Effective equations and the inverse cascade theory for Kolmogorov flows

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A detailed study is presented of the inverse cascade process in the Kolmogorov flows in the limit as the forcing wave number goes to infinity. Extensive numerical results reveal that energy is transferred from high wave numbers to low ones through two distinctive stages. At the early stage, the low wave-number spectrum evolves to a universal \( k^{-4} \) decay law. At the same time, enough scales are generated between the forcing scale (small) and the size of the system (large scale) so that there is no longer separation of scales in the system.

In the next stage the flow undergoes a transition to a continuum of scales. This transition proceeds through the formation of a hierarchy of layers of elongated vortices at increasingly larger scales with alternating orientation. To explain these phenomena, effective equations governing the evolution of the large-scale quantities are derived. Strong numerical evidence is presented that even with smooth initial data, the solution to the effective equation develops a \( k^{-4} \) type singularity at a finite time. The effective equation also exhibits a weak instability which suggest that gradients in the direction of the forcing will grow much quicker than gradients in other directions.

I. INTRODUCTION

Toward the end of the 1950s, Kolmogorov proposed to study the two-dimensional incompressible flow with high wave-number forcing as a model to help understand the transition to turbulence and the inverse cascade process in a turbulent flow. These Kolmogorov flows are described by the following equation:

\[
\begin{align*}
\dot{u}_r + (u \cdot \nabla) u + \nabla p &= \nu \Delta u + \nu k^2 \sin(k x_2) \\
\nabla \cdot u &= 0.
\end{align*}
\]

Here \( u \) is the velocity, \( k \) is the forcing wave number, \( u = (u_1, u_2) \). Since then this problem has attracted a lot of attention. Considerable progress has been made on both the experimental and theoretical side. We refer to Obukhov's review article\(^1\) for a discussion on the laboratory realization of the Kolmogorov flows. The theoretical side of this problem has been centered at the dynamical stability viewpoint. Linear stability of the basic Kolmogorov flow

\[
u = [\sin(k x_2), 0]
\]

was studied by Green\(^2\) and Meshlkin and Sinai.\(^3\) The amplitude equations for the weakly nonlinear theory were derived by Nepomnyachtchyi\(^4\) and Sivashinsky.\(^5\) It was found that this problem belongs to a wider class of problems which exhibit large-scale instability. Later stage development of the Kolmogorov flow was extensively studied more recently\(^6\) from a dynamical systems viewpoint. Several scenarios of the transition to chaos and turbulence were identified by resorting to careful numerical simulations.

In this paper, we take a different approach and study Kolmogorov's problem in the asymptotic limit: \( k_f \to \infty \).

Our main interest lies in the inverse cascade process in the Kolmogorov flow. Our extensive numerical results reveal that energy is transferred from high wave numbers to low ones through two distinctive stages. At the early stage, even if we start with smooth initial data with an exponentially decaying energy spectrum, the high wave-number forcing pumps energy into lower wave numbers. As a result, the low wave-number energy spectrum saturates to a universal \( k^{-4} \) algebraic decay law. At the same time, enough scales are generated between the forcing scale and the size of the system so that there is no longer separation of scales in the system. In the next stage the flow undergoes a transition to a continuum of scales. This transition proceeds through the formation of a hierarchy of layers of elongated vortices at increasingly larger scales with alternating orientation. More precisely, if we filter out the forcing wave number (and higher ones), we observe a drastic growth of gradients in the \( x_1 \) direction. (Notice that the overall flow has much larger gradients in the \( x_2 \) direction as a result of the forcing.) This corresponds to the fact that very elongated vortices are formed at the forcing scale, and these vortices split into vortices with smaller aspect ratio and then line up in the transversal direction. The growth of the \( x_1 \) gradients eventually saturates as a result of the non-linearity and settles down at the subharmonics of \( k_f \). This process continues as a cascade since the variations in the \( x_1 \) direction at these subharmonics act as high wave-number forcing to even larger scales. Consequently, larger vortices are formed which line up in the \( x_2 \) direction, and the flow develops gradients in the \( x_2 \) direction at larger scales.

To explain this phenomenon, we derive the effective equations governing the evolution of the large-scale flow quantities. We present strong numerical evidence that the solution of the effective equation with smooth initial data form a singularity at a finite time. Furthermore, the energy
spectrum at the time of singularity formation is precisely \( k^{-4} \). Since we expect that before singularity forms in the solutions of the effective equation, the low wave-number spectrum of the Kolmogorov flow is closely approximated by the spectrum associated with the solutions of the effective equation, this gives a satisfactory explanation for the observed \( k^{-4} \) decaying behavior. The effective equations also exhibit a weak instability which suggest that gradients in the direction of the forcing will grow much quicker than gradients in other directions.

The difference between the present work and previous work of Sulem et al.\(^8\) and Platt et al.\(^6\) is that we study systems with significantly more spatial degrees of freedom at early and intermediate times, whereas they study systems with relatively few degrees of freedom at much longer time scales.

