Russian Math. Surveys **55**:4 635–666

Uspekhi Mat. Nauk **55**:4 25–58
DOI 10.1070/RM2000v055n04ABEH000313

Recent results on mathematical and statistical hydrodynamics

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Abstract. This paper is a survey of recent results of the authors and their collaborators on stochastic partial differential equations in hydrodynamics. We discuss the stochastic Burgers equation, the stochastic Navier–Stokes equation, and the stochastic passive scalar transport equation. In contrast to previous publications on this subject (see, for example, [25], which is mainly devoted to existence problems for stochastic dynamics), the work surveyed here emphasizes qualitative properties of solutions, including the existence and uniqueness of an invariant measure under certain physical assumptions, the asymptotic behaviour of the statistics of these solutions, and so on. We also discuss new investigations concerning the deterministic Navier–Stokes equation.

Contents

AMS 2000 *Mathematics Subject Classification.* Primary 35R60, 35Q53, 35Q30; Secondary 35L67, 76D03, 76D05, 76N10, 60H15.

PART I

THE BURGERS SYSTEM

1. Definition and general properties of the Burgers system

The Burgers system is a simplification of the Navier–Stokes system. In the d-dimensional case it is written for d unknown functions $u(x,t) = \{u_j(x,t), 1 \leqslant j \leqslant d\}$ of d spatial variables $x = \{x_j,\, 1 \leqslant j \leqslant d\}$ and the time as follows:

$$\frac{\partial u_j}{\partial t} + \sum_{\ell=1}^{d} u_\ell \frac{\partial u_j}{\partial x_\ell} = \nu \sum_{\ell=1}^{d} \frac{\partial^2 u_j}{\partial x_\ell^2} + f_j(x,t). \tag{1}$$

It differs from the Navier–Stokes system in the absence of pressure and of the incompressibility condition. For this reason the problems involving the Burgers system sometimes go under the name of pressureless hydrodynamics. The coefficient ν is the viscosity, and the case $\nu = 0$ is referred to as the inviscid Burgers system or the Hopf system; for $d = 1$, it is also called the Riemann equation, which gives the simplest quasi-linear equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = f(x,t).$$

In general, the vector $f = \{f_j, 1 \leqslant j \leqslant d\}$ in (1) is an external force. Zeldovitch suggested regarding the inviscid free system ($f = 0$) as an equation describing the evolution of a rarefied gas of non-interacting particles [70]. According to his idea, the pure kinematics of the underlying particles can lead to singularities in the distribution of mass and is responsible for the non-uniformity of matter in the universe.

On the space of gradient-like solutions, the system (1) is simpler than the full Navier–Stokes system. Namely, if $u(x,0) = -2\nu\nabla\varphi(x,0)/\varphi(x,0)$ for some positive function $\varphi(x,0)$ and if $f_j = \partial F(x,t)/\partial x_j$, then $u(x,t) = -2\nu\nabla\varphi(x,t)/\varphi(x,t)$ for all $t > 0$, where the function $\varphi(x,t)$ satisfies the heat equation

$$\frac{\partial \varphi(x,t)}{\partial t} = \nu\Delta\varphi(x,t) - \frac{1}{2\nu} F(x,t)\varphi. \tag{2}$$

The transition from the equation for u to that for φ is called the Hopf–Cole substitution [11], [31]. However, it was seemingly known much earlier [68].

In many respects, the case $\nu > 0$ is simpler than the case $\nu = 0$; strictly speaking, the latter is the limit as $\nu \to 0$. As a rule, when discussing the theory for $\nu = 0$, we only mention related results for $\nu > 0$. As usual, the questions of continuity at $\nu = 0$ are difficult.

We mainly discuss two sets of problems for the Burgers system.

A. The inviscid system (1) with random initial conditions. In his monograph [8], Burgers considered the case $d = 1$, $f = 0$ and the initial condition $u(x,0) = B(x)$, where B is a white noise, that is, a generalized random process. He showed that the solution $u(x,t)$ is a piecewise linear function of x for $t > 0$. Discontinuities of solutions of quasi-linear equations are called shocks. One can say that irregularities of the initial conditions produce irregularities of solutions, that is, shocks. We discuss below the work of Avellaneda and E on the asymptotic behaviour of statistics of the shocks [1], [3], [4]. Frisch was the first to understand that an interesting situation arises if $B(x)$ is replaced by a homogeneous random process with independent increments (like a Brownian motion or a Lévy-stable process). In the joint paper [56] of Frisch, She, and Aurell the authors constructed solutions numerically and discovered the appearance of infinitely many shocks for any $t > 0$. This gives an interesting example in which the dynamics transforms a continuous distribution of mass into a discrete one. The mathematical theory for the case in [56] was constructed in [58]. We also discuss related problems below.

B. Statistically homogeneous regimes for the inviscid Burgers equation with random forcing. We mainly discuss the results of W. E, K. Khanin, A. Mazel, and Ya. Sinai [18], [19] established for $d = 1$ and mention some related problems. We also consider the master equation approach of W. E and E. Vanden Eijnden [14], [15] to the study of statistical properties of the Burgers system.

2. The inviscid Burgers equation and its modifications with random initial conditions

In the one-dimensional case, the problem is purely classical. We have the equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0$$

and under irregular initial conditions we want to construct solutions that are typical realizations of a random process. The usual method of characteristics does not work because of irregularities (since the characteristics intersect each other for arbitrarily small $t > 0$). We must use the so-called Lax–Oleinik variational principle [42], [52], [65], [68], according to which we take the function

$$W(x,t) = \int_0^x [u(y,0) + y/t]\,dy = \int_0^x u(y,0)\,dy + \frac{x^2}{2t}.$$

If $u(y,0)$ is a realization of a random process such that the growth of $|u(y,0)|$ is less than that of $|y|$, then W is a parabola-like function perturbed by a random process. The next step is to consider the convex *minorant* C of W, that is, the largest convex function bounded above by W. The graph of $C(x,t)$ consists of line segments and a closed set outside these segments. The derivative

$$F(x,t) = \frac{\partial C(x,t)}{\partial x}$$

is constant on any interval on which $C(x,t)$ is linear. For this reason F is a (devil's) staircase-type function. The solution is

$$u(x,t) = \frac{x}{t} - \frac{F^{-1}(x/t,t)}{t}\,, \tag{3}$$

where $F^{-1}(x,t)$ is the inverse function. The easiest way to understand (3) is to regard F as a continuous curve on the plane and F^{-1} as the continuous curve which arises when interchanging the axes.

The solution $u(x,t)$ can also be regarded as a continuous curve on the plane with vertical segments corresponding to the shocks.

From the mathematical point of view, the main problem can be formulated as an analysis of convex minorants of random processes. Let us give an example of a simple general statement that can be proven here. We set $t=1$ for simplicity and consider the random process

$$W(x)=W(x,1)=\int_0^x [u(y,0)+y]\,dy=\int_0^x u(y,0)\,dy+\frac{x^2}{2}.$$

Suppose that $u(\,\cdot\,,0)$ is continuous with probability 1. Then W is differentiable almost everywhere, and $W'(x)=u(x,0)+x$. A point $\bar{x}$ is said to be *special* (for $W(x)$ or for $u(\,\cdot\,,0)$) if there is a neighbourhood $(\bar{x}-\delta,\bar{x}+\delta)$ such that either

$$W(x)\geqslant W(\bar{x})+(x-\bar{x})(u(\bar{x},0)+\bar{x}) \tag{4$'$}$$

for all $x\in(\bar{x}-\delta,\bar{x}+\delta)$ or

$$W(x)\leqslant W(\bar{x})+(x-\bar{x})(u(\bar{x},0)+\bar{x}) \tag{4$''$}$$

for all $x\in(\bar{x}-\delta,\bar{x}+\delta)$, that is, the graph of W lies either above or below its tangent line passing through $\bar{x}$. The event described by these inequalities depends on the behaviour of the process $u(x,0)$ in an arbitrarily small neighbourhood of $\bar{x}$, that is, it belongs to the local σ-algebra of the process u. In many cases (Brownian motion, fractional Brownian motion, Lévy-stable processes, diffusion processes) it is known that this σ-algebra is trivial, that is, has only events of probability 0 or 1. In this case one can prove that our event (4$'$) or (4$''$) has probability zero, which holds in all the situations mentioned above.

Let us return to the graph of F. We recall the following definition.

Definition 1. A devil's staircase is said to be *complete* if the union of the intervals on which it is constant is a set of full measure.

We assume that the probability that a point $\bar{x}$ is special is zero for every $\bar{x}$. Using Fubini's theorem, one can readily conclude that the graph of F is a complete devil's staircase with probability 1 (that is, the set of special points has measure zero for almost all u). Frisch raised the problem of estimating the Hausdorff dimension of this set [56]. In the case of Brownian motion, the answer was obtained in [58], and it is equal to $\frac{1}{2}$. The proof is based on the estimate

$$\mathsf{P}\biggl\{\int_0^t b(s)\,ds>-a_0+a_1t\quad\text{for all}\quad 0\leqslant t\leqslant T\biggr\}\sim\frac{C(a_0,a_1)}{T^{1/4}}$$

as $T\to\infty$. Its derivation can be found in several publications, see, for example, [33], [59]. In a more general situation, Handa [30] obtained a simple proof of a lower bound for the Hausdorff dimension.

The case of a Lévy-stable process was studied numerically and qualitatively by Janicki and Woyczynski [34]. A complete answer was obtained recently by Bertoin [7].

Next we consider the situation with white-noise initial data. This is the problem Burgers himself was interested in and to which he devoted most of his book [8], where many important statistical quantities for the problem are calculated. Burgers' solutions were often expressed in the form of series. For this reason it was hard to extract explicit information on the asymptotics of statistical quantities such as the distribution of shock strength. However, in a recent paper [23] Frachebourg and Martin gave simpler formulas for many of these statistical quantities, from which the asymptotics can readily be extracted; [23] provides very valuable exact results for this classical problem.

The crudest question that can be asked in this problem is about dynamic space-time scaling relations. More precisely, since the distribution of white noise is invariant under the scaling $\delta^{\frac{1}{2}} u_0(\delta x) \overset{D}{=} u_0(x)$, we can consider a rescaling of the variables

$$\delta x = x', \qquad \delta^{\frac{1}{2}} u = u', \qquad \delta^{\gamma} t = t'$$

and ask for the value of γ such that the equation in the rescaled variables (x', t', u') has non-trivial dynamics as $\delta \to 0$. This amounts to asking whether the dynamics at large scales is dominated by the convective term or by the diffusive one. For the generalized Burgers equation

$$u_t + \left(\tfrac{1}{p}|u|^p\right)_x = u_{xx},$$

the answer is given by the following assertion.

Theorem 1.

$$\gamma = \begin{cases} 1 + \frac{1}{2}(p-1) & \textit{for} \quad p \leqslant 3, \\ 2 & \textit{for} \quad p \geqslant 3, \end{cases}$$

and the effective equations in the rescaled variables are (*omitting the primes*)

$$\begin{aligned} u_t + \left(\tfrac{1}{p}|u|^p\right)_x &= 0 & \textit{for} \quad & p < 3, \\ u_t + \left(\tfrac{1}{p}|u|^p\right)_x &= u_{xx} & \textit{for} \quad & p = 3, \\ u_t &= u_{xx} & \textit{for} \quad & p > 3. \end{aligned}$$

This shows that the problem has diffusive behaviour at large scales for $p > 3$ and convective behaviour at large scales for $p < 3$. Similar questions can also be asked for initial data with different statistics; see [29], [68]. Through the variational characterization of the solutions the question often reduces to studying the statistics of extreme values of the initial data. Such issues are dealt with in the book [40], and the result depends on the tail behaviour of one-point statistics.

