Another note on forced burgers turbulence*

Weinan E[a] and Eric Vanden Eijnden[b]

Courant Institute of Mathematical Sciences, New York University, New York, New York 10012

(Received 25 March 1999; accepted 7 September 1999)

The power law range for the velocity gradient probability density function in forced Burgers turbulence has been an issue of intense discussion recently. It is shown in E and Vanden Eijnden, Phys. Rev. Lett. 83, 2572 (1999) that the negative exponent in the assumed power law range has to be strictly larger than 3. Here we give another direct argument for that result, working with finite viscosity. At the same time we discuss viscous corrections to the power law range. This should answer the questions raised in Kraichnan, Phys. Fluids 11, 3738 (1999) regarding the results of E and Vanden Eijnden, Phys. Rev. Lett. 83 2572 (1999).

The expression for \( F^* \) satisfies

\[
F^* = \xi Q^* + (\xi^2 Q^*) \xi + B_1 Q_{\xi \xi}^* + F'^*.
\]

We emphasize that there is an important distinction between the strict inequality \( \alpha > 3 \) and the bound \( \alpha \geq 3 \) advanced in Ref. 2: (6) rules out all the predictions in the literature (including those of Refs. 3–5) except that of Ref. 6 with \( \alpha = 7/2 \). As is discussed in Ref. 2, \( \alpha = 3 \) and \( \alpha > 3 \) imply qualitatively different pictures concerning the contribution of the \( F^r \) term in (4) in the viscous range.

In Ref. 1, (6) was derived from (5). Here we will work directly with (4). Consider statistical steady state \( (Q^r)' = 0 \). For \( \xi \ll -\xi_0(-B_1^{1/3}) \), the \( B_1 \) term in (4) can be neglected giving

\[
\xi Q^r + (\xi^2 Q^r) \xi + F'^r = 0,
\]

or, equivalently,

\[
(\xi^3 Q^r) \xi = -F'^r.
\]

Because of the exponential decay in the viscous range, we get from (8)

\[
Q^r \sim |\xi|^{-3} \int_{-\infty}^{\xi} d\xi' \xi' F'^r(\xi') \quad \text{for} \quad \xi \ll -\xi_0.
\]

Using the same analysis as in Ref. 1, for small \( \nu \) we can obtain an explicit expression for \( Q^r \) from (9). The calculation is performed in Appendix A and gives

\[
Q^r \sim 2\rho|\xi|^{-3} \int_{-\infty}^{\xi} d\xi' \int_{-s}^{s} ds A(s, \xi - \xi') V(s, \xi')
\]

\[
-2\nu\rho|\xi|^{-2} \int_{-\infty}^{\xi} d\xi' \int_{-s}^{s} ds \frac{(\xi - \xi')}{A(s, \xi - \xi')} V(s, \xi')
\]

for \( \xi \ll -\xi_0 \),

where

\[
A(s, \xi) = (\frac{1}{8} s^2 + 2\nu \xi)^{1/2},
\]

\( \rho \) is the number density of shocks, \( s = -2(2 \nu (\xi' - \xi))^{1/2} \), \( V(s, \xi) = (V_-(s, \xi) + V_+(s, \xi))/2 \), \( V_+(s, \xi) \) are the PDFs of \( s(y, t) \) (shock amplitude, \( s \leq 0 \)), and \( \xi_\pm(y, t) \) (gradients at the left and right of the shock in the inviscid limit), conditional on the property that there is a shock at \( x = y \).

Without further information about \( V(s, \xi) \) it is difficult to carry out asymptotics on (10), and we shall not dwell on this problem [see however (A19)–(A22)]. What is easier and

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*aThis is a companion paper to R. H. Kraichnan, Phys. Fluids 11, 3738 (1999).

[a]Electronic mail: weinan@cims.nyu.edu

[b]Electronic mail: eve2@cims.nyu.edu
more instructive is to actually take the limit as $\nu \to 0$ for fixed $\xi$ in (10). Then, only the first term at the right hand side of (10) survives, and $Q^\nu$ converges to

$$Q(\xi) \sim \xi^{-3} \int_{\xi}^{+\infty} d\xi' \xi' F(\xi') \quad \text{for} \quad \xi \ll -\xi_0,$$

with

$$F(\xi) = \lim_{\nu \to 0} F^\nu(\xi) = \rho \int_{-\infty}^{0} ds \, sV(s, \xi).$$

At this stage, we use the fact that

$$A = \int_{-\infty}^{+\infty} d\xi \, \xi F = \frac{\rho}{2} ((s\xi_-) + (s\xi_+)) = 0.$$  

This is a steady state consequence of

$$\frac{d}{dt}(\rho(s)) = -\frac{\rho}{2} ((s\xi_-) + (s\xi_+)),$$

which is proven in Appendix B. As a result of (14), (12) may be rewritten as

$$Q(\xi) \sim |\xi|^{-3} \int_{-\infty}^{\xi} d\xi' \xi' F(\xi') \quad \text{for} \quad \xi \ll -\xi_0.$$  

(6) then follows because $\int_{-\infty}^{\xi} d\xi' \xi' F(\xi') \to 0$ as $\xi \to -\infty$. We stress that this argument, and in particular the use of (15) at steady state, does not imply that $Q \sim A |\xi|^{-3}$ at transient states because the whole analysis here rests on (9), which is only valid at statistical steady state. In fact, as we will now show, (6) also holds for transient states.\textsuperscript{7} Again in Ref. 1 this argument was presented for $Q$. Here we reinterpret this using $Q^\nu$ for small $\nu$.

Multiplying (4) by $\xi$ and integrating from $\xi_+ \to +\infty$ (for fixed $\xi_+ \ll -\xi_0$), we get

$$\int_{\xi_+}^{+\infty} d\xi \, \xi Q^\nu = -\xi_+^3 Q^\nu(\xi_+, t) + \int_{\xi_+}^{+\infty} d\xi \, \xi F^\nu.$$  

Here we neglected contribution from the $B_1$ term since it is small compared with the remaining terms. Taking the limit as $\nu \to 0$, we get:

$$\int_{\xi_+}^{+\infty} d\xi \, \xi Q = -\xi_+^3 Q(\xi_+, t) + \int_{\xi_+}^{+\infty} d\xi \, \xi F.$$  

Therefore

$$\lim_{\nu \to 0} \xi_+^3 Q(\xi_+, t) = \int_{-\infty}^{+\infty} d\xi \, \xi F - \frac{d}{dt} \int_{-\infty}^{+\infty} d\xi \, \xi Q.$$  

Notice that even though by homogeneity

$$\langle \xi \rangle_\nu = \int_{-\infty}^{+\infty} d\xi \, \xi Q^\nu = 0,$$

in the limit as $\nu \to 0$,

$$\langle \xi \rangle = \int_{-\infty}^{+\infty} d\xi \, \xi Q = -\rho(s) \neq 0.$$  

In other words, a finite amount of $\xi = u_x$ has gone to the shocks in the limit as $\nu \to 0$. Hence, using (21) at the right-hand side of (19), this equation becomes

$$\lim_{\nu \to 0} \xi_+^3 Q(\xi_+, t) = \frac{\rho}{2} ((s\xi_-) + (s\xi_+)) - \frac{d}{dt} \langle \xi \rangle = \frac{\rho}{2} ((s\xi_-) + (s\xi_+)) + \frac{d}{dt}(\rho(s)) = 0.$$  

The last equality follows from (15).

One main message in Ref. 2 is the claim that the argument in Ref. 1 which led to the strict bound (6) is insufficient. Here we paraphrase the argument which led to this claim. One may always write (9) as

$$|\xi|^3 Q^\nu(\xi) = \int_{-\infty}^{E_M} d\xi' \xi' F^\nu(\xi') + \int_{E_M}^{E} d\xi' \xi' F^\nu(\xi'),$$

where $E_M$ is defined as the value at which the $-\alpha$ range is masked by the viscous range. Assuming the latter behaves as $\nu C|\xi|^{-1}$ [see Refs. 2 and (A20)] then $E_M$ is determined from solving

$$C \nu |\xi|^{-\alpha} = \nu C|\xi|^{-1},$$  

which gives $E_M = -C_0 \nu^{-1/(\alpha - 1)}$ with $C_0 = (C/C_\infty)^{-1/(\alpha - 1)}$. In the limit as $\nu \to 0$, the second term at the right-hand side of (23) gives

$$\lim_{\nu \to 0} \int_{E_M}^{E} d\xi' \xi' F^\nu(\xi') = \int_{-\infty}^{E_M} d\xi' \xi' F^\nu(\xi'),$$

which goes to 0 as $\xi \to -\infty$. Therefore, whether $\alpha > 3$ depends on whether

$$\lim_{\nu \to 0} \int_{E_M}^{E} d\xi' \xi' F^\nu(\xi') = 0,$$

holds. Since

$$E_M^\nu(\xi_M) = \int_{-\infty}^{E_M} d\xi' \xi' F^\nu(\xi'),$$

an equivalent form of (26) is

$$\lim_{\nu \to 0} \xi_M^3 Q^\nu(\xi_M) = 0.$$  

This argument gives another way of appreciating the difference between $\alpha = 3$ and $\alpha > 3$. However, it does not allow one to address the issue of whether $\alpha = 3$, for the simple reason that the validity of (26) and (28) depends sensitively on the value of $\xi_M$. which cannot be known prior to knowing $\alpha$. Unlike what is claimed in Ref. 2, $\xi_M$ cannot be an arbitrary choice that satisfies $\xi_0 + \xi_M \to 0$, $\nu \xi_M \to 0$ as $\nu \to 0$. For example if we choose $\xi_M = \xi_N = -C_1 \nu^{-1/2}$, from (A20)

$$\lim_{\nu \to 0} \xi_N^3 Q^\nu(\xi_N) = C C_1^3 \neq 0,$$

regardless of the value of $\alpha$. On the other hand, it is easy to see that if $\xi_M = o(\nu^{-1/2})$, then (26) and (28) hold. The important technical point in Ref. 1 is to find ways to circumvent this.
path. That was done by studying directly the inviscid limit of $Q''$. Here we have presented a direct argument based on the integral expression for $Q''$. As a by-product, we have

$$\xi_M = o(\nu^{-1/2}). \quad (30)$$

It is also worth stressing that even though (9) and (16) look similar, they are not equivalent. (9) is meant for the case of finite $\nu$ and is derived using straightforward integration from (7). (16) is valid for the limiting PDF $Q$ and its derivation is much more nontrivial. In Appendix C, we show that (16) can be derived directly from (5) using the realizability constraint over the whole $\xi$ line, as well as the additional information provided by (15). It shows that the only realizable steady state solutions of (5) has the form (16). Other solutions violate non-negativity either for $\xi \to -\infty$ or for $\xi \to +\infty$. The argument in Ref. 1 was a global argument, not localized at very large negative values of $\xi$.

The characterization that `the analysis in (Ref. 1) is done in terms of a split of $u$ and $\xi$ into a part exterior to shocks and a part interior to shocks’ also needs more clarification. What was done in Ref. 1 was a derivation of an approximation to $\nu \xi_{xx}$ (or $\nu \xi_x^2$) using boundary layer analysis and matched asymptotics in order to evaluate the limit of $F''$ as $\nu \to 0$. The same technique was used here to evaluate directly the limit of $Q''$ (for large negative $\xi$ as $\nu \to 0$). This approximation is uniformly valid except at shock creation and collision whose contributions to $F''$ is of lower order. It can also be systematically improved if additional information is required.

Going back to the statistical stationary state, what is the actual value of $\alpha$? Reference 6 predicted that $\alpha = 7/2$ under the assumption that the main contribution to $Q$ for large negative values of $\xi$ comes from neighborhoods of shock creation points (preshocks). This geometric argument was expanded in Ref. 1 in the context of (5) and in particular the form of $F$: under the geometric argument $F(\xi) \sim C |\xi|^{-5/2}$ with $C < 0$ for $\xi \ll -\xi_0$. Reference 2 further expanded the geometric argument and obtained values of $\alpha$ in (3, 7/2) by considering special singular data. Polyakov gave an example that gives $\alpha = 3$. However these are rather pathological situations that lie outside the regime of interest here, i.e., the case of smooth forcing. By studying the master equation for the environment of shocks, Ref. 9 verifies that indeed the main contribution does come from shock creation points, and thereby confirms $\alpha = 7/2$.

In conclusion, we stress that there are many ways to exclude the possibility of having $|\xi|^{-3}$ behavior for the left tail of $Q$ in the case of smooth Gaussian force. The discussion in Ref. 2 provides yet another way of understanding the different consequences of $\alpha = 3$ and $\alpha > 3$, but it is not the right way to address the issue of whether $\alpha = 3$.

ACKNOWLEDGMENTS

We benefitted from our frequent e-mail exchanges with R. H. Kraichnan and A. M. Polyakov. The work of W. E. is supported by a Presidential Faculty Fellowship from the National Science Foundation. The work of E. Vanden Eijnden is supported by U.S. Department of Energy Grant No. DE-FG02-86ER-53223.

APPENDIX A: EVALUATION OF $Q''$

In this appendix we perform the computation of $Q''$ to $O(\nu)$. We first put (9) in a form which is more convenient for the calculation. Notice that

$$F''(\xi,t) = G''_{\xi}(\xi,t),$$

(A1)

where

$$G''(\xi,t) = -\nu(\xi_x^2(t) \delta(\xi - \xi(x,t))).$$

(A2)

Thus, from (9),

$$Q'' \sim -|\xi|^{-3}G'' - |\xi|^{-2}G''_{\xi} \quad \text{for } \xi \ll -\xi_0. \quad (A3)$$

We now evaluate $G''$. For statistically homogeneous situations, the averages at the right-hand side of (A2) can be evaluated upon resorting to spatial ergodicity to replace the ensemble average by the space average:

$$G''(\xi,t) = -\nu \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} dx \xi_x^2(\xi - \xi(x,t)).$$

(A4)

In the limit of small $\nu$ only small intervals around the shocks will contribute to this integral. In these layers, we use boundary layer analysis to evaluate $\xi(x,t)$. This analysis was outlined in Ref. 1 (for details see Ref. 9): Let $y$ be a shock position. Near $y$, $u$ can be expressed as

$$u(x,t) = u^m(x,t) = \sqrt{\frac{x - y}{\nu}}, \quad (A5)$$

and the expression for $v(z,t)$ can be obtained via a series expansion in $\nu$. It yields $v = v_0 + \nu v_1 + O(\nu^2)$, with

$$v_0(z,t) = \bar{u} - \frac{s}{2} \tanh \left( \frac{sz}{4} \right). \quad (A6)$$

$V_1$ is a solution of

$$v_0 + (v_0 - \bar{u}) v_1 + v_1 v_0 = v_1 + f. \quad (A7)$$

The actual expression of $v_1$ is rather complicated, and the only information really needed about $V_1$ to evaluate (A4) is the values of $V_1$ as $z \to \pm \infty$. Let $\lim_{z \to \pm \infty} v_1 = \xi \pm$. Then

$$\xi_\pm = \pm \frac{2\bar{u}_t}{s} \frac{s_t}{s} \pm \frac{2f}{s}, \quad (A8)$$

or, equivalently,

$$s_t = -\frac{s}{2}(\xi_\pm + \xi_\pm), \quad \bar{u}_t = \frac{s}{4}(\xi_\pm - \xi_\pm) + f. \quad (A9)$$

In these expressions, the values of $s$ and $\bar{u} = dy/dt$ must be obtained by matching $u^m(x,t)$ with the solution of the Burgers equation outside the shock layer, say, $u^{m0}(x,t)$: this eventually produces an approximation for $u(x,t)$ uniformly valid except at shock creation and collision.

Using the results of the boundary layer analysis, to $O(\nu)$ (A4) can be estimated as $E_{\xi}(u^m)$.
\[
G'(\xi,t) = - \nu \lim_{L \to \infty} \frac{1}{N} \sum_{j} \int_{\text{layer}} ds \left( \xi_{s}^{2} \delta(\xi - \xi_{n}(x,t)) \right),
\]

or, picking any particular shock layer, going to the stretched variable \( z = (x - y)/\nu \), and taking the limit as \( L \to \infty \),

\[
G'(\xi,t) = - \rho \int_{-\infty}^{+\infty} ds \, d\xi \, d\xi \cdot T(\bar{u},s,\xi_{-},\xi_{+},s,t)
\times \int_{-\infty}^{+\infty} dz \, \eta_{s}^{2} \delta(\xi - \eta(z,t)),
\]

(A10)

where we defined \( \eta = \nu^{-1} \bar{u}_{0} + \bar{v}_{1} = \nu^{-1} \bar{u}_{0} + \xi_{z} \). Here \( T(\bar{u},s,\xi_{-},\xi_{+},t) \) is the PDF of \( \bar{u},s,\xi_{-},\xi_{+} \) conditional on the existence of a shock at \( x = y \); it arises because \( \eta(z,t) \) depends parametrically on these (random) variables. To perform the \( z \) integral in (A11), we change the integration variable to \( \xi' = \eta - \xi_{z} \) for \( z < 0 \) and to \( \xi' = \eta - \xi_{z} \) for \( z > 0 \). Then, using the equation for \( v_{0} - \bar{u} = v_{0} + \bar{v}_{1} \), as well as (A6), we have for \( z < 0 \),

\[
\eta_{z} = ((v_{0} - \bar{u})(\eta - \xi_{z})) = \left( \frac{1}{2} z^{2} + 2\nu(\eta - \xi_{z}) \right) \frac{1}{2}(\eta - \xi_{z}).
\]

(A12)

and for \( z > 0 \),

\[
\eta_{z} = ((v_{0} - \bar{u})(\eta - \xi_{z})) = -\left( \frac{1}{2} z^{2} + 2\nu(\eta - \xi_{z}) \right) \frac{1}{2}(\eta - \xi_{z}).
\]

(A13)

Thus

\[
d\eta_{z} = \left\{ \begin{array}{ll}
\int_{-\infty}^{+\infty} d\eta \, \delta(\eta - \eta(z,t)) & \text{for } z < 0 \\
-\int_{-\infty}^{+\infty} d\eta \, \delta(\eta - \eta(z,t)) & \text{for } z > 0.
\end{array} \right.
\]

(A14)

Also, \( \xi' \in [-s^{2}/8\nu,0] \) as \( z \in [-\infty,0] \) or \( z \in [0,\infty] \). Combining these results gives

\[
\int_{-\infty}^{+\infty} d\eta \, \delta(\eta - \eta(z,t))
\]

\[
= \int_{-s^{2}/8\nu}^{0} d\xi \, \xi' \left( \frac{1}{2} s^{2} + 2\nu \xi' \right) \frac{1}{2} \delta(\xi - \xi' - \xi_{-})
\]

\[
+ \int_{r^{2}/8\nu}^{0} d\xi \, \xi' \left( \frac{1}{2} s^{2} + 2\nu \xi' \right) \frac{1}{2} \delta(\xi - \xi' - \xi_{+})
\]

\[
= (\xi - \xi_{-}) \left( \frac{1}{2} s^{2} + 2\nu(\xi - \xi_{-}) \right) \frac{1}{2} \times (H(\xi - \xi_{-})
\]

\[
- H(\xi_{-} - s^{2}/8\nu - \xi_{+}) + (\xi - \xi_{+}) \left( \frac{1}{2} s^{2} + 2\nu(\xi' - \xi_{+}) \right) \frac{1}{2}
\]

\[
\times (H(\xi_{+} - \xi) - H(\xi_{+} - s^{2}/8\nu - \xi)),
\]

(A15)

where \( H(x) \) is the Heaviside function. Inserting this expression in (A11) then results in

\[
G'(\xi,t) = 2\rho \int_{-\infty}^{+\infty} d\xi' \int_{-\infty}^{+\infty} ds(\xi - \xi') A(s,\xi,\xi') V(s,\xi',t),
\]

(A16)

where \( s_{\pm} = -2(2\nu(\xi' - \xi) \left( \frac{1}{2} s^{2} + 2\nu(\xi - \xi_{-}) \right) \frac{1}{2}, A(s,\xi) \) was given in (11) and we used the consistency constraint

\[
\int_{-\infty}^{+\infty} d\nu d\xi \nu \delta(\nu(t - t_{1})).
\]

(A17)

We also have

\[
G'(\xi,t) = 2\rho \int_{-\infty}^{+\infty} d\xi' \int_{-\infty}^{+\infty} ds \frac{\xi - \xi'}{A(s,\xi,\xi')} V(s,\xi',t).
\]

(A18)

At statistical steady state, \( V(s,\xi,t) = V(s,\xi) \), and \( G'(\xi,t) = G'(\xi) \). Then inserting (A16), (A18) in (A3) gives (10).

Note that if one neglects the \( O(1) \) term in the expansion for \( F(\xi) \) (an assumption we do not make), then \( V(s,\xi,t) = S(s,t) \delta(\xi) \), where \( S(s,t) \) is the conditional PDF of \( s(y,t) \). At statistical steady state this is \( V(s,\xi) = S(s) \delta(\xi) \) and (10) reduces to

\[
Q^r \sim 2\rho |s|^{-1} \left( \frac{2(2\nu |s|)}{(\nu^{2} + 2\nu)^{1/2}} \right).
\]

(A19)

This is the expression obtained by Gotoh and Kraichnan for \( Q^r \) in the viscous range. (A19) is hard to justify since it amounts to assessing the accuracy of the approximation \( V(s,\xi,t) \approx S(s,t) \delta(\xi) \) for large negative \( \xi \). Granting (A19), \( Q^r \) may be further simplified if one assumes that the first inverse moment of the shock amplitude \( s \) is finite, i.e.,

\[
|s|^{-1} = \int_{-\infty}^{0} ds \delta(s) = \int_{-\infty}^{0} ds \delta(s)
\]

(A20)

\[
C = 4\rho |s|^{-1}.
\]

(A21)

(A20) describes the well-known \( |x|^{-1} \) viscous range (the \(-1 \) range). This expression can be combined with (16) to give

\[
Q^r \sim C_{-} |\xi|^{-\alpha} + \nu C |\xi|^{-1},
\]

(A22)

on the range \( \xi = o(\nu^{-1}) \). Here \( C_{-} = |\xi|^{-\alpha} = |\xi|^{-\alpha} \int_{-\infty}^{\xi} d\xi' F(\xi') \).

**APPENDIX B: DERIVATION OF (14)**

To get (15) we need to evaluate the time derivative of (using ergodicity)

\[
\rho(s) = \lim_{L \to \infty} \frac{1}{2L} \sum_{i=1}^{N} s(y_{j},t).
\]

(B1)

where \( N \) is the number of shocks in \([-L,L] \) and the \( y_{j} \)'s are their locations. Using (A9), clearly the time derivative of (B1) will give (15) if the time dependence of \( N \) does not make any contribution. \( N \) varies due to shock creation or shock collision. Consider the creation first, and assume a shock is created at position \( y_{1} \) at time \( t_{1} \). Then one has in (B1) a term like (disregarding the factor \( 1/2L \))

\[
T_{1} = s(y_{1},t)H(t-t_{1}).
\]

(B2)
where $H(x)$ is the Heaviside function. Time differentiation of (B2) gives
\[ \frac{dT_1}{dt} = \frac{d}{dt} s(y_1,t)H(t-t_1) + s(y_1,t)\delta(t-t_1). \] (B3)

The second term accounts for the time dependence of $N$. Since the shock amplitude is zero at creation, $s(y_1,t)\delta(t-t_1) = s(y_1,t_1)\delta(t-t_1) = 0$. This means that the term $dN/dt$ makes no contribution to the time derivative of (B1) at shock creation. Consider now the merging events. Assume the shocks located at position $y_2$ and $y_3$ merge into one shock located at position $y_1$ at time $t_1$. By definition $y_1(t_1) = y_2(t_1) = y_3(t_1)$. Such an event contributes in (B1) by a term like
\[ T_2 = s(y_1,t)H(t-t_1) + (s(y_2,t) + s(y_3,t))H(t_1-t). \] (B4)

Time differentiation of (B4) gives
\[ \frac{dT_2}{dt} = \frac{d}{dt} s(y_1,t)H(t-t_1) + \frac{d}{dt} (s(y_2,t) + s(y_3,t))H(t_1-t) \]
\[ + s(y_1,t)\delta(t-t_1) - (s(y_2,t) + s(y_3,t))\delta(t-t_1). \] (B5)

Since shock amplitudes add up at collision:
\[ \lim_{t\to 0^+} s(y_1(t_1+t),t_1+t) = \lim_{t\to 0^+} (s(y_2(t_1-t),t_1-t) + s(y_3(t_1-t),t_1-t)). \] (B6)

Thus the terms in (B5) involving $\delta$ functions vanish. This means that the term $dN/dt$ makes no contribution to the time derivative of (B1) at shock collision. Hence (15).

**APPENDIX C: GLOBAL REALIZABILITY CONSTRAINTS**

Here we study (5) at steady state
\[ 0 = \xi Q + (\xi^2)Q \xi + B_1 Q \xi + F. \] (C1)

We will show that the only non-negative solution of (C1) is
\[ Q_j(\xi) = \frac{1}{B_1} \left[ \int_{-\infty}^{\xi} d\xi' \xi' F(\xi') - \frac{\xi e^{-A}}{B_1} \int_{-\infty}^{\xi} d\xi' e^{A'} G(\xi'), \right] \] (C2)

where $A = \xi^3/3B_1$ and
\[ G(\xi) = F(\xi) + \frac{\xi}{B_1} \int_{-\infty}^{\xi} d\xi' \xi' F(\xi'). \] (C3)

We will also show that for large positive $\xi$, $F$ decays in such a way that
\[ \lim_{\xi \to +\infty} \xi^{-2} e^{A} F(\xi) = 0, \] (C4)

assuming the limit exists. Note that from (14), the first moment of $F$ exists. Hence $\xi F(\xi)$ is integrable at $-\infty$.

Before proving (C2) and (C4), let us consider some simple consequences. For $\xi < 0$, integrating by parts in (C2) gives
\[ Q_2(\xi) = -\frac{\xi e^{-A}}{B_1} \int_{-\infty}^{\xi} d\xi' e^{A'} F(\xi'). \] (C13)

We now show that the realizability constraint requires that $C_1 = C_2 = 0$. First, one readily checks that $Q_1$ grows unbounded as $\xi \to -\infty$, while $Q_2 \to \xi^{-3} Q_3 = 0$. Hence in order that $Q$ be integrable we must set $C_1 = 0$ and the general solution of (C1) is
\[ Q(\xi) = C_2 Q_2(\xi) + Q_4(\xi). \] (C14)

For large negative $\xi$, this leads to the expansion
\[ Q(\xi) \sim C_2 B_1 |\xi|^{-3} + |\xi|^{-3} \int_{-\infty}^{\xi} d\xi' \xi' F(\xi'). \] (C15)

For large positive $\xi$, we must distinguish two cases. If (C4) does not hold, then [using (C8)]
\[ Q(\xi) \sim -C_2 B_1 \xi^{-3} + \xi^{-3} \int_{\xi}^{+\infty} d\xi' \xi' F(\xi'). \] (C16)

In contrast, if (C4) holds, then
\[ Q(\xi) \sim - C_2 B_1 \xi^{-3} + C_4 \xi e^{-\lambda}. \]  
(C17)

In (C15)–(C17), if nonzero the \( C_2 \) term at the right-hand side will dominate the second term. However, since the \( C_2 \) term has opposite sign as \( \xi \to \pm \infty \), it must be zero, i.e., we must set \( C_2 = 0 \). This proves that \( Q = Q_s \). Furthermore, since the \( F \) term at the right-hand side of (C16) is negative [recall that from (13), \( F \approx 0 \)], this solution must be rejected in order that \( Q \) be non-negative. Thus (C4) must hold.


\(^7\)This is true for smooth forcing. Notice however that for some situations (like, e.g., for piecewise linear forcing, or for zero forcing and piecewise linear initial data) shocks may be created at finite amplitude. In this case, a straightforward generalization of the argument in Appendix C shows that \( A \) is nonzero in (14), hence \( Q - A |\xi|^{-1} \) as \( \xi \to -\infty \).

\(^8\)A. M. Polyakov (private communication).