This paper is organized as follows. In the next section we present results of direct numerical simulation of the Kolmogorov flow in the limit as the forcing wave number goes to infinity. In Sec. III, we derive the effective equations for the averaged quantities using multiple scale asymptotics. In Sec. IV, we present our numerical evidence for the singularity formation in the solutions of the effective equations. We conclude this paper in Sec. V by commenting on the relevance of our findings for more general flows. We will use bold-faced letters to denote vectors. Components of a vector will be indicated using subscripts.

II. THE INVERSE CASCADE PHENOMENON IN THE KOLMOGOROV FLOWS

As we mentioned in the Introduction, we are interested in studying the inverse cascade phenomena associated with the Kolmogorov flow in the asymptotic limit as the forcing wave number goes to infinity. There are two control parameters in this problem: the viscosity \( \nu \) and the forcing wave number \( k_f \). We are interested in the situation when both the viscous term and the high wave-number forcing contribute to the evolution of the large-scale quantities in the leading order. This forces us to study the following distinguished limit:

\[ \nu = 1/k_f = \varepsilon. \]

As can be easily seen from the derivations presented in the next section (see Remark 3.1), if the viscosity diminishes slower than \( k_f^{-1} \), then the effect of the forcing will not show up in the leading-order equations for the large-scale quantities. On the other hand, if the viscosity diminishes faster than \( k_f^{-1} \), then the viscous term will not contribute in the leading order, and the effective equation will have a singular flux.

Under the scaling (3), Eq. (1) takes the following form:

\[ u^e + (u^e \cdot \nabla) u^e + \nabla p^e = \varepsilon \Delta u^e + \frac{1}{\varepsilon} \left( \sin \frac{\xi_2}{\varepsilon} \right), \]

\[ \nabla \cdot u^e = 0. \]

To study the influence of the small-scale forcing on the evolution of the large scales, we have numerically integrated (4) using a number of different initial velocity fields and studied the limit as \( \varepsilon \to 0 \). We discretized (4) using a Fourier collocation method in space, and Runge–Kutta methods of various order in time. More details of the numerical method will be given in Sec. IV. Our calculation differs from the previous calculations of Sulem et al.\(^8\) and Platt et al.\(^6\) in that we studied as far as we could the limit as \( \varepsilon \to 0 \). In particular, we computed the solutions of (4) with forcing wave numbers \( k_f = 10, 25, 50, \) and \( 100 \) (corresponds to \( \varepsilon = 0.1, 0.04, 0.02, \) and \( 0.01 \)). In all these cases the solutions exhibit a clear separation of scales for some fixed amount of time.

There are two important scales in this problem. One is the small scale at which the system is forced and energy is constantly fed in. The other is the large scale determined by the size of the system. In most cases we studied, we chose the initial energy to reside only on the large scales. Here we report two examples of our computational results. The main example we will dwell on has an initial vorticity field

\[ \omega_0(x) = \frac{\partial u_0^0}{\partial x_2} - \frac{\partial u_0^0}{\partial x_1} = \sin^2 \frac{x_1}{2} \sin^2 \frac{x_2}{2}. \]

We have experimented with other sets of initial data. In all cases we observe the same kind of phenomena as we report below.

To get a rough idea about the time evolution of the Kolmogorov flow, we display in Fig. 1 the energy spectrum computed using the following formula:

\[ E(k) = \sum_{\|k_1 + k_2 - k\| < 1/2} \left[ |\hat{u}_1(k_1, k_2)|^2 + |\hat{u}_2(k_1, k_2)|^2 \right], \]

where \( \hat{u}(k_1, k_2) \) is the \((k_1, k_2)\)th Fourier mode of \( u \). Figure 1 is a log–log plot of the energy spectrum from time \( t = 0 \) to time \( t = 10 \) with \( \Delta t = 0.3 \) between consecutive curves.
In,

\[ \ln E(k) \]

FIG. 2. Portions of Fig. 1 from \( t=0 \) to 5. The energy spectrum is also compared to a straight line with slope \(-4\). An inertial range forms between \( t=1 \) and 5.

\( \epsilon=0.02, k_f=50, \) and the computation is done on a \( 256^2 \) grid. As expected, the energy spectrum consists of two parts: The high wave-number part is dominated by the forcing wave number and its higher harmonics. The low wave-number part corresponds to the large-scale structure of the flow. Figure 1 reveals two important pieces of information. At early times, the low wave-number spectrum saturates and forms an envelop. This happens between \( t=1 \) and 5. At later times, a considerable amount of energy is shifted to the subharmonics of \( k_P \). Below we will study these two aspects in detail.

**A. Early stage evolution: Formation of an inertial range**

To better appreciate the early time evolution, we plotted in Fig. 2 the energy spectrum from \( t=0 \) to 5. For \( t<1 \), the low wave-number spectrum is very well separated from the high wave-number part. At about \( t=1 \), the high end of the low wave-number spectrum begins to merge with the low end of the high wave-number spectrum. At the same time, the low end of the low wave-number spectrum forms an envelop. This envelope is compared to a straight line with slope \(-4\). They are close over the initial two and one-half decades of wave numbers.

A word about the sufficiency of the numerical resolution is in order. The results described here were also checked on a \( 512^2 \) grid. It was found that the low wave-number energy spectrum did not change at all while the spectral peak at \( k=2k_f \) changed. Instead of having two peaks at the end of the spectrum as in Fig. 2, the \( 512^2 \) calculation gives rise to four peaks at the high harmonics of \( k_f=50, 100, 150, \) and \( 200 \).

Another way of checking the numerical resolution is to compute the dissipation wave number defined by

\[ k_d=\eta^{1/6}\nu^{1/2}, \]

where \( \eta \) is the enstrophy dissipation rate:

\[ \eta=\nu \int |\nabla \omega|^2 \, dx_1 \, dx_2. \]

For the present problem, the enstrophy dissipation rate can be estimated from the basic Kolmogorov flow (2). This gives \( \eta=(2\epsilon)^{-1} \) and \( k_d \sim (\epsilon)^{-1} k_f \). As long as we resolve the forcing scale, we are reasonably sure that all significant scales are captured. In all our computations, we have taken the numerical cut-off wave number to be much larger than the forcing wave number.

To examine the flow structures in the physical space accompanying the formation of an inertial range, we studied the contour curves of various quantities as time evolves. It turns out that the usual diagnostic, namely, the contour curves of vorticity and streamfunction, hardly gives us any information. The contour curves of vorticity consist of crowded and more or less parallel strips. When \( \epsilon=0.02, \) there are 100 of these strips. The contour curves of streamfunction consist of very slowly deformed concentric circles with small oscillations added. The former is dominated by structures at the forcing wave number which do not defer much from the basic shear flow (2). The latter is dominated by structures at the lowest wave number whose evolution at early times is effectively locked by the small-scale forcing. We will return to this point later.

We turned next to the behavior of vorticity but with the high wave numbers filtered out. More precisely, we examined the quantity

\[ \omega_\Lambda=\sum_{\sqrt{k_1^2+k_2^2}<k_f/2+1} \omega_\kappa e^{i\kappa \cdot \mathbf{x}}. \]

The observed phenomena did not change much as we varied the cutoff wave number between \( k_f/2+1 \) and \( k_f-1 \). At early times, \( \omega_\Lambda \) is just a single eddy with the size of the box, as can be seen from the initial data (5). As time evolves, the eddy first undergoes slow variation. After about \( t=1, \) the eddy begins to deform strongly, and at the same time polarizes in the \( x_3 \) direction. Figure 3 shows the

\[ \omega_\Lambda \]

FIG. 3. Contour plot of \( \omega_\Lambda \) at \( t=3 \). Notice the appearance of elongated eddies whose major axes are essentially locked with the \( x_3 \) axis.

\[ k_d \]

FIG. 2. Portions of Fig. 1 from \( t=0 \) to 5. The energy spectrum is also compared to a straight line with slope \(-4\). An inertial range forms between \( t=1 \) and 5.
FIG. 4. Log-log plot of the time evolution of the energy spectrum with initial data (9) from $t=0$ to 1. The time increment between consecutive curves is $\Delta t=0.1$. Other parameters are $c=0.01$, $k_f=50$, $N=400$. Time increases upward. The initial spectrum decays like $k^{-3}$. The spectrum at later times is compared to straight lines with slopes $-4$ and $-5$, and is seen to be much closer to the former.

Contour curves of $\omega_A$ at $t=3$. A very large gradient has developed along the principal axis of the eddy. The eddy is deformed so much that the local flow structure resembles that of a vortex sheet. (Note that some features of this can already be seen at $t=1$.) Since $\omega_A$ stays bounded, we obtain a $k^{-4}$ decay in the spectrum of $\omega_A$, consistent with what was described earlier.

In the second example we choose a nonsmooth initial vorticity:

$$\omega(x_1,x_2) = H(x_1)H(x_2) - \pi^4,$$

(9)

where $H(x)$ is the hat function on $[0,2\pi]$ with height $\pi$. The energy spectrum at $t=0,0.1,0.2,...,1$ is plotted in Fig. 4. We observe basically the same phenomena as in Fig. 2 except that here the initial data are much less smooth, and the low wave-number spectrum settles down to an envelope much faster. The envelope is also compared to a straight line of slope $-4$ and is seen to be very close to that. It is easy to see that the spectrum of the initial data decays like $k^{-5}$. From the plot, it is tempting to conclude that at later times the low wave-number spectrum is also close to $k^{-5}$. But a detailed comparison shows that $k^{-4}$ is a much better approximation at later times.

B. Subsequent evolution: Cascade to a hierarchy of elongated vortices

We will concentrate on the first set of data to examine the subsequent evolution and comment on the generality of our findings later.

As time evolves, the principal eddy continues to deform. Its orientation stays close to the $x_3$ axis. Between $t=4$ and 5, the eddy undergoes a fission process and splits into two eddies. This is clear from the contour curves of $\omega_A$ at $t=5$ shown in Fig. 5. This process is more or less replicated at the corners of the box. The resulting eddies undergo further deformation and fission until the subharmonic wavelength of $k_f\approx 2\varepsilon$ is reached. At $t=10$ (Fig. 6), the flow field of $\omega_A$ is almost layered with a layer width equal to $2\varepsilon$.

What accompanies the formation of these layers is the appearance and lineup of the small-scale vortices. This is clear from Fig. 7 which is the contour plot of

FIG. 5. Contour plot of $\omega_A$ at $t=5$. The elongated vortex in the middle has now split into two vortices.

FIG. 6. Contour plot of $\omega_A$ at $t=10$. The flow field of $\omega_A$ basically consists of layers of elongated vortices whose major axes are parallel to the $x_3$ axis. The layer width is approximately $2\varepsilon$. 
FIG. 7. Contour plot of $\omega_z$ at $t=5$. The flow field consists of elongated vortices whose major axes are parallel to the $x_1$ axis. The elongated vortices break up and form vortices with aspect ratio close to 2. These vortices then line up in the $x_2$ direction. This figure should be compared to Fig. 5.

$$\omega_z = \sum_{\sqrt{k_1^2+k_2^2}<k_f} \omega_k e^{i k \cdot x}$$  \hspace{1cm} (10)

at $t=5$. This should be compared to Fig. 5 which displays $\omega_\lambda$ at the same time. Vortices that appear in Fig. 7 are distinguished by their aspect ratios. The ones whose aspect ratio is approximately 2 are responsible for the layers in $\omega_\lambda$. The ones which have a very large aspect ratio are still at their early stage of evolution. These vortices are unstable to large-scale perturbations and will split into vortices of smaller aspect ratios. We can expect by $t=10$, most of the vortices should have an aspect ratio close to 2.

We make a remark here about the time scale we are considering. The enstrophy of the initial data is equal to 0.0781. In the absence of the forcing, the turnover time of the largest eddy is $0.0781^{-1/2} = 3.5777$. At time $t=10$, the largest eddy should have made more than two turns. The small-scale forcing considerably inhibited the motion of the largest scale. As we have shown, the orientation of the largest eddy is locked with the $x_2$ axis before it has hardly made any turns. It will remain locked at least until the transition to a continuum of scales is completed. This is the reason why contour plots of the streamfunction did not reveal much information since it is dominated by the structures at the largest scale whose motion is locked by the small scales.

To get an idea about the wavelength involved, we plot in Figs. 8(a), 8(b), and 8(c), respectively, the slice of $\omega$ at $x_1=\pi/2$, the slice of $\omega$ at $x_1=\pi/2$, and the slice of $\omega_\lambda$ at $x_2=\pi$, all at $t=10$. Figures 8(a) and 8(c) should be compared to Fig. 8(b) which reveals the scale of the forcing $\epsilon$.

It is quite clear that Fig. 8(c) contains oscillations at the wavelength $4\epsilon$ with three modulations, whereas Fig. 8(a) picks up two small scales: $2\epsilon$ and $4\epsilon$. Besides the forcing scale $\epsilon$, Fig. 8(b) also contains another wavelength which is roughly $8\epsilon$ since there are six modulations.

If we now consider scales that are much larger than $2\epsilon$, then the oscillations in $\omega_\lambda$ will set again as small-scale forcing. The new effective forcing is in a direction that is perpendicular to the direction of the original forcing. Based on what we described above, we expect that clon-
gated vortices should form in $\omega_{\Lambda}$, and these vortices should split and line up. As a result, the quantity

$$\omega_{\Lambda\Lambda} = \sum \omega_k e^{k \cdot x},$$

will have layers that are orientated in the $x_1$ direction. It is already quite clear from Figs. 5 and 6 that elongated vortices are formed and they line up in the $x_1$ direction. But the effect is more drastically shown in Fig. 9 which is the contour plot of $\omega_{\Lambda\Lambda}$. At this scale the layers are still predominantly in the $x_2$ direction. One should therefore go to even larger scales to see layers in the $x_1$ direction. A much smaller value of $\epsilon$ is required to produce clear evidence of that. But partial evidence can already be seen from Figs. 8(a) and 8(b) where variations on the wavelength $8\epsilon$ can be clearly seen in Fig. 8(b) but not in Fig. 8(a). This implies that if we look at scales that are larger or equal to $8\epsilon$, then the layers will be predominantly aligned with the $x_1$ axis. We intend to pursue this matter further in future publications.

The process described above should continue and we obtain a hierarchy of layers and lineup of elongated vortices at increasingly larger scales with alternating orientation.

At a smaller value of $\epsilon$, $\epsilon = 0.01$, the phenomenon described above repeats itself. The contour curves of $\omega_{\Lambda}$ consist of layers of width $2\epsilon$ aligned with the $x_2$ direction. Moreover, the slice of $\omega$ in the $x_1$ direction contains approximately 12 modulations, whereas the slice in the $x_1$ direction contains mainly wavelengths $2\epsilon$ and $4\epsilon$. This implies that at the scale $8\epsilon$, the variations are predominantly in the $x_2$ direction.

This occurs also in the other sets of data that we have tested, including random initial data with exponentially decaying spectra. In all cases we observe that the fully developed vorticity field varies at scales $\epsilon, 8\epsilon, \ldots$, in the $x_2$ direction, and $2\epsilon, 4\epsilon, \ldots$, in the $x_1$ direction. Symbolically, we can write the following tentative ansatz for the Kolmogorov flows:

$$\omega^*(x,t) = \omega_0 \left( x, \frac{x_2}{\epsilon}, t \right) + \omega_1 \left( x, \frac{x_1}{2\epsilon}, t \right) + \omega_2 \left( x, \frac{x_1}{4\epsilon}, t \right) + \omega_3 \left( x, \frac{x_1}{8\epsilon}, t \right) + \cdots. \quad (12)$$

In the next two sections, we will seek an explanation for the phenomena described in this section. We first derive effective equations governing the evolution of the large-scale quantities. We will show that the effective equation has a weak instability which implies that variations in the $x_1$ direction will grow much faster than variations in the $x_2$ direction. Then in Sec. IV we will present numerical results which suggest that due to the formation of singularities, the spectrum of the solutions of the effective equation evolves to a $k^{-4}$ decay law, even with smooth initial data. The time that this happens is close to the time when an inertial range begins to form at the low wave-number part of the spectrum of the Kolmogorov flows.

### III. EFFECTIVE EQUATIONS FOR THE LARGE SCALES

In this section, we derive effective equations governing the evolution of large-scale flow quantities. Enormous efforts have been devoted to the derivation of similar equations, the so-called "Reynolds equation," for general turbulent flows. The fact that turbulent flow has a continuum range of scales presents a serious difficulty to this objective. Our problem is drastically simplified since there is a separation of the scales which are active in the flow. As we pointed out in the last section, eventually separation of scales no longer holds and the flow exhibits a transition to a continuum of scales. This signals the breakdown of the effective equation, an issue which will be looked at more carefully in the next section.

We begin with the following ansatz:

$$u^e(x,t) = \bar{u}(x,t) + w(x,t, \frac{x_2}{\epsilon}) + \cdots, \quad (13)$$

$$p^e(x,t) = \bar{p}(x,t) + P(x,t, \frac{x_2}{\epsilon}) + \cdots,$$

where $w(x,t,y_2)$, $P(x,t,y_2)$, $u_1(x,t,y_1)$, $y_2(x,t,y_1)$, $P_1(x,t,y_2)$, etc., are periodic functions of $y_2$ with period $2\pi$, and the averages of $w(x,t,y_2)$, $P(x,t,y_2)$ over their period are zero: $\langle w(x,t,y_2) \rangle = 0$, $\langle P(x,t,y_2) \rangle = 0$, for every $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$. Here $y_2 = x_2/\epsilon$ is the fast variable. We could have started with a more general ansatz where dependence on $y_1 = x_1/\epsilon$ is also introduced, but the asymptotic procedure would automatically imply that all functions involved are inde-
pendent of $y$. Substituting (13) into (4) and collecting equal orders of $\epsilon$, we obtain a hierarchy of equations. The $O(\epsilon^{-1})$ equations are

$$(u_2 + w_2)(\bar{u}_1 + w_1) - w_2 = \sin y_2,$$

$$v_2 = 0. \quad (20)$$

This set of equations can be easily solved explicitly:

$$w_2 = 0, \quad \Pi = 0,$$

$$w_1 = \frac{1}{\bar{u}_2 + 1} (\sin y_2 - \bar{u}_2 \cos y_2). \quad (21)$$

We next come to the $O(1)$ equations:

$$u_1 + w_1 + (\bar{u} + w) \cdot \nabla_x (\bar{u} + w) + \nabla_x \Pi =$$

$$+ \left( \left( \bar{u}_1 + w_1 \right) u_1^2 + \left( \bar{u}_2 + w_2 \right) u_2^2 \right)_{y_2} + \frac{1}{\bar{u}_2 + 1} \left( \frac{1}{\bar{u}_2 + 1} \right)_{y_2} + \left( \frac{1}{\bar{u}_2 + 1} \right)_{y_2} = 0,$$

$$\nabla_x \cdot (\bar{u} + w) + u_2^1 = 0. \quad (22)$$

Averaging both sides of (22) with respect to $y_2$, we obtain

$$\bar{u}_1 + (\bar{u} \cdot \nabla) \bar{u} + \left( \frac{1}{2(\bar{u}_2 + 1)} \right)_{x_1} + \nabla_x \bar{p} = 0,$$

$$\nabla \cdot \bar{u} = 0. \quad (23)$$

The difference between this and the usual Euler equations is in the so-called Reynolds stress terms. These terms can be computed using (21). We obtain

$$\langle u_1^2 \rangle = \frac{1}{2(\bar{u}_2 + 1)}, \quad \langle w_1 u_2 \rangle = 0, \quad \langle w_2^2 \rangle = 0. \quad (24)$$