Let us return to the (inviscid) Burgers equation with white-noise initial data. The solution has the scaling property

$$u(x,t) \overset{D}{=} t^{-2/3} \overline{u}\left(\frac{x}{t^{2/3}}\right),$$

where $\overline{u}(x) = u(x,1)$. It can be shown that $\overline{u}$ consists of straight lines (the ramps) of slope 1, separated by shocks. The study of $\overline{u}$ amounts to a study of the convexification of the function $F(y) = y^2/2 + B(y)$, where $B(y)$ is a two-sided Wiener process.

The next question we address is the statistics of the strength of shocks. The following result was proved by Avellaneda and E [3].

Theorem 2. *There exist constants C_1 and C_2 such that*

$$C_1\sqrt{s} \leqslant \mathsf{P}\{S < s\} \leqslant C_2\sqrt{s}$$

for $s \ll 1$.

Here P is the conditional probability that there is a shock at x_0, and S is the strength of that shock. Since the process is spatially homogeneous, P does not depend on x_0.

This result was proved by adapting the Pitman–Groenboom method for studying the convex hull of a Wiener process by using time-reversal and conditional diffusion-process techniques, see [28], [53]. A more precise result was recently obtained in [23].

Finally, we consider the statistics of large shocks. It turns out that the leading-order asymptotics for the distribution of large shocks is given by the tail behaviour of the primitive function of the initial data, which is a two-sided Wiener process in the present case. The following argument captures the heart of the matter. To have a shock of amplitude λ at $x = 0$, we must have $F(y) = y^2/2 + B(y) \leqslant 0$ for $0 \leqslant y \leqslant \lambda$. The probability of such an event is smaller than

$$\mathsf{P}\left\{\frac{\lambda^2}{2} + B(\lambda) \leqslant 0\right\} \leqslant e^{-\frac{1}{2\lambda}(\frac{\lambda^2}{2})^2} = e^{-\frac{\lambda^3}{8}}.$$

It turns out that, aside from algebraic factors, this also provides a lower bound. This was first recognized in [2]–[4].

Theorem 3. *There exist constants C_1 and C_2 such that*

$$C_1 \leqslant s^{-3} \ln \mathsf{P}\{S > s\} \leqslant C_2$$

as $s \to \infty$.

In this form, the result was proved by Molchan and Ryan independently [51], [55]. They also proved various generalizations of this statement. Finding rigorous bounds amounts to estimating certain large deviation probabilities associated with the Wiener process, see [51], [55]. This argument can be generalized to initial data with more general statistics. For example, for the above Brownian-motion initial data, the leading-order asymptotics for the distribution of large shocks is given by

$$\mathsf{P}\left\{\min_{y>\lambda} F(y) < 0\right\} \propto e^{-\frac{1}{8}\lambda}$$

for $\lambda \gg 1$. To the best of the authors' knowledge, this is not yet rigorously proved.

Another interesting question that has been studied recently is the fluctuation of the shock position under a white noise perturbation [20], [67]. For the Burgers equation, it was proved by Wehr and Xin [67] that the statistic of fluctuation is Gaussian.

3. Adhesion dynamics and generalizations of the Burgers equation

The Burgers equation is related to the dynamics of so-called sticky particles. Suppose that we have a set of particles (x_i, v_i, m_i), $i = 1, \dots, I$, at $t = 0$. Each particle moves with its own velocity until it collides with another particle. At the collision a new particle is created whose mass and momentum are equal to the sum of the masses and the sum of the momenta of the colliding particles. The entire process preserves the total mass and momentum. It can be described in the one-dimensional case by the system of two conservation laws

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2)_x &= 0. \end{aligned} \tag{5}$$

We also consider a more general case of particles interacting through gravitational forces. In the one-dimensional case the gravitational force acting on a particle is proportional to the difference between the mass on the right and that on the left of the particle. Formally it can be described by the system of equations

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2)_x &= -\rho \mathcal{R}_x, \\ \mathcal{R}_{xx} &= \rho. \end{aligned} \tag{6}$$

However, the form of (5) and (6) is not general enough because in principle we have to consider arbitrary distributions of masses, which may have no densities. A general solution of (5) or (6) must be a family of finite measures P_t describing the distribution of mass. The distribution of momentum is given by a signed measure I_t absolutely continuous with respect to P_t, so that the Radon–Nikodym derivative $dI_t(x)/dP_t(x) = u(x,t)$ is the velocity at x, t.

Definition 2 [17]. The family $\{P_t, I_t, t > 0\}$ is called a weak solution of (5) if

(i) it is continuous with respect to t in the weak topology,

(ii) for any $f, g \in C_0^1(R^1)$ ($C_0^1(R^1)$ stands for the space of C^1-functions with compact support) and any t_1, t_2 with $t_2 > t_1$ we have

$$\begin{aligned} \int f(x)\, dP_{t_2}(x) - \int f(x)\, dP_{t_1}(x) &= \int_{t_1}^{t_2} d\tau \int F'(y)\, dI_\tau(y), \\ \int g(x)\, dI_{t_2}(x) - \int g(x)\, dI_{t_1}(x) &= \int_{t_1}^{t_2} d\tau \int g'(y) u(y,\tau)\, dI_\tau(y). \end{aligned}$$

In the case of (6), the definition is analogous if we note that the third equation can be replaced by

$$\mathcal{R}_x(x,t) = P_t(x,\infty) - P_t(-\infty, x).$$

Then instead of the second equation in (ii) we obtain

$$\begin{aligned} &\int g(y)\, dI_{t_2}(y) - \int g(y)\, dI_{t_1}(y) \\ &\qquad = \int_{t_1}^{t_2} d\tau \int g'(y) u(y,\tau)\, dI_\tau(y) + \int_{t_1}^{t_2} d\tau \int \mathcal{R}_y(y,\tau)\, dP_\tau(y). \end{aligned}$$

A remarkable property of equations (5) and (6) is their integrability (in a certain sense), which was discovered by Martin and Piasecki [47]. In [17] we rediscovered this fact, not knowing about the paper [47].

The idea is as follows. Since the dynamics has an adhesive character, it follows that for any $t > 0$ the set of particles which are stuck together is a closed segment on $\mathbb{R}^1$. We denote the partition of $\mathbb{R}^1$ into these segments (which can also be points) by ξ_t.

The crucial remark is that the position of all the particles $x \in C_t$ at time t, where C_{ξ_t} is an element of the partition ξ_t, is the position $x(t)$ of the center of mass of C_{ξ_t} moving uniformly (in the case of (5)) with velocity $\bar{v}$ such that

$$X(t) = X(0) + I(C_{\xi_t})t, \qquad I(C_{\xi_t}) = \int_{C_{\xi_t}} u(x,0)\, dP_0.$$

In the case of particles with gravitational forces the rule is the same except for the trajectory of the center of mass, which is a quadratic function of t describing a motion with constant acceleration.

Using this picture, we can introduce the Lagrangian map $\mathcal{L}_t$ that sends x to the position of the center of mass of the element $C_{\xi_t}(x)$ containing x at time t. The generalized variational principle [47], [17] for the systems (5) and (6) says that $\mathcal{L}_t(x)$ is the trajectory of x.

The partition ξ_t can be constructed explicitly from the initial condition, and (5) and (6) are integrable in this sense. We shall explain this for the system (5) (it is quite similar in the case of (6)). It turns out to be possible to characterize the points x that are left endpoints of the partition ξ_t. Namely, x is a left endpoint if $\bar{y}_1 + tI_t((y_1, x)) < \bar{y}_2 + tI_t([x, y_2])$ for every $y_1 < x < y_2$, where $\bar{y}_1$ and $\bar{y}_2$ are the centers of mass of the sets (y_1, x) and $[x, y_2]$, respectively.

In [47] Martin and Piasecki considered an ensemble of n gravitating points with coordinates ka/n, $k = 1, \ldots, n$, masses $1/n$, and random initial velocities v_i having symmetric Gaussian distribution. They showed that there is a non-random instant τ^* such that all the masses are macroscopically small before this instant with probability tending to 1 as $n \to \infty$, while after τ^* a so-called almost collapse takes place in the sense that a particle appears whose mass tends to 1 as $n \to \infty$. T. Suidan (Princeton) recently proved a similar statement for a much broader class of velocity distributions.

4. The one-dimensional Burgers equation with random forcing

Let us consider the equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + \sum_k f_k'(x) B_k(t). \tag{7}$$

Here the sum is finite, the $B_k(t)$ are independent standard white noises, and the $f_k \in C^2$ are linearly independent deterministic functions. We consider (7) with periodic boundary condition on $[0, 1)$ requiring periodicity of f_k with period 1. Extension of the theory described below to the entire line is apparently a very difficult problem. Using the Hopf–Cole substitution, we can reduce it to a problem about statistics of directed polymers, which is known to be difficult.

Equation (7) is in fact a stochastic differential equation in a function space. The basic problem is the existence and uniqueness of a stationary measure for the Markov process that arises. The case $\nu > 0$ was treated in [60], where the existence and uniqueness of a stationary measure were proved by using methods of statistical mechanics.

The case $\nu = 0$, which must be regarded as the limiting case as $\nu \to +0$, is much more complicated. Formally, we have again the equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = \sum_k f_k'(x)B_k(t). \tag{8}$$

We restrict ourselves to the case $\int_0^1 u(x,t)\,dx = 0$. The general case can be treated in a similar way [19].

Equation (8) has a hidden dissipation. To see this, consider (7) for $\nu > 0$ and write (using Itô's formula)

$$\begin{aligned} d\int_0^1 \frac{u^2(x,t)}{2}\,dx &= \int_0^1 \frac{u^2(x,t+dt)-u^2(x,t)}{2}\,dx \\ &= \int_0^1 u(x,t)(u(x,t+dt)-u(x,t))\,dx \\ &\quad + \frac{1}{2}\int_0^1 (u(x,t+dt)-u(x,t))^2\,dx \\ &= -\frac{1}{2}\int_0^1 u(x,t)\frac{\partial}{\partial x}u^2(x,t)\,dx\,dt - \nu\int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx\,dt \\ &\quad - \int_0^1 u'(x,t)\sum_k f_k(x)B_k(t)\,dt\,dx + \frac{1}{2}\sum_k\int_0^1 f_k^2(x)\,dx\,dt \\ &= -\nu\int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx\,dt - \sum_k\int_0^1 u_k'(x,t)f_k(x)\,dx\,B_k(t)\,dt \\ &\quad + \frac{1}{2}\sum_k\int_0^1 f_k^2(x)\,dx\,dt. \end{aligned}$$

Taking the expectation, we obtain

$$\frac{d}{dt}\mathsf{E}\int_0^1 \frac{u^2(x,t)}{2}\,dx \leqslant -8\pi^2\nu\mathsf{E}\int_0^1 \frac{u^2(x,t)}{2}\,dx + \frac{1}{2}\sum_k\int_0^1 f_k^2(x)\,dx.$$

This shows that $\mathsf{E}\int \frac{1}{2}u^2(x,t)\,dx$ remains bounded. As $\nu \to 0$, the main contribution to $\nu\int_0^1 (\partial u/\partial x)^2\,dx$ comes from small neighbourhoods of shocks. The theory discussed below shows that

$$\lim_{\nu\to 0}\nu\int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx = \frac{1}{12}\sum_i \bigl[u(x_i-0,t)-u(x_i+0,t)\bigr]^3,$$

where the last sum is taken over all shocks. It follows from the so-called entropy condition [52], [63] that each expression in the square brackets is positive.

