Q CURVATURE ON A CLASS OF 3 MANIFOLDS

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Abstract. Motivated by the strong maximum principle for Paneitz operator in dimension 5 or higher found in [GM] and the calculation of the second variation of the Green's function pole's value on S^3 in [HY2], we study Riemannian metric on 3 manifolds with positive scalar and Q curvature. Among other things, we show it is always possible to find a constant Q curvature metric in the conformal class. Moreover the Green's function is always negative away from the pole and the pole's value vanishes if and only if the Riemannian manifold is conformal diffeomorphic to the standard S^3. Compactness of constant Q curvature metrics in a conformal class and the associated Sobolev inequality are also discussed.

1. Introduction

The study of Paneitz operator and Q curvature has improved our understanding of four dimensional conformal geometry ([CGY]). On three dimensional Riemannian manifold, much less is known. However the Paneitz operator and Q curvature may contain valuable information besides those related to conformal Laplacian which is associated with the scalar curvature. These additional information may help us distinguish some conformal classes from others. The aim of this article is to understand this fourth order operator for a class of metrics on three manifolds.

Recall on three manifolds, the Q curvature is given by

$$Q=-\frac{1}{4}\Delta R-2\left|Rc\right|^{2}+\frac{23}{32}R^{2}, \tag{1.1}$$

and the fourth order Paneitz operator is defined as

$$P\varphi=\Delta^{2}\varphi+4\operatorname{div}\left[Rc\left(\nabla\varphi,e_{i}\right)e_{i}\right]-\frac{5}{4}\operatorname{div}\left(R\nabla\varphi\right)-\frac{1}{2}Q\varphi. \tag{1.2}$$

Here e_1,e_2,e_3 is a local orthonormal frame with respect to the metric (see [B, P]). Under conformal transformation of the metric, the operator satisfies

$$P_{\rho^{-4}g}\varphi=\rho^{7}P_{g}\left(\rho\varphi\right). \tag{1.3}$$

Note this is similar to the conformal Laplacian operator. As a consequence we know

$$P_{\rho^{-4}g}\varphi\cdot\psi d\mu_{\rho^{-4}g}=P_{g}\left(\rho\varphi\right)\cdot\rho\psi d\mu_{g}. \tag{1.4}$$

Here μ_g is the measure associated with metric g. Moreover

$$\ker P_{g}=0\Leftrightarrow\ker P_{\rho^{-4}g}=0,$$

and under this assumption, the Green's functions satisfy the transformation law

$$G_{\rho^{-4}g}\left(p,q\right)=\rho\left(p\right)^{-1}\rho\left(q\right)^{-1}G_{g}\left(p,q\right). \tag{1.5}$$

Assume (M,g) is a smooth compact three dimensional Riemannian manifold, for $u,v\in C^{\infty}(M)$, we denote

$$E(u,v) = \int_M Pu\cdot v d\mu = \int_M \left[\Delta u\Delta v - 4Rc(\nabla u,\nabla v) + \frac{5}{4}R\nabla u\cdot\nabla v - \frac{1}{2}Quv\right] d\mu \tag{1.6}$$

and

$$E(u) = E(u,u). \tag{1.7}$$

By the integration by parts formula in (1.6) we know $E(u,v)$ also makes sense for $u,v\in H^2(M)$.

By (1.3) the scaling invariant quantity

$$\mu_g(M)^{\frac{1}{3}}\int_M Q_g d\mu_g$$

satisfies

$$\mu_{\rho^{-4}g}(M)^{\frac{1}{3}}\int_M Q_{\rho^{-4}g} d\mu_{\rho^{-4}g} = -2E_g(\rho)\left\|\rho^{-1}\right\|^2_{L^6(M,g)}.$$

Hence

$$\begin{aligned} & \sup_{\rho\in C^{\infty},\rho>0}\mu_{\rho^{-4}g}(M)^{\frac{1}{3}}\int_M Q_{\rho^{-4}g}d\mu_{\rho^{-4}g} \\ = & -2\inf_{\rho\in C^{\infty},\rho>0}E(\rho)\left\|\rho^{-1}\right\|^2_{L^6} \\ = & -2\inf_{u\in H^2(M),u>0}E(u)\left\|u^{-1}\right\|^2_{L^6}. \end{aligned}$$

As in [HY1], we denote

$$I(u) = E(u)\left\|u^{-1}\right\|^2_{L^6} \tag{1.8}$$

and

$$Y_4^3(M,g) = \inf_{u\in H^2(M),u>0}E(u)\left\|u^{-1}\right\|^2_{L^6}. \tag{1.9}$$

From above discussion we see $Y_4^3(\rho^2 g) = Y_4^3(g)$ for every positive smooth ρ i.e. it is a conformal invariant quantity like the Yamabe invariant $Y(g)$ ([LP]).

An interesting question is whether $Y_4^3(g)$ is finite and achieved by some particular metrics. This inequality is analytically different from the $Y(g)$ case due to the negative power involved. On the other hand, the critical point for I satisfies

$$Pu = const\cdot u^{-7},$$

i.e. $Q_{u^{-4}g} = const$.

In [YZ] it was shown that $Y_4^3(S^3, g_{S^3})$ is achieved by the standard metric g_{S^3} itself (see [H, HY1] for different approaches). The closely related condition NN was introduced in [HY1, Definition 5.1]. We recall the definition here for reference:

Definition 1.1. *Given a smooth compact* 3 *dimensional Riemannian manifold* (M,g)*, if for any* $u\in H^2(M)$*,* $u(p)=0$ *for some* p *implies* $E(u)\geq 0$*, then we say* (M,g) *satisfies condition NN. If for any nonzero* $u\in H^2(M)$*,* $u(p)=0$ *for some* p *implies* $E(u)>0$*, then we say* (M,g) *satisfies condition P.*

For detailed discussion of condition NN, we refer to [HY1, section 5]. Here we simply note that both condition NN and P are conformally invariant properties. The main use of condition NN is (see [HY1, Lemma 5.3]): if g satisfies condition NN, $u \in H^2(M)$, $u(p) = 0$ and $E(u) = 0$, then $Pu = const \cdot \delta_p$. In particular, when $\ker P = 0$, u is simply a constant multiple of the Green function G_p. If (M, g) satisfies condition P, then $Y_4^3(g)$ is achieved and modulus scaling the set of minimizers are compact in C^∞ topology.

(S^3, g_{S^3}) satisfies condition NN but not condition P. Indeed, let x be the coordinate given by the stereographic projection with respect to north pole N, then the Green's function of P at N is given by

$$G_N = -\frac{1}{4\pi}\frac{1}{\sqrt{|x|^2+1}}.$$

In particular, $E(G_N) = G_N(N) = 0$. In general condition NN is hard to check. In [HY1] by explicit calculation of the eigenvalues and Green's function pole's value for Paneitz operator on Berger's spheres, condition NN was verified for all these special metrics. The main aim of this note is to prove the following

Theorem 1.1. *Let M be a smooth compact 3 manifold, consider*

$$\mathcal{M} = \left\{ g : \begin{array}{l} g \text{ is a smooth metric such that there exists a positive} \\ \text{smooth function } \rho \text{ with } R_{\rho^2 g} > 0, Q_{\rho^2 g} > 0 \end{array} \right\}.$$

Endow $\mathcal{M}$ with C^∞ topology. Then we have

- *For every $g \in \mathcal{M}$, $\ker P = 0$, the Greens function $G_p < 0$ on $M \backslash \{p\}$. Moreover if there exists a p with $G_p(p) = 0$, then (M, g) is conformal diffeomorphic to the standard S^3.*
- *For every $g \in \mathcal{M}$, there exists $\rho \in C^\infty(M)$, $\rho > 0$ such that $Q_{\rho^2 g} = 1$ and $R_{\rho^2 g} > 0$. Moreover as long as (M, g) is not conformal diffeomorphic to the standard S^3, the set*
$$\left\{\rho \in C^\infty(M) : \rho > 0, Q_{\rho^2 g} = 1\right\}$$
is compact in C^∞ topology, and any such ρ must satisfy $R_{\rho^2 g} > 0$.
- *Let $\mathcal{N}$ be a path connected component of $\mathcal{M}$. If there is a metric in $\mathcal{N}$ satisfying condition NN, then every metric in $\mathcal{N}$ satisfies condition NN. Hence as long as the metric is not conformal to the standard S^3, it satisfies condition P. As a consequence, for any metric in $\mathcal{N}$,*
$$\inf\left\{E(u)\left\|u^{-1}\right\|_{L^6(M)}^2 : u \in H^2(M), u > 0\right\} > -\infty$$
and is always achieved.

Note that positive Q curvature Berger's sphere are contained in the path connected component of the standard sphere and condition NN follows. Theorem 1.1 is motivated by recent study of Paneitz operator in dimension 5 or higher in [GM], where they found strong maximum principle and constant Q curvature metrics in conformal class of metrics with positive scalar and Q curvature, and the calculation of second variation of Green's function pole's value near the standard sphere in [HY2].

To prove Theorem 1.1, the first step is to show Green's function must be nonpositive. In another word, for any $u \in C^\infty(M)$, $Pu \leq 0$ implies $u \geq 0$. To achieve this we use the method of continuity with the path used in [GM] for dimension 5 or higher. In dimension 3, the argument in [GM] will not run through since certain function which is supharmonic in higher dimension becomes subharmonic and one can not apply strong maximum principle. Interestingly we found that use of weak Harnack principle for another function would resolve the obstruction. As a consequence we know $\ker P = 0$ and the Green's function exists. Next note that the argument in [HR] can be adapted to dimension three to show the Greens function pole value is negative when the metric is not conformally equivalent to the standard sphere. In particular, in this case we know the Green's function are strictly negative everywhere. Based on this fact and apriori estimate for solutions of some integral equations we can show the existence of constant Q curvature metrics in the conformal class. It is not clear in general the minimizer for functional I (see (1.8)) exists. But when the metric can be connected to another metric satisfying condition NN through a good path, we know that metric satisfies condition NN and $Y_4^3(g)$ is achieved hence finite.

We would like to thank M. Gursky for sending us their preprint [GM].

2. Green function's sign and pole's value

In this section we will show that for metrics with positive scalar curvature and positive Q curvature, the Green's function are in fact negative everywhere except at the pole. Moreover, if the value at pole vanishes, then the metric is conformally equivalent to the standard sphere. The first step is the following

Proposition 2.1. *Let (M,g) be a smooth compact Riemannian 3 manifold with $R>0$, $Q\geq 0$. If $u\in C^\infty(M)$, $u\neq const$ and $Pu\leq 0$, then $u>0$ and $R_{u^{-4}g}>0$.*

This proposition is in the same spirit as [GM, Theorem 2.2] and we will adapt their proof for dimension 5 or higher to dimension 3. The main new ingredient is when the strong maximum principle does not apply, we can replace it by the weak Harnack principle.

Proof of Proposition 2.1. For $\lambda\geq 0$, let $u_\lambda = u+\lambda$, then $Pu_\lambda = Pu - \frac{\lambda}{2}Q\leq 0$. Let

$$\lambda_0 = \inf\{\lambda\geq 0 : u_\lambda > 0\}. \tag{2.1}$$

For $u_\lambda>0$, let $g_\lambda = u_\lambda^{-4}g$, then

$$Q_\lambda = -2P_{u_\lambda^{-4}g}1 = -2u_\lambda^7 Pu_\lambda \geq 0.$$

This implies

$$-\frac{1}{4}\Delta_\lambda R_\lambda - 2\left|Rc_\lambda\right|^2 + \frac{23}{32}R_\lambda^2 \geq 0$$

and hence $-\Delta_\lambda R_\lambda + \frac{23}{8}R_\lambda^2 \geq 0$. In particular by strong maximum principle (or use the weak Harnack inequality [GT, Theorem 8.18, p194]) if $R_\lambda\geq 0$, then either $R_\lambda > 0$ or $R_\lambda\equiv 0$. The latter case is impossible since the Yamabe invariant $Y(g)>0$ ([LP]). Hence $R_\lambda>0$. Now for $\lambda\gg 1$, we have

$$R_\lambda = \lambda^4 R_{\left(1+\lambda^{-1}u\right)^{-4}g} > 0.$$

Let

$$\lambda_1 = \inf\{\lambda>\lambda_0 : \text{for all } \tau\geq\lambda,\ R_\tau>0\}.$$

Then $\lambda_1=\lambda_0$. If not, then $R_{\lambda_1}\geq 0$ and hence $R_{\lambda_1}>0$. This contradicts with the definition of λ_1. Hence for any $\lambda>\lambda_0$, $R_\lambda>0$. Because

$$R_\lambda = u_\lambda^5\left[-8\Delta\left(u_\lambda^{-1}\right) + Ru_\lambda^{-1}\right],$$

we know

$$-\Delta\left(u_\lambda^{-1}\right)+\frac{R}{8}u_\lambda^{-1}>0.$$

It follows from weak Harnack principle [GT, Theorem 8.18, p194] that for any $0<p<3$,

$$\inf_M u_\lambda^{-1}\geq C\left(M,g,p\right)\left\|u_\lambda^{-1}\right\|_{L^p(M)}.$$

Because $u\neq const$, we know $\inf_M u_\lambda^{-1}\leq C$, independent of $\lambda>\lambda_0$, and hence

$$\left\|u_\lambda^{-1}\right\|_{L^p(M)}\leq C.$$

Let $\lambda\downarrow\lambda_0$, we see

$$\left\|u_{\lambda_0}^{-1}\right\|_{L^p(M)}\leq C<\infty. \tag{2.2}$$

Hence u_{λ_0} can not touch 0 i.e. $u_{\lambda_0}>0$. Indeed if $u_{\lambda_0}(p)=0$, then under the normal coordinate at p, by smoothness of u we have $u_{\lambda_0}=O\left(r^2\right)$, here r is the distance to p. In particular $\left\|u_{\lambda_0}^{-1}\right\|_{L^p(M)}=\infty$ for $\frac{3}{2}\leq p<3$, this contradicts with (2.2). Hence $\lambda_0=0$, $u>0$. It follows that $R_{u^{-4}g}\geq 0$ and hence $R_{u^{-4}g}>0$. □

Corollary 2.1. *Assume $R>0$, $Q\geq 0$, then* $\ker P\subset\{$*constant functions*$\}$. *If in addition Q is not identically* 0, *then* $\ker P=0$ *i.e.* 0 *is not in the spectrum of P.*

Proof. Assume $Pu=0$, then $u\equiv const$. If not, by applying the previous proposition to u and $-u$, we see $u>0$ and $-u>0$, a contradiction. □

Next we will show the Green's function is always negative away from the pole.

Proposition 2.2. *Assume $R>0$, $Q\geq 0$ and not identically zero.*

- *For any $p\in M$, let G_p be the Green's function of P at p i.e. $PG_p=\delta_p$, then*

  $$G_p|_{M\setminus\{p\}}<0$$

  and

  $$-8\Delta\left(G_p^{-1}\right)+RG_p^{-1}\leq 0.$$

- *If $u\in H^2(M)$ such that $Pu\leq 0$ in distribution sense and u is not identically zero, then either $u>0$ or $u=-cG_p$ for some $c>0,p\in M$, moreover*

  $$-8\Delta\left(u^{-1}\right)+Ru^{-1}\geq 0$$

  in distribution sense.

Proof. First we will show that if $u\in H^2(M)$ such that $Pu\leq 0$ in distribution sense and u is not identically zero, then $u\geq 0$ and for any $0<p<3$, $\left\|u^{-1}\right\|_{L^p(M)}<\infty$, moreover

$$-8\Delta\left(u^{-1}\right)+Ru^{-1}\geq 0$$

in distribution sense.

Indeed we can find a sequence of smooth nonzero functions f_i such that $f_i\to Pu$ in $H^{-2}(M)=H^2(M)'$ and $f_i\leq 0$. Because 0 is not in spectrum of P, we can find $u_i\in C^\infty(M)$ such that $Pu_i=f_i$. This implies $u_i\to u$ in $H^2(M)$. It follows from the Proposition 2.1 that $u_i>0$ and $R_{u_i^{-4}g}>0$ i.e.

$$-8\Delta\left(u_i^{-1}\right)+Ru_i^{-1}>0.$$

Because u is not identically constant and $u_i\rightrightarrows u$ we see $\inf_M u_i^{-1}\leq C$ independent of i. Hence by weak Harnack inequality we have $\left\|u_i^{-1}\right\|_{L^p(M)}\leq C$ for any fixed

$p \in (1,3)$. By Fatou's lemma we have $\left\|u^{-1}\right\|_{L^p(M)} \leq C$ and $u_i^{-1} \rightharpoonup u^{-1}$ in L^p, hence $-8\Delta\left(u^{-1}\right) + Ru^{-1} \geq 0$ in distribution sense.

It follows that $G_p \leq 0$ and $\left\|G_p^{-1}\right\|_{L^q(M)} < \infty$ for any $1 < q < 3$. Because G_p is smooth away from p, we see $G_p < 0$ away from p. Now if $Pu \leq 0$, then $Pu = -\nu$ for some nonnegative measure ν. It follows that

$$u(p) = -\int_M G_p(q)\, d\nu(q).$$

It ν is not a constant multiple of Dirac mass, then clearly $u(p) > 0$ for every p. If ν is a constant multiple of Dirac mass, then u is simply a constant multiple of Green's function. □

For the Green's function pole's value, we adapt the argument in [HR] to prove the following

Proposition 2.3. *Assume the Yamabe invariant $Y(g) > 0$, $\ker P = 0$. If $p \in M$ such that $G_p < 0$ on $M\setminus\{p\}$, then $G_p(p) < 0$ except when (M,g) is conformal equivalent to the standard S^3.*

Proof. Under the conformal normal coordinate with respect to p, we have after multiplying 32π, the Green's function of conformal Laplacian $L = -8\Delta + R$ can be written as (see [LP, section 6], indeed since we are in odd dimension, the remain term belongs to $O^{(\infty)}(1)$)

$$\Gamma = \frac{1}{r} + O^{(4)}(1).$$

Here we write $f = O^{(4)}(1)$ to mean f is C^4 away from origin with $\partial_{i_1\cdots i_k} f = O\left(r^{-k}\right)$ for $0 \leq k \leq 4$. We write $G = G_p$, then (see [HY1, section 4])

$$G = A + O^{(4)}(r).$$

Let $\widetilde{g} = \Gamma^4 g$ on $M\setminus\{p\}$, then

$$\widetilde{R} = 0, \quad \widetilde{Q} = -2\left|\widetilde{Rc}\right|^2.$$

On $M\setminus\{p\}$,

$$0 = P_g G = P_{\Gamma^{-4}\widetilde{g}} G = \Gamma^7 \widetilde{P}(\Gamma G).$$

Denote $u = \Gamma G < 0$, then

$$0 = \widetilde{\Delta}^2 u + 4\widetilde{\operatorname{div}}\left[\widetilde{Rc}\left(\widetilde{\nabla} u, e_i\right) e_i\right] + \left|\widetilde{Rc}\right|^2 u.$$

Here e_1, e_2, e_3 is a local orthonormal base with respect to $\widetilde{g}$.

For $\delta > 0$ small, let $B_\delta = B_\delta(p)$ with respect to the conformal normal coordinate, then integrate the above equation on $M\setminus B_\delta$ we see

$$0 = -\int_{\partial B_\delta} \frac{\partial \widetilde{\Delta} u}{\partial \widetilde{\nu}} d\widetilde{S} - 4\int_{\partial B_\delta} \widetilde{Rc}\left(\widetilde{\nabla} u, \widetilde{\nu}\right) d\widetilde{S} + \int_{M\setminus B_\delta} \left|\widetilde{Rc}\right|^2 u\, d\widetilde{\mu}.$$

We will let $\delta \to 0^+$ and calculate all the limits.

Claim 2.1.

$$\lim_{\delta\to 0^+} \int_{\partial B_\delta} \frac{\partial \widetilde{\Delta} u}{\partial \widetilde{\nu}} d\widetilde{S} = 8\pi A.$$

Indeed since $\widetilde{g}=\Gamma^4 g$, we see

$$\begin{aligned}\widetilde{\nu} &= \Gamma^{-2}\nu=\Gamma^{-2}\partial_r,\\ d\widetilde{S} &= \Gamma^4 dS.\end{aligned}$$

On the other hand,

$$\begin{aligned}\widetilde{\Delta}u &= \Gamma^{-4}\Delta u+2\Gamma^{-4}\nabla\log\Gamma\cdot\nabla u\\ &= 2Ar+O^{(2)}\left(r^2\right).\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial\widetilde{\Delta}u}{\partial\widetilde{\nu}}d\widetilde{S} &= \Gamma^2\partial_r\widetilde{\Delta}u\cdot dS\\ &= \left(\frac{2A}{r^2}+O^{(1)}\left(\frac{1}{r}\right)\right)dS.\end{aligned}$$

This implies

$$\int_{\partial B_\delta}\frac{\partial\widetilde{\Delta}u}{\partial\widetilde{\nu}}d\widetilde{S}=8\pi A+O\left(\delta\right).$$

Claim 2.2.

$$\lim_{\delta\to 0^+}\int_{\partial B_\delta}\widetilde{Rc}\left(\widetilde{\nabla}u,\widetilde{\nu}\right)d\widetilde{S}=0.$$

Indeed we have

$$\begin{aligned}\widetilde{Rc} &= Rc-2D^2\log\Gamma+4d\log\Gamma\otimes d\log\Gamma-\left(2\Delta\log\Gamma+4\left|\nabla\log\Gamma\right|^2\right)g,\\ \log\Gamma &= -\log r+O^{(4)}\left(r\right),\\ \widetilde{\nabla}u &= -Ar^2\partial_r+O\left(r^3\right),\\ \widetilde{\nu} &= \Gamma^{-2}\partial_r,\end{aligned}$$

hence $\widetilde{Rc}=O\left(\frac{1}{r}\right)$ and

$$\widetilde{Rc}\left(\widetilde{\nabla}u,\widetilde{\nu}\right)d\widetilde{S}=O\left(\frac{1}{r}\right)dS.$$

The claim follows.

By letting $\delta\to 0^+$ we get

$$\int_{M\backslash\{p\}}\left|\widetilde{Rc}\right|^2 u\,d\widetilde{\mu}=8\pi A.$$

Note $A=G_p\left(p\right)$. If $A=0$, then $\widetilde{Rc}=0$. Since $\left(M\backslash\left\{p\right\},\widetilde{g}\right)$ is asymptotically flat, it follows from relative volume comparison theorem that $\left(M\backslash\left\{p\right\},\widetilde{g}\right)$ is isometric to the standard $\mathbb{R}^3$. In particular, $\left(M,g\right)$ must be locally conformally flat and simply connected compact manifold, hence it is conformal to the standard S^3 by [K]. □

3. Proof of Theorem 1.1

Based on preparations in section 2, we will complete the proof for Theorem 1.1. Assume $\left(M,g\right)$ is not conformal to the standard S^3, then the Green's function $G\left(p,q\right)<0$ for all $p,q\in M$. For $u\in C^\infty\left(M\right)$, $u>0$,

$$Q_{u^{-4}g}=1$$

if and only if

$$Pu = -\frac{1}{2} u^{-7}$$

and this is equivalent to

$$u(p) = -\frac{1}{2} \int_M G(p,q) u(q)^{-7} d\mu(q). \tag{3.1}$$

For convenience we write $K(p,q) = -G(p,q)$. We will derive the existence of solution by degree theory. Indeed for $0 \le t \le 1$ let

$$K_t(p,q) = 1 - t + tK(p,q). \tag{3.2}$$

Then for some $\alpha_0, \alpha_1 > 0$, we have $\alpha_0^{-1} \le K_t(p,q) \le \alpha_1$. Consider the equation

$$u(p) = \frac{1}{2} \int_M K_t(p,q) u(q)^{-7} d\mu(q) \tag{3.3}$$

for $u \in C(M)$ and $u > 0$.

Claim 3.1.

$$c_0 \le u(p) \le c_1.$$

here

$$c_0 = \left(\frac{\mu(M)}{2}\right)^{\frac{1}{8}} \alpha_0^{-1} \alpha_1^{-\frac{7}{8}}, \quad c_1 = \left(\frac{\mu(M)}{2}\right)^{\frac{1}{8}} \alpha_0^{\frac{7}{8}} \alpha_1.$$

Indeed by the equation we see

$$\frac{1}{2\alpha_0} \int_M u^{-7} d\mu \le u(p) \le \frac{\alpha_1}{2} \int_M u^{-7} d\mu.$$

Hence for $p_1, p_2 \in M$,

$$u(p_1) \le \alpha_0 \alpha_1 u(p_2).$$

It follows that

$$u(p) \le \frac{\alpha_1}{2} \int_M u^{-7} d\mu \le \frac{\mu(M)}{2} \alpha_0^7 \alpha_1^8 u(p)^{-7},$$

hence the upper bound follows. Lower bound can be proven similarly.

Let

$$\Omega = \left\{ u \in C(M) : \frac{c_0}{2} \le u \le \frac{c_1}{2} \right\} \subset C(M)$$

with uniform convergence topology. Let

$$(T_t u)(p) = \frac{1}{2} \int_M K_t(p,q) u(q)^{-7} d\mu(q).$$

Claim 3.1 tells us

$$\deg(I - T_1, \Omega, 0) = \deg(I - T_0, \Omega, 0) = 1.$$

The existence of solution follows.

Let

$$\mathcal{M}_0 = \{ g \in \mathcal{M} : (M,g) \text{ satisfies condition NN} \}.$$

It is clear that $\mathcal{M}_0$ is closed. We only need to show $\mathcal{M}_0$ is open. Indeed assume $g \in \mathcal{M}_0$, if (M,g) is not conformal diffeomorphic to the standard S^3, then $G_p(p) < 0$ for all $p \in M$. It follows that (M,g) satisfies condition P, and hence nearby metric also satisfies condition P (see [HY1, Lemma 5.1]). Assume (M,g) is conformal diffeomorphic to the standard S^3, we may assume $M = S^3$ and $g = g_{S^3}$, the standard metric. Recall we have the so called Berger's metric g_t for $t > 0$, with $g_1 = g_{S^3}$ (see [HY1, section 8]).

Claim 3.2. *If $\widetilde{g}$ is close to g_{S^3} in the C^∞ topology and $\left(S^3, \widetilde{g}\right)$ is not conformally equivalent to $\left(S^3, g_{S^3}\right)$, then there exists a $t \neq 1$, $|t-1|$ small, and a continuous path (with respect to C^∞ topology) of metrics $h = h(s)$, with $h(0) = \widetilde{g}$, $h(1) = g_t$, $\left(S^3, h(s)\right)$ is not conformally equivalent to $\left(S^3, g_{S^3}\right)$ and $h(s)$ is close to g_{S^3} in C^∞ topology.*

Assume the claim has been proven, then we will show $\widetilde{g}$ satisfies condition P. Indeed the set $\{s \in [0,1] : h(s) \text{ satisfies condition } P\}$ is clearly open, but it is also closed since for any limit point s, $h(s)$ satisfies condition NN, and the Green's function pole value does not vanish (by Proposition 2.3) imply it satisfies condition P.

To prove Claim 3.2, we recall in three dimension, the metric is locally conformally flat if and only if the Cotton tensor equals to zero. Note $\left|C_{g_t}\right|^2 = 192t^2\left(t^2-1\right)^2 \neq 0$ for $t \neq 1$ (see [HY1, section 8]). Since $\left(S^3, \widetilde{g}\right)$ is not conformally equivalent to $\left(S^3, g_{S^3}\right)$, by [K] we know it is not locally conformally flat, hence it's Cotton tensor $\widetilde{C} \neq 0$ somewhere. By rotation we can assume $\widetilde{C}(N) \neq 0$, here N is the north pole. To continue we fix a $\eta \in C^\infty\left(S^3\right)$ with $0 \leq \eta \leq 1$, $\eta = 1$ near north pole N and 0 near the south pole S. Then the path

$$h(s) = \begin{cases} (1-2s)\,\widetilde{g} + 2s\left[\eta\widetilde{g} + (1-\eta)\,g_t\right], & 0 \leq s \leq \frac{1}{2} \\ (2-2s)\left[\eta\widetilde{g} + (1-\eta)\,g_t\right] + (2s-1)\,g_t, & \frac{1}{2} \leq s \leq 1 \end{cases}.$$

satisfies all the requirement. This finishes proof of the claim.

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