Hence, we obtain

$$\bar{u}_1 + (\bar{u} \cdot \nabla) \bar{u} + \left( \frac{1}{2(\bar{u}_2 + 1)} \right)_{x_1} + \nabla_x \bar{p} = 0,$$

$$\nabla \cdot \bar{u} = 0. \quad (25)$$

This is the effective equation governing the evolution of the large-scale quantities for the Kolmogorov flow. To verify the self-consistency of this asymptotic argument, we have to produce a candidate $(\bar{u}', \bar{p}')$ which solves the remainder of (22). To this end we write (22) as

$$\left( \left( \bar{u}_1 + w_1 \right) u_2^1 + \bar{u}_2 w_2 \right)_{y_2} + \frac{1}{\bar{u}_2 + 1} \left( \frac{1}{\bar{u}_2 + 1} \right)_{y_2} = f,$$

$$u_2^1 = -\nabla_x \cdot (\bar{u} + w) = g, \quad (26)$$

where

$$f = -\bar{v}_1 - w_1 - (\bar{u} + w) \cdot \nabla_x (\bar{u} + w) - \nabla_x \bar{p} + 2 w_2 \bar{u}_2. \quad (27)$$

In (20) and (21), $x$ and $t$ are viewed as parameters. The constructions above for $w$, $\Pi$, $\bar{u}$, and $\bar{p}$ guarantee that the averages of $f$ and $g$ in $y_2$ over the period $[0, 2\pi]$ are zero. Therefore we can solve $u_2$ from (21). The first equation of (20) will then give the solution $u_1^1$ whereas the second equation will give the solution $p^1$. It is easy to see that if we define

$$\bar{u}^e(x, t) = \bar{u}(x, t) + w \left( x, t, \frac{x_2}{\epsilon} \right) + cu^1 \left( x, t, \frac{x_2}{\epsilon} \right),$$

$$\bar{p}^e(x, t) = \bar{p}(x, t) + \epsilon p^1 \left( x, t, \frac{x_2}{\epsilon} \right),$$

then Eqs. (4) are satisfied by $(\bar{u}^e, \bar{p}^e)$ with an error of order $\epsilon$. The remainder consists of derivatives up to the third order of $\bar{u}$ and $\bar{p}$. Therefore, we expect that as long as $\bar{u}$ and $\bar{p}$ remain three times differentiable, $(\bar{u}', \bar{p}')$ will be closely approximated by $(\bar{u}^e, \bar{p}^e)$.

Remark 3.1. It is clear from these derivations that $\nu = \kappa = 1 = \epsilon$ is the only distinguished limit in which both the viscous term and forcing contributes to the leading order of $(\bar{u}', \bar{p}')$. If we choose $\nu = \epsilon^2$, $\alpha \geq 0$, then for $\alpha < 1$, the forcing will not contribute to the leading order. As a result, we have $w_1 = 0$, implying that to the leading order, $(\bar{u}', \bar{p}')$ is not oscillatory. The effective equation will then be the same as the Euler equation. If $\alpha > 1$, then the viscous term will not contribute to the leading order and we have:

$$w_1 = -\frac{1}{\bar{u}_2} \cos y_2, \quad \langle w_1 \rangle = 0, \quad \langle w_2 \rangle = 0. \quad (28)$$

The new flux term in the effective equation becomes singular in $\bar{u}_2$.

Remark 3.2. We notice that (19) conserves energy:

$$\frac{1}{2} \int \frac{1}{d} \int \left| \bar{u} \right|^2 \frac{1}{d} x_1 \frac{1}{d} x_2 = - \int \bar{u}_1 \left( \frac{1}{2(\bar{u}_2 + 1)} \right) \frac{1}{d} x_1 \frac{1}{d} x_2$$

$$+ \int \bar{u}_2 \left( \frac{1}{2(\bar{u}_2 + 1)} \right) \frac{1}{d} x_1 \frac{1}{d} x_2$$

$$= 0. \quad (29)$$

An interesting feature of (19) is that it exhibits a weak instability. To see this we linearize (19) around a constant state $\bar{u} = (\bar{u}_1, 0)$. Infinitesimal perturbations $\bar{u} = (\bar{u}_1', \bar{u}_2')$ and $\bar{p}$ will obey

$$\left( \bar{u}_1' \right)_{x_2} + \left( \bar{u}_2' \right)_{x_1} = 0, \quad \left( \bar{u}_1' \right)_{x_1} + \left( \bar{u}_2' \right)_{x_2} = 0, \quad \left( \bar{u}_1' \right)_{x_1} + \left( \bar{u}_2' \right)_{x_2} = 0, \quad (30)$$

where

$$\bar{f} = -\bar{v}_1 - w_1 - (\bar{u} + w) \cdot \nabla_x (\bar{u} + w) - \nabla_x \bar{p} + 2 w_2 \bar{u}_2. \quad (31)$$

Notice that the first matrix has a nontrivial Jordan block. From standard theory of partial differential equations
(PDE's), this means that disturbances will grow algebraically in their wave number. The easiest way of seeing this in the present situation is to neglect the pressure term and write (25) in the Fourier space:

$$\frac{d}{dt} \hat{u}(k, t) = \left( \begin{array}{cc} ik_1 u_1^0 + ik_2 u_2^0 & -ik_1 u_1^0 \\ 0 & ik_1 u_1^0 + ik_2 u_2^0 \end{array} \right) \hat{u}(k, t) = 0.$$  \hfill (26)

The solution of this ordinary differential equation (ODE) is given by

$$\hat{u}_2(k, t) = e^{-ik_1 u_1^0 + ik_2 u_2^0 t} \hat{u}_2(k, 0),$$

$$\hat{u}_1(k, t) = e^{-ik_1 u_1^0 + ik_2 u_2^0 t} \hat{u}_1(k, 0) + ik_1 t e^{-ik_1 u_1^0 + ik_2 u_2^0} \hat{u}_2(k, 0).$$  \hfill (27)

For large $k_1$, we have

$$\left| \frac{\hat{u}_1(k, t)}{\hat{u}_1(k, 0)} \right| \sim \text{const} \cdot |k_1| t.$$  \hfill (28)

The amplification is linear in $k_1$. As a result, we expect that gradients in the $x_1$ direction will grow much faster than gradients in the $x_2$ direction.

### IV. BEHAVIOR OF SOLUTIONS OF THE EFFECTIVE EQUATIONS

In this section, we present numerical evidence which suggests that the solutions of (19) with analytic initial data develop singularities in finite time. There is a simple, yet quite strong argument in favor of a finite time singularity formation. If the solutions of (19) stay smooth, then by the asymptotics presented in the last section, we expect that $u^\ell$ will be close to $\bar{u}^\ell$. In particular, the scales of $u^\ell$ will stay separated. We have seen in Sec. II that this cannot be true.

The numerical results presented below were computed using a Fourier collocation method. We checked these results using both a second-order centered differencing scheme and a fourth-order ENO-type scheme.\(^9\) We remark that searching for singularities numerically is a very difficult undertaking. It is important to check the numerical results using different methods since each method is likely to exhibit some artifacts when it comes to the computation of singularities.

Strictly speaking, demonstrating finite time singularity formation is an analytical problem. The ultimate answer has to come analytically, either by giving specific examples or by establishing a lower bound for the life span of the smooth solutions. Computers are limited to finite arithmetic and finite capacities. A numerical approach can at best give partial evidence, not the complete answer. For the moment though, at best we can present numerical evidence to shed some light on the problem.

### A. Description of the numerical method

For the spatial discretization, we used the Fourier collocation method.\(^10\) Roughly speaking, the differentiation operator is approximated in the Fourier space, while the nonlinear operations such as multiplication are done in the physical space. We used the intrinsic Cray fast Fourier transform (FFT) routines which considerably enhanced the performance of the code.

We observed that even for the computation of smooth solutions of the standard two-dimensional incompressible Euler equation, a certain amount of filtering is needed if the numerical method is to be stable, and it is crucial to add the filters correctly. A robust way of adding the filters\(^11\) is to replace the Fourier multiplier $ik_j$, for the differentiation operator $\partial/\partial x_j$, by $ik_j \rho(|k_j|)$, where

$$\rho(k) = e^{-\alpha(k/N)^m}, \quad \text{for } |k| < N.$$  \hfill (29)

Here $N$ is the numerical cutoff for the Fourier modes, $m_f$ is the order of the filter, and $\alpha$ is chosen so that $\rho(N) = e^{-\alpha} = \text{machine accuracy}$. The machine accuracy on Cray-YMP with single precision is roughly $10^{-14}$. Denote by $F$ and $F^{-1}$, respectively, the forward and backward Fourier transform operators, then the numerical derivative is evaluated as

$$D_{nf} F = F^{-1} [ik \rho(|k|)] F f.$$  \hfill (30)

The accuracy of such an approximation scheme depends on the parameter $m_f$. For smooth functions $f(x)$, we have

$$\|f'(x) - D_{nf} f(x)\| = O(N^{-m_f}).$$  \hfill (31)

Unless otherwise stated, the results presented in Sec. IV B 4.2 were computed with $m_f = 14$.

For the temporal discretization, we used Runge–Kutta methods of various order designed in Ref. 9. No major difference between the third-, fourth-, and fifth-order methods was found in the numerical results. It seems to be a general fact that temporal accuracy is much less important than the spatial accuracy. We used the third-order version most often since it only requires three auxiliary arrays, whereas the fourth-order version requires five auxiliary arrays. We take initial data that is periodic with period $D$ where $D = [0, 2\pi] \times [0, 2\pi]$. The results reported below were computed using CFL equal to 0.5. This is very much within the stability region of these methods.

### B. Numerical results

We present our numerical results for the case when the initial vorticity is the same as in the first example of Sec. II: $\omega_0(x, y) = \sin^2(x/2)\sin^2(y/2) = \Delta_0^2$. As a first hint for the singularity formation, we plotted the time history of the quantity $H_1(t) = \int D\{\nabla \omega(x, t)\}^2 dx_1 dx_2$ computed on different grids: $256^2$, $256^3$, and $400^2$. Figure 10 suggests that as $N \to +\infty$, $H_1(t)$ grows without bound, for $t > 0.9$. On the $400^2$ grid, $H_1(t)$ at $t = 1$ is amplified by more than 45 times of its initial value.

This information alone cannot be used as solid evidence for singularity formation, since it only says that the quantity $H_1(t)$ is poorly resolved for $t > 0.9$. Next we plotted in Fig. 11 the time evolution of the energy spectrum (on a log–log scale) associated with the numerical solutions obtained on $256^2$ and $512^2$ grids. We observe that at $t = 1$, a bump has developed near the cutoff wave number
on the energy spectrum. This indicates that at this time, the numerical solutions are not very well resolved. Figure 12 displays separately the energy spectrum at $t=0.9$. It is clear that at this time, the energy spectrum is well represented by the numerical solutions obtained on these grids, and it fits very well with a straight line with slope $-4$. This is a stronger indication that at this time, the energy spectrum of the solution no longer decays exponentially, but algebraically as $k^{-4}$.

To determine more precisely when algebraic decay sets in, we use a technique proposed by Sulem et al. Let $f(z)$ be a function which is analytic in the strip $|\text{Im } z| < \alpha$, and has a branch-cut singularity $z(t) \in \mathbb{C}$ on $|\text{Im } z| = \alpha$, then its Fourier transform $\hat{f}(k)$ decays like

$$\hat{f}(k) \sim Ck^{-\beta}e^{-\alpha k}, \quad |k| \to +\infty.$$ (32)

Imagine the following scenario for the singularity formation in the solutions of (19). Instead of solving (19) in the real space $\mathbb{R}^2$, we solve (19) in the complex space $\mathbb{C}^2$. Initially the solution is an entire function. At a later time the solution develops branch point of order $\beta(t)$ at $z(t)$ with $|\text{Im } z(t)| = \alpha(t) > 0$, but it is analytic in the strip $|\text{Im } z| < \alpha(t)$. If this branch point travels to the real axis at a finite time $t^*$, then the solution of (19), solved in the real space, develops a singularity at $t=t^*$.

This scenario has been proposed for a number of problems in fluid mechanics, notably the formation of singularities in the evolution of a vortex sheet (see Refs. 13–16). In this case, there is strong numerical and analytical evidence to confirm this picture.

To check whether this provides a plausible picture for our problem, we checked numerically the validity of the ansatz (32) for the energy spectrum. To do that we pick three consecutive wave numbers $(k-1,k,k+1)$, and solve

$$E(k,t) = Ck^{-\beta}e^{-\alpha k}, \quad k=k-1,k,k+1,$$ (33)

with double precision arithmetic (on Cray-YMP). At each fixed time, we obtain three functions $C(k,t), \beta(k,t), \alpha(k,t)$. These functions depend on $k$. In order that (32) be a good ansatz for the energy spectrum of the solutions of (19), asymptotically as $k \to \infty$, the functions $C(k,t), \beta(k,t), \alpha(k,t)$ should not depend on $k$ in this limit. In Fig. 13 we plotted the functions $\alpha(k,t)$ from $t=0.4$ to $1$ with constant time increment $\Delta t=0.02$ between consecutive curves. The computation is done on a $256^2$ grid. It is clear from Fig. 13 that for fixed $t$, $\alpha(k,t)$ is almost a constant for $20 < k < 60$. For $k \approx 20$, $\alpha(k,t)$ exhibits fluctuations since $k$ is not yet in the asymptotic regime. For $80 < k < 128$, the energy density is at the order of $10^{-32}$, which is below machine accuracy even if double precision is used. These wave numbers are needed only to enforce smooth decay for the Fourier coefficients down to the machine accuracy. Figure 14 displays similar results for a computation using a $400^2$ grid. The fluctuations at high wave number is a manifestation of trying to fