n}{n+4}}(\mathbb{R}^{n})}^{2}}$$

$$= \frac{1}{2n(n-2)(n-4)\omega_{n}} \frac{\int_{\mathbb{R}^{n} \times \mathbb{R}^{n}} \frac{f_{1}(x)f_{1}(y)}{|x-y|^{n-4}} dx dy}{\|f_{1}\|_{L^{\frac{2n}{n+4}}(\mathbb{R}^{n})}^{2}}$$

$$= \frac{1}{Y_{4}(S^{n})}.$$

 ${\rm Here}$

$$f_1(x) = \left(|x|^2 + 1\right)^{-\frac{n+4}{2}}.$$
(2.17)

For $\lambda > 0$, let

$$f_{\lambda}\left(x\right) = \lambda^{-\frac{n+4}{2}} f_1\left(\frac{x}{\lambda}\right) = \left(\frac{\lambda}{\left|x\right|^2 + \lambda^2}\right)^{\frac{n+4}{2}},$$
(2.18)

 then

$$\Delta^2 u_{\lambda} = n (n+2) (n-2) (n-4) f_{\lambda}.$$
(2.19)

2.3. Almost sharp Sobolev inequalities. Note by (2.16) for $u \in C_c^{\infty}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \left(\Delta u\right)^2 dx \le \Theta_4\left(S^n\right) \left\|\Delta^2 u\right\|_{L^{\frac{2n}{n+4}}(\mathbb{R}^n)}^2.$$
(2.20)

The aim of this subsection is to derive the following almost sharp Sobolev inequality, which will be useful when applying the concentration compactness principle to extremal problem of $\Theta_4(g)$ in section 3.

Lemma 2.3. Assume M is a smooth compact Riemannian manifold with dimension $n \ge 5$. Then for any $\varepsilon > 0$, we have

$$\|\Delta u\|_{L^{2}(M)}^{2} \leq (\Theta_{4}(S^{n}) + \varepsilon) \|Pu\|_{L^{\frac{2n}{n+4}}(M)}^{2} + C(\varepsilon) \|u\|_{L^{\frac{2n}{n+4}}(M)}^{2}$$
(2.21)

for all $u \in W^{4,\frac{2n}{n+4}}(M)$.

Before proving the inequality, we recall some basic facts. The Sobolev embedding theorem tells us

$$W^{4,\frac{2n}{n+4}}(M) \subset W^{3,\frac{2n}{n+2}}(M) \subset W^{2,2}(M) \subset W^{1,\frac{2n}{n-2}}(M) \subset L^{\frac{2n}{n-4}}(M).$$
(2.22)

Moreover, the embedding becomes compact if we are willing to lower the integrable power a little bit, for example

$$W^{4,\frac{2n}{n+4}}(M) \subset W^{3,q}(M)$$
 (2.23)

is a compact embedding for any $1 \le q < \frac{2n}{n+2}$. These facts can be used to get the interpolation inequalities.

We will frequently use the following fact: for $1 \le p < \infty$ and $a, b \ge 0, \varepsilon > 0$,

$$(a+b)^{p} \le (1+\varepsilon) a^{p} + C(\varepsilon, p) b^{p}.$$
(2.24)

Indeed we can choose

$$C(\varepsilon, p) = \sup_{t \ge 0} \left(\left(t + 1\right)^p - \left(1 + \varepsilon\right) t^p \right) < \infty.$$

By standard elliptic estimates we have for every $u \in W^{4,\frac{2n}{n+4}}(M)$,

$$\|u\|_{W^{4,\frac{2n}{n+4}}} \le C\left(\|Pu\|_{L^{\frac{2n}{n+4}}} + \|u\|_{L^{\frac{2n}{n+4}}}\right)$$
(2.25)

and

$$\|u\|_{W^{4,\frac{2n}{n+4}}} \le C\left(\left\|\Delta^{2}u\right\|_{L^{\frac{2n}{n+4}}} + \|u\|_{L^{\frac{2n}{n+4}}}\right).$$
(2.26)

On the other hand, the usual compactness argument tells us for $\varepsilon > 0$,

$$\|u\|_{W^{3,\frac{2n}{n+4}}} \le \varepsilon \|u\|_{W^{4,\frac{2n}{n+4}}} + C(\varepsilon) \|u\|_{L^{\frac{2n}{n+4}}}.$$
(2.27)

Hence

$$\left\|u\right\|_{W^{3,\frac{2n}{n+4}}} \le \varepsilon \left\|Pu\right\|_{L^{\frac{2n}{n+4}}} + C\left(\varepsilon\right) \left\|u\right\|_{L^{\frac{2n}{n+4}}}$$
(2.28)

and

$$\left\|u\right\|_{W^{3,\frac{2n}{n+4}}} \le \varepsilon \left\|\Delta^{2}u\right\|_{L^{\frac{2n}{n+4}}} + C\left(\varepsilon\right) \left\|u\right\|_{L^{\frac{2n}{n+4}}}.$$
(2.29)

To prove the Lemma 2.3 we only need to show for $\varepsilon > 0$,

$$\|\Delta u\|_{L^{2}}^{\frac{2n}{n+4}} \le (1+\varepsilon) \Theta_{4} (S^{n})^{\frac{n}{n+4}} \|\Delta^{2} u\|_{L^{\frac{2n}{n+4}}}^{\frac{2n}{n+4}} + C(\varepsilon) \|u\|_{L^{\frac{2n}{n+4}}}^{\frac{2n}{n+4}}.$$
 (2.30)

In fact, once (2.30) is known we have

$$\|\Delta u\|_{L^{2}} \leq (1+\varepsilon)^{\frac{n+4}{2n}} \Theta_{4} (S^{n})^{\frac{1}{2}} \|\Delta^{2} u\|_{L^{\frac{2n}{n+4}}} + C(\varepsilon) \|u\|_{L^{\frac{2n}{n+4}}}.$$

In another way it is

$$\|\Delta u\|_{L^{2}} \leq (1+\varepsilon_{1}) \Theta_{4} (S^{n})^{\frac{1}{2}} \|\Delta^{2} u\|_{L^{\frac{2n}{n+4}}} + C(\varepsilon_{1}) \|u\|_{L^{\frac{2n}{n+4}}}$$

for any $\varepsilon_1 > 0$. Hence

$$\begin{split} \|\Delta u\|_{L^{2}} \\ &\leq (1+\varepsilon_{1}) \Theta_{4} \left(S^{n}\right)^{\frac{1}{2}} \|Pu\|_{L^{\frac{2n}{n+4}}} + C \|u\|_{W^{2,\frac{2n}{n+4}}} + C \left(\varepsilon_{1}\right) \|u\|_{L^{\frac{2n}{n+4}}} \\ &\leq (1+2\varepsilon_{1}) \Theta_{4} \left(S^{n}\right)^{\frac{1}{2}} \|Pu\|_{L^{\frac{2n}{n+4}}} + C \left(\varepsilon_{1}\right) \|u\|_{L^{\frac{2n}{n+4}}} \,. \end{split}$$

Taking square on both sides we get

$$\|\Delta u\|_{L^{2}}^{2} \leq (1+2\varepsilon_{1})^{3} \Theta_{4}(S^{n}) \|Pu\|_{L^{\frac{2n}{n+4}}}^{2} + C(\varepsilon_{1}) \|u\|_{L^{\frac{2n}{n+4}}}^{2}$$

and Lemma 2.3 follows.

For any $\varepsilon > 0$, we can find a $\delta > 0$ such that if u is supported in $B_{\delta}(p)$, then

$$\|\Delta u\|_{L^{2}}^{\frac{2n}{n+4}} \leq (1+\varepsilon) \Theta_{4} \left(S^{n}\right)^{\frac{n}{n+4}} \left\|\Delta^{2} u\right\|_{L^{\frac{2n}{n+4}}}^{\frac{2n}{n+4}} + C\left(\varepsilon\right) \|u\|_{L^{\frac{2n}{n+4}}}^{\frac{2n}{n+4}}.$$
(2.31)

Indeed, let x_1, \dots, x_n be the normal coordinate at p, then $g = g_{ij}dx_i dx_j$ with $g_{ij}(p) = \delta_{ij}$ and the Euclidean metric $g_0 = \delta_{ij}dx_i dx_j$. We have

$$\left|\Delta u - \Delta_0 u\right| \le \varepsilon_1 \left| D^2 u \right| + C \left| D u \right|$$

and

$$\left|\Delta^2 u - \Delta_0^2 u\right| \le \varepsilon_1 \left|D^4 u\right| + C\left(\left|D^3 u\right| + \left|D^2 u\right| + \left|D u\right|\right)$$

if δ is small enough. Then

$$\begin{split} \|\Delta u\|_{L^{2}} &\leq \|\Delta_{0}u\|_{L^{2}} + \varepsilon_{1} \|D^{2}u\|_{L^{2}} + C \|Du\|_{L^{2}} \\ &\leq \|\Delta_{0}u\|_{L^{2}} + +C\varepsilon_{1} \|D^{4}u\|_{L^{\frac{2n}{n+4}}} + C \|u\|_{W^{3,\frac{2n}{n+4}}} \\ &\leq (1+\varepsilon_{1}) \Theta_{4} (S^{n})^{\frac{1}{2}} \|\Delta_{0}^{2}u\|_{L^{\frac{2n}{n+4}}} + C\varepsilon_{1} \|D^{4}u\|_{L^{\frac{2n}{n+4}}} + C \|u\|_{W^{3,\frac{2n}{n+4}}} \\ &\leq (1+\varepsilon_{1}) \Theta_{4} (S^{n})^{\frac{1}{2}} \|\Delta^{2}u\|_{L^{\frac{2n}{n+4}}} + C\varepsilon_{1} \|D^{4}u\|_{L^{\frac{2n}{n+4}}} + C \|u\|_{W^{3,\frac{2n}{n+4}}} \\ &\leq (1+C\varepsilon_{1}) \Theta_{4} (S^{n})^{\frac{1}{2}} \|\Delta^{2}u\|_{L^{\frac{2n}{n+4}}} + C(\varepsilon_{1}) \|u\|_{L^{\frac{2n}{n+4}}} \,. \end{split}$$

Hence

$$\left\|\Delta u\right\|_{L^{2}}^{\frac{2n}{n+4}} \leq \left(1+C\varepsilon_{1}\right)^{\frac{2n}{n+4}+1} \Theta_{4}\left(S^{n}\right)^{\frac{n}{n+4}} \left\|\Delta^{2} u\right\|_{L^{\frac{2n}{n+4}}}^{\frac{2n}{n+4}} + C\left(\varepsilon_{1}\right) \left\|u\right\|_{L^{\frac{2n}{n+4}}}^{\frac{2n}{n+4}}.$$

(2.31) follows.

To continue, following [DHL] we choose $\eta_1, \dots, \eta_m \in C^{\infty}(M)$ such that $0 \leq \eta_i \leq 1, \ \eta_i^{\alpha} \in C^{\infty}(M)$ for any $\alpha > 0, \ \eta_i$ is supported in $B_{\delta}(p_i)$ for some p_i and

$$\begin{split} \sum_{i=1}^{m} \eta_{i} &= 1. \text{ We have} \\ \|\Delta u\|_{L^{2}}^{\frac{2n}{n+4}} &= \left\| |\Delta u|^{\frac{2n}{n+4}} \right\|_{L^{\frac{n+4}{n}}} \\ &\leq \sum_{i=1}^{m} \left\| \eta_{i} |\Delta u|^{\frac{2n}{n+4}} \right\|_{L^{\frac{n+4}{n}}} \\ &= \sum_{i=1}^{m} \left\| \eta_{i}^{\frac{n+4}{2n}} \Delta u \right\|_{L^{2}}^{\frac{2n}{n+4}} \\ &\leq \sum_{i=1}^{m} \left\| \left\| \Delta \left(\eta_{i}^{\frac{n+4}{2n}} u \right) \right\| + C \left(|Du| + |u| \right) \right\|_{L^{2}}^{\frac{2n}{n+4}} \\ &\leq \left(1 + \varepsilon_{1} \right) \sum_{i=1}^{m} \left\| \Delta \left(\eta_{i}^{\frac{n+4}{2n}} u \right) \right\|_{L^{2}}^{\frac{2n}{n+4}} + C \left(\varepsilon_{1} \right) \left\| u \right\|_{W^{1,2}}^{\frac{2n}{n+4}} \\ &\leq \left(1 + \varepsilon_{1} \right)^{2} \Theta_{4} \left(S^{n} \right)^{\frac{n}{n+4}} \sum_{i=1}^{m} \left\| \Delta^{2} \left(\eta_{i}^{\frac{n+4}{2n}} u \right) \right\|_{L^{\frac{2n}{n+4}}}^{\frac{2n}{n+4}} + C \left(\varepsilon_{1} \right) \left\| u \right\|_{W^{3,\frac{2n}{n+4}}}^{\frac{2n}{n+4}} \\ &\leq \left(1 + \varepsilon_{1} \right)^{3} \Theta_{4} \left(S^{n} \right)^{\frac{n}{n+4}} \sum_{i=1}^{m} \left\| \eta_{i}^{\frac{n+4}{2n}} \Delta^{2} u \right\|_{L^{\frac{2n}{n+4}}}^{\frac{2n}{n+4}} + C \left(\varepsilon_{1} \right) \left\| u \right\|_{W^{3,\frac{2n}{n+4}}}^{\frac{2n}{n+4}} \\ &= \left(1 + \varepsilon_{1} \right)^{3} \Theta_{4} \left(S^{n} \right)^{\frac{n}{n+4}} \left\| \Delta^{2} u \right\|_{L^{\frac{2n}{n+4}}}^{\frac{2n}{n+4}} + C \left(\varepsilon_{1} \right) \left\| u \right\|_{W^{3,\frac{2n}{n+4}}}^{\frac{2n}{n+4}} \\ &\leq \left(1 + \varepsilon_{1} \right)^{4} \Theta_{4} \left(S^{n} \right)^{\frac{n}{n+4}} \left\| \Delta^{2} u \right\|_{L^{\frac{2n}{n+4}}}^{\frac{2n}{n+4}} + C \left(\varepsilon_{1} \right) \left\| u \right\|_{W^{3,\frac{2n}{n+4}}}^{\frac{2n}{n+4}} \\ &\leq \left(1 + \varepsilon_{1} \right)^{4} \Theta_{4} \left(S^{n} \right)^{\frac{n}{n+4}} \left\| \Delta^{2} u \right\|_{L^{\frac{2n}{n+4}}}^{\frac{2n}{n+4}} + C \left(\varepsilon_{1} \right) \left\| u \right\|_{L^{\frac{2n}{n+4}}}^{\frac{2n}{n+4}}. \end{split}$$

This proves (2.30).

2.4. Expansion of Green's function of Paneitz operator. In [LP], the expansion formula of Green's function of conformal Laplacian operator plays important role. Here we determine the expansion formulas for Green's function of Paneitz operator. These formulas will be crucial in the choice of test function in section 4.

We use the same strategy as [LP, section 6], but since there are more lower order terms, some efforts are needed in doing the algebra. Let us introduce some notations. For $m \in \mathbb{Z}_+$, let

$$\mathcal{P}_m = \{\text{homogeneous degree } m \text{ polynomials on } \mathbb{R}^n \}, \qquad (2.32)$$

and

 $\mathcal{H}_m = \{\text{harmonic degree } m \text{ homogeneous polynomials} \}.$ (2.33)

Let f be a function defined on a neighborhood of 0 except at 0, namely $U \setminus \{0\}$, m be nonnegative integer, and $\theta \in \mathbb{R}$. Then we write $f = O^{(m)}(r^{\theta})$ as $r \to 0$ if

$$f \in C^m(U \setminus \{0\})$$
 and $\partial_{i_1 \cdots i_k} f(x) = O(r^{\theta - k})$ as $r \to 0$ (2.34)

for $k = 0, 1, \dots, m$. Here r = |x|.

Another useful notation is as follows. Let f be a function defined on a neighborhood of 0, namely U, m and k be nonnegative integers. Then we write $f = O_m(r^k)$ if $f \in C^m(U)$ and $f(x) = O(r^k)$ as $r \to 0$.

Let M be a smooth compact manifold with a conformal class of Riemannian metrics. For a point $p \in M$, choose a conformal normal coordinate at p, namely

 x_1, \dots, x_n . Let the metric $g = g_{ij} dx_i dx_j$. Then we have([LP])

$$J(p) = 0, \quad J_i(p) = 0, \quad \Delta J(p) = -\frac{|W(p)|^2}{12(n-1)}, \quad (2.35)$$

$$A_{ij}(p) = 0, \quad A_{ijk}(p) x_i x_j x_k = 0, \tag{2.36}$$

and

$$A_{ijkl}(p) x_i x_j x_k x_l = -\frac{2}{9(n-2)} \sum_{kl} \left(W_{ikjl}(p) x_i x_j \right)^2 - \frac{r^2}{n-2} J_{ij}(p) x_i x_j. \quad (2.37)$$

Proposition 2.1. Assume $n \ge 5$ and kerP = 0. Then under the conformal normal coordinate at p, we have the following statements:

• If the original conformal class is conformal flat on a neighborhood of p, then we may choose g such that it is flat near p, and

$$2n(2-n)(4-n)\omega_n G_{P,p} = r^{4-n} + O_{\infty}(1).$$
(2.38)

• If n is odd, then

$$2n(2-n)(4-n)\omega_n G_{P,p} = r^{4-n}\left(1+\sum_{i=4}^n \psi_i\right) + O_4(1).$$
 (2.39)

Here $\psi_i \in \mathcal{P}_i$.

• If n is even and larger than or equal to 8, then

$$2n(2-n)(4-n)\omega_{n}G_{P,p}$$

$$= r^{4-n}\left(1+\sum_{i=4}^{n}\psi_{i}\right)+r^{4-n}\log r\sum_{i=n-4}^{n}\psi_{i}'+r^{4-n}\log^{2}r\sum_{i=n-2}^{n}\psi_{i}''$$

$$+r^{4-n}\log^{3}r\cdot\psi_{n}'''+O_{4}(1).$$

$$(2.40)$$

Here $\psi_i, \psi'_i, \psi''_i, \psi'''_i \in \mathcal{P}_i$.

• If n = 6, then

$$96\omega_6 G_{P,p} = r^{-2} \left(1 + \psi_4 + \psi_5 + \psi_6 \right) + r^{-2} \log r \left(\psi'_4 + \psi'_5 + \psi'_6 \right) \quad (2.41)$$
$$+ r^{-2} \log^2 r \cdot \psi''_6 + O_4 \left(1 \right).$$

Here $\psi_i, \psi'_i, \psi''_i \in \mathcal{P}_i$.

In another way, we have

• If n = 5, 6, 7 or M is conformal flat near p, then

$$2n(2-n)(4-n)\omega_n G_{P,p} = r^{4-n} + A + O^{(4)}(r).$$
(2.42)

Here A is a constant.

• If n = 8, then

$$384\omega_8 G_{P,p} = r^{-4} - \frac{|W(p)|^2}{1440} \log r + O^{(4)}(1).$$
(2.43)

• If $n \ge 9$, then

$$2n(2-n)(4-n)\omega_n G_{P,p} = r^{4-n} + r^{4-n}\psi_4 + O^{(4)}(r^{9-n}), \qquad (2.44)$$

here $\psi_4 \in \mathcal{P}_4$ and in fact ψ_4

$$= \frac{1}{40(n-2)} \left[\frac{2}{9} \sum_{kl} \left(W_{ikjl}(p) x_i x_j \right)^2 - \frac{2r^2}{9(n+4)} \sum_{jkl} \left(W_{ijkl}(p) x_i + W_{ilkj}(p) x_i \right)^2 + \frac{|W(p)|^2}{r^4} r^4 \right] + \frac{r^2}{r^4} \left[\frac{4}{r^4} \sum_{jkl} \left(W_{ijkl}(p) x_j + W_{ilkj}(p) x_j \right)^2 \right]$$

$$+ \frac{1}{3(n+2)(n+4)} \int_{-1}^{1} \frac{1}{48(n-6)} \left[\frac{9(n+4)}{9(n+4)} \sum_{jkl} (W_{ijkl}(p) x_i + W_{ilkj}(p) x_i) \right]$$

$$- 2(n-6) J_{ij}(p) x_i x_j - \frac{(n^2+6n-32)|W(p)|^2}{6n(n+4)(n-1)} r^2 \right]$$

$$+ r^4 \cdot \frac{(n-4)(3n^2-2n-64)|W(p)|^2}{576n(n+2)(n-1)(n-6)(n-8)}.$$

The terms in the square brackets are harmonic polynomials.

To derive these expansions, we need some algebraic preparations. Note that \mathcal{P}_m has the following decomposition (see [S])

$$\mathcal{P}_m = \bigoplus_{k=0}^{\left[\frac{m}{2}\right]} \left(r^{2k} \mathcal{H}_{m-2k} \right).$$
(2.46)

(2.45)

Under this decomposition, we have

$$(r^{2}\Delta)|_{r^{2k}\mathcal{H}_{m-2k}} = 2k(2m-2k+n-2) \text{ for } k = 0, 1, 2, \cdots, \left[\frac{m}{2}\right].$$
 (2.47)

Here Δ denotes the Laplacian operator with respect to the Euclidean metric.

For $\alpha \in \mathbb{R}$, let

$$A_{\alpha} = r^{2} \Delta + 2\alpha r \partial_{r} + \alpha \left(\alpha + n - 2\right), \qquad (2.48)$$

and

$$B_{\alpha} = \frac{\partial}{\partial \alpha} A_{\alpha} = 2r\partial_r + (2\alpha + n - 2), \qquad (2.49)$$

 then

$$\begin{aligned} \Delta (r^{\alpha} \varphi) &= r^{\alpha-2} A_{\alpha} \varphi, \\ A_{\alpha} (r^{\beta} \varphi) &= r^{\beta} A_{\alpha+\beta} \varphi, \\ A_{\alpha} (\varphi \log r) &= (A_{\alpha} \varphi) \log r + B_{\alpha} \varphi, \\ B_{\alpha} (r^{\beta} \varphi) &= r^{\beta} B_{\alpha+\beta} \varphi, \\ B_{\alpha} (\varphi \log r) &= (B_{\alpha} \varphi) \log r + 2\varphi. \end{aligned}$$

In addition,

$$A_{\alpha}|_{\mathcal{P}_m} = r^2 \Delta + \alpha \left(2m + \alpha + n - 2\right), \qquad (2.50)$$

$$B_{\alpha}|_{\mathcal{P}_m} = 2m + 2\alpha + n - 2,$$
 (2.51)

and

$$A_{\alpha}|_{r^{2k}\mathcal{H}_{m-2k}} = (\alpha+2k)(2m-2k+\alpha+n-2)$$
(2.52)

for $k = 0, 1, 2, \cdots, \left[\frac{m}{2}\right]$. In particular,

$$(A_{2-n}A_{4-n})|_{r^{2k}\mathcal{H}_{m-2k}}$$

$$= (2m-2k)(2m-2k+2)(2k+2-n)(2k+4-n),$$
(2.53)

for $k = 0, 1, 2, \cdots, \left[\frac{m}{2}\right]$.

Lemma 2.4. For any real numbers α and β , and any nonnegative integer k, we have

$$B_{\alpha}\left(\varphi \log^{k} r\right) = B_{\alpha}\varphi \cdot \log^{k} r + 2k\varphi \log^{k-1} r,$$

$$A_{\alpha}\left(\varphi \log^{k} r\right) = A_{\alpha}\varphi \cdot \log^{k} r + kB_{\alpha}\varphi \cdot \log^{k-1} r + k\left(k-1\right)\varphi \log^{k-2} r,$$

and

$$A_{\alpha}A_{\beta}\left(\varphi \log^{k} r\right)$$

$$= A_{\alpha}A_{\beta}\varphi \cdot \log^{k} r + k\left(A_{\alpha}B_{\beta}\varphi + B_{\alpha}A_{\beta}\varphi\right)\log^{k-1} r$$

$$+k\left(k-1\right)\left(A_{\alpha}\varphi + A_{\beta}\varphi + B_{\alpha}B_{\beta}\varphi\right)\log^{k-2} r$$

$$+k\left(k-1\right)\left(k-2\right)\left(B_{\alpha}\varphi + B_{\beta}\varphi\right)\log^{k-3} r$$

$$+k\left(k-1\right)\left(k-2\right)\left(k-3\right)\varphi\log^{k-4} r.$$

Proof. Observe

$$\frac{\partial}{\partial \alpha} B_{\alpha} \varphi = 2\varphi, \quad \frac{\partial^2}{\partial \alpha^2} B_{\alpha} \varphi = 0.$$

Now since $B_{\alpha}\left(r^{\beta}\varphi\right) = r^{\beta}B_{\alpha+\beta}\varphi$, we know

$$B_{\alpha}\left(\varphi \log^{k} r\right) = \frac{\partial^{k}}{\partial \beta^{k}} \bigg|_{\beta=0} B_{\alpha}\left(r^{\beta}\varphi\right)$$
$$= \frac{\partial^{k}}{\partial \beta^{k}} \bigg|_{\beta=0} \left(r^{\beta}B_{\alpha+\beta}\varphi\right)$$
$$= B_{\alpha}\varphi \cdot \log^{k} r + 2k\varphi \log^{k-1} r,$$

here we have used the Newton-Lebniz formula. For the second equation, we start with

$$\frac{\partial}{\partial \alpha} A_{\alpha} \varphi = B_{\alpha} \varphi, \quad \frac{\partial^2}{\partial \alpha^2} A_{\alpha} \varphi = 2\varphi, \quad \frac{\partial^3}{\partial \alpha^3} A_{\alpha} \varphi = 0,$$

then

$$\begin{aligned} A_{\alpha}\left(\varphi \log^{k} r\right) &= \left. \frac{\partial^{k}}{\partial \beta^{k}} \right|_{\beta=0} A_{\alpha}\left(r^{\beta}\varphi\right) \\ &= \left. \frac{\partial^{k}}{\partial \beta^{k}} \right|_{\beta=0} \left(r^{\beta}A_{\alpha+\beta}\varphi\right) \\ &= \left. A_{\alpha}\varphi \cdot \log^{k} r + kB_{\alpha}\varphi \cdot \log^{k-1} r + k\left(k-1\right)\varphi \log^{k-2} r. \end{aligned}$$

Define an operator

$$M_g \varphi = 4 \operatorname{div} \left(A \left(\nabla_g \varphi, e_i \right) e_i \right) + (2 - n) \operatorname{div} \left(J \nabla_g \varphi \right).$$
(2.54)

The Paneitz operator can be written as

$$P_g\varphi = \Delta_g^2\varphi + M_g\varphi + \frac{n-4}{2}Q\varphi.$$
(2.55)

For any $\alpha \in \mathbb{R}$, define

$$N_{\alpha,g}\varphi = r^{4}M_{g}\varphi + 8\alpha r^{2}A(r\partial_{r}, \nabla_{g}\varphi) + 2(2-n)\alpha r^{2}J \cdot r\partial_{r}\varphi \qquad (2.56)$$
$$+4\alpha r^{2}\operatorname{div}\left(A(r\partial_{r}, e_{i})e_{i}\right)\varphi + (2-n)\alpha r^{2} \cdot r\partial_{r}J \cdot \varphi$$
$$+4\alpha(\alpha-2)A(r\partial_{r}, r\partial_{r})\varphi + (2-n)\alpha(\alpha+n-2)r^{2}J\varphi,$$

then

$$M_g\left(r^{\alpha}\varphi\right) = r^{\alpha-4}N_{\alpha,g}\varphi.$$
(2.57)

At first, we claim that

$$P_g(r^{4-n}) = 2n(2-n)(4-n)\omega_n\delta_p + fr^{-n}, \qquad (2.58)$$

with $f = O_{\infty}(r^4)$. Indeed, because r^{4-n} is radial, we have

$$\Delta_g^2 \left(r^{4-n} \right) = 2n \left(2 - n \right) \left(4 - n \right) \omega_n \delta_p.$$
 (2.59)

On the other hand,

$$M_g(r^{4-n}) = r^{-n} N_{4-n,g} 1.$$

In view of the facts

$$\operatorname{div} \left(A\left(r\partial_{r}, e_{i}\right) e_{i} \right)$$

$$= \partial_{k} \left(x_{i}A_{ij}g^{jk} \right)$$

$$= g^{ij}A_{ij} + x_{i}\partial_{k}A_{ij}g^{jk} + O_{\infty}\left(r^{2}\right)$$

$$= J + x_{i}A_{ijk}\left(p\right)\delta_{jk} + O_{\infty}\left(r^{2}\right)$$

$$= x_{i}J_{i}\left(p\right) + O_{\infty}\left(r^{2}\right)$$

$$= O_{\infty}\left(r^{2}\right),$$

and

$$A(r\partial_r, r\partial_r) = A_{ij}x_ix_j = A_{ijk}(p)x_ix_jx_k + O_{\infty}(r^4) = O_{\infty}(r^4),$$

we see $N_{4-n,g} 1 \in O_{\infty}(r^4)$, (2.58) follows. To continue, first we introduce a notation. For any $\alpha \in \mathbb{R}$, let

$$A_{\alpha,g} = r^2 \Delta_g + 2\alpha r \partial_r + \alpha \left(\alpha + n - 2\right), \qquad (2.60)$$

 then

$$\begin{split} \Delta_g \left(r^{\alpha} \varphi \right) &= r^{\alpha-2} A_{\alpha,g} \varphi, \\ A_{\alpha,g} \left(r^{\beta} \varphi \right) &= r^{\beta} A_{\alpha+\beta,g} \varphi, \\ A_{\alpha,g} \left(\varphi \log r \right) &= A_{\alpha,g} \varphi \cdot \log r + B_{\alpha} \varphi. \end{split}$$

Note that

$$A_{\alpha,g} = A_{\alpha} + r^2 \left(\Delta_g - \Delta\right) = A_{\alpha} + r^2 \partial_i \left(\left(g^{ij} - \delta_{ij}\right) \partial_j \right).$$
(2.61)

Computation shows

$$P_g\left(r^{\alpha}\varphi\right) = r^{\alpha-4}\left(A_{\alpha-2}A_{\alpha}\varphi + K_{\alpha}\varphi\right),\qquad(2.62)$$

where

$$K_{\alpha}\varphi \qquad (2.63)$$

$$= A_{\alpha-2}\left(r^{2}\left(\Delta_{g}-\Delta\right)\varphi\right) + r^{2}\left(\Delta_{g}-\Delta\right)A_{\alpha,g}\varphi + N_{\alpha,g}\varphi + \frac{n-4}{2}r^{4}Q\varphi.$$

We easily see that for any nonnegative integer k, $\varphi = O_{\infty}(r^k)$ implies $K_{\alpha}\varphi = O_{\infty}(r^{k+2})$.

We also introduce the following two operators,

$$K_{\alpha}^{(1)}\varphi = \frac{\partial}{\partial\alpha}K_{\alpha}\varphi \qquad (2.64)$$

$$= B_{\alpha-2}\left(r^{2}\left(\Delta_{g}-\Delta\right)\varphi\right) + r^{2}\left(\Delta_{g}-\Delta\right)B_{\alpha}\varphi + 8r^{2}A\left(r\partial_{r},\nabla_{g}\varphi\right) + 2\left(2-n\right)r^{2}J\cdot r\partial_{r}\varphi + 4r^{2}\operatorname{div}\left(A\left(r\partial_{r},e_{i}\right)e_{i}\right)\varphi + (2-n)r^{2}\cdot r\partial_{r}J\cdot\varphi + 8\left(\alpha-1\right)A\left(r\partial_{r},r\partial_{r}\right)\varphi + (2-n)\left(2\alpha+n-2\right)r^{2}J\varphi,$$

and

$$K_{\alpha}^{(2)}\varphi = \frac{\partial}{\partial\alpha}K_{\alpha}^{(1)}\varphi \qquad (2.65)$$
$$= 4r^{2}\left(\Delta_{g}-\Delta\right)\varphi + 8A\left(r\partial_{r},r\partial_{r}\right)\varphi + 2\left(2-n\right)r^{2}J\varphi$$
$$= K^{(2)}\varphi$$

because it is independent of α . Clearly, $\varphi = O_{\infty}(r^k)$ for some nonnegative integer would imply $K_{\alpha}^{(1)}\varphi, K^{(2)}\varphi = O_{\infty}(r^{k+2})$. In addition, they satisfy the following

$$\begin{aligned}
K_{\alpha} \left(r^{\beta} \varphi \right) &= r^{\beta} K_{\alpha+\beta} \varphi, \\
K_{\alpha} \left(\varphi \log r \right) &= K_{\alpha} \varphi \cdot \log r + K_{\alpha}^{(1)} \varphi, \\
K_{\alpha}^{(1)} \left(r^{\beta} \varphi \right) &= r^{\beta} K_{\alpha+\beta}^{(1)} \varphi, \\
K_{\alpha}^{(1)} \left(\varphi \log r \right) &= K_{\alpha}^{(1)} \varphi \cdot \log r + K_{\alpha}^{(2)} \varphi, \\
K^{(2)} \left(r^{\beta} \varphi \right) &= r^{\beta} K^{(2)} \varphi, \\
K^{(2)} \left(\varphi \log r \right) &= K^{(2)} \varphi \cdot \log r.
\end{aligned}$$

More generally, we have

Lemma 2.5. For any nonnegative integer k, we have

$$K_{\alpha}^{(1)}\left(\varphi \log^{k} r\right) = K_{\alpha}^{(1)}\varphi \cdot \log^{k} r + kK^{(2)}\varphi \cdot \log^{k-1} r,$$

$$K_{\alpha}\left(\varphi \log^{k} r\right) = K_{\alpha}\varphi \cdot \log^{k} r + kK_{\alpha}^{(1)}\varphi \cdot \log^{k-1} r + \frac{k\left(k-1\right)}{2}K^{(2)}\varphi \cdot \log^{k-2}\varphi.$$

This follows for the set of the set

This follows from the same proof of Lemma 2.4.

Case 2.1. The dimension n is odd.

In this case, we claim that we may find a $\psi = \sum_{i=1}^{n} \psi_i$, with $\psi_i \in \mathcal{P}_i$ such that $A_{2-n}A_{4-n}\psi + K_{4-n}\psi + f = O_{\infty}\left(r^{n+1}\right).$ (2.66)

Once this has been done, then we have

$$r^{-n}(A_{2-n}A_{4-n}\psi + K_{4-n}\psi + f) \in C^{\alpha}$$
 for any $0 < \alpha < 1$

If the domain is small enough, then we may find a $\overline{\psi} \in C^{4,\alpha}$ such that

$$P_{g}\overline{\psi} = -r^{-n} \left(A_{2-n}A_{4-n}\psi + K_{4-n}\psi + f \right).$$

Then

$$P_g\left(r^{4-n}\left(1+\psi\right)+\overline{\psi}\right) = 2n\left(2-n\right)\left(4-n\right)\omega_n\delta_p.$$
 (2.67)

Hence the Green's function satisfies

$$2n(2-n)(4-n)\omega_n G_p = r^{4-n}(1+\psi) + \overline{\psi} + O_\infty(1).$$
 (2.68)

To define ψ_1, \dots, ψ_n , we let $\psi_1 = 0, \psi_2 = 0$ and $\psi_3 = 0$. One easily see

$$f_{3} = A_{2-n}A_{4-n}(\psi_{1} + \psi_{2} + \psi_{3}) + K_{4-n}(\psi_{1} + \psi_{2} + \psi_{3}) + f \qquad (2.69)$$

= $f = O_{\infty}(r^{4}).$

Assume we have found $\psi_1, \psi_2, \cdots, \psi_k$ for $3 \le k \le n-1$, such that $\psi_i \in \mathcal{P}_i$ and

$$f_k = A_{2-n} A_{4-n} \left(\sum_{i=1}^k \psi_i \right) + K_{4-n} \left(\sum_{i=1}^k \psi_i \right) + f = O_\infty \left(r^{k+1} \right),$$

then we write $f_k = \phi_{k+1} + O_{\infty}(r^{k+2}), \phi_{k+1} \in \mathcal{P}_{k+1}$. Since

$$A_{2-n}A_{4-n}|_{r^{2j}\mathcal{H}_{k+1-2j}} = (2(k+1)-2j)(2(k+1)-2j+2)(2j+2-n)(2j+4-n) \neq 0$$

for $j = 0, 1, 2, \dots, \left[\frac{k+1}{2}\right]$, $A_{2-n}A_{4-n}$ is invertible on \mathcal{P}_{k+1} . We may find a unique $\psi_{k+1} \in \mathcal{P}_{k+1}$, such that

$$A_{2-n}A_{4-n}\psi_{k+1} + \phi_{k+1} = 0. (2.70)$$

Then

$$f_{k+1} = A_{2-n}A_{4-n}\left(\sum_{i=1}^{k+1}\psi_i\right) + K_{4-n}\left(\sum_{i=1}^{k+1}\psi_i\right) + f$$
$$= f_k + A_{2-n}A_{4-n}\psi_{k+1} + K_{4-n}\psi_{k+1} = O_{\infty}\left(r^{k+2}\right).$$

This finishes the induction.

Case 2.2. *n* is even and larger than or equal to 8.

In this case, we first set $\psi_1 = 0$, $\psi_2 = 0$ and $\psi_3 = 0$. Since $A_{2-n}A_{4-n}$ is invertible on \mathcal{P}_k for $0 \le k \le n-5$, by the same induction procedure as Case 2.1, we can find $\psi_4, \cdots, \psi_{n-5}$ such that $\psi_i \in \mathcal{P}_i$ and

$$f_{n-5} = A_{2-n}A_{4-n}\left(\sum_{i=1}^{n-5}\psi_i\right) + K_{4-n}\left(\sum_{i=1}^{n-5}\psi_i\right) + f = O_{\infty}\left(r^{n-4}\right).$$

To continue, we write

$$f_{n-5} = \phi_{n-4} + O_{\infty}\left(r^{n-3}\right), \quad \phi_{n-4} \in \mathcal{P}_{n-4}.$$

Let $\psi_{n-4}^{(0)} = \alpha_{n-4}^{(0)} + \beta_{n-4}^{(0)} \log r$ with $\alpha_{n-4}^{(0)}, \beta_{n-4}^{(0)} \in \mathcal{P}_{n-4}$, then $A_{2-n}A_{4-n}\psi_{n-4}^{(0)}$

$$= A_{2-n}A_{4-n}\alpha_{n-4}^{(0)} + (A_{2-n}B_{4-n} + B_{2-n}A_{4-n})\beta_{n-4}^{(0)} + A_{2-n}A_{4-n}\beta_{n-4}^{(0)} \cdot \log r$$

Let $\beta_{n-4}^{(0)} \in r^{n-4} \mathcal{H}_0$, then since

$$(A_{2-n}B_{4-n} + B_{2-n}A_{4-n})|_{r^{n-4}\mathcal{H}_0} = -2(n-2)(n-4) \neq 0,$$

and

$$A_{2-n}A_{4-n}|_{r^{2k}\mathcal{H}_{n-4-2k}} = (2(n-4)-2k)(2(n-4)-2k+2)(2k+2-n)(2k+4-n) \neq 0,$$

for $0 \le k \le \frac{n}{2} - 3$, we may find a $\alpha_{n-4}^{(0)} \in \mathcal{P}_{n-4}$ and a $\beta_{n-4}^{(0)} \in r^{n-4}\mathcal{H}_0$ such that $A_{2-n}A_{4-n}\psi_{n-4}^{(0)} + \phi_{n-4} = 0.$

This implies

$$f_{n-4}$$

$$= A_{2-n}A_{4-n}\left(\sum_{i=1}^{n-5}\psi_i + \psi_{n-4}^{(0)}\right) + K_{4-n}\left(\sum_{i=1}^{n-5}\psi_i + \psi_{n-4}^{(0)}\right) + f$$

$$= f_{n-5} + A_{2-n}A_{4-n}\psi_{n-4}^{(0)} + K_{4-n}\psi_{n-4}^{(0)}$$

$$= O_{\infty}\left(r^{n-3}\right) + O_{\infty}\left(r^{n-2}\right)\log r.$$

Next we write

r

$$f_{n-4} = \phi_{n-3} + O_{\infty} \left(r^{n-2} \right) \log r + O_{\infty} \left(r^{n-2} \right), \quad \phi_{n-3} \in \mathcal{P}_{n-3}$$

Again by similar arguments, we can find a $\psi_{n-3}^{(0)} \in \mathcal{P}_{n-3} + r^{n-4}\mathcal{H}_1 \log r$ such that

$$A_{2-n}A_{4-n}\psi_{n-3}^{(0)} + \phi_{n-3} = 0.$$

Then

$$J_{n-3} = A_{2-n}A_{4-n}\left(\sum_{i=1}^{n-5}\psi_i + \psi_{n-4}^{(0)} + \psi_{n-3}^{(0)}\right) + K_{4-n}\left(\sum_{i=1}^{n-5}\psi_i + \psi_{n-4}^{(0)} + \psi_{n-3}^{(0)}\right) + f$$

= $f_{n-4} + A_{2-n}A_{4-n}\psi_{n-3}^{(0)} + K_{4-n}\psi_{n-3}^{(0)}$
= $O_{\infty}\left(r^{n-2}\right)\log r + O_{\infty}\left(r^{n-2}\right).$

We write

$$f_{n-3} = \phi_{n-2}^{(1)} \log r + O_{\infty} \left(r^{n-2} \right) + O_{\infty} \left(r^{n-1} \right) \log r.$$

Similar as before, we may find a

$$\psi_{n-2}^{(1)} \in \mathcal{P}_{n-2}\log r + \left(r^{n-2}\mathcal{H}_0 + r^{n-4}\mathcal{H}_2\right)\log^2 r$$

such that

$$A_{2-n}A_{4-n}\psi_{n-2}^{(1)} + \phi_{n-2}^{(1)}\log r \in \mathcal{P}_{n-2}.$$

Indeed, for $\psi_{n-2}^{(1)} = \alpha_{n-2}^{(1)} \log r + \beta_{n-2}^{(1)} \log^2 r$, with $\alpha_{n-2}^{(1)}, \beta_{n-2}^{(1)} \in \mathcal{P}_{n-2}$, we have

$$= \left(A_{2-n}A_{4-n}\varphi_{n-2}^{(1)} + 2\left(A_{2-n}B_{4-n} + B_{2-n}A_{4-n}\right)\beta_{n-2}^{(1)}\right)\log r + A_{2-n}A_{4-n}\beta_{n-2}^{(1)} \cdot \log^2 r + \mathcal{P}_{n-2}.$$

Let $\beta_{n-2}^{(1)} \in r^{n-2}\mathcal{H}_0 + r^{n-4}\mathcal{H}_2$. Since

$$2(A_{2-n}B_{4-n} + B_{2-n}A_{4-n})|_{r^{n-2}\mathcal{H}_0} = 4n(n-2) \neq 0,$$

$$2(A_{2-n}B_{4-n} + B_{2-n}A_{4-n})|_{r^{n-4}\mathcal{H}_2} = -4n(n+2) \neq 0,$$

and

$$A_{2-n}A_{4-n}|_{r^{2k}\mathcal{H}_{n-2-2k}} = (2(n-2)-2k)(2(n-2)-2k+2)(2k+2-n)(2k+4-n) \neq 0$$

for $0 \le k \le \frac{n}{2} - 3$, we may find the above needed $\psi_{n-2}^{(1)}$. Then

$$f_{n-2}^{(1)}$$

$$= A_{2-n}A_{4-n} \left(\sum_{i=1}^{n-5} \psi_i + \psi_{n-4}^{(0)} + \psi_{n-3}^{(0)} + \psi_{n-2}^{(1)} \right)$$

$$+ K_{4-n} \left(\sum_{i=1}^{n-5} \psi_i + \psi_{n-4}^{(0)} + \psi_{n-3}^{(0)} + \psi_{n-2}^{(1)} \right) + f$$

$$= f_{n-3} + A_{2-n}A_{4-n}\psi_{n-2}^{(1)} + K_{4-n}\psi_{n-2}^{(1)}$$

$$= O_{\infty} \left(r^{n-2} \right) + O_{\infty} \left(r^{n-1} \right) \log r + O_{\infty} \left(r^n \right) \log^2 r.$$

The next step is to remove the \mathcal{P}_{n-2} term in $O_{\infty}(r^{n-2})$, then the $\mathcal{P}_{n-1}\log r$ term in $O_{\infty}(r^{n-1})\log r$ and so on, until we reach $O_{\infty}(r^{n+1})\log^2 r + O_{\infty}(r^{n+1})\log r + O_{\infty}(r^{n+1}) + O_{\infty}(r^{n+2})\log^3 r$. That is, we find

$$\begin{split} \psi_{n-4}^{(0)} &\in \mathcal{P}_{n-4} + r^{n-4}\mathcal{H}_0 \log r, \\ \psi_{n-3}^{(0)} &\in \mathcal{P}_{n-3} + r^{n-4}\mathcal{H}_1 \log r, \\ \psi_{n-2}^{(0)} &\in \mathcal{P}_{n-2} \log r + \left(r^{n-2}\mathcal{H}_0 + r^{n-4}\mathcal{H}_2\right) \log^2 r, \\ \psi_{n-2}^{(0)} &\in \mathcal{P}_{n-2} + \left(r^{n-2}\mathcal{H}_0 + r^{n-4}\mathcal{H}_2\right) \log r, \\ \psi_{n-1}^{(1)} &\in \mathcal{P}_{n-1} \log r + \left(r^{n-2}\mathcal{H}_1 + r^{n-4}\mathcal{H}_3\right) \log^2 r, \\ \psi_{n-1}^{(0)} &\in \mathcal{P}_{n-1} + \left(r^{n-2}\mathcal{H}_1 + r^{n-4}\mathcal{H}_3\right) \log r, \\ \psi_n^{(2)} &\in \mathcal{P}_n \log^2 r + \left(r^{n-2}\mathcal{H}_2 + r^{n-4}\mathcal{H}_4\right) \log^3 r, \\ \psi_n^{(1)} &\in \mathcal{P}_n \log r + \left(r^{n-2}\mathcal{H}_2 + r^{n-4}\mathcal{H}_4\right) \log^2 r, \end{split}$$

and

$$\psi_n^{(0)} \in \mathcal{P}_n + \left(r^{n-2}\mathcal{H}_2 + r^{n-4}\mathcal{H}_4 \right) \log r,$$

such that

$$f_n = A_{2-n}A_{4-n} \left(\sum_{i=1}^{n-5} \psi_i + \sum_{i=n-4}^n \psi_i^{(0)} + \sum_{i=n-2}^n \psi_i^{(1)} + \psi_n^{(2)} \right) + K_{4-n} \left(\sum_{i=1}^{n-5} \psi_i + \sum_{i=n-4}^n \psi_i^{(0)} + \sum_{i=n-2}^n \psi_i^{(1)} + \psi_n^{(2)} \right) + f = O_{\infty} \left(r^{n+1} \right) \log^2 r + O_{\infty} \left(r^{n+1} \right) \log r + O_{\infty} \left(r^{n+1} \right) + O_{\infty} \left(r^{n+2} \right) \log^3 r.$$

Clearly $r^{-n}f_n \in C^{\alpha}$ for any $0 < \alpha < 1$. This implies locally we may find a $\overline{\psi} \in C^{4,\alpha}$ such that $P_g \overline{\psi} = -r^{-n}f_n$. Let

$$\psi = \sum_{i=1}^{n-5} \psi_i + \sum_{i=n-4}^n \psi_i^{(0)} + \sum_{i=n-2}^n \psi_i^{(1)} + \psi_n^{(2)},$$

 then

$$P_g\left(r^{4-n}\left(1+\psi\right)+\overline{\psi}\right) = 2n\left(2-n\right)\left(4-n\right)\omega_n\delta_p$$

on a small disk. Hence

$$2n(2-n)(4-n)\omega_n G_p = r^{4-n}(1+\psi) + \overline{\psi} + O_{\infty}(1).$$

Case 2.3. n = 6.

This case can be done similarly as Case 2.2. That is, we can find

$$\begin{split} \psi_4^{(0)} &\in \mathcal{P}_4 + \left(r^4 \mathcal{H}_0 + r^2 \mathcal{H}_2\right) \log r, \\ \psi_5^{(0)} &\in \mathcal{P}_5 + \left(r^4 \mathcal{H}_1 + r^2 \mathcal{H}_3\right) \log r, \\ \psi_6^{(1)} &\in \mathcal{P}_6 \log r + \left(r^4 \mathcal{H}_2 + r^2 \mathcal{H}_4\right) \log^2 r, \end{split}$$

and

$$\psi_6^{(0)} \in \mathcal{P}_6 + \left(r^4 \mathcal{H}_2 + r^2 \mathcal{H}_4 \right) \log r,$$

such that

.

$$f_{6} = A_{-4}A_{-2}\left(\psi_{4}^{(0)} + \psi_{5}^{(0)} + \psi_{6}^{(0)} + \psi_{6}^{(1)}\right) + K_{-2}\left(\psi_{4}^{(0)} + \psi_{5}^{(0)} + \psi_{6}^{(0)} + \psi_{6}^{(1)}\right) + f$$

= $O_{\infty}\left(r^{7}\right)\log r + O_{\infty}\left(r^{7}\right) + O_{\infty}\left(r^{8}\right)\log^{2} r.$

The remaining argument can be done as before.

Case 2.4. *M* is conformal flat near *p*.

In this case, we may take the metric g such that it is flat near p. This implies $P_g = \Delta^2$, and hence

$$P_g\left(r^{4-n}\right) = 2n\left(2-n\right)\left(4-n\right)\omega_n\delta_p.$$

It follows that

$$2n(2-n)(4-n)\omega_n G_{P,p} = r^{4-n} + O_{\infty}(1).$$

Finally, to get the leading terms in the expansion for $n \ge 8$, by computation we have $f_3 = f = \phi_4 + O_\infty (r^5)$, with $\phi_4 \in \mathcal{P}_4$ and

$$= -\frac{4(n-4)}{9} \sum_{kl} (W_{ikjl}(p) x_i x_j)^2 + 2(n-4)(n-6) r^2 J_{ij}(p) x_i x_j + \frac{(n-4) |W(p)|^2}{24(n-1)} r^4.$$
(2.71)

From this, we can compute the leading terms of $G_{P,p}$ directly from the arguments in Case 2.2.

3. A CRITERION FOR THE EXISTENCE OF MAXIMIZERS

Here we apply the concentration compactness principle in [Ln] to extremal problem (2.2).

Lemma 3.1. Let M be a smooth compact Riemannian manifold with dimension $n \geq 5$, ker P = 0, $f_i \in L^{\frac{2n}{n+4}}(M)$ such that $f_i \rightharpoonup f$ weakly in $L^{\frac{2n}{n+4}}$. Let $u_i, u \in W^{4,\frac{2n}{n+4}}(M)$ such that $Pu_i = f_i$, Pu = f. Assume

$$|f_i|^{\frac{2n}{n+4}} d\mu \rightharpoonup \sigma \ in \ \mathcal{M}(M) \tag{3.1}$$

and

$$\left|\Delta u_{i}\right|^{2} d\mu \rightharpoonup \nu \text{ in } \mathcal{M}\left(M\right), \qquad (3.2)$$

here $\mathcal{M}(M)$ is the space of all Radon measures on M. Then there exists countably many points $p_i \in M$ such that

$$\sigma \ge |f|^{\frac{2n}{n+4}} d\mu + \sum_{i} \sigma_i \delta_{p_i} \tag{3.3}$$

and

$$\nu = |\Delta u|^2 d\mu + \sum_i \nu_i \delta_{p_i}, \qquad (3.4)$$

here $\sigma_i = \sigma\left(\{p_i\}\right), \nu_i = \nu\left(\{p_i\}\right)$. Moreover

$$\nu_i \le \Theta_4 \left(S^n \right) \sigma_i^{\frac{n+4}{n}}. \tag{3.5}$$

Proof. First assume f = 0, then $f_i \to 0$ weakly in $L^{\frac{2n}{n+4}}$, $u_i \to 0$ weakly in $W^{4,\frac{2n}{n+4}}$ and $u_i \to 0$ in $W^{3,\frac{2n}{n+4}}$. Fix a $\varphi \in C^{\infty}(M)$, then

$$P\left(\varphi u_i\right) = \varphi f_i + g_i \tag{3.6}$$

with $g_i \to 0$ in $L^{\frac{2n}{n+4}}$. Let $v_i \in W^{4,\frac{2n}{n+4}}$ such that $Pv_i = g_i$, then $v_i \to 0$ in $W^{4,\frac{2n}{n+4}}$. We have

$$P\left(\varphi u_i - v_i\right) = \varphi f_i. \tag{3.7}$$

By Lemma 2.3 we know for any $\varepsilon > 0$,

$$\left\|\Delta\left(\varphi u_{i}-v_{i}\right)\right\|_{L^{2}}^{2} \leq \left(\Theta_{4}\left(S^{n}\right)+\varepsilon\right)\left\|\varphi f_{i}\right\|_{L^{\frac{2n}{n+4}}}^{2}+C\left(\varepsilon\right)\left\|\varphi u_{i}-v_{i}\right\|_{L^{\frac{2n}{n+4}}}^{2}.$$
(3.8)

Let $i \to \infty$, using $u_i \to 0$ weakly in $W^{4,\frac{2n}{n+4}}$, $u_i \to 0$ in $W^{3,\frac{2n}{n+4}}$ and $v_i \to 0$ in $W^{4,\frac{2n}{n+4}}$, we see $u_i \to 0$ weakly in $W^{2,2}$, $u_i \to 0$ in $W^{1,2}$ and $v_i \to 0$ in $W^{2,2}$, hence

$$\int_{M} \varphi^{2} d\nu \leq \left(\Theta_{4}\left(S^{n}\right) + \varepsilon\right) \left(\int_{M} \left|\varphi\right|^{\frac{2n}{n+4}} d\sigma\right)^{\frac{n+4}{n}}$$

Let $\varepsilon \to 0$ we get

$$\int_{M} \varphi^{2} d\nu \leq \Theta_{4} \left(S^{n} \right) \left(\int_{M} \left| \varphi \right|^{\frac{2n}{n+4}} d\sigma \right)^{\frac{n+4}{n}}.$$
(3.9)

Since φ is an arbitrary smooth function,

$$\nu(E) \le \Theta_4(S^n) \,\sigma(E)^{\frac{n+4}{n}} \tag{3.10}$$

for any Borel set E. Now we can follow the argument in [Ln] to determine the structure of σ and ν . Indeed by the fact ν is absolutely continuous with respect to σ , let $\chi = \frac{d\nu}{d\sigma}$. Define

$$\mathcal{B} = \left\{ p \in M : \sigma\left(\{p\}\right) > 0 \right\}.$$

Then \mathcal{B} is countable and we write points in it as p_i . On the other hand for σ a.e. $p \notin \mathcal{B}$, we have

$$\chi(p) = \lim_{r \to 0} \frac{\nu(B_r(p))}{\sigma(B_r(p))} \le \lim_{r \to 0} \inf \Theta_4(S^n) \sigma(B_r(p))^{\frac{4}{n}} = 0.$$

Hence $\nu = \sum_{i} \nu_i \delta_{p_i}, \sigma \ge \sum_{i} \sigma_i \delta_{p_i}$ and $\nu_i \le \Theta_4(S^n) \sigma_i^{\frac{n+4}{n}}$. In general f may not be zero, we can apply the previous discussion to $f_i - f$.

In general f may not be zero, we can apply the previous discussion to $f_i - f$ After passing to a subsequence we have

$$|f_i - f|^{\frac{2n}{n+4}} d\mu \rightharpoonup \widetilde{\sigma} \text{ and } (\Delta u_i - \Delta u)^2 d\mu \rightharpoonup d\widetilde{\nu} \text{ in } \mathcal{M}(M),$$

moreover

$$\widetilde{\sigma} \geq \sum_{i} \widetilde{\sigma}_{i} \delta_{p_{i}}, \quad \widetilde{\nu} = \sum_{i} \widetilde{\nu}_{i} \delta_{p_{i}}$$

with $\widetilde{\sigma}_{i} = \widetilde{\sigma}(\{p_{i}\}), \widetilde{\nu}_{i} = \widetilde{\nu}(\{p_{i}\})$ and $\widetilde{\nu}_{i} \leq \Theta_{4}(S^{n})\widetilde{\sigma}_{i}^{\frac{n+4}{n}}$. On the other hand, for any $\varphi \in C(M)$,

$$\int_{M} \varphi \left(\Delta u_{i} - \Delta u \right)^{2} d\mu$$

$$= \int_{M} \left[\varphi \left(\Delta u_{i} \right)^{2} - 2\varphi \Delta u_{i} \Delta u + \varphi \left(\Delta u \right)^{2} \right] d\mu$$

$$\rightarrow \int_{M} \varphi d\nu - \int_{M} \varphi \left(\Delta u \right)^{2} d\mu.$$

Hence $\tilde{\nu} = \nu - (\Delta u)^2 d\mu$. In another way,

$$\nu = \left(\Delta u\right)^2 d\mu + \sum_i \nu_i \delta_{p_i}$$

and $\nu_i = \nu(\{p_i\}) = \widetilde{\nu}_i$. For any $\rho \in C(M)$ we

For any $\varphi \in C(M)$ we have

$$\left| \left\| \varphi \left(f_i - f \right) \right\|_{L^{\frac{2n}{n+4}}} - \left\| \varphi f_i \right\|_{L^{\frac{2n}{n+4}}} \right| \le \left\| \varphi f \right\|_{L^{\frac{2n}{n+4}}}.$$

Let $i \to \infty$ we get

$$\left| \left(\int_{M} |\varphi|^{\frac{2n}{n+4}} d\widetilde{\sigma} \right)^{\frac{n+4}{2n}} - \left(\int_{M} |\varphi|^{\frac{2n}{n+4}} d\sigma \right)^{\frac{n+4}{2n}} \right| \le \left(\int_{M} |\varphi|^{\frac{2n}{n+4}} |f|^{\frac{2n}{n+4}} d\mu \right)^{\frac{n+4}{2n}}.$$

Hence with respect to μ , σ and $\tilde{\sigma}$ have the same singular part. In particular,

$$\sigma \geq \sum_i \sigma_i \delta_{p_i}$$

with $\sigma_i = \sigma(\{p_i\}) = \widetilde{\sigma}_i$. Because $\sigma \ge |f|^{\frac{2n}{n+4}} d\mu$, we get

$$\sigma \ge |f|^{\frac{2n}{n+4}} \, d\mu + \sum_i \sigma_i \delta_{p_i}$$

and $\nu_i \leq \Theta_4(S^n) \sigma_i^{\frac{n+4}{n}}$.

Now we are ready to derive a criterion for the existence of maximizers. Such kind of criterion is an analog statement for those of Yamabe problems ([LP]) and integral equations considered in [HWY1, HWY2].

Proposition 3.1. Assume (M, g) is a smooth compact n dimensional Riemannian manifold with $n \ge 5$, ker P = 0. Let

$$\Theta_4\left(g\right) = \sup_{f \in L^{\frac{2n}{n+4}}(M) \setminus \{0\}} \frac{\int_M G_P f \cdot f d\mu}{\left\|f\right\|_{L^{\frac{2n}{n+4}}}^2}.$$

If $\Theta_4(g) > \Theta_4(S^n)$ and $f_i \in L^{\frac{2n}{n+4}}$ satisfies $\|f_i\|_{L^{\frac{2n}{n+4}}} = 1$, $\int_M G_P f_i \cdot f_i d\mu \to \Theta_4(g)$, then after passing to a subsequence, we can find a $f \in L^{\frac{2n}{n+4}}$ such that $f_i \to f$ in $L^{\frac{2n}{n+4}}$. In particular, $\|f\|_{L^{\frac{2n}{n+4}}} = 1$ and $\int_M G_P f \cdot f d\mu = \Theta_4(g)$, f is a maximizer for $\Theta_4(g)$.

Proof. After passing to a subsequence we can assume $f_i \rightharpoonup f$ weakly in $L^{\frac{2n}{n+4}}$. Let $u_i, u \in W^{4,\frac{2n}{n+4}}$ such that $Pu_i = f_i, Pu = f$. Then $u_i \rightharpoonup u$ weakly in $W^{4,\frac{2n}{n+4}}$, $u_i \rightarrow u$ in $W^{3,\frac{2n}{n+4}}$ and $u_i \rightarrow u$ in $W^{1,2}$. After passing to another subsequence we have

$$|f_i|^{\frac{dn}{n+4}} d\mu \rightharpoonup d\sigma \text{ and } (\Delta u_i)^2 d\mu \rightharpoonup d\nu \text{ in } \mathcal{M}(M),$$

moreover it follows from Lemma 3.1 that

$$\sigma \ge |f|^{\frac{2n}{n+4}} d\mu + \sum_{i} \sigma_i \delta_{p_i}, \quad \nu = (\Delta u)^2 d\mu + \sum_{i} \nu_i \delta_{p_i},$$

here $\sigma_{i} = \sigma\left(\left\{p_{i}\right\}\right), \nu_{i} = \nu\left(\left\{p_{i}\right\}\right)$ and

$$\nu_i \le \Theta_4\left(S^n\right) \sigma_i^{\frac{n+4}{n}}.$$

It follows that $\sigma(M) = 1$ and

$$\int_{M} G_{P} f_{i} \cdot f_{i} d\mu$$

$$= \int_{M} u_{i} P u_{i} d\mu = E(u_{i})$$

$$= \int_{M} \left((\Delta u_{i})^{2} - 4A(\nabla u_{i}, \nabla u_{i}) + (n-2)J |\nabla u_{i}|^{2} + \frac{n-4}{2}Qu_{i}^{2} \right) d\mu$$

$$\rightarrow E(u) + \sum_{i} \nu_{i}.$$

Hence

$$\begin{aligned} \Theta_{4}\left(g\right) &= E\left(u\right) + \sum_{i} \nu_{i} \\ &\leq \Theta_{4}\left(g\right) \left\|f\right\|_{L^{\frac{2n}{n+4}}}^{2} + \Theta_{4}\left(S^{n}\right) \sum_{i} \sigma_{i}^{\frac{n+4}{n}} \\ &\leq \Theta_{4}\left(g\right) \left[\left(\left\|f\right\|_{L^{\frac{2n}{n+4}}}^{\frac{2n}{n+4}}\right)^{\frac{n+4}{n}} + \sum_{i} \sigma_{i}^{\frac{n+4}{n}}\right] \\ &\leq \Theta_{4}\left(g\right) \left(\left\|f\right\|_{L^{\frac{2n}{n+4}}}^{\frac{2n}{n+4}} + \sum_{i} \sigma_{i}\right)^{\frac{n+4}{n}} \\ &\leq \Theta_{4}\left(g\right). \end{aligned}$$

Hence all inequalities become equalities. In particular, $\sigma_i = 0$, $\nu_i = 0$, $\|f\|_{L^{\frac{2n}{n+4}}} = 1$. Hence $f_i \to f$ in $L^{\frac{2n}{n+4}}$, $E(u) = \int_M G_P f \cdot f d\mu = \Theta_4(g)$.

4. EXISTENCE AND REGULARITY OF MAXIMIZERS

The main aim of this section is to show the strict inequality between $\Theta_4(g)$ and $\Theta_4(S^n)$ in the assumption of Proposition 3.1 is valid as long as (M,g) is not conformal equivalent to the standard sphere. As in the Yamabe problem case ([LP]), this is achieved by a careful choice of test function. More precisely we have

Proposition 4.1. Assume (M, g) is a smooth compact n dimensional Riemannian manifold with $n \ge 5$, Y(g) > 0, $Q \ge 0$ and not identically zero, then

$$\Theta_4\left(g\right) \ge \Theta_4\left(S^n\right) \tag{4.1}$$

and equality holds if and only if (M,g) is conformal equivalent to the standard sphere.

Before we start the proof of Proposition 4.1, we list several basic identities which will facilitate the calculations. For b > -n and 2a - b > n,

$$\int_{\mathbb{R}^n} \frac{|x|^b}{\left(|x|^2+1\right)^a} dx = \frac{n\omega_n}{2} \frac{\Gamma\left(\frac{b+n}{2}\right)\Gamma\left(a-\frac{b+n}{2}\right)}{\Gamma\left(a\right)} = \pi^{\frac{n}{2}} \frac{\Gamma\left(\frac{b+n}{2}\right)\Gamma\left(a-\frac{b+n}{2}\right)}{\Gamma\left(a\right)\Gamma\left(\frac{n}{2}\right)}.$$
 (4.2)

If we fix an orthonormal frame at p, then

$$\Delta \sum_{k,l} (W_{ikjl}(p) x_i x_j)^2$$

$$= 4W_{ikjl}(p) W_{ikml}(p) x_j x_m + 4W_{ikjl}(p) W_{ilmk}(p) x_j x_m$$

$$= 2\sum_{ikl} (W_{ikjl}(p) x_j + W_{iljk}(p) x_j)^2$$

$$= 2\sum_{jkl} (W_{ijkl}(p) x_i + W_{ilkj}(p) x_i)^2,$$
(4.3)

and

$$\Delta^{2} \sum_{kl} \left(W_{ikjl}(p) \, x_{i} x_{j} \right)^{2} = 8 \left(\left| W(p) \right|^{2} + W_{ikjl}(p) \, W_{iljk}(p) \right) = 12 \left| W(p) \right|^{2}, \quad (4.4)$$

here we have used

$$W_{ikjl}(p) W_{iljk}(p) = \frac{1}{2} |W(p)|^2, \qquad (4.5)$$

which follows from the usual Bianchi identity. Hence

$$\sum_{kl} (W_{ikjl}(p) x_i x_j)^2 \qquad (4.6)$$

$$= \left[\sum_{kl} (W_{ikjl}(p) x_i x_j)^2 - \frac{r^2}{n+4} \sum_{jkl} (W_{ijkl}(p) x_i + W_{ilkj}(p) x_i)^2 + \frac{3}{2(n+2)(n+4)} |W(p)|^2 r^4 \right] + r^2 \cdot \left[\frac{1}{n+4} \sum_{jkl} (W_{ijkl}(p) x_i + W_{ilkj}(p) x_i)^2 - \frac{3}{n(n+4)} |W(p)|^2 r^2 \right] + r^4 \cdot \frac{3}{2n(n+2)} |W(p)|^2.$$

The polynomials in the square brackets are harmonic. In particular,

$$\int_{S^{n-1}} \sum_{kl} \left(W_{ikjl}(p) \, x_i x_j \right)^2 dS = \frac{3\omega_n}{2 \, (n+2)} \, |W(p)|^2 \,. \tag{4.7}$$

Recall

$$\Theta_{4}(g) = \sup_{f \in L^{\frac{2n}{n+4}}(M) \setminus \{0\}} \frac{\int_{M} G_{P} f \cdot f d\mu}{\|f\|_{L^{\frac{2n}{n+4}}}^{2}} = \sup_{u \in W^{4,\frac{2n}{n+4}}(M) \setminus \{0\}} \frac{\int_{M} Pu \cdot u d\mu}{\|Pu\|_{L^{\frac{2n}{n+4}}}^{2}}.$$
 (4.8)

Fix a function $\eta_1 \in C^{\infty}(\mathbb{R}, \mathbb{R})$ such that $\eta_1|_{(-\infty,1)} = 0$, $\eta_1|_{(2,\infty)} = 1$ and $0 \le \eta_1 \le 1$. Denote $\eta_2 = 1 - \eta_1$.

Case 4.1. *M* is conformally flat near $p, n \ge 5$.

In this case we may assume the metric g is flat near p. Under the Euclidean coordinate at p, namely x_1, \dots, x_n we have

$$2n(2-n)(4-n)\omega_n G_{P,p} = r^{4-n} + A_0 + \alpha.$$
(4.9)

Here A_0 is a constant, $\alpha = O_{\infty}(r)$ is a biharmonic function (with respect to Euclidean metric). For convenience we denote

$$H = 2n (2 - n) (4 - n) \omega_n G_{P,p}.$$
(4.10)

For $0 < \lambda < \delta$, let

$$u_{\lambda} = \left(\frac{\lambda}{\left|x\right|^{2} + \lambda^{2}}\right)^{\frac{n-4}{2}} \tag{4.11}$$

and

$$\beta = \lambda^{\frac{n-4}{2}} r^{4-n} - u_{\lambda}. \tag{4.12}$$

Denote $\phi(x) = |x|^{4-n} - \left(|x|^2 + 1\right)^{\frac{4-n}{2}}$, then $\beta = \lambda^{\frac{4-n}{2}} \phi\left(\frac{x}{\lambda}\right)$. Define

$$\varphi_{\lambda} = \begin{cases} u_{\lambda} + \eta_1\left(\frac{r}{\delta}\right)\beta + \lambda^{\frac{n-4}{2}}A_0 + \lambda^{\frac{n-4}{2}}\alpha, & \text{on } B_{3\delta}\left(p\right), \\ \lambda^{\frac{n-4}{2}}H, & \text{on } M \setminus B_{3\delta}\left(p\right). \end{cases}$$
(4.13)

It is clear that $\varphi_{\lambda} \in C^{\infty}(M)$. Note that

$$P\varphi_{\lambda}$$

$$= \begin{cases} n(n+2)(n-2)(n-4)\lambda^{\frac{n+4}{2}}(|x|^{2}+\lambda^{2})^{-\frac{n+4}{2}}, & \text{on } B_{\delta}(p), \\ O(\lambda^{\frac{n}{2}}), & \text{on } B_{2\delta}(p)\setminus B_{\delta}(p), \\ 0, & \text{on } M\setminus B_{2\delta}(p). \end{cases}$$

$$(4.14)$$

Hence

$$\int_{M} |P\varphi_{\lambda}|^{\frac{2n}{n+4}} d\mu$$

$$(n(n+2)(n-2)(n-4))^{\frac{2n}{n+4}} \frac{\Gamma\left(\frac{n}{2}\right)\pi^{\frac{n}{2}}}{(n-1)!} + O\left(\lambda^{\frac{n^{2}}{n+4}}\right).$$
(4.15)

It follows that

=

$$\|P\varphi_{\lambda}\|_{L^{\frac{2n}{n+4}}}^{2}$$

$$= (n(n+2)(n-2)(n-4))^{2} \frac{\Gamma\left(\frac{n}{2}\right)^{\frac{n+4}{n}} \pi^{\frac{n+4}{2}}}{((n-1)!)^{\frac{n+4}{n}}} + O\left(\lambda^{\frac{n^{2}}{n+4}}\right).$$

$$(4.16)$$

On the other hand,

$$\int_{M} P\varphi_{\lambda} \cdot \varphi_{\lambda} d\mu$$

$$= n (n+2) (n-2) (n-4) \frac{\Gamma\left(\frac{n}{2}\right) \pi^{\frac{n}{2}}}{(n-1)!} + \frac{4 (n-2) (n-4) \pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} A_{0} \lambda^{n-4} + o \left(\lambda^{n-4}\right).$$
(4.17)

Hence

$$\frac{\int_{M} P\varphi_{\lambda} \cdot \varphi_{\lambda} d\mu}{\left\| P\varphi_{\lambda} \right\|_{L^{\frac{2n}{n+4}}}^{2}}$$

$$= \Theta_{4} \left(S^{n} \right) + \frac{4 \left(\left(n-1 \right)! \right)^{\frac{n+4}{n}}}{n^{2} \left(n+2 \right)^{2} \left(n-2 \right) \left(n-4 \right) \Gamma \left(\frac{n}{2} \right)^{\frac{2n+4}{n}} \pi^{2}} A_{0} \lambda^{n-4} + o \left(\lambda^{n-4} \right).$$

$$(4.18)$$

If (M, g) is not conformal diffeomorphic to the standard sphere, then it follows from the arguments in [HY4, section 6] that $A_0 > 0$. Fix δ small and let $\lambda \downarrow 0$, we see $\Theta_4(g) > \Theta_4(S^n)$.

Case 4.2. n = 5, 6, 7.

In this case by conformal change of the metric we can assume \exp_p preserves the volume near p. Under the normal coordinate at p, namely x_1, \dots, x_n , we have

$$2n(2-n)(4-n)\omega_n G_{P,p} = r^{4-n} + A_0 + \alpha.$$
(4.19)

Here A_0 is a constant and $\alpha = O^{(4)}(r)$. Denote

$$H = 2n(2-n)(4-n)\omega_n G_{P,p}.$$
(4.20)

For $0 < \lambda < \delta$, let $\phi(x) = |x|^{4-n} - \left(|x|^2 + 1\right)^{\frac{4-n}{2}}$, $\beta = \lambda^{\frac{4-n}{2}} \phi\left(\frac{x}{\lambda}\right) = \lambda^{\frac{n-4}{2}} r^{4-n} - u_{\lambda}; \qquad (4.21)$

$$u_{\lambda} = \left(\frac{\lambda}{|x|^2 + \lambda^2}\right)^{\frac{n-4}{2}} \tag{4.22}$$

and

$$\varphi_{\lambda} = \begin{cases} u_{\lambda} + \eta_1\left(\frac{r}{\delta}\right)\beta + \lambda^{\frac{n-4}{2}}A_0 + \lambda^{\frac{n-4}{2}}\alpha, & \text{on } B_{3\delta}\left(p\right), \\ \lambda^{\frac{n-4}{2}}H, & \text{on } M \setminus B_{3\delta}\left(p\right). \end{cases}$$
(4.23)

then $\varphi_{\lambda} \in W^{4,\frac{2n}{n+4}}(M)$. On $B_{\delta}(p) \setminus \{p\}$,

$$P\varphi_{\lambda}$$

$$= Pu_{\lambda} - \lambda^{\frac{n-4}{2}} P(r^{4-n})$$

$$= \Delta^{2}u_{\lambda} - 4 \operatorname{div} \left(A(\nabla\beta, e_{i}) e_{i}\right) + (n-2) \operatorname{div} (J\nabla\beta) - \frac{n-4}{2}Q\beta$$

$$= n(n+2)(n-2)(n-4)\lambda^{\frac{n+4}{2}} \left(|x|^{2} + \lambda^{2}\right)^{-\frac{n+4}{2}} + O\left(\lambda^{\frac{n}{2}} |x|^{2-n}\right).$$

$$W_{\lambda} = \lambda \left(225\right) - \lambda \left(225\right) - 2 \left(2$$

Here we will need to use (2.35) and (2.36). On $B_{2\delta}(p) \setminus B_{\delta}(p)$,

$$P\varphi_{\lambda} = -P\left(\eta_2\left(\frac{r}{\delta}\right)\beta\right) = O\left(\lambda^{\frac{n}{2}}\right) \tag{4.25}$$

and on $M \setminus B_{2\delta}(p), P\varphi_{\lambda} = 0$. Hence

$$\int_{M} |P\varphi_{\lambda}|^{\frac{2n}{n+4}} d\mu$$

= $(n(n+2)(n-2)(n-4))^{\frac{2n}{n+4}} \frac{\Gamma(\frac{n}{2})\pi^{\frac{n}{2}}}{(n-1)!} + o(\lambda^{n-4}),$

and

$$\begin{aligned} \|P\varphi_{\lambda}\|_{L^{\frac{2n}{n+4}}}^{2} &= (n(n+2)(n-2)(n-4))^{2} \frac{\Gamma\left(\frac{n}{2}\right)^{\frac{n+4}{n}} \pi^{\frac{n+4}{2}}}{((n-1)!)^{\frac{n+4}{n}}} + o\left(\lambda^{n-4}\right) \end{aligned}$$

On the other hand,

$$\int_{M} P\varphi_{\lambda} \cdot \varphi_{\lambda} d\mu \tag{4.26}$$

$$= n(n+2)(n-2)(n-4)\frac{\Gamma(\frac{n}{2})\pi^{\frac{n}{2}}}{(n-1)!} + \frac{4(n-2)(n-4)\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}A_0\lambda^{n-4} + o(\lambda^{n-4}).$$

Sum up we have

$$\frac{\int_{M} P\varphi_{\lambda} \cdot \varphi_{\lambda} d\mu}{\left\| P\varphi_{\lambda} \right\|_{L^{\frac{2n}{n+4}}}^{2}}$$

$$= \Theta_{4} \left(S^{n} \right) + \frac{4 \left(\left(n-1 \right)! \right)^{\frac{n+4}{n}}}{n^{2} \left(n+2 \right)^{2} \left(n-2 \right) \left(n-4 \right) \Gamma \left(\frac{n}{2} \right)^{\frac{2n+4}{n}} \pi^{2}} A_{0} \lambda^{n-4} + o \left(\lambda^{n-4} \right).$$

$$(4.27)$$

By [HY4, section 6] we know when (M, g) is not conformal diffeomorphic to the standard sphere, A_0 is strictly positive. Letting $\lambda \downarrow 0$, we get $\Theta_4(g) > \Theta_4(S^n)$ in this case.

Case 4.3. (M, g) is not locally conformally flat and n = 8.

In this case we can choose p such that $W(p) \neq 0$. By conformal change of the metric we can assume \exp_p preserves the volume near p. Under the normal coordinate at p, namely x_1, \dots, x_8 , we have

$$384\omega_8 G_{P,p} = r^{-4} - \frac{|W(p)|^2}{1440} \log r + \alpha.$$
(4.28)

Here $\alpha = O^{(4)}(1)$. Denote

$$H = 384\omega_8 G_{P,p}.$$
 (4.29)

For $0 < \lambda < \delta$, let $\phi(x) = |x|^{-4} - \left(|x|^2 + 1\right)^{-2}$, $\beta = \lambda^{-2}\phi\left(\frac{x}{\lambda}\right) = \lambda^2 r^{-4} - u_\lambda$, $u_\lambda = \left(\frac{\lambda}{|x|^2 + \lambda^2}\right)^2$ (4.30)

and

$$\varphi_{\lambda} = \begin{cases} u_{\lambda} + \eta_1\left(\frac{r}{\delta}\right)\beta - \frac{|W(p)|^2}{1440}\lambda^2\log r + \lambda^2\alpha, & \text{on } B_{3\delta}\left(p\right), \\ \lambda^2 H, & \text{on } M \setminus B_{3\delta}\left(p\right). \end{cases}$$
(4.31)

Then $\varphi \in W^{4,\frac{4}{3}}(M)$. On $B_{\delta}(p) \setminus \{p\}$,

$$P\varphi_{\lambda} = Pu_{\lambda} - \lambda^{2} P(r^{-4})$$

$$= 1920\lambda^{6} \left(|x|^{2} + \lambda^{2}\right)^{-6} - 4 \operatorname{div} \left(A(\nabla\beta, e_{i}) e_{i}\right) + 6 \operatorname{div} \left(J\nabla\beta\right) - 2Q\beta$$

$$= 1920\lambda^{6} \left(|x|^{2} + \lambda^{2}\right)^{-6} + O(\beta) + O(\beta'r) + O(\beta''r^{2}).$$

$$(4.32)$$

Here we have used (2.35) and (2.36). On $B_{2\delta}\left(p\right)\setminus B_{\delta}\left(p\right)$,

$$P\varphi_{\lambda} = -P\left(\eta_2\left(\frac{r}{\delta}\right)\beta\right) = O\left(\lambda^4\right) \tag{4.33}$$

and on $M \setminus B_{2\delta}(p)$, $P \varphi_{\lambda} = 0$. Note that

$$\beta = \lambda^{2} r^{-4} - \lambda^{2} (r^{2} + \lambda^{2})^{-2},$$

$$\beta' = -4\lambda^{2} r^{-5} + 4\lambda^{2} (r^{2} + \lambda^{2})^{-3} r,$$

$$\beta'' = 20\lambda^{2} r^{-6} - 24\lambda^{2} (r^{2} + \lambda^{2})^{-4} r^{2} + 4\lambda^{2} (r^{2} + \lambda^{2})^{-3}.$$

Hence we have

$$\int_{M} |P\varphi_{\lambda}|^{\frac{4}{3}} d\mu = \frac{1920^{\frac{4}{3}}\pi^{4}}{840} + O\left(\lambda^{4}\right), \qquad (4.34)$$

and

$$\int_{M} P\varphi_{\lambda} \cdot \varphi_{\lambda} d\mu = \frac{1920\pi^{4}}{840} + \frac{\pi^{4} \left|W\left(p\right)\right|^{2}}{90} \lambda^{4} \log \frac{1}{\lambda} + O\left(\lambda^{4}\right).$$

It follows that

$$\frac{\int_{M} P\varphi_{\lambda} \cdot \varphi_{\lambda} d\mu}{\left\| P\varphi_{\lambda} \right\|_{L^{\frac{4}{3}}}^{2}} = \Theta_{4}\left(S^{8}\right) + \frac{210^{\frac{3}{2}}}{41472000\pi^{2}} \left| W\left(p\right) \right|^{2} \lambda^{4} \log \frac{1}{\lambda} + O\left(\lambda^{4}\right).$$

Hence $\Theta_4(g) > \Theta_4(S^8)$.

Case 4.4. *M* is not conformally flat and n = 9.

In this case we can choose p such that $W(p) \neq 0$. By a conformal change of metric we can assume \exp_p preserves the volume near p. Under the normal coordinate at p, namely x_1, \dots, x_9 , we have

$$630\omega_9 G_{P,p} = r^{-5} + r^{-5}\psi_4 + \alpha. \tag{4.35}$$

Here $\alpha = O^{(4)}(1)$ and

$$\psi_{4}$$

$$= \frac{1}{280} \left[\frac{2}{9} \sum_{kl} \left(W_{ikjl}(p) x_{i}x_{j} \right)^{2} - \frac{2}{117} r^{2} \sum_{jkl} \left(W_{ijkl}(p) x_{i} + W_{ilkj}(p) x_{i} \right)^{2} \right. \\ \left. + \frac{\left| W(p) \right|^{2}}{429} r^{4} \right] + \frac{r^{2}}{144} \left[\frac{4}{117} \sum_{jkl} \left(W_{ijkl}(p) x_{i} + W_{ilkj}(p) x_{i} \right)^{2} \right. \\ \left. - 6J_{ij}(p) x_{i}x_{j} - \frac{103}{5616} \left| W(p) \right|^{2} r^{2} \right] + \frac{805}{1368576} \left| W(p) \right|^{2} r^{4}.$$

$$(4.36)$$

Denote

$$H = 630\omega_9 G_{P,p}.$$
 (4.37)

For
$$0 < \lambda < \delta$$
, let $\phi(x) = |x|^{-5} - \left(|x|^2 + 1\right)^{-\frac{5}{2}}$, $\beta = \lambda^{-\frac{5}{2}}\phi\left(\frac{x}{\lambda}\right) = \lambda^{\frac{5}{2}}r^{-5} - u_{\lambda}$,
$$u_{\lambda} = \left(\frac{\lambda}{|x|^2 + \lambda^2}\right)^{\frac{5}{2}}$$
(4.38)

and

$$\varphi_{\lambda} = \begin{cases} u_{\lambda} + \eta_1\left(\frac{r}{\delta}\right)\beta + \lambda^{\frac{5}{2}}r^{-5}\psi_4 + \lambda^{\frac{5}{2}}\alpha, & \text{on } B_{3\delta}\left(p\right), \\ \lambda^{\frac{5}{2}}H, & \text{on } M \setminus B_{3\delta}\left(p\right). \end{cases}$$
(4.39)

Then $\varphi \in W^{4,\frac{18}{13}}(M)$. On $B_{\delta}(p) \setminus \{p\}$,

$$P\varphi_{\lambda} = Pu_{\lambda} - \lambda^{\frac{5}{2}} P(r^{-5})$$

$$= 3465\lambda^{\frac{13}{2}} \left(|x|^{2} + \lambda^{2} \right)^{-\frac{13}{2}} - 4 \operatorname{div} \left(A(\nabla\beta, e_{i}) e_{i} \right) + 7 \operatorname{div} \left(J\nabla\beta \right) - \frac{5}{2}Q\beta$$

$$= 3465\lambda^{\frac{13}{2}} \left(|x|^{2} + \lambda^{2} \right)^{-\frac{13}{2}} - 2 \left(\frac{\beta'}{r} \right)' \frac{A_{ijkl}(p) x_{i}x_{j}x_{k}x_{l}}{r}$$

$$+ \frac{7}{2} \left(\frac{\beta'}{r} \right)' r J_{ij}(p) x_{i}x_{j} + \frac{65}{2} \frac{\beta'}{r} J_{ij}(p) x_{i}x_{j} - \frac{5}{192} |W(p)|^{2}\beta$$

$$+ O(\beta r) + O(\beta' r^{2}) + O(\beta'' r^{3}).$$

$$(4.40)$$

On $B_{2\delta}(p) \setminus B_{\delta}(p)$,

$$P\varphi_{\lambda} = -P\left(\eta_2\left(\frac{r}{\delta}\right)\beta\right) = O\left(\lambda^{\frac{9}{2}}\right) \tag{4.41}$$

and on $M \setminus B_{2\delta}(p), P \varphi_{\lambda} = 0$. Note that

$$\begin{split} \beta &= \lambda^{\frac{5}{2}} r^{-5} - \lambda^{\frac{5}{2}} \left(r^2 + \lambda^2 \right)^{-\frac{5}{2}}, \\ \beta' &= -5\lambda^{\frac{5}{2}} r^{-6} + 5\lambda^{\frac{5}{2}} \left(r^2 + \lambda^2 \right)^{-\frac{7}{2}} r, \\ \frac{\beta'}{r} &= -5\lambda^{\frac{5}{2}} r^{-7} + 5\lambda^{\frac{5}{2}} \left(r^2 + \lambda^2 \right)^{-\frac{7}{2}}, \\ \beta'' &= 30\lambda^{\frac{5}{2}} r^{-7} - 35\lambda^{\frac{5}{2}} \left(r^2 + \lambda^2 \right)^{-\frac{9}{2}} r^2 + 5\lambda^{\frac{5}{2}} \left(r^2 + \lambda^2 \right)^{-\frac{7}{2}}, \\ \frac{\beta'}{r} \\ \end{pmatrix}' &= 35\lambda^{\frac{5}{2}} r^{-8} - 35\lambda^{\frac{5}{2}} \left(r^2 + \lambda^2 \right)^{-\frac{9}{2}} r. \end{split}$$

Calculation shows

$$\int_{M} |P\varphi_{\lambda}|^{\frac{18}{13}} d\mu \qquad (4.42)$$

$$= \frac{3465^{\frac{18}{13}}\pi^{5}}{6144} \left[1 + \left(\frac{94208}{4459455} \frac{1}{\pi} - \frac{41}{9009} \right) |W(p)|^{2} \lambda^{4} + o\left(\lambda^{4}\right) \right],$$

hence

$$\|P\varphi_{\lambda}\|_{L^{\frac{18}{13}}}^{2}$$

$$= \frac{3465^{2}\pi^{\frac{65}{9}}}{6144^{\frac{13}{9}}} \left[1 + \left(\frac{94208}{3087315}\frac{1}{\pi} - \frac{41}{6237}\right)|W(p)|^{2}\lambda^{4} + o\left(\lambda^{4}\right)\right]$$

$$(4.43)$$

On the other hand,

$$\int_{M} P\varphi_{\lambda} \cdot \varphi_{\lambda} d\mu$$

$$= \frac{1155}{2048} \pi^{5} \left[1 + \left(\frac{94208}{3087315} \frac{1}{\pi} - \frac{41}{12474} \right) |W(p)|^{2} \lambda^{4} + o(\lambda^{4}) \right]$$
(4.44)

Sum up we get

$$\frac{\int_{M} P\varphi_{\lambda} \cdot \varphi_{\lambda} d\mu}{\left\| P\varphi_{\lambda} \right\|_{L^{\frac{18}{13}}}^{2}} = \Theta_{4}\left(S^{9}\right) \left(1 + \frac{41}{12474} \left| W\left(p\right) \right|^{2} \lambda^{4} + o\left(\lambda^{4}\right) \right).$$
(4.45)

Hence we see $\Theta_4(g) > \Theta_4(S^9)$.

Case 4.5. *M* is not conformally flat and $n \ge 10$.

We can find a point p such that $W(p) \neq 0$. For $\lambda > 0$, denote

$$u_{\lambda}\left(x\right) = \left(\frac{\lambda}{\left|x\right|^{2} + \lambda^{2}}\right)^{\frac{n-4}{2}}.$$
(4.46)

Let x_1, \dots, x_n be a conformal normal coordinate at p, δ be a small fixed positive number,

$$\varphi_{\lambda} = u_{\lambda}(x) \eta_2\left(\frac{|x|}{\delta}\right).$$
 (4.47)

Then on $B_{2\delta}(p) \setminus B_{\delta}(p)$,

$$P\varphi_{\lambda} = O\left(\lambda^{\frac{n-4}{2}}\right). \tag{4.48}$$

On $B_{\delta}(p)$,

$$P\varphi_{\lambda}$$

$$= n (n+2) (n-2) (n-4) \lambda^{\frac{n+4}{2}} (|x|^{2} + \lambda^{2})^{-\frac{n+4}{2}}$$

$$-\frac{4}{9} (n-4) \lambda^{\frac{n-4}{2}} (|x|^{2} + \lambda^{2})^{-\frac{n}{2}} \sum_{kl} (W_{ikjl}(p) x_{i}x_{j})^{2}$$

$$+\frac{n-4}{2} \lambda^{\frac{n-4}{2}} (|x|^{2} + \lambda^{2})^{-\frac{n}{2}} (4 (n-6) |x|^{2} + (n^{2} - 16) \lambda^{2}) J_{ij}(p) x_{i}x_{j}$$

$$+\frac{n-4}{24 (n-1)} \lambda^{\frac{n-4}{2}} |W(p)|^{2} (|x|^{2} + \lambda^{2})^{-\frac{n-4}{2}}$$

$$+O \left(\lambda^{\frac{n-4}{2}} (|x|^{2} + \lambda^{2})^{-\frac{n-4}{2}} |x| \right).$$

$$(4.49)$$

Using the basic inequality

$$\left| |1+t|^{\frac{2n}{n+4}} - 1 - \frac{2n}{n+4}t \right| \le C |t|^{\frac{2n}{n+4}}$$
(4.50)

we see on $B_{\delta}(p)$,

$$\begin{split} &|P\varphi_{\lambda}|^{\frac{2n}{n+4}} \\ = & (n\left(n+2\right)\left(n-2\right)\left(n-4\right)\right)^{\frac{2n}{n+4}}\lambda^{n}\left(|x|^{2}+\lambda^{2}\right)^{-n} \cdot \\ & \left[1-\frac{8}{9}\frac{\lambda^{-4}\left(|x|^{2}+\lambda^{2}\right)^{2}}{\left(n+2\right)\left(n+4\right)\left(n-2\right)}\sum_{kl}\left(W_{ikjl}\left(p\right)x_{i}x_{j}\right)^{2} \right. \\ & \left.+\frac{\lambda^{-4}\left(|x|^{2}+\lambda^{2}\right)^{2}}{\left(n+2\right)\left(n+4\right)\left(n-2\right)}\left(4\left(n-6\right)|x|^{2}+\left(n^{2}-16\right)\lambda^{2}\right)J_{ij}\left(p\right)x_{i}x_{j} \right. \\ & \left.+\frac{\lambda^{-4}\left|W\left(p\right)\right|^{2}}{12\left(n+2\right)\left(n+4\right)\left(n-1\right)\left(n-2\right)}\left(|x|^{2}+\lambda^{2}\right)^{4} \right. \\ & \left.+O\left(\lambda^{-4}\left(|x|^{2}+\lambda^{2}\right)^{4}|x|\right)+O\left(\lambda^{-\frac{8n}{n+4}}\left(|x|^{2}+\lambda^{2}\right)^{\frac{8n}{n+4}}\right) \\ & \left.+O\left(\lambda^{-\frac{8n}{n+4}}\left(|x|^{2}+\lambda^{2}\right)^{\frac{8n}{n+4}}|x|^{\frac{2n}{n+4}}\right)\right]. \end{split}$$

Calculation shows

$$\int_{M} |P\varphi_{\lambda}|^{\frac{2n}{n+4}} d\mu$$

$$= (n (n+2) (n-2) (n-4))^{\frac{2n}{n+4}} \frac{\pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)}{(n-1)!} \cdot \left(1 - \frac{1}{3} \frac{n^2 - 4n - 4}{(n+2) (n+4) (n-2) (n-6) (n-8)} |W(p)|^2 \lambda^4 + o\left(\lambda^4\right)\right).$$
(4.51)

Hence

$$\left\| P\varphi_{\lambda} \right\|_{L^{\frac{2n}{n+4}}}^{2}$$

$$= (n(n+2)(n-2)(n-4))^{2} \frac{\pi^{\frac{n+4}{2}} \Gamma\left(\frac{n}{2}\right)^{\frac{n+4}{n}}}{((n-1)!)^{\frac{n+4}{n}}} \cdot \left(1 - \frac{1}{3} \frac{n^{2} - 4n - 4}{n(n+2)(n-2)(n-6)(n-8)} \left| W(p) \right|^{2} \lambda^{4} + o\left(\lambda^{4}\right) \right).$$

$$(4.52)$$

On the other hand,

$$\int_{M} P\varphi_{\lambda} \cdot \varphi_{\lambda} d\mu$$

$$= n (n+2) (n-2) (n-4) \frac{\pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)}{(n-1)!} \cdot \left(1 - \frac{n^2 - 4n - 4}{6n (n+2) (n-2) (n-6) (n-8)} |W(p)|^2 \lambda^4 + o \left(\lambda^4\right)\right).$$
(4.53)

Sum up we get

=

$$\frac{\int_{M} P\varphi_{\lambda} \cdot \varphi_{\lambda} d\mu}{\|P\varphi_{\lambda}\|_{L^{\frac{2n}{n+4}}}^{2}}$$
(4.54)
$$= \Theta_{4} \left(S^{n}\right) \left(1 + \frac{n^{2} - 4n - 4}{6n\left(n+2\right)\left(n-2\right)\left(n-6\right)\left(n-8\right)} \left|W\left(p\right)\right|^{2} \lambda^{4} + o\left(\lambda^{4}\right)\right).$$

It follows that $\Theta_4(g) > \Theta_4(S^n)$.

Next we turn to the regularity issue for maximizers of $\Theta_4(g)$ in (1.16). Assume $f \in L^{\frac{2n}{n+4}}(M), f \geq 0$ and not identically zero, and it is a maximizer for $\Theta_4(g)$, then after scaling we have

$$G_P f = \frac{2}{n-4} f^{\frac{n-4}{n+4}}.$$
(4.55)

Note this equation is critical in the sense that if we start with $f \in L^{\frac{2n}{n+4}}$ and use the equation, the usual bootstrap method simply ends with $f \in L^{\frac{2n}{n+4}}$ again. Approaches in deriving further regularity for such kind of equations has been well understood (see for example [DHL, ER, R, V] and so on). Here we state a result particularly tailored for our purpose. To facilitate our discussion of compactness of solutions later, we also sketch a proof.

Proposition 4.2. Assume (M, g) is a smooth compact n dimensional Riemannian manifold with $n \geq 5$, $\frac{2n}{n+4} < q < \frac{n}{4}$, $u \in W^{4, \frac{2n}{n+4}}(M)$, $b \in L^{\frac{n}{4}}(M)$, $f \in L^{q}(M)$ such that

$$\Delta^2 u = bu + f, \tag{4.56}$$

then $u \in W^{4,q}(M)$.

Proof. First we assume u is supported in $B_R(p)$ for some R > 0 small. Let x_1, \dots, x_n be a normal coordinate at p, then using the integral expression of u we have for |x| < 2R,

$$|u(x)| \le C \int_{B_{2R}} \frac{|b(y)| |u(y)|}{|x-y|^{n-4}} dy + C \int_{B_{2R}} \frac{|f(y)|}{|x-y|^{n-4}} dy.$$

Let r be chosen as $\frac{1}{r} = \frac{1}{q} - \frac{4}{n}$, then $\int_{B_{2R}} \frac{|f(y)|}{|x-y|^{n-4}} dy \in L^r(B_{2R})$. If R is small enough, then $\|b\|_{L^{\frac{n}{4}}(B_{2R})}$ is small and it follows from [L, Theorem 1.3] that $u \in L^r(B_R)$. It follows that $\Delta^2 u \in L^q(M)$ and hence $u \in W^{4,q}(M)$.

In general assume $\eta \in C^{\infty}(M)$ is a smooth cut-off function supported in a small ball, then

$$\Delta^2 \left(\eta u \right) = b\eta u + \eta f + f_1.$$

Here $f_1 \in L^{\frac{2n}{n+2}}$. If $q \leq \frac{2n}{n+2}$, then by previous discussion we know $\eta u \in W^{4,q}(M)$. Since this is true for any cut-off function with small support, we get $u \in W^{4,q}(M)$. If $q > \frac{2n}{n+2}$, then we can apply the usual bootstrap method. In fact we have $\eta u \in W^{4,\frac{2n}{n+2}}(M)$, hence $u \in W^{4,\frac{2n}{n+2}}(M)$. By Sobolev embedding theorem we have $f_1 \in L^2$. If $q \leq 2$, then $\eta u \in W^{4,q}(M)$ and $u \in W^{4,q}(M)$. If q > 2, then we have $u \in W^{4,2}(M)$ and go back to the bootstrap process. Eventually we arrive at $u \in W^{4,q}(M)$.

Corollary 4.1. Assume (M, g) is a smooth compact n dimensional Riemannian manifold with $n \ge 5$, Y(g) > 0, $Q \ge 0$ and not identically zero, $f \in L^{\frac{2n}{n+4}}(M)$, $f \ge 0$ and not identically zero, moreover

$$G_P f = \frac{2}{n-4} f^{\frac{n-4}{n+4}}.$$
(4.57)

Then $f \in C^{\infty}(M), f > 0.$

Proof. It follows from [HY4, Proposition 1.1] that $G_P > 0$. Let $u = G_P f$, then $u \ge c > 0, u \in W^{4, \frac{2n}{n+4}}(M) \subset L^{\frac{2n}{n-4}}(M), u = \frac{2}{n-4}f^{\frac{n-4}{n+4}}$ and

$$Pu = \left(\frac{n-4}{2}\right)^{\frac{n+4}{n-4}} u^{\frac{n+4}{n-4}}.$$
(4.58)

In another way, it is

$$\Delta^2 u = \left(\frac{n-4}{2}\right)^{\frac{n+4}{n-4}} u^{\frac{8}{n-4}} u + f_1,$$

where

$$f_1 = -4\operatorname{div}\left(A\left(\nabla u, e_i\right)e_i\right) + (n-2)\operatorname{div}\left(J\nabla u\right) - \frac{n-4}{2}Qu$$

Since $u^{\frac{8}{n-4}} \in L^{\frac{n}{4}}(M)$ and $f_1 \in W^{2,\frac{2n}{n+4}}(M) \subset L^2(M)$, it follows from Proposition 4.2 that $u \in W^{4,\frac{2n}{n+4}+\varepsilon} \subset L^{\frac{2n}{n-4}+\varepsilon_1}(M)$ for some $\varepsilon, \varepsilon_1 > 0$. Now the standard bootstrap method and elliptic theory together with the fact $u \ge c > 0$ tell us $u \in C^{\infty}(M)$ and u > 0. Hence $f \in C^{\infty}(M)$, f > 0.

On the other hand, assume $u \in H^2(M)$ is a minimizer for $Y_4(g)$ in (1.13), after scaling we can assume $||u||_{L^{\frac{2n}{n-4}}(M)} = 1$, then u satisfies

$$Pu = Y_4(g) |u|^{\frac{8}{n-4}} u. (4.59)$$

Corollary 4.2. Assume (M, g) is a smooth compact n dimensional Riemannian manifold with $n \geq 5$, $u \in H^2(M)$ satisfies (4.59), then $u \in C^{4,\alpha}(M)$ for all $\alpha \in (0, 1)$.

Proof. Since $u \in W^{4, \frac{2n}{n+4}}(M) \subset L^{\frac{2n}{n-4}}(M)$, we see $|u|^{\frac{8}{n-4}} u \in L^{\frac{2n}{n+4}}(M)$. Hence $u \in W^{4, \frac{2n}{n+4}}(M)$. (4.59) becomes

$$\Delta^{2} u = Y_{4}(g) |u|^{\frac{8}{n-4}} u + f_{1}$$

with

$$f_1 = -4\operatorname{div}\left(A\left(\nabla u, e_i\right)e_i\right) + (n-2)\operatorname{div}\left(J\nabla u\right) - \frac{n-4}{2}Qu$$

Since $|u|^{\frac{8}{n-4}} \in L^{\frac{n}{4}}(M)$ and $f_1 \in W^{2,\frac{2n}{n+4}}(M) \subset L^2(M)$, it follows from Proposition 4.2 that $u \in W^{4,\frac{2n}{n+4}+\varepsilon} \subset L^{\frac{2n}{n-4}+\varepsilon_1}(M)$ for some $\varepsilon, \varepsilon_1 > 0$. Standard bootstrap method and elliptic theory implies $u \in C^{4,\alpha}(M)$ for any $\alpha \in (0,1)$.

Now we have all the ingredients to prove Theorem 1.3. Theorem 1.1 clearly follows from Theorem 1.3.

Proof of Theorem 1.3. If (M, g) is conformal equivalent to the standard sphere, then everything follows from discussions in section 2.2. From now on we assume (M, g) is not conformal equivalent to the standard sphere. By Proposition 4.1 we know $\Theta_4(g) > \Theta_4(S^n)$. [HY4, Proposition 1.1] tells us ker P = 0 and $G_P > 0$. By Proposition 3.1 we know the set

$$\mathcal{M} = \left\{ f \in L^{\frac{2n}{n+4}}(M) : \|f\|_{L^{\frac{2n}{n+4}}(M)} = 1, \int_{M} G_{P}f \cdot fd\mu = \Theta_{4}(g) \right\}$$

is nonempty and compact in $L^{\frac{2n}{n+4}}(M)$. If $f \in \mathcal{M}$, we can assume $f^+ \neq 0$, then f^- must be equal to zero. Indeed

$$\Theta_{4}(g)$$

$$= \int_{M} G_{P}f \cdot f d\mu$$

$$= \int_{M} \left(G_{P}f^{+} \cdot f^{+} - 2G_{P}f^{+} \cdot f^{-} + G_{P}f^{-} \cdot f^{-} \right) d\mu$$

$$\leq \int_{M} G_{P} |f| \cdot |f| d\mu$$

$$\leq \Theta_{4}(g).$$

Hence $\int_M G_P f^+ \cdot f^- d\mu = 0$. Using the fact $G_P > 0$ and $f^+ \neq 0$, we see $f^- = 0$. In another word, f does not change sign. It follows from Corollary 4.1 that $f \in C^{\infty}(M)$ and f > 0. Moreover the compactness of \mathcal{M} under $C^{\infty}(M)$ topology follows from its compactness in $L^{\frac{2n}{n+4}}(M)$ and the proofs of Proposition 4.2 and Corollary 4.1.

5. Some discussions

5.1. $Y_4(g)$ revisited. Recall

$$Y_4(g) = \inf_{u \in H^2(M) \setminus \{0\}} \frac{E(u)}{\|u\|_{L^{\frac{2n}{n-4}}}^2},$$
(5.1)

here E(u) is given in (1.10) and (1.11).

Proposition 5.1. Let (M,g) be a smooth compact n dimensional Riemannian manifold with $n \ge 5$, Y(g) > 0, $Q \ge 0$ and not identically zero, then

- (1) $Y_4(g) \leq Y_4(S^n)$, here S^n has the standard metric. $Y_4(g) = Y_4(S^n)$ if and only if (M, g) is conformal diffeomorphic to the standard sphere.
- (2) $Y_4(g)$ is always achieved. Let

$$\mathcal{M}_{P} = \left\{ u \in H^{2}(M) : \|u\|_{L^{\frac{2n}{n-4}}(M)} = 1 \text{ and } E(u) = Y_{4}(g) \right\},$$
 (5.2)

then \mathcal{M}_P is not empty. For any $\alpha \in (0,1)$, $\mathcal{M}_P \subset C^{4,\alpha}(M)$ and when (M,g) is not conformal diffeomorphic to the standard sphere, \mathcal{M}_P is compact under $C^{4,\alpha}$ topology.

We start with some basic well known facts on compact Riemannian manifolds (see for example [DHL]).

Lemma 5.1. For any $u \in H^2(M)$,

$$\|u\|_{H^2} \le C \left(\|\Delta u\|_{L^2} + \|u\|_{L^2} \right). \tag{5.3}$$

This follows from standard elliptic estimate.

Lemma 5.2. Assume $u \in H^2(M)$, then for any $\varepsilon > 0$, we have

$$\left\|u\right\|_{H^{1}} \le \varepsilon \left\|D^{2}u\right\|_{L^{2}} + C\left(\varepsilon\right) \left\|u\right\|_{L^{2}}$$

$$(5.4)$$

and

$$\|u\|_{H^1} \le \varepsilon \|\Delta u\|_{L^2} + C(\varepsilon) \|u\|_{L^2}.$$

$$(5.5)$$

Proof. Using compact embedding $H^{2}(M) \subset H^{1}(M)$, standard compactness argument shows

$$\|u\|_{H^1} \le \varepsilon \, \|u\|_{H^2} + C\left(\varepsilon\right) \|u\|_{L^2}$$

Hence

$$\left\|u\right\|_{H^{1}} \leq \varepsilon \left\|D^{2}u\right\|_{L^{2}} + \varepsilon \left\|u\right\|_{H^{1}} + C\left(\varepsilon\right) \left\|u\right\|_{L^{2}}.$$

(5.4) follows. On the other hand, by (5.3) we know

$$||u||_{H^1} \le C\varepsilon \left(||\Delta u||_{L^2} + ||u||_{L^2} \right) + C(\varepsilon) ||u||_{L^2}.$$

(5.5) follows.

By (2.12) for any $u \in C_c^{\infty}(\mathbb{R}^n)$,

$$\|u\|_{L^{\frac{2n}{n-4}}(\mathbb{R}^n)}^2 \le \frac{1}{Y_4(S^n)} \|\Delta u\|_{L^2(\mathbb{R}^n)}^2.$$
(5.6)

Here is a well known almost sharp Sobolev inequality on compact manifolds. We present a proof for reader's convenience and completeness.

Lemma 5.3. For any $\varepsilon > 0$, we have

$$\|u\|_{L^{\frac{2n}{n-4}}(M)}^{2} \leq \frac{1+\varepsilon}{Y_{4}(S^{n})} \|\Delta u\|_{L^{2}(M)}^{2} + C(\varepsilon) \|u\|_{L^{2}(M)}^{2}$$
(5.7)

for all $u \in H^2(M)$.

Proof. The derivation follows the same line as arguments in [DHL] or the proof of Lemma 2.3. First we claim that for any $\varepsilon > 0$, we can find a $\delta > 0$ such that if u is supported in $B_{\delta}(p)$, then

$$\|u\|_{L^{\frac{2n}{n+4}}(M)}^{2} \leq \frac{1+\varepsilon}{Y_{4}(S^{n})} \|\Delta u\|_{L^{2}(M)}^{2} + C(\varepsilon) \|u\|_{L^{2}(M)}^{2}.$$
(5.8)

Indeed let x_1, \dots, x_n be a normal coordinate at p, then we have $g = g_{ij} dx_i dx_j$ with $g_{ij}(p) = \delta_{ij}$ and the Euclidean metric $g_0 = \delta_{ij} dx_i dx_j$. If δ is small enough, then

$$\left|\Delta u - \Delta_0 u\right| \le \varepsilon_1 \left|D^2 u\right| + C \left|D u\right|.$$

Here ε_1 is a small positive number. Then using (5.6) we have

$$\begin{aligned} \|u\|_{L^{\frac{2n}{n-4}}} &\leq \frac{1+\varepsilon_{1}}{\sqrt{Y_{4}(S^{n})}} \|\Delta_{0}u\|_{L^{2}} \\ &\leq \frac{1+\varepsilon_{1}}{\sqrt{Y_{4}(S^{n})}} \|\Delta u\|_{L^{2}} + C\varepsilon_{1} \|D^{2}u\|_{L^{2}} + C \|Du\|_{L^{2}} \\ &\leq \frac{1+C\varepsilon_{1}}{\sqrt{Y_{4}(S^{n})}} \|\Delta u\|_{L^{2}} + C(\varepsilon_{1}) \|u\|_{L^{2}}. \end{aligned}$$

Hence

$$\|u\|_{L^{\frac{2n}{n-4}}} \le \frac{1+C\varepsilon_1}{Y_4(S^n)} \|\Delta u\|_{L^2}^2 + C(\varepsilon_1) \|u\|_{L^2}^2$$

(5.8) follows.

To continue, following [DHL] we choose $\eta_1, \dots, \eta_m \in C^{\infty}(M)$ such that $0 \leq \eta_i \leq 1, \sqrt{\eta_i} \in C^{\infty}(M), \eta_i$ is supported in $B_{\delta}(p_i)$ for some p_i and $\sum_{i=1}^m \eta_i = 1$. Then

$$\begin{split} \|u\|_{L^{\frac{2n}{n-4}}}^{2} &= \|u^{2}\|_{L^{\frac{n}{n-4}}} \\ &\leq \sum_{i=1}^{m} \|\eta_{i}u^{2}\|_{L^{\frac{n}{n-4}}} \\ &= \sum_{i=1}^{m} \|\sqrt{\eta_{i}}u\|_{L^{\frac{2n}{n-4}}}^{2} \\ &\leq \frac{1+\varepsilon_{1}}{Y_{4}\left(S^{n}\right)} \sum_{i=1}^{m} \left\|\Delta\left(\sqrt{\eta_{i}}u\right)\right\|_{L^{2}}^{2} + C\left(\varepsilon_{1}\right) \|u\|_{L^{2}}^{2} \\ &\leq \frac{\left(1+\varepsilon_{1}\right)^{2}}{Y_{4}\left(S^{n}\right)} \|\Delta u\|_{L^{2}}^{2} + C\left(\varepsilon_{1}\right) \|u\|_{H^{1}}^{2} \\ &\leq \frac{\left(1+\varepsilon_{1}\right)^{2}}{Y_{4}\left(S^{n}\right)} \|\Delta u\|_{L^{2}}^{2} + C\left(\varepsilon_{1}\right) \|u\|_{L^{2}}^{2} . \end{split}$$

(5.7) follows.

Lemma 5.4. Let

$$\mathcal{M}_{P} = \left\{ u \in H^{2}(M) : \left\| u \right\|_{L^{\frac{2n}{n-4}}(M)} = 1 \text{ and } E(u) = Y_{4}(g) \right\}.$$

If $Y_4(g) < Y_4(S^n)$, then \mathcal{M}_P is nonempty. Moreover for any $\alpha \in (0,1)$, $\mathcal{M}_P \subset C^{4,\alpha}(M)$ and it is compact in $C^{4,\alpha}$ topology.

Proof. If $u_i \in H^2(M)$ such that $||u_i||_{L^{\frac{2n}{n-4}}} = 1$ and $E(u_i) \to Y_4(g)$. Since

$$E(u_i) \ge \frac{1}{C} \|u_i\|_{H^2}^2 - C \|u\|_{L^2}^2$$

it follows from Holder inequality that $\sup_i ||u_i||_{H^2} < \infty$. Hence after passing to a subsequence, we may find a $u \in H^2$ such that $u_i \rightharpoonup u$ weakly in H^2 , $u_i \rightarrow u$ in H^1 and $u_i \rightarrow u$ a.e.. We have

$$\begin{aligned} \|\Delta u_{i} - \Delta u\|_{L^{2}}^{2} \\ &= \|\Delta u_{i}\|_{L^{2}}^{2} - \|\Delta u\|_{L^{2}}^{2} + o(1) \\ &= E(u_{i}) - E(u) + o(1) \\ &\leq Y_{4}(g) - Y_{4}(g) \|u\|_{L^{\frac{2n}{n-4}}}^{2} + o(1). \end{aligned}$$

36

On the other hand

$$1 - \|u\|_{L^{\frac{2n}{n-4}}}^{2}$$

$$= \|u_i\|_{L^{\frac{2n}{n-4}}}^{2} - \|u\|_{L^{\frac{2n}{n-4}}}^{2}$$

$$\leq \left\|\|u_i\|_{L^{\frac{2n}{n-4}}}^{\frac{2n}{n-4}} - \|u\|_{L^{\frac{2n}{n-4}}}^{\frac{2n}{n-4}}\right|^{\frac{n-4}{n}}$$

$$= \left(\|u_i - u\|_{L^{\frac{2n}{n-4}}}^{\frac{2n}{n-4}} + o(1)\right)^{\frac{n-4}{n}}$$

$$= \|u_i - u\|_{L^{\frac{2n}{n-4}}}^{2} + o(1),$$

hence

$$\begin{split} &\|\Delta u_{i} - \Delta u\|_{L^{2}}^{2} \\ \leq & Y_{4}\left(g\right) \|u_{i} - u\|_{L^{\frac{2n}{n-4}}}^{2} + o\left(1\right) \\ \leq & \left(1 + \varepsilon\right) \frac{Y_{4}\left(g\right)^{+}}{Y_{4}\left(S^{n}\right)} \|\Delta u_{i} - \Delta u\|_{L^{2}(M)}^{2} + o\left(1\right). \end{split}$$

Here $Y_4(g)^+ = \max \{Y_4(g), 0\}$. Choosing ε small enough such that

 $(1+\varepsilon) Y_4(g)^+ < Y_4(S^n),$

then we have $\|\Delta u_i - \Delta u\|_{L^2}^2 \to 0$, this implies $u_i \to u$ in $H^2(M)$. It follows that $\|u\|_{L^{\frac{2n}{n-4}}} = 1$ and $E(u) = Y_4(g)$. Hence $u \in \mathcal{M}_P$. The above discussion implies \mathcal{M}_P is nonempty and compact in $H^2(M) \subset L^{\frac{2n}{n-4}}(M)$. On the other hand for any $u \in \mathcal{M}_P$, $Pu = Y_4(g) |u|^{\frac{8}{n-4}} u$. It follows from this and the proof of Proposition 4.2 and Corollary 4.2 that $\mathcal{M}_P \subset C^{4,\alpha}(M)$ and it is compact in $C^{4,\alpha}(M)$ for any $\alpha \in (0,1)$.

Now we are ready to deduce Proposition 5.1.

Proof of Proposition 5.1. If (M, g) is conformal to the standard sphere, then the conclusion follows from discussions in section 2.2. Assume (M, g) is not conformal diffeomorphic to the standard sphere, then it follows from Lemma 2.2 and Proposition 4.1 that

$$Y_{4}\left(g\right) \leq \frac{1}{\Theta_{4}\left(g\right)} < \frac{1}{\Theta_{4}\left(S^{n}\right)} = Y_{4}\left(S^{n}\right).$$

Here we want to point out that the fact $Y_4(g) < Y_4(S^n)$ can be verified, with the help of positive mass theorem for Paneitz operator ([HuR, GM, HY4]), by choosing a particular test function in (1.13) (see [ER, R, GM]). In fact the corresponding calculation is easier than what we have in the proof of Proposition 4.1, but the statement in Proposition 4.1 is stronger. By Lemma 5.4, we know \mathcal{M}_P is nonempty and $\mathcal{M}_P \subset C^{4,\alpha}(M)$ and it is compact in $C^{4,\alpha}(M)$ for any $\alpha \in (0,1)$.

Assume ker P = 0, then we have

$$\Theta_{4}(g) = \sup_{f \in L^{\frac{2n}{n+4}}(M) \setminus \{0\}} \frac{\int_{M} G_{P} f \cdot f d\mu}{\|f\|_{L^{\frac{2n}{n+4}}}^{2}} = \sup_{u \in W^{4,\frac{2n}{n+4}}(M) \setminus \{0\}} \frac{\int_{M} Pu \cdot u d\mu}{\|Pu\|_{L^{\frac{2n}{n+4}}}^{2}}.$$
 (5.9)

Proposition 5.2. Let (M, g) be a smooth compact n dimensional Riemannian manifold with $n \ge 5$, Y(g) > 0, $Y_4(g) > 0$, $Q \ge 0$ and not identically zero. Denote

$$\mathcal{M}_{P} = \left\{ u \in H^{2}(M) : \left\| u \right\|_{L^{\frac{2n}{n-4}}(M)} = 1 \text{ and } E(u) = Y_{4}(g) \right\}$$

and

$$\mathcal{M}_{\Theta} = \left\{ u \in W^{4, \frac{2n}{n+4}}(M) : \left\| u \right\|_{L^{\frac{2n}{n-4}}(M)} = 1 \text{ and } \frac{E(u)}{\left\| Pu \right\|_{L^{\frac{2n}{n+4}}}^{2}} = \Theta_{4}(g) \right\}.$$

then

(1) $\mathcal{M}_P \subset C^{\infty}(M)$ and for any $u \in \mathcal{M}_P$, either u > 0 or -u > 0. (2) $Y_4(g) \Theta_4(g) = 1$. (3) $\mathcal{M}_P = \mathcal{M}_{\Theta}$.

Proof. By Proposition 5.1 we know \mathcal{M}_P is nonempty and for any $\alpha \in (0, 1)$, $\mathcal{M}_P \subset C^{4,\alpha}(M)$. By [HY4, Proposition 1.1] we know ker P = 0 and $G_P > 0$. Assume $u \in \mathcal{M}_P$, without losing of generality we can assume $u^+ \neq 0$. Now we will use an observation in [R] to show u > 0. In fact u satisfies $||u||_{L^{\frac{2n}{2n-4}}} = 1$ and

$$Pu = Y_4(g) |u|^{\frac{8}{n-4}} u$$

Let $v = G_P(|Pu|)$, then $v \in C^{4,\alpha}(M)$, v > 0 and $|u| \le v$. We have

$$Y_{4}(g) \leq \frac{E(v)}{\left\|v\right\|_{L^{\frac{2n}{n-4}}}^{2}} = Y_{4}(g) \frac{\int_{M} |u|^{\frac{n+4}{n-4}} v d\mu}{\left\|v\right\|_{L^{\frac{2n}{n-4}}}^{2}} \leq Y_{4}(g) \left\|v\right\|_{L^{\frac{2n}{n-4}}}^{-1} \leq Y_{4}(g).$$

Hence all the inequalities become equalities. In particular $\|v\|_{L^{\frac{2n}{n-4}}} = 1 = \|u\|_{L^{\frac{2n}{n-4}}}$. Since $v \ge |u|$, we see v = |u|. This together with $u^+ \ne 0$ implies u = v > 0. Standard bootstrap method shows $u \in C^{\infty}(M)$. Hence $\mathcal{M}_P \subset C^{\infty}(M)$, moreover when (M,g) is not conformal diffeomorphic to the standard sphere, \mathcal{M}_P is compact in $C^{\infty}(M)$.

For $u \in \mathcal{M}_P$, we can assume u > 0, then $\|u\|_{L^{\frac{2n}{n-4}}} = 1$ and

$$Pu = Y_4\left(g\right) u^{\frac{n+4}{n-4}}.$$

It follows that from this equation and Lemma 2.2 that

$$\Theta_{4}(g) \ge \frac{E(u)}{\|Pu\|_{L^{\frac{2n}{n+4}}}^{2}} = \frac{1}{Y_{4}(g)} \ge \Theta_{4}(g).$$

Hence $Y_4(g) \Theta_4(g) = 1$ and $u \in \mathcal{M}_{\Theta}$.

On the other hand, if $u \in \mathcal{M}_{\Theta}$, let f = Pu, then

$$\Theta_{4}(g) = \frac{\int_{M} Pu \cdot ud\mu}{\|Pu\|_{L^{\frac{2n}{n+4}}}^{2}} = \frac{\int_{M} G_{P}f \cdot fd\mu}{\|f\|_{L^{\frac{2n}{n+4}}}^{2}}.$$

Hence it follows from Theorem 1.3 that $f \in C^{\infty}(M)$ and either f > 0 or -f > 0. Without losing of generality we assume f > 0, then $u = G_P f \in C^{\infty}(M)$, u > 0 and

$$Pu = \kappa u^{\frac{n+4}{n-4}}$$

for some positive constant κ . Using $||u||_{L^{\frac{2n}{n-4}}} = 1$ we see

$$\Theta_4\left(g\right) = \frac{E\left(u\right)}{\left\|Pu\right\|_{L^{\frac{2n}{n+4}}}^2} = \frac{1}{\kappa},$$

and hence $\kappa = Y_4(g)$. It follows that $E(u) = Y_4(g)$ and hence $u \in \mathcal{M}_P$. Sum up we see $\mathcal{M}_P = \mathcal{M}_{\Theta}$.

Now we are ready to derive Theorem 1.2.

Proof of Theorem 1.2. It is clear Theorem 1.2 follows from Proposition 5.1 and 5.2. The compactness of \mathcal{M}_P in C^{∞} topology was shown in the proof of Proposition 5.2.

5.2. Yamabe problem revisited. In this subsection we will show the above approach to the Q curvature equation gives another way to find constant scalar curvature metric in a conformal class with positive Yamabe invariant. Here we always assume (M, g) is a smooth compact n dimensional Riemannian manifold with $n \ge 3$ and Y(g) > 0.

The conformal Laplacian is given by

$$L\varphi = -\frac{4(n-1)}{n-2}\Delta\varphi + R\varphi.$$
(5.10)

Under the conformal change of metrics, we have

$$L_{\rho^{\frac{4}{n-2}}g}\varphi = \rho^{-\frac{n+2}{n-2}}L_g\left(\rho\varphi\right).$$
(5.11)

In particular,

$$R_{\rho^{\frac{4}{n-2}}g} = \rho^{-\frac{n+2}{n-2}} L_g \rho.$$
(5.12)

Hence to find a conformal metric with scalar curvature 1 is the same as solving

$$L_g \rho = \rho^{\frac{n+2}{n-2}}, \quad \rho \in C^{\infty}(M), \rho > 0.$$
 (5.13)

For any $u \in C^{\infty}(M)$ we write

$$E_2(u) = \int_M Lu \cdot ud\mu$$

$$= \int_M \left(\frac{4(n-1)}{n-2} |\nabla u|^2 + Ru^2\right) d\mu.$$
(5.14)

Note this formula also makes sense for $u \in H^1(M)$. To solve (5.13), people consider the variational problem (see [LP])

$$Y(g) = \inf_{u \in H^1(M) \setminus \{0\}} \frac{E_2(u)}{\|u\|_{L^{\frac{2n}{n-2}}}^2}.$$
(5.15)

Note that

$$Y\left(g\right) = \inf_{\substack{u \in H^{1}(M) \setminus \{0\}\\ u \ge 0}} \frac{E_{2}\left(u\right)}{\left\|u\right\|_{L^{\frac{2n}{n-2}}}^{2}} = \inf_{\substack{u \in C^{\infty}(M)\\ u > 0}} \frac{\int_{M} Lu \cdot ud\mu}{\left\|u\right\|_{L^{\frac{2n}{n-2}}}^{2}} = \inf_{\widetilde{g} \in [g]} \frac{\int_{M} Rd\widetilde{\mu}}{\left(\widetilde{\mu}\left(M\right)\right)^{\frac{n-2}{n}}}.$$

Denote

$$\mathcal{M}_{L} = \left\{ u \in H^{1}(M) : \left\| u \right\|_{L^{\frac{2n}{n-2}}} = 1 \text{ and } E_{2}(u) = Y(g) \right\},$$
(5.16)

then it is well known that \mathcal{M}_L is always nonempty, $\mathcal{M}_L \subset C^{\infty}(M)$ and for any $u \in \mathcal{M}_L$, either u > 0 or -u > 0. If u > 0, then after scaling u solves (5.13).

Moreover when (M, g) is not conformal diffeomorphic to the standard sphere, we have $Y(g) < Y(S^n)$ and \mathcal{M}_L is compact in C^{∞} topology.

Now we turn to another approach to solve (5.13). Since Y(g) > 0, we know the Green's function of L exists and it is always positive. We can define an operator

$$(G_L f)(p) = \int_M G_L(p,q) f(q) d\mu(q).$$
 (5.17)

This is the inverse operator of L. Let $f = \rho^{\frac{n+2}{n-2}}$, then (5.13) becomes

$$G_L f = f^{\frac{n+2}{n-2}}, \quad f \in C^{\infty}(M), f > 0.$$
 (5.18)

Let

$$\Theta_{2}(g) = \sup_{f \in L^{\frac{2n}{n+2}}(M) \setminus \{0\}} \frac{\int_{M} G_{L} f \cdot f d\mu}{\|f\|_{L^{\frac{2n}{n+2}}}^{2}} = \sup_{u \in W^{2,\frac{2n}{n+2}}(M) \setminus \{0\}} \frac{\int_{M} Lu \cdot u d\mu}{\|Lu\|_{L^{\frac{2n}{n+2}}}^{2}}.$$
 (5.19)

Note that this functional is considered in [DZ].

Lemma 5.5. Let (M, g) be a smooth compact n dimensional Riemannian manifold with $n \ge 3$, Y(g) > 0, then

$$\Theta_{2}(g) = \sup_{\widetilde{g} \in [g]} \frac{\int_{M} \widetilde{R} d\widetilde{\mu}}{\left\|\widetilde{R}\right\|_{L^{\frac{2n}{n+2}}(M, d\widetilde{\mu})}^{2}}.$$
(5.20)

Proof. Using the fact $G_L > 0$, we have

$$\Theta_{2}(g) = \sup_{\substack{f \in L^{\frac{2n}{n+2}}(M) \setminus \{0\} \\ f \ge 0}} \frac{\int_{M} G_{L} f \cdot f d\mu}{\|f\|_{L^{\frac{2n}{n+2}}}^{2}} \le \sup_{\substack{u \in W^{2, \frac{2n}{n+2}}(M) \setminus \{0\} \\ u \ge 0}} \frac{\int_{M} Lu \cdot u d\mu}{\|Lu\|_{L^{\frac{2n}{n+2}}}^{2}} \le \Theta_{2}(g).$$

Hence

$$\Theta_{2}(g) = \sup_{\substack{u \in W^{2,\frac{2n}{n+2}}(M) \setminus \{0\}\\ u \ge 0}} \frac{\int_{M} Lu \cdot ud\mu}{\left\|Lu\right\|_{L^{\frac{2n}{n+2}}}^{2}}}$$
$$= \sup_{\substack{u \in C^{\infty}(M)\\ u > 0}} \frac{\int_{M} Lu \cdot ud\mu}{\left\|Lu\right\|_{L^{\frac{2n}{n+2}}}^{2}}}$$
$$= \sup_{\widetilde{g} \in [g]} \frac{\int_{M} \widetilde{R}d\widetilde{\mu}}{\left\|\widetilde{R}\right\|_{L^{\frac{2n}{n+2}}(M,d\widetilde{\mu})}^{2}}.$$

With the solution to Yamabe problem ([LP]) we can easily deduce

Lemma 5.6. Let (M, g) be a smooth compact n dimensional Riemannian manifold with $n \ge 3$, Y(g) > 0. Denote

$$\mathcal{M}_{\Theta_{2}} = \left\{ u \in W^{2,\frac{2n}{n+2}}(M) : \left\| u \right\|_{L^{\frac{2n}{n-2}}(M)} = 1 \text{ and } \frac{E_{2}(u)}{\left\| Lu \right\|_{L^{\frac{2n}{n+2}}}^{2}} = \Theta_{2}(g) \right\}.$$

Then

(1)
$$Y(g)\Theta_2(g) = 1.$$

(2) $\mathcal{M}_L = \mathcal{M}_{\Theta_2}$.

Since the proof is essentially the same as the one for Proposition 5.2, we omit it here. Roughly speaking Lemma 5.6 tells us the maximization problem for $\Theta_2(g)$ will not produce new constant scalar curvature metrics other than those by minimizing problem for Y(g). However, without using the solution to Yamabe problem, we can use the same argument as for Theorem 1.3 to show $\Theta_2(g) \geq \Theta_2(S^n)$, with equality holds if and only if (M,g) is conformal diffeomorphic to the standard sphere (here one needs to use the positive mass theorem); \mathcal{M}_{Θ_2} is always nonempty, $\mathcal{M}_{\Theta_2} \subset C^{\infty}(M)$ and any $u \in \mathcal{M}_{\Theta_2}$ must be either positive or negative; \mathcal{M}_{Θ_2} is compact in $C^{\infty}(M)$ when (M,g) is not conformal diffeomorphic to the standard sphere. In particular, this gives another way to solve (5.13). The details are left to interested readers.

References

- [B] T. Branson. Differential operators canonically associated to a conformal structure. Math. Scand. 57 (1985), no. 2, 293–345.
- [CGY] S. Y Chang, M. J. Gursky and P. C. Yang. An equation of Monge-Ampere type in conformal geometry, and four-manifolds of positive Ricci curvature. Ann. of Math. (2) 155 (2002), 709-787.
- [CLO] W. X. Chen, C. M. Li and B. Ou. Classification of solutions for an integral equation. Comm Pure Appl Math, 59 (2006), no. 3, 330–343.
- [CY] S. Y. Chang and P. C. Yang. Extremal metrics of zeta function determinants on 4 manifolds. Ann. of Math. (2) 142 (1995), no. 1, 171–212.
- [DHL] Z. Djadli, E. Hebey and M. Ledoux. Paneitz-type operators and applications. Duke Math. Jour. 104 (2000), 129–169.
- [DM] Z. Djadli and A. Malchiodi. Existence of conformal metrics with constant Q curvature. Ann. of Math. (2) 168 (2008), no. 3, 813–858.
- [DZ] J. B. Dou and M. J. Zhu. Sharp Hardy-Littlewood-Sobolev inequality on the upper half space. Preprint, (2013), arXiv:1309.2341.
- [ER] P. Esposito and F. Robert. Mountain pass critical points for Paneitz-Branson operators. Calc. Var. Partial Differential Equations 15 (2002), no. 4, 493–517.
- [FR] A. Fardoun and R. Regbaoui. Q-curvature flow for GJMS operators with non-trivial kernel. J. Geom. Phys. 62 (2012), no. 12, 2321–2328.
- [GM] M. J. Gursky and A. Malchiodi. A strong maximum principle for the Paneitz operator and a nonlocal flow for the Q curvature. Preprint (2014).
- [HWY1] F. B. Hang, X. D. Wang and X. D. Yan. Sharp integral inequalities for harmonic functions. Comm. Pure Appl. Math. 61 (2008), no. 1, 54–95.
- [HWY2] F. B. Hang, X. D. Wang and X. D. Yan. An integral equation in conformal geometry. Ann. Inst. H. Poincare Anal. Non Lineaire 26 (2009), no. 1, 1–21.
- [HY1] F. B. Hang and P. Yang. The Sobolev inequality for Paneitz operator on three manifolds. Calculus of Variations and PDE. 21 (2004), 57–83.
- [HY2] F. B. Hang and P. Yang. Paneitz operator for metrics near S^3 . Preprint (2014).
- [HY3] F. B. Hang and P. Yang. Q curvature on a class of 3 manifolds. Comm Pure Appl Math, to appear.
- [HY4] F. B. Hang and P. Yang. Sign of Green's function of Paneitz operators and the Q curvature. Preprint (2014).
- [HeR1] E. Hebey and F. Robert. Compactness and global estimates for the geometric Paneitz equation in high dimensions. Electron Res Ann Amer Math Soc. 10 (2004), 135–141.
- [HeR2] E. Hebey and F. Robert. Asymptotic analysis for fourth order Paneitz equations with critical growth. Adv Calc Var. 4 (2011), no. 3, 229–275.
- [HuR] E. Humbert and S. Raulot. Positive mass theorem for the Paneitz-Branson operator. Calculus of Variations and PDE. 36 (2009), 525–531.
- [LP] J. M. Lee and T. H. Parker. The Yamabe problem. Bull AMS. 17 (1987), no. 1, 37–91.
- [L] Y. Y. Li. Remark on some conformally invariant integral equations: the method of moving spheres. Journal of European Mathematical Society. 6 (2004), 153–180.

- [Li] E. H. Lieb. Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. Annals of Mathematics. Second Series. 118, no. 2, (1983), 349–374.
- [Ln] P. L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. I. Rev. Mat. Iberoamericana 1 (1985), no. 1, 145–201.
- [P] S. Paneitz. A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds. Preprint (1983).
- [QR1] J. Qing and D. Raske. Compactness for conformal metrics with constant Q curvature on locally conformally flat manifolds. Calc. Var. Partial Differential Equations 26 (2006), no. 3, 343-356.
- [QR2] J. Qing and D. Raske. On positive solutions to semilinear conformally invariant equations on locally conformally flat manifolds. Int Math Res Not. Art. id 94172 (2006).
- [R] F. Robert. Fourth order equations with critical growth in Riemannian geometry. Unpublished notes. Available at http://www.iecn.u-nancy.fr/~frobert/LectRobertFourth.pdf.
- [SY] R. Schoen and S. T. Yau. Conformally flat manifolds, Kleinian groups and scalar curvature. Inventiones Mathematicae 92 (1988) no. 1, 47–71.
- [S] E. M. Stein. Singular integrals and differentiability properties of functions. Princeton University Press, Princeton, New Jersey, 1970.
- [V] R. C. A. M. Van der Vorst. Best constant for the embedding of the space $H^2 \cap H^1_0(\Omega)$ into $L^{\frac{2N}{N-4}}(\Omega)$. Differential Integral Equations 6 (1993), no. 2, 259–276.
- [XY] X. W. Xu and P. Yang. On a fourth order equation in 3-D. ESAIM: Control, optimization and Calculus of Variations 8 (2002) 1029–1042.
- [YZ] P. Yang and M. J. Zhu. Sharp inequality for Paneitz operator on S³. ESAIM: Control, Optim. and Calc. of Var. 10 (2004) 211-223.

251 MERCER STREET, NEW YORK NY 10012 *E-mail address*: fengbo@cims.nyu.edu

Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton NJ 08544

E-mail address: yang@math.princeton.edu