In [19] we construct a stationary measure for (8) explicitly. To describe our approach, we need some notation. By $\mathcal{F}_{t_1}^{t_2}$, $t_1 < t_2$, we denote the σ-algebra generated by all the $B_k(t)$ with $t_1 \leqslant t \leqslant t_2$. If $\{S^\tau\}$ is the one-parameter group of shifts, $S^\tau B_k(t) = B_k(t+\tau)$, then $\{S^\tau\}$ preserves the natural measure P defined on all the $\mathcal{F}_{t_1}^{t_2}$ and is ergodic and mixing with respect to P.

Assume now that we have succeeded in constructing a special functional $M(B)$ which is measurable with respect to $\mathcal{F}_{\infty}^{0}$ and takes values in the Skorokhod space $\mathcal{D}_0$ of functions with discontinuities of the first kind and with integral zero. This space is a natural space for solutions of (8). The main property of this functional is that $u(x,t) = MS^t(B)$ is a weak solution of (8). Then, due to the invariance of P under the group $\{S^\tau\}$ of shifts, the probability distribution of $M(B)$ gives rise to a stationary measure for the Markov process (8).

The functional M has a special form described below.

Let us consider an arbitrary piecewise-C^1 function $x(t)\colon R^- \to S^1$ and take the formal integral

$$A(\{x(t)\}) = \int_{-\infty}^{0} \left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \sum_k f_k(x) B_k(t)\right] dt.$$

We call it the *action*. The last part is a stochastic Itô integral. However, the well-known difficulties of stochastic calculus do not occur here because we can integrate by parts and reduce the calculation to integration of piecewise continuous functions.

Definition 3. A function $\bar{x}(t)$ is called a *one-sided minimizer* if $A\{x(t)\}$ is minimal with respect to all local continuous perturbations, that is,

$$\begin{aligned} A(x(t)) - A(\bar{x}(t)) = \int_{t'}^{0} \Bigg[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 &+ \sum_k f_k(x) B_k(t) \\ &- \frac{1}{2}\left(\frac{d\bar{x}}{dt}\right)^2 - \sum_k f_k(\bar{x}) B_k(t)\Bigg] dt \geqslant 0 \end{aligned}$$

for any $x(t)$ such that $x(t) = \bar{x}(t)$ for all $t \leqslant t'$.

It is easy to show that each one-sided minimizer satisfies the Euler–Lagrange equation; in our case, this equation becomes

$$\frac{dx}{dt} = v, \qquad \frac{dv}{dt} = \sum f_k'(x) B_k(t), \tag{9}$$

and it must also be regarded as a stochastic differential equation.

Not every solution of (9) is a one-sided minimizer. We describe below some special properties of minimizers.

Theorem 4. *With probability* 1 *there exists for every point $x \in S^1$ at least one one-sided minimizer $\bar{x}(t)$ for which $\bar{x}(0) = x$.*

It is a general fact of the calculus of variations that any two one-sided minimizers cannot have two intersections. In our probabilistic setting a stronger statement holds.

Theorem 5. *With probability* 1 *any two one-sided minimizers do not intersect. More precisely, if two one-sided minimizers intersect at some instant, then their continuations as solutions of* (9) *are no longer one-sided minimizers.*

It readily follows from Theorem 2 that the set of $x \in S^1$ that are endpoints of several minimizers is at most countable. Let us describe the functional M. We set $M(B)(x) = d\bar{x}/dt$ if the one-sided minimizer $\bar{x}(t)$ for which $\bar{x}(0) = x$ is unique. For all other points, the limits $\lim_{x' \to x-0} M(B)(x')$ and $\lim_{x'' \to x+0} M(B)(x'')$ exist, and $\lim_{x' \to x-0} M(B)(x') > \lim_{x'' \to x+0} M(B)(x'')$. In other words, the points of non-uniqueness of one-sided minimizers are shocks. We write $M(B) = \bar{u}(x, 0)$.

It follows from the Lax–Oleinik variational principle [42], [52] that $\bar{u}(x,t) = M(S^t B)$, $-\infty < t < \infty$, is a weak solution of the inviscid Burgers equation (8) [19]. This gives the existence of at least one stationary measure for (8). The uniqueness follows from other properties of (8).

The following theorem gives the existence of the so-called main shock.

Theorem 6. *There exists a unique continuous curve* $\bar{y}(t)$, $-\infty < t < \infty$, *such that* $\bar{u}(x,t)$ *is a solution that is discontinuous at* $x = \bar{y}(t)$.

This curve is called the main shock of our solution $\bar{u}(x,t)$. It is possible to show that all other shocks eventually merge with the main shock. In this sense, the main shock serves as the attractor of all one-sided minimizers.

Definition 4. A curve $\bar{\bar{x}}(t)\colon R^1 \to S^1$ is called a two-sided minimizer if it provides a minimum to the two-sided action

$$A(x) = \int_{-\infty}^{\infty} \left[\frac{1}{2} \left(\frac{dx}{d\tau} \right)^2 + \sum f_k(x) B_k(\tau) \right] d\tau$$

with respect to all continuous local perturbations.

Theorem 7. *With probability* 1 *the two-sided minimizer* $\bar{\bar{x}}(t)$ *exists and is unique.*

This curve plays a very special role in our theory. It turns out that it is a hyperbolic trajectory of the system (9) in the sense that it has stable and unstable manifolds, which are one-dimensional curves in our situation. The main conclusion is the following theorem [19].

Theorem 8. *The solution* $\{\bar{u}(x,0)\} = M(B)$ *is a subset of the unstable manifold of the two-sided minimizer. More precisely, if* $\gamma^{(u)}$ *is the unstable manifold, then* $(x, \bar{u}(x,0)) \in \gamma^{(u)}$. *The number of shocks of* $\bar{u}(x,0)$ *is finite with probability* 1.

5. Several asymptotics for the stationary distribution

By μ_0 we denote the stationary probability measure for (8) that is the distribution of $\bar{u}(x,0)$. It has several properties following more or less immediately from the construction.

1°. If μ_ν is the analogous measure for the viscous Burgers equation, then

$$\lim_{\nu \to 0} \mu_\nu = \mu_0,$$

where the convergence is understood in the sense of the weak topology [19].

2°. The Markov process (8) has a unique stationary measure [19].

The theory in §4 allows us to obtain some quantitative results [18].

6. Asymptotic behaviour of the probability distribution density functions

Next we consider the asymptotic behaviour of the probability distribution functions (PDFs) of u_x and $\delta u = u(x+r,t) - u(x)$ for our stationary measure.

6.1. PDF of the velocity gradient.

This question has recently attracted a lot of attention. Let $Q(\xi)$ be the PDF of the regular part of $\xi = u_x$. This function is well-defined because, as proved in [19], the shocks are isolated, and the solution is smooth away from the shocks. It was suggested for the first time by Polyakov [54] that Q has the following asymptotics:

$$Q(\xi) \sim \begin{cases} C_-|\xi|^{-\alpha} & \text{as} \quad \xi \to -\infty, \\ C_+\xi^{\beta} e^{-\xi^3/3B_1} & \text{as} \quad \xi \to +\infty, \end{cases} \tag{10}$$

where B_1 is a constant associated with the forcing. However, a variety of values for the exponents α and β were predicted in various papers as a consequence of the variety of techniques used. By invoking the operator product expansion, Polyakov [54] proposed that $\alpha = \frac{5}{2}$ and $\beta = \frac{1}{2}$. E and others [18] suggested that the main contribution for large negative values of ξ comes from the pre-shocks, that is, the small regions near the points of shock creation. This argument gives the value $\alpha = \frac{7}{2}$. Gotoh and Kraichnan argued that the viscous term can be neglected to leading order, giving rise to the values $\alpha = 3$ and $\beta = 1$ [26].

Let Q^ν be the PDF for the statistical stationary state of the viscous problem. If we assume statistical homogeneity, then Q^ν satisfies the following master equation:

$$B_1 Q^\nu_{\xi\xi} + \xi Q^\nu + (\xi^2 Q^\nu)_\xi - \nu(\langle \xi_{xx}|\xi\rangle Q^\nu)_\xi = 0 \tag{11}$$

where $\langle \xi_{xx}|\xi\rangle$ is the conditional average of ξ_{xx} for given ξ, and B_1 is the same parameter as above. This equation is not closed because of the viscous term, and the crux of the matter is how to estimate this term. In [14], [15] E and Vanden Eijnden estimated the term by using formal matched asymptotics, and in the limit as $\nu \to 0$ they obtained

$$F(\xi) = -\lim_{\nu\to 0} \nu(\langle \xi_{xx}|\xi\rangle Q^\nu)_\xi = \varrho \int_{-\infty}^0 sV(s,\xi)\,ds, \tag{12}$$

where ϱ is the mean number of shocks, $V(s,\xi) = \frac{1}{2}(V_+(s,\xi) + V_-(s,\xi))$, and $V_\pm(s,\xi)$ are the conditional PDFs of $(s(y_0,t), \xi_\pm(y_0,t) = u_x(y_0\pm,t))$ for given y_0, which is a shock location. The equation for $Q(\xi) = \lim_{\nu\to 0} Q^\nu(\xi)$ becomes

$$B_1 Q_{\xi\xi} + \xi Q + (\xi^2 Q)_\xi + F(\xi) = 0. \tag{13}$$

An alternative definition of Q is $Q(\xi) = \lim_{\delta\to 0} Q^\delta(\xi)$, where Q^δ is the PDF of the divided difference $(u(x+\delta,t) - u(x,t))/\delta$. It is not yet rigorously proved that these two definitions give the same answer, although the calculation in [15] using matched asymptotics strongly suggests that they do. Nevertheless, for the second definition of Q it can be proved by using the BV-calculus developed by Vol'pert [66] (and, more generally, geometric measure theory) that Q satisfies (13).

To proceed further, E and Vanden Eijnden made the following assumptions:

1. Solutions of (7) are piecewise smooth in the (x, t) plane.
2. Shocks are created at zero amplitude, and the shock strengths are added at any collision.

These statements are slightly stronger than the ones proved in [19]. It is believed that the techniques developed in [19] (with some improvement) are sufficient to prove the statements, but this has not yet been done. Nevertheless, under these assumptions, E and Vanden Eijnden proved that Q has the following representation:

$$Q(\xi) = |\xi|^{-3} \int_{\infty}^{\xi} \xi' F(\xi')\, d\xi' + O(|\xi|^{-6}). \tag{14}$$

Moreover, $G(\xi) = \xi F(\xi)$ is absolutely integrable on $\mathbb{R}^1$. This immediately implies that

$$\lim_{\xi \to -\infty} |\xi|^3 Q(\xi) = 0. \tag{15}$$

If we assume that Q is of the form (10), then the possibility $\alpha \leqslant 3$ is eliminated, and the only candidate is $\alpha = \frac{7}{2}$.

In order to proceed further, E and Vanden Eijnden derived master equations for $V_{\pm}(s, \xi)$. Based on that, a self-consistent asymptotic argument suggests that the main contributions to $F(\xi)$ come from shock creation, which confirms the main assumption made in [18] and their prediction that $\alpha = \frac{7}{2}$. This circle of ideas is not fully rigorous, but strongly suggests that the correct answer is $\alpha = \frac{7}{2}$ and $\beta = 1$. Furthermore, this is among the very few cases where specific answers are obtained from the master equation without making uncontrolled closure assumptions. It should also be noted that, even though Polyakov's predictions for the exponents α and β were incorrect [54], he wrote down equations of the type (11) for the first time and made it clear that the difficulty with closing the master equations is associated with the so-called "dissipative anomaly" rather than with the non-linear term. This observation has led to other very interesting predictions.

6.2. PDF for the velocity difference.

Now let us consider

$$\lambda = \delta u(r) = u(x + r, t) - u(x, t).$$

It is fairly clear that the PDF $Q(\lambda, r)$ for the right-hand tail of λ should behave in the same way as Q; see (10). It is also clear that the extreme left-hand tail of $Q(\lambda, r)$ is dominated by large shocks, which should give $e^{-C|\lambda|^3}$. In the middle there are at least two more regimes. Near the origin, $\delta u(r) \approx r u_x = r\xi$ for small values of λ. Therefore, this regime must be scaled as $|\lambda|^{-\frac{7}{2}}$. Moving to the left, between this regime and the far left tail, there must be a regime dominated by the statistics of small shocks, and very little is known about this regime. Any insight, including careful numerical results, is very welcome!

7. Concluding remarks

It is important to understand the sense in which the theory of the inviscid Burgers equation (8) is special.

In the theory of dynamical systems a beautiful theory of twist maps, also known as the Aubry–Mather theory, was developed twenty years ago [1], [48], [62]. It can be described as related to two-dimensional maps of the cylinder $\mathbb{R}^1 \times S^1$ that are of the following form (in the simplest case):

$$T(x, \varphi) = (z', \varphi'),$$

where $z' = z + V'(\varphi)$, $\varphi' = \varphi + z' \mod 1$, and V is a smooth periodic function. The discrete version of the Burgers equation with random forcing can be described as a sequence (z_n, φ_n), where

$$z_{n+1} = z_n + \xi_n, \qquad \varphi_{n+1} = \varphi_n + z_{n+1},$$

and ξ_n is a sequence of independent identically distributed random variables. However, it should be stressed that the transition from (8) to (10) is not straightforward, and so far there is no analogue of the Aubry–Mather theory for (10).

Bec, Frisch, and Khanin [6] considered the case which they called a *kicked* Burgers equation, where the forcing acts at discrete instants, and they derived results similar to those in [18]; see also [24].

Our theory described in §§ 4, 5 is also related to the Baxendale theory of stochastic dynamical systems [5]. It follows from the general theory that such systems have several positive (that is, unstable) Lyapunov exponents and several negative (that is, stable) Lyapunov exponents. Due to the hidden dissipation which was explained at the beginning of §4, the Burgers system (8) actually has negative Lyapunov exponents only. One can hope that our approach works for the entire class of such systems. For example, there are some reasons to expect that it can work for the multi-dimensional Burgers equation. At the same time, Navier–Stokes systems with arbitrary viscosity can presumably have arbitrarily many positive Lyapunov exponents, and the situation there is quite different. We discuss the Navier–Stokes system in the second part of this text.

PART II

NAVIER–STOKES SYSTEMS WITH PERIODIC BOUNDARY CONDITIONS

8. General remarks concerning the NSS

The general d-dimensional Navier–Stokes system (NSS) is written for d unknown functions $u(x,t) = (u_1(x,t), \dots, u_d(x,t))$ of d variables $x = (x_1, \dots, x_d)$ and time t and for the pressure $p(x,t)$ as follows:

$$\frac{\partial u_i}{\partial t} + \sum u_k \frac{\partial u_i}{\partial x_k} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f^{(i)}. \tag{16}$$

We consider an incompressible liquid with density $\rho \equiv 1$. In this case we have an additional equation

$$\operatorname{div} u = \sum_{i=1}^{d} \frac{\partial u_i}{\partial x_i} \equiv 0. \tag{17}$$

The functions $f^{(i)}$ are the components of the external forcing, and $\nu > 0$ is the viscosity.

Equations (16), (17) play the central role in hydrodynamics, especially in connection with turbulence. In the mathematical theory the basic question is about how complicated the solutions of (16), (17) can be.

The integral $E(u) = \frac{1}{2}\int u^2(x,t)\,dx = \frac{1}{2}\int \sum_{i=1}^{d} u_i^2(x,t)\,dx$ is the energy. People are usually interested in solutions of (16), (17) with finite $E(u)$.

The first simplification is to consider (16), (17) with periodic boundary conditions, which allows us to use Fourier series. We write

$$u(x,t) = \sum_{k\in\mathbb{Z}^d} u_k(t) e^{2\pi i(k,x)}.$$

Here $u_k \in R^d$ for each k, and the incompressibility condition reduces to the requirement that $u_k \perp k$ for $k \in \mathbb{Z}^d$. Instead of (16), (17) we have

$$\frac{du_k}{dt} = -2\pi i \sum_{k_1\in\mathbb{Z}^d} (u_{k_1}, k) \sqcap_k u_{k-k_1} - \nu |k|^2 u_k + f_k. \tag{18}$$

Here $\sqcap_k$ is the operator of orthogonal projection onto the $(d-1)$-dimensional plane orthogonal to k, and the f_k are the components of the external forcing. For simplicity we assume that only finitely many of the f_k are non-zero. In the finite-energy case we have $\sum_k |u_k|^2 < \infty$. Sometimes the system (18) is called the *Galerkin system* for the initial NSS. Finite-dimensional Galerkin approximations for (18) arise if $k_1, k, k-k_1 \in O$ in (18), where O is a finite convex domain which is symmetric (this means that k and $-k$ belong to O simultaneously), and we use the notation (18_O) for the corresponding system. Let us formulate the following simple lemma.

Lemma 1. *If $E(u) < \infty$, then the right-hand side of* (18) *is finite for each k.*

This lemma shows that, at least formally, the vector field corresponding to (18) is well-defined on the entire Hilbert space $L^2(\mathbb{Z}^d)$ of functions u with finite $E(u)$. Of course, the lemma does not imply that the right-hand side also belongs to $L^2(\mathbb{Z}^d)$. Let us introduce the following definition.

Definition 5. A *strong solution* of (18) on an interval $[0,T]$ is a one-parameter family $\{u_k(t)\} \in L^2(\mathbb{Z}^d)$ such that the function $u_k(t)$ is differentiable with respect to t and (18) holds for each $k \in \mathbb{Z}^d$.

Local existence of strong solutions means that, for a given $\{u_k(0), k \in \mathbb{Z}^d\}$, one can find a closed interval $[0,t_0]$ such that there is a strong solution $\{u_k(t), k \in \mathbb{Z}^d\}$ on $[0,t_0]$. In other words, there is a finite piece of a trajectory of the NSS that goes out of $\{u_k(0)\}$. Global existence means that there is an entire trajectory $\{u_k(t), t \geqslant 0\}$ going out of $\{u_k(0)\}$. The local existence statement is certainly much simpler (see below). One can introduce analogous definitions for any finite Galerkin approximation (18_O).

The existence and uniqueness problem for solutions of the NSS was treated for the first time by Leray in his classical 1934 paper [44]. He proved the global existence of weak solutions for the Cauchy problem in $\mathbb{R}^3$. Later Hopf [32] extended these results to bounded domains with zero boundary conditions. The existence of strong solutions was established in the two-dimensional case by Ladyzhenskaya in the sixties [41]. Other methods were developed in work of Foiaş and Temam (see [22] and the survey [64]), Yudovich [69], Constantin–Foiaş [12], and others. However, the main problem of global existence of strong solutions in the three-dimensional case remains open. Some people, including one of the authors of the present paper, do not even believe the answer is positive.

9. Existence theorems for global solutions of the NSS

The methods of proof of the global existence theorem are based on energy and enstrophy estimates. Let us take any system (18_O) and set $E_O(u) = \sum_{k\in O} |u_k|^2$.

Lemma 2 (energy estimate). *Let $\{u_O(t)\}, t \in [t_1, t_2]$, be a strong solution of* (18_O), *where O is either finite or equal to $\mathbb{Z}^d$. Then*

$$\frac{d}{dt} E(u_O(t)) \leqslant -4\pi^2 \nu E(u_O(t)) + 2F\sqrt{E(u_O(t))},$$

where $F = \sum_k |f_k|^2$.

The proof is straightforward (see, for example, [61]). We have

$$\begin{aligned}\frac{d}{dt}E(u(t)) &= \frac{d}{dt}\frac{1}{2}\sum_{k\in O}(u_k,u_k) = \frac{1}{2}\sum_{k\in O}\left(\frac{du_k}{dt},u_k\right) + \frac{1}{2}\sum_{k\in O}\left(u_k,\frac{du_k}{dt}\right)\\ &= -\pi i \sum_{k,k_1\in O}(u_{k_1},k)(u_{k-k_1},u_k) - \pi i \sum_{k,k_1\in O}(u_{k_1},u_{k-k_1})(u_{k_1},k)\\ &\qquad - 4\pi^2\nu\sum_{k\in O}|k|^2(u_k,u_k) + \sum_{k\in O}(f_k,u_k) + \sum_{k\in O}(u_k,f_k).\end{aligned}$$

We can readily see that the first two sums are identically zero. A simple estimate of the other terms gives the result.

Lemma 2 shows that, if the initial energy is finite, then it remains bounded on the entire interval on which a strong solution exists.

Theorem 9 (local existence theorem). *Suppose that $\{u_k(0)\}$ satisfies the relation $\sum_{k\in\mathbb{Z}^d} |u_k(0)| = h < \infty$. Then there is a strong solution of* (18) *on the interval $[0, t_0]$, $t_0 = c\nu/h$, where c is a dimension-dependent absolute constant.*

There are stronger results that are not covered by this theorem [35], [45]. Another example is given by the following assertion.

Theorem 10 (Kaloshin and Sannikov). *Let $d = 3$ and let $|u_k(0)| \leqslant c/|k|^\gamma$, where $\gamma \geqslant 2$. If c is small enough, then there is a strong solution of* (18) *for all $t > 0$.*

There are several methods to obtain local existence results. However, the values of t_0 are of the same order of magnitude.

The proof of the global existence theorem is based on the notion of enstrophy. In what follows we assume that $d = 2$ or 3.

Definition 6. The *enstrophy* of $\{u_k, k \in O\}$ is

$$V_O(\{u_k, k \in O\}) = \sum_{k \in O} |k|^2 |u_k|^2.$$

For $O = \mathbb{Z}^d$ we write $V(\{u_k\})$. The basic result says that, if the enstrophy $V_O(\{u_k(t),\ k \in O\})$ remains bounded for a strong solution of (18) or (18_O), then the modes u_k decay exponentially. Here is a more precise formulation.

Theorem 11. *Let us assume that* $\{u_k(0),\ k \in O \text{ or } \mathbb{Z}^d\}$ *is such that*

$$|u_k(0)| \leqslant \frac{\mathcal{D}}{|k|^\gamma}, \qquad k \in 0 \quad or \quad \mathbb{Z}^d, \tag{19}$$

for some $\mathcal{D} < \infty$ *and* $\gamma > \frac{d}{2} + 1$, *and that*

$$V(\{u_k(t), k \in O \text{ or } \mathbb{Z}^d\}) \leqslant V_0, \qquad 0 \leqslant t \leqslant T,$$

for a strong solution $\{u_k(t)\}$. *Then there are positive numbers* α *and* $\mathcal{D}_0$ *such that*

$$|u_k(t)| \leqslant \frac{\mathcal{D}_0 e^{-\alpha t \cdot |k|}}{|k|^\gamma}, \qquad 0 \leqslant t \leqslant T, \quad for\ all \quad k. \tag{20}$$

This theorem was proved by Foiaş and Temam [22]. The proof below follows the papers [50] and [61]. We begin with the following preliminary statement.

Lemma 3. *If* $\{u_k(0)\}$ *satisfies* (19) *and the enstrophy is bounded, then one can find another constant* $\mathcal{D}_1$ *such that* $|u_k(t)| \leqslant \mathcal{D}_1/|k|^\gamma$ *for all* $0 \leqslant t \leqslant T$. *This constant does not depend on* O.

Proof. For simplicity we consider the real version of (18) in this and subsequent lemmas:

$$\frac{du_k}{dt} = 2\pi \sum (u_{k_1}, k) \sqcap_k u_{k-k_1} - \nu k^2 u_k + f_k. \tag{21}$$

We also assume that all the u_k are one-dimensional. This also is just a technical simplification. Let us take K large enough (the value of K will be specified below). Since $V(\{u_k(t), k \in \mathbb{Z}^d\}) \leqslant V_0$, we can find a constant $\mathcal{D}_1(K)$ such that

$$|u_k(t)| \leqslant \frac{\sqrt{V_0}}{|k|} \leqslant \frac{\mathcal{D}_1(K)}{|k|^\gamma}, \qquad |k| \leqslant K.$$

We prove that $|u_k(t)| < \mathcal{D}_1(K)|k|^{-\gamma}$ for all k with $|k| > K$, provided that K is large enough. Suppose that this is wrong, and

$$|u_{\bar{k}}(\bar{t})| = \frac{\mathcal{D}_1(K)}{|\bar{k}|^\gamma}$$

for some values of $\bar{k}$ with $|\bar{k}| > K$ and $\bar{t} \leqslant T$. Let us take the least value of $\bar{t}$ and consider the case

$$u_{\bar{k}}(\bar{t}) = \frac{\mathcal{D}_1(K)}{|\bar{k}|^{\gamma}}.$$

The other case can be treated similarly. We must have

$$\frac{du_{\bar{k}}(\bar{t})}{dt} \geqslant 0. \tag{22}$$

Our arguments below show that this is impossible because the viscous term dominates. Namely,

$$\nu|\bar{k}|^2 u_{\bar{k}} > \nu\mathcal{D}_1(K)\cdot|\bar{k}|^{2-\gamma}. \tag{23}$$

We may always take K so large that $f_k = 0$ for $|k| \geqslant K$. In order to estimate the sum

$$\left|\sum_{k_1}(u_{k_1},\bar{k})\cdot \sqcap_{\bar{k}} u_{\bar{k}-k_1}\right| \leqslant \sum_{k_1}\left|u_{k_1}\right|\cdot\left|\bar{k}-k_1\right|\cdot\left|u_{\bar{k}\cdot k_1}\right|,$$

we consider three cases.

Case I. $|k_1| \leqslant \frac{1}{2}|\bar{k}|$.

Here $|\bar{k}-k_1| \geqslant \frac{1}{2}|\bar{k}|$, and therefore $|u_{\bar{k}-k_1}|\cdot|\bar{k}-k| \leqslant 2^{\gamma}\cdot\mathcal{D}_1(K)|\bar{k}|^{1-\gamma}$. On the other hand, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\sum_{|k_1|\leqslant\frac{1}{2}\bar{k}}|u_{k_1}| &\leqslant \sum_{|k_1|\leqslant\frac{1}{2}|\bar{k}|}|k_1|\cdot|u_{k_1}|\cdot\frac{1}{|k_1|} \leqslant \sqrt{\sum|k_1|^2|u_{k_1}|^2}\cdot\sqrt{\sum\frac{1}{|k_1|^2}}\\ &\leqslant \text{const}\sqrt{V}\cdot|\bar{k}|^{\frac{1}{2}}.\end{aligned}$$

The last inequality holds for $d \leqslant 3$. We see that the viscous term in (21) dominates if K is large enough.

Case II. $\frac{1}{2}|\bar{k}| < |k_1| \leqslant 2|\bar{k}|$.

In this case $|u_{k_1}| \leqslant 2^{\gamma}\mathcal{D}_1(K)\cdot|k_1|^{-\gamma}$, and again

$$\sum_{\frac{1}{2}|\bar{k}|\leqslant|k_1|\leqslant2|\bar{k}|}|u_{\bar{k}_1}|\cdot|k-k_1|\cdot|u_{k-k_1}| \leqslant \frac{2^{\gamma}\mathcal{D}_1(K)}{|\bar{k}|^{\gamma-1}}\cdot\sum|u_{\bar{k}-k_1}|\cdot|\bar{k}-k_1|.$$

The same arguments as in **I** give

$$\sum|u_{\bar{k}-k_1}|\cdot|\bar{k}-k_1| \leqslant \text{const}\sqrt{\sum_{|k|\leqslant3|\bar{k}|}|u_k|^2|k|^2}\cdot|\bar{k}|^{3/2} \leqslant \text{const}\sqrt{V}|\bar{k}|^{3/2}.$$

We see again that the viscous term dominates for $d \leqslant 3$ if K is large enough.

Case III. $|k_1| > 2|\bar{k}|$.

Here $|\bar{k} - k_1| > |\bar{k}|$. Therefore,

$$\begin{aligned}\sum |u_{k_1}|\cdot|\bar{k}-k_1|\cdot|u_{\bar{k}-k_1}| &\leqslant \frac{1}{|\bar{k}|}\sum |u_{k_1}|\cdot|k_1|\cdot|\bar{k}-k_1|\cdot|u_{\bar{k}-k_1}| \\ &\leqslant \frac{1}{|\bar{k}|}\cdot\sqrt{\sum |u_{k_1}|^2\cdot|k_1|^2}\cdot\sqrt{\sum|\bar{k}-k_1|^2\cdot|u_{\bar{k}-k_1}|^2} \\ &\leqslant \frac{\sqrt{V}}{|\bar{k}|}\cdot\mathcal{D}_1(K)\sqrt{\sum_{|k_1|\geqslant 2|\bar{k}|}\frac{1}{|k_1|^{2\gamma-2}}} \\ &\leqslant \frac{\mathrm{const}\sqrt{V}\cdot\mathcal{D}_1(K)}{|\bar{k}|}\cdot\frac{|\bar{k}|^{1+d/2}}{|\bar{k}|^{\gamma}}.\end{aligned}$$

The viscous term dominates again, and this completes the proof of Lemma 3.

Now we complete the proof of Theorem 6. Let us consider the case $O = \mathbb{Z}^d$. In view of Lemma 3,

$$|u_k(t)| \leqslant \frac{\mathcal{D}_1}{|k|^{\gamma}}, \qquad 0 \leqslant t \leqslant T,$$

for all $k \in \mathbb{Z}^d$. Therefore, for any K we can find an $\alpha(K) = \alpha$ such that

$$|u_k(t)| \leqslant \frac{2\mathcal{D}_1}{|k|^{\gamma}}e^{-\alpha t|k|}, \qquad 0 \leqslant t \leqslant T, \tag{24}$$

for all k, $|k| < K$. Let us consider again the real version of (18),

$$\frac{du_k}{dt} = 2\pi\sum_{k_1}(u_{k_1}, k)\sqcap_k u_{k-k_1} - \nu|k|^2 u_k + f_k. \tag{18$'$}$$

We set $v_k(t) = e^{\alpha|k|t}u_k(t)$. Then we have the following system of equations for v_k:

$$\frac{dv_k(t)}{dt} = \sum(v_{k_1}, k)\sqcap_k v_{k-k_1}\cdot\frac{e^{-\alpha|k_1|t-\alpha|k-k_1|t}}{e^{-\alpha|k|t}} - (\nu|k|^2 - \alpha|k|)v_k + e^{\alpha|k|t}f_k. \tag{25}$$

Our arguments here are practically the same as in the proof of Lemma 3. For $|k| \leqslant K$ we have the needed estimates, which can be rewritten as

$$|v_k(t)| \leqslant \frac{2\mathcal{D}_1}{|k|^{\gamma}}.$$

Let us assume that, for some $\bar{k}$ with $|\bar{k}| > K$ there is a t_0 with $0 < t_0 \leqslant T$ for which $|v_{\bar{k}}(t_0)| = 2\mathcal{D}_1|\bar{k}|^{-\gamma}$. For definiteness, we again consider the case $v_{\bar{k}}(t_0) = 2\mathcal{D}_1|\bar{k}|^{-\gamma}$. We may also assume that t_0 is the smallest number with the required properties. Then we must have $(dv_{\bar{k}}/dt)(t_0) \geqslant 0$. A contradiction to this inequality is obtained in the same way as in Lemma 3. We note that $e^{-\alpha|k_1|t-\alpha|k-k_1|t} \leqslant e^{-\alpha|k|t}$ because $|k_1| + |k-k_1| \geqslant |k|$. Instead of finiteness of the enstrophy, we must everywhere use the estimates $|v_k(t_0)| \leqslant 2\mathcal{D}_1/|k|^{\gamma}$. The details are left to the reader.

In the two-dimensional case one can prove the existence of strong solutions for all $t > 0$ under very mild assumptions on the initial conditions. For $d = 2$ the components $u^{(1)}(x_1, x_2, t)$, $u^{(2)}(x_1, x_2, t)$ of the velocity satisfy the incompressibility condition $\partial u^{(1)}/\partial x_1 + \partial u^{(2)}/\partial x_2 = 0$. Therefore, we can introduce the vorticity

$$\omega = \frac{\partial u^{(1)}}{\partial x_2} - \frac{\partial u^{(2)}}{\partial x_1}$$

so that the enstrophy becomes proportional to the L^2-norm of the vorticity. It satisfies the so-called enstrophy inequality, which is analogous to the energy inequality.

Lemma 4. *For any solution of* (18) *or* (18_O)

$$\frac{dV(\{u_k(t)\})}{dt} \leqslant -4\pi^2 \nu V(\{u_k(t)\}) + 2F\sqrt{V(\{u_k(t)\})}.$$

The proof can be found in many textbooks; see also [61].

This lemma shows that, if the enstrophy at $t = 0$ is finite, then it remains bounded on any time interval. Now Theorem 6 immediately implies the existence of global strong solutions for $d = 2$.

As we mentioned above, the existence problem for global strong solutions of the NSS for $d = 3$ remains a challenging open problem.

10. The two-dimensional Navier–Stokes system with periodic boundary conditions and random forcing

The Navier–Stokes system with random forcing is regarded as the most popular model of turbulence. However, some people express concern about its relevance to actual hydrodynamics. We begin with the case $d = 2$ because it has been more thoroughly studied for the reason that existence problems are simpler, even in the random case.

The system of equations has the same form as above,

$$\begin{gathered}
\frac{\partial u^{(1)}}{\partial t} + u^{(1)}\frac{\partial u^{(1)}}{\partial x_1} + u^{(2)}\frac{\partial u^{(1)}}{\partial x_2} = \nu\Delta u^{(1)} + g^{(1)}, \\
\frac{\partial u^{(2)}}{\partial t} + u^{(1)}\frac{\partial u^{(2)}}{\partial x_1} + u^{(2)}\frac{\partial u^{(2)}}{\partial x_2} = \nu\Delta u^{(2)} + g^{(2)}, \\
\frac{\partial u^{(1)}}{\partial x_1} + \frac{\partial u^{(2)}}{\partial x_2} = 0,
\end{gathered} \tag{26}$$

where $g = (g^{(1)}, g^{(2)})$ is a random function of space and time. It is convenient to consider the vorticity $\omega = \partial u^{(1)}/\partial x_2 - \partial u^{(2)}/\partial x_1$ and to write the equation for ω,

$$\frac{\partial \omega}{\partial t} + u^{(1)}\frac{\partial \omega}{\partial x_1} + u^{(2)}\frac{\partial \omega}{\partial x_2} = \nu\Delta\omega + f,$$

where $f = \partial g^{(1)}/\partial x_2 - \partial g^{(2)}/\partial x_1$. Expanding ω in the corresponding Fourier series $\omega = \sum_{k\in\mathbb{Z}^2} w_k e^{2\pi i(k,x)}$, we obtain the system of equations for w_k,

$$\frac{dw_k}{dt} = -2\pi i \sum_{k_1\in\mathbb{Z}^2} w_{k_1} w_{k-k_1} \frac{(k, k_1^{\perp})}{(k_1, k_1)} - \nu|k|^2 w_k + f_k(t), \qquad w_{-k} = \bar{w}_k. \tag{27}$$

We assume that the f_k are independent white noises except for the relations $f_{-k} = \bar{f}_k$, which imply that the solutions of (27) are real. Equations (27) can be written as an infinite-dimensional system of stochastic differential equations. For this situation various existence theorems can be found in [13], [39], [25].

We restrict ourselves to general properties of (27) and make a simplifying assumption that $f_k = 0$ for $|k| > K_0$. We need the notion of determining modes introduced by Foiaş (see [64]). Let us take a value K and arbitrary functions $\bar{w}_k(t)$, where $-\infty < t \leqslant 0$ and $|k| \leqslant K$. We consider the system (27) for $|k| \geqslant K$ and $t \geqslant -T$ with initial conditions $w_k(-T) = 0$, $k > K$, denote the solution by $w_k(t; -T)$, and assume the existence of the limit $\lim_{T\to\infty} w_k(t; -T) = w_k(t, -\infty)$. This can hold if K is large enough and the viscous term dominates. Of course, the functions $w_k(t)$, $|k| \leqslant K$, should not grow too fast as $t \to -\infty$.

If $w_k(t)$, $|k| \leqslant K$, are determining modes, then each high mode $w_k(t, -\infty)$, $|k| > K$, can be written as $w_k(t, -\infty) = \Phi_k(w_{k_1}(\zeta),\ \ s \leqslant t,\ |k_1| \leqslant K)$, where Φ_k is a functional depending on the entire pre-history of high modes with $|k| \leqslant K$. Thus, the complete system (27) is effectively reduced to the finite-dimensional system of stochastic differential equations

$$dw_k = +\left[2\pi i \sum_{k_1\in\mathbb{Z}^2} w_{k_1} w_{k-k_1} \frac{(k, k_1^{\perp})}{(k_1, k_1)} - \nu|k|^2 w_k\right] dt + f_k\, db_k(t), \tag{28}$$

where the high modes w_{k_1}, $|k_1| > K$, are functionals of low modes, and the b_k are independent Brownian motions except for the relations $b_{-k} = b_k$.

The first results concerning the existence of an invariant measure for the Markov process in the function space corresponding to (28) were obtained by Fursikov and Vishik [25] and by Flandoli and Maslowski [21]. The argument is simple. Let us consider the enstrophy $V(t) = \sum_k w_k(t)^2$. Its differential can be written with the help of the Itô formula,

$$dV(t) = \sum dw_k(t)\, w_{-k}(t) + \sum w_k(t)\, dw_{-k} + \sum |f_k|^2.$$

If we substitute the right-hand side of (28), then we easily come to the inequality [21], [49]

$$EV(t) \leqslant C_1 + C_2(EV(t_0) - C_3)$$

for some constants $C_1 < \infty$, $C_2 < 1$, and $C_3 < \infty$. This immediately implies that $EV(t)$ is bounded. General results in the theory of Markov processes now imply the existence of an invariant measure.

Uniqueness is much more difficult. Flandoli and Maslowski [21] proved the uniqueness of an invariant measure under the assumption that the coefficients f_k are non-zero for all k and have regular decay at infinity. Mattingly [49] proved uniqueness under less restrictive assumptions on the f_k and for large ν. Uniqueness was established for the case in which $f_k \neq 0$ for all determining modes with $|k| \leqslant K$ and $f_k = 0$ for $|k| > K$, in a recent paper by Kuksin and Shirikyan [38].

PART III

THE STOCHASTIC PASSIVE SCALAR TRANSPORT EQUATION

Consider the passive scalar transport equation for a scalar field $\theta^\kappa(\vec{x},t)$ in $\mathbb{R}^d$,

$$\frac{\partial\theta^\kappa}{\partial t}+(\vec{u}(\vec{x},t)\cdot\nabla)\theta^\kappa=\kappa\Delta\theta^\kappa, \tag{29}$$

where $\vec{u}$ is a white-in-time random process (infinite-dimensional Brownian motion) defined on a probability space $(\Omega,\mathcal{F},\mathsf{P})$. It is well known that this problem is closely related to the stochastic ODE

$$d\varphi^\omega_{s,t}(\vec{x})=\vec{u}(\varphi^\omega_{s,t}(\vec{x}),t)\,dt,\qquad \varphi^\omega_{s,s}(\vec{x})=\vec{x}. \tag{30}$$

In connection with the transport equation, it is most natural to consider this stochastic ODE in the Stratonovich sense. It is known [39] that if the local characteristic of $\vec{u}$ is spatially twice continuously differentiable, then the system in (30) has a unique solution. Such conditions are not satisfied by typical turbulent velocity fields on the scale of interest. In fact, Kolmogorov's theory of turbulence in the three-dimensional case suggests that $\vec{u}$ is only Hölder continuous with an exponent roughly equal to $\frac{1}{3}$. If the regularity condition on $\vec{u}$ fails, then there are at least two natural ways to regularize (29). The first is to add diffusion,

$$d\varphi^{\omega;\kappa}_{s,t}(\vec{x})=\vec{u}(\varphi^{\omega;\kappa}_{s,t}(\vec{x}),t)\,dt+\sqrt{2\kappa}\,d\vec{\beta}(t), \tag{31}$$

and consider the limit as $\kappa\to0$, as in (29). We call this case the *κ-limit.* The other is to smooth out the velocity field. Let ψ_ε be defined as

$$\psi_\varepsilon(\vec{x})=\frac{1}{\varepsilon^d}\psi\left(\frac{\vec{x}}{\varepsilon}\right),$$

where ψ is a standard mollifier, so that $\psi\geqslant0$, $\int_{\mathbb{R}^d}\psi d\vec{x}=1$, and ψ decays rapidly at infinity. Let us set $\vec{u}^\varepsilon=\vec{u}\star\psi_\varepsilon$ and consider

$$d\varphi^{\omega,\varepsilon}_{s,t}(\vec{x})=\vec{u}^\varepsilon(\varphi^{\omega,\varepsilon}_{s,t}(\vec{x}),t)\,dt \tag{32}$$

in the limit as $\varepsilon\to0$. We call this case the *ε-limit.* Physically, κ plays the role of molecular diffusivity, and ε can be thought of as a crude model of the viscous cut-off scale. Therefore, the κ-limit corresponds to the situation in which the Prandtl number tends to zero, $\mathrm{Pr}\to0$, whereas the ε-limit corresponds to the situation in which the Prandtl number diverges, $\mathrm{Pr}\to\infty$.

Before proceeding further, we relate the regularized flows in (31), (32) to the solution of the transport equation. Let us consider κ-regularization first. It is convenient to introduce the backward transition probability

$$g^\kappa_\omega(\vec{x},t\mid d\vec{y},s)=\mathsf{E}_\beta\delta(\vec{y}-\varphi^{\omega,\kappa}_{t,s}(\vec{x}))\,d\vec{y},\quad s<t, \tag{33}$$

where the expectation is taken with respect to $\vec{\beta}(t)$, and $\varphi_{t,s}^{\omega,\kappa}(\vec{x})$ is the flow inverse to the flow $\varphi_{s,t}^{\omega,\kappa}(\vec{x})$ defined in (31) (that is, $\varphi_{s,t}^{\omega,\kappa}(\vec{x})$ is the forward flow, and $\varphi_{t,s}^{\omega,\kappa}(\vec{x})$ is the backward flow). The action of g_ω^κ generates a semigroup of transformations

$$S_{t,s}^{\omega;\kappa}\psi(\vec{x}) = \int_{\mathbb{R}^d} \psi(\vec{y}) g_\omega^\kappa(\vec{x}, t \mid d\vec{y}, s), \tag{34}$$

for all test functions ψ, and the function $\theta_\omega^\kappa(\vec{x}, t) = S_{t,s}^{\omega,\kappa}\psi(\vec{x})$ solves the transport equation in (29) for the initial condition $\theta_\omega^\kappa(\vec{x}, s) = \psi(\vec{x})$. Similarly, for the flow in (32) we set

$$S_{t,s}^{\omega,\varepsilon}\psi(\vec{x}) = \psi(\varphi_{t,s}^{\omega,\varepsilon}(\vec{x})), \quad s < t, \tag{35}$$

and then $\theta_\omega^\varepsilon(\vec{x}, t) = S_{t,s}^{\omega,\varepsilon}\psi(\vec{x})$ solves the transport equation

$$\frac{\partial \theta^\varepsilon}{\partial t} + (\vec{u}^\varepsilon(\vec{x}, t) \cdot \nabla)\theta^\varepsilon = 0 \tag{36}$$

with initial condition $\theta(\vec{x}, s) = \psi(\vec{x})$. Similar definitions can be given for forward flows, but we restrict our attention to backward ones because we are primarily interested in scalar transport. The results below can trivially be generalized to forward flows.

11. Kraichnan's model

We consider a generalization of Kraichnan's model [37] for which $\vec{u}$ is assumed to be a statistically homogeneous, isotropic, and stationary Gaussian field with zero mean and covariance

$$\mathsf{E} u_\alpha(\vec{x}, t) u_\beta(\vec{y}, s) = (C_0 \delta_{\alpha\beta} - c_{\alpha\beta}(\vec{x} - \vec{y}))\delta(t - s). \tag{37}$$

We assume that $\vec{u}$ has correlation length ℓ_0, that is, the covariance in (37) decays rapidly for $|\vec{x} - \vec{y}| > \ell_0$. As is consistent, $c_{\alpha\beta}(\vec{x}) \to C_0\delta_{\alpha\beta}$ as $|\vec{x}|/\ell_0 \to \infty$. On the other hand, we are mainly interested in small-scale phenomena for which $|\vec{x}| \ll \ell_0$. In this range we take $c_{\alpha\beta}(\vec{x}) = d_{\alpha\beta}(\vec{x}) + O(|\vec{x}|^2/\ell_0^2)$ with

$$d_{\alpha\beta}(\vec{x}) = A d_{\alpha\beta}^P(\vec{x}) + B d_{\alpha\beta}^S(\vec{x}), \tag{38}$$

where

$$\begin{aligned} d_{\alpha\beta}^P(\vec{x}) &= D\biggl(\delta_{\alpha\beta} + \xi \frac{x_\alpha x_\beta}{|\vec{x}|^2}\biggr)|\vec{x}|^\xi, \\ d_{\alpha\beta}^S(\vec{x}) &= D\biggl((d + \xi - 1)\delta_{\alpha\beta} - \xi \frac{x_\alpha x_\beta}{|\vec{x}|^2}\biggr)|\vec{x}|^\xi, \end{aligned} \tag{39}$$

and D is a parameter having dimension $[\text{length}]^{2-\xi}[\text{time}]^{-1}$. The dimensionless parameters A and B measure the divergence and the rotation of the field $\vec{u}$, respectively; the case $A = 0$ corresponds to incompressible fields with $\nabla \cdot \vec{u} = 0$, and the case $B = 0$ corresponds to irrotational fields with $\nabla \times \vec{u} = 0$. The parameter ξ measures the spatial regularity of $\vec{u}$. For $\xi \in (0, 2)$, the local characteristic of $\vec{u}$ fails to be twice differentiable, and this fact has important consequences for both the transport equation in (29) and the systems of ODEs in (30) or (31).

The related physics literature is concentrated on the κ-limit for Kraichnan's model [10], [26], [57]. Let $\mathcal{S}^2 = A + (d-1)B$, $\mathcal{C}^2 = A$, and $\mathcal{P} = \mathcal{C}^2/\mathcal{S}^2$. The value $\mathcal{P} \in [0,1]$ measures the degree of compressibility of $\vec{u}$. The pioneering work of Gawędzki and Vergassola [27] identifies the following two different regimes for the κ-limit.

1. The strongly compressible regime in which
$$\mathcal{P} \geqslant \frac{d}{\xi^2}\,. \tag{40}$$
In this regime, the family g_ω^κ is convergent to a flow of maps, that is, there is a two-parameter family $\{\varphi_{t,s}^\omega(\vec{x})\}$ of maps such that
$$g_\omega^\kappa(\vec{x}, t \mid d\vec{y}, s) \to \delta(\vec{y} - \varphi_{t,s}^\omega(\vec{x}))\, d\vec{y}. \tag{41}$$
Moreover, particles have a finite probability of coalescing under the action of the flow $\{\varphi_{t,s}^\omega(\vec{x})\}$.
2. For
$$\mathcal{P} \leqslant \frac{d}{\xi^2}\,, \tag{42}$$
the family g_ω^κ is convergent to a "generalized stochastic flow" [16],
$$g_\omega^\kappa(\vec{x}, t \mid d\vec{y}, s) \to g_\omega(\vec{x}, t \mid d\vec{y}, s), \tag{43}$$
and the limit g_ω is a non-trivial probability distribution with respect to $\vec{y}$. This means that the image of a particle under the flow defined by the velocity field $\vec{u}$ is non-unique and has non-trivial distribution. In other words, particle trajectories branch.

The following result of E and Vanden Eijnden provides a rigorous justification of these predictions and also extends the result to the ε-limit. In particular, it points out three regimes in which both the κ- and the ε-limits are considered.

Theorem 12. *For a strongly compressible regime in which*
$$\mathcal{P} \geqslant \frac{d}{\xi^2}\,, \tag{44}$$
there exists a two-parameter family $\{\varphi_{t,s}^\omega(\vec{x})\}$ of random maps such that
$$\mathsf{E}\big(S_{t,s}^{\omega,\kappa}\psi(\vec{x}) - \psi(\varphi_{t,s}^\omega(\vec{x}))\big)^2 \to 0 \tag{45}$$
as $\kappa \to 0$ and
$$\mathsf{E}\big(\psi(\varphi_{t,s}^{\omega,\varepsilon}(\vec{x})) - \psi(\varphi_{t,s}^\omega(\vec{x}))\big)^2 \to 0 \tag{46}$$
as $\varepsilon \to 0$ for all smooth test functions ψ and for all $(s,t,\vec{x})$, $s < t$. Moreover, the limiting flow $\{\varphi_{t,s}^\omega(\vec{x})\}$ is coalescent in the sense that, for almost all $(s,\vec{x},\vec{y})$, $\vec{x} \neq \vec{y}$, there is a time τ, $0 < \tau < \infty$, such that
$$\varphi_{s,t}^\omega(\vec{x}) = \varphi_{s,t}^\omega(\vec{y}) \tag{47}$$
for $t \geqslant \tau$.

For a weakly compressible regime in which

$$\mathcal{P} \leqslant \frac{d+\xi-2}{2\xi}\,, \tag{48}$$

there exists a random family of generalized flows $g_\omega(\vec{x},t|d\vec{y},s)$ *such that for all test functions* ψ *the family*

$$S^\omega_{t,s}\psi(\vec{x}) = \int_{\mathbb{R}^d} \psi(\vec{y}) g_\omega(\vec{x},t \mid d\vec{y},s) \tag{49}$$

satisfies the relation

$$\mathsf{E}\big(S^{\omega,\kappa}_{t,s}\psi(\vec{x}) - S^\omega_{t,s}\psi(\vec{x})\big)^2 \to 0 \tag{50}$$

as $\kappa \to 0$ *for all* $(s,t,\vec{x})$, $s<t$, *and*

$$\mathsf{E}\bigg(\int_{\mathbb{R}^d} \eta(\vec{x})\,\big(\psi(\varphi^{\omega,\varepsilon}_{t,s}(\vec{x})) - S^\omega_{t,s}\psi(\vec{x})\big)\,d\vec{x}\bigg)^2 \to 0 \tag{51}$$

as $\varepsilon \to 0$ *for all* (s,t), $s<t$, *and for all test functions* η. *Moreover,* $g_\omega(\vec{x},t \mid d\vec{y},s)$ *is non-degenerate in the sense that*

$$S^\omega_{t,s}\psi^2(\vec{x}) - \big(S^\omega_{t,s}\psi(\vec{x})\big)^2 > 0 \quad a.s. \tag{52}$$

For an intermediate regime in which

$$\frac{d+\xi-2}{2\xi} \leqslant \mathcal{P} \leqslant \frac{d}{\xi^2}\,, \tag{53}$$

there exists a random family $\{\varphi^\omega_{t,s}(\vec{x})\}$ *of maps and a random family of generalized flows* $g_\omega(\vec{x},t|d\vec{y},s)$ *such that*

$$\mathsf{E}\big(S^{\omega,\kappa}_{t,s}\psi(\vec{x}) - S^\omega_{s,t}\psi(\vec{x})\big)^2 \to 0 \tag{54}$$

as $\kappa \to 0$ *for all test functions* ψ *and for all* $(s,t,\vec{x})$, $s<t$. *In the* ε*-limit the flow* $\varphi^{\omega,\varepsilon}_{t,s}(x)$ *is convergent in the sense of distributions, that is, there is a family* $\{G_n(x_1,\dots,x_n,t \mid y_1,\dots,y_n,s)dy_1\cdots dy_n\}$, $n=1,2,\dots,$ *of probability densities such that*

$$\begin{aligned}\mathsf{E}\psi(\varphi^{\omega,\varepsilon}_{t,s}(x_1),\dots,\varphi^{\omega,\varepsilon}_{t,s}(x_n)) \to \int\limits_{\mathbb{R}^d\times\cdots\times\mathbb{R}^d} &\psi(y_1,\dots,y_n)\\ \times\, G_n(x_1,\dots,x_n,t \mid y_1,&\dots,y_n,s)\,dy_1\cdots dy_n,\end{aligned} \tag{55}$$

as $\varepsilon \to 0$ *for any continuous function* ψ *with compact support. Furthermore, the* ε*-limit is coalescent in the sense that*

$$G_2(x_1,x_2,t \mid y_1,y_2,s) = \tilde{G}_2(x_1,x_2,t \mid y_1,y_2,s) + A(y_1,x_1,x_2,t,s)\delta(y_1-y_2), \tag{56}$$

with $A>0$ *for* $t>s$. *Here* $\tilde{G}_2$ *is the absolutely continuous part of* G_2 *with respect to Lebesgue measure. Similar statements hold for the other densities* G_n. *In particular, the densitites* G_n *differ from the moments of the* κ*-limit* g_ω *determined in* (54).

12. The "one force–one solution" principle

We now turn to the issue of an invariant measure for a transport equation that is suitably forced. We first restrict our attention to the non-degenerate case. This includes the weakly compressible regime and the intermediate regime in the κ-limit. The non-degeneracy of $g_\omega(\vec{x}, t \mid d\vec{y}, s)$ regarded as a probability distribution with respect to $\vec{y}$ implies a dissipation of energy or, in other words, a decay in the memory of the semigroup $S^\omega_{t,s}$ generated by $\{g_\omega\}$. This is the main reason that the forced transport equation has a unique invariant measure, as we now explain.

We consider (compare with (29))

$$\frac{\partial \theta}{\partial t} + (\vec{u}(\vec{x}, t) \cdot \nabla)\theta = b(\vec{x}, t), \tag{57}$$

where b is a white-noise forcing such that

$$\mathsf{E} b(\vec{x}, t) b(\vec{y}, s) = B(|\vec{x} - \vec{y}|)\delta(t - s). \tag{58}$$

here $B(r)$ is assumed to be smooth and rapidly decaying to zero for $r > L$, and L is referred to as the forcing scale. The solution of (57) for the initial condition $\theta_\omega(\vec{x}, s) = \theta_0(\vec{x})$ is understood as

$$\theta_\omega(\vec{x}, t) = S_{t,s}\theta_0(\vec{x}) + \int_s^t S_{t,\tau} b(\vec{x}, \tau)\, d\tau. \tag{59}$$

Let us define the product probability space $(\Omega_u \times \Omega_b, \mathcal{F}_u \times \mathcal{F}_b, \mathsf{P}_u \times \mathsf{P}_b)$, and the shift operator

$$T_\tau \omega(t) = \omega(t + \tau), \tag{60}$$

where $\omega = (\omega_u, \omega_b)$.

Theorem 13 (one force–one solution, I). *For almost all ω there is a unique solution of (57) defined on $\mathbb{R}^d \times (-\infty, \infty)$. This solution can be expressed as*

$$\theta^\star_\omega(\vec{x}, t) = \int_{-\infty}^t S_{t,s} b(\vec{x}, s)\, ds. \tag{61}$$

Furthermore, the map $\omega \to \theta^\star_\omega$ satisfies the invariance property

$$\theta^\star_{T_\tau \omega}(\vec{x}, t) = \theta^\star_\omega(\vec{x}, t + \tau). \tag{62}$$

Theorem 13 is the "one force–one solution" principle; it was articulated in [19]. Because of the invariance property (62), the map in (61) leads to a natural invariant measure. As a consequence we have the following result.

Theorem 14. *There is a unique invariant measure on $L^2_{\mathrm{loc}}(\mathbb{R}^d \times \Omega)$ for the dynamics defined by* (57).

The connection between the map (61) and the invariant measure, together with uniqueness, is explained in [19].

We sketch the proof of Theorem 13. Basically, it amounts to verifying that the dissipation and the loss of memory in the system are strong enough in the sense that

$$\mathsf{E}\Big(\int_{T_1}^{T_2}\int_{\mathbb{R}^d} b(\vec{y},s)g(\vec{x},0 \mid d\vec{y},s)ds\Big)^2 \to 0 \tag{63}$$

as T_1, $T_2 \to -\infty$ for fixed $\vec{x}$ and t. The average in (63) is given explicitly by

$$\int_{T_1}^{T_2}\int_0^\infty B(r)P(0 \mid r,s)\,dr\,ds. \tag{64}$$

Here $P(\rho \mid r,s)$ is defined for all test functions η as

$$\int_0^\infty \eta(r)P(\rho \mid r,s-t)\,dr = \int_{\mathbb{R}^d\times\mathbb{R}^d} \eta(|x-x'|)\mathsf{E}\big(g_\omega(y,t \mid x,s)g_\omega(z,t \mid x',s)\big)\,dx\,dx' \tag{65}$$

in the non-degenerate case, and as

$$\int_0^\infty \eta(r)P(\rho \mid r,s-t)\,dr = \mathsf{E}\eta(|\varphi_{t,s}(y)-\varphi_{t,s}(z)|) \tag{66}$$

in the coalescent case, where $\rho = |y-z|$ and $s<t$, and $P(\rho \mid r,s)$ can be thought of as the probability density that two particles are at distance r at time $s<t$ if their final distance at time t is equal to ρ. For the Kraichnan model, P satisfies the backward equation

$$-\frac{\partial P}{\partial s} = -\frac{\partial}{\partial r}(b(r)P) + \frac{\partial^2}{\partial r^2}(a(r)P) \tag{67}$$

with the initial condition $\lim_{s\to 0-} P(\rho|r,s) = \delta(r-\rho)$, and with $a(r)$ and $b(r)$ such that

$$\begin{aligned} a(r) &= D(\mathcal{S}^2+\xi\mathcal{C}^2)r^\xi + O(r^2/\ell_0^2),\\ b(r) &= D((d-1+\xi)\mathcal{S}^2-\xi\mathcal{C}^2)r^{\xi-1} + O(r/\ell_0^2). \end{aligned} \tag{68}$$

For $r \gg \ell_0$, $a(r)$ tends to C_0, $b(r)$ tends to $C_0(d-1)/r$, and the equation in (67) reduces to a diffusion equation with constant coefficient. The equation in (67) is singular at $r=0$. The convergence of the integral in (63) depends on the rate of decay of $P(0 \mid r,s)$ with respect to $|s|$. Since the integral with respect to r in (64) has a cut-off at the forcing scale L due to $B(r)$, we can restrict our attention to the behaviour of $P(0 \mid r,s)$ for $r<L$. For r not too large, it follows from (67) that P can be approximated as follows:

$$P = C(s)r^{d-1} + o(r^{d-1}), \tag{69}$$

where $C(s)$ is yet to be determined. The range of values of r for which the approximation in (69) is valid increases with $|s|$. For $|s|$ large enough, most of the mass of $P(0 \mid r,s)$ is in the range $r \gg \ell_0$, where P satisfies the diffusion equation with constant diffusion coefficient C_0, whose exact solution is known. A standard matching argument between this solution and the approximation in (69) can be used to estimate $C(s)$. This gives

$$P(0|r,s) = \frac{Cr^{d-1}}{|s|^{d/2}} + o(r^{d-1}). \tag{70}$$

Using (70), we obtain the following leading-order estimate for the average in (63):

$$C\int_0^\infty B(r)r^{d-1}\,dr\int_{T_1}^{T_2}|s|^{-d/2}\,ds. \tag{71}$$

The integral with respect to s in this expression tends to zero as T_1, $T_2 \to -\infty$ for $d > 2$. It follows that an invariant measure in (61) exists if $d > 2$. The dimension restriction can be relaxed by confining attention to a forcing such that $\int_0^\infty B(r)r^{d-1}\,dr = 0$, as can readily be shown by computing the next-order term in the expansion in (70).

Let us consider now the coalescent case, that is, the strongly compressible regime and the intermediate regime in the ε-limit. Since no anomalous dissipation is present in this case, it follows that no invariant measure like the one in (61) can exist for the temperature field. However, it makes sense to ask about the existence of an invariant measure for the temperature difference, that is, to consider

$$\delta\theta^\star_\omega(\vec{x},\vec{y},t) = \int_{-\infty}^t S_{t,s}(b(\vec{x},s) - b(\vec{y},s))\,ds. \tag{72}$$

If $\theta^\star_\omega$ exists, then one has $\delta\theta^\star_\omega(\vec{x},\vec{y},t) = \theta^\star_\omega(\vec{x},t) - \theta^\star_\omega(\vec{y},t)$, but it is conceivable that $\delta\theta^\star_\omega$ exists in the coalescent case even though $\theta^\star_\omega$ is not defined. The reason is that coalescence of the generalized flow implies that the temperature field flattens with time, which is a dissipation mechanism as far as the temperature difference is concerned. Of course, this effect has to overcome the fluctuations produced by the forcing, and the existence of an invariant measure such as (72) will depend on how fast particles are coalesced under the flow.

At this point a difficulty arises. If we were to consider two particles separated by a much larger distance than the correlation length ℓ_0, then the dynamics of their distance under the flow is governed by the equation (67) with $\eta \approx C_0$, that is, by a diffusion equation with constant diffusion coefficient on the scale of interest. It follows that no tendency to coalesce is observed before the distance becomes smaller than ℓ_0, which, as shown below, does not happen fast enough to overcome the fluctuations produced by the forcing. In other words, we have the following.

Lemma 5. *In the coalescent case and for finite ℓ_0 there is no invariant measure with finite energy for the temperature difference.*

The obvious question to ask next is: What happens if we let $\ell_0 \to \infty$? However, this question must be treated carefully because the velocity field with covariance in (37) diverges as $\ell_0 \to \infty$. The right way to proceed is to consider an alternative velocity $\vec{v}$, taken to be Gaussian and white-in-time but *non-homogeneous*, with covariance

$$\mathsf{E}v_\alpha(\vec{x},t)v_\beta(\vec{y},s) = \big(c_{\alpha\beta}(\vec{x}) + c_{\alpha\beta}(\vec{y}) - c_{\alpha\beta}(\vec{x}-\vec{y})\big)\delta(t-s). \tag{73}$$

For finite ℓ_0 one has $\vec{v}(\vec{x},t) = \vec{u}(\vec{x},t) - \vec{u}(\vec{a},t)$, where $\vec{a}$ is arbitrary but fixed. However, $\vec{v}$ makes sense in the limit as $\ell_0 \to \infty$. By $\vartheta_\omega(\vec{x},t)$ we denote the temperature field advected by $\vec{v}$, that is, the solution of the transport equation (57) with $\vec{u}$

replaced by $\vec{v}$. We restrict ourselves to the zero initial condition; then it follows from the homogeneity of the forcing that the single-time moments of θ_ω and ϑ_ω coincide for finite ℓ_0 but, in contrast to θ_ω, ϑ_ω makes sense as $\ell_0 \to \infty$. Thus, ϑ_ω is a natural process for studying the limit as $\ell_0 \to \infty$, and from now on we restrict our attention to this case. Let $\delta\vartheta_\omega(\vec{x},\vec{y},t) = \vartheta_\omega(\vec{x},t) - \vartheta_\omega(\vec{y},t)$. The temperature difference $\delta\vartheta_\omega$ satisfies the transport equation

$$\frac{\partial \delta\vartheta}{\partial t} + \big(\vec{v}(\vec{x},t)\cdot \nabla_x + \vec{v}(\vec{y},t)\cdot \nabla_y\big)\delta\vartheta = b(\vec{x},t) - b(\vec{y},t). \tag{74}$$

Theorem 15 (one force–one solution, II). *For almost all ω there exists in the strongly and the weakly compressible regimes, as well as in the intermediate regime if the flow is non-degenerate, a unique solution of* (74) *defined on $\mathbb{R}^d \times (-\infty,\infty)$. This solution can be expressed as*

$$\delta\vartheta^\star_\omega(\vec{x},\vec{y},t) = \int_{-\infty}^{t} S_{t,s}(b(\vec{x},s) - b(\vec{y},s))\, ds. \tag{75}$$

Furthermore, the map $\omega \to \delta\vartheta^\star_\omega$ satisfies the invariance property

$$\delta\vartheta^\star_{T_\tau\omega}(\vec{x},\vec{y},t) = \delta\vartheta^\star_\omega(\vec{x},\vec{y},t+\tau). \tag{76}$$

In contrast, there is no solution like (75) *with finite covariance in the intermediate regime if the flow is coalescent* (ε*-limit*).

This theorem has an immediate consequence.

Theorem 16. *In the strongly and weakly compressible regimes, as well as in the intermediate regime if the flow is non-degenerate, there exists a unique invariant measure on $L^2_{\mathrm{loc}}(\mathbb{R}^d \times \Omega)$ for the dynamics determined by* (74). *In the intermediate regime there is no invariant measure for the equation in* (74) *with finite energy if the generalized flow is coalescent.*

In regimes for which the generalized flow is non-degenerate, Theorem 15 follows from Theorem 13. In the coalescent cases one proceeds as in the proof of Theorem 13 and studies the convergence as T_1, $T_2 \to -\infty$ of

$$\mathsf{E}\bigg(\int_{T_1}^{T_2} (b(\varphi^\omega_{t,s}(\vec{x},s) - b(\varphi^\omega_{t,s}(\vec{y},s))\, ds\bigg)^2. \tag{77}$$

The average in (63) is given explicitly by

$$2\int_{T_1}^{T_2}\int_0^\infty (B(0) - B(\rho))P(r \mid \rho, s)\, dr\, ds, \tag{78}$$

where $r = |\vec{x}-\vec{y}|$, and P satisfies (67) with $\ell_0 = \infty$. Since $r = 0$ is an exit boundary in the coalescent case, it follows that P loses mass at $r = 0+$. The convergence of the time integral in (78) depends on the rate at which mass is lost (that is, the rate at which particles are coalescent). Analysis of the equation (67) shows that the process is fast enough, and thus the integral over s in (78) tends to zero as T_1, $T_2 \to -\infty$ in the strongly compressible regime. In contrast, the integral is divergent in the weakly compressible regime in the coalescent case.

Acknowledgments. We are grateful to our colleagues, M. Avellaneda, K. Khanin, J. Mattingly, A. Mazel, and E. Vanden Eijnden for their contributions to the work surveyed here. We thank U. Frisch and M. Molchan for suggestions on improving the first draft of this paper. The first author was partially supported by a Presidential Faculty Fellowship from the NSF. The second author was partially supported by NSF grant DMS-97067994 and RFBR grant 99-01-00314.

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Received 3/APR/00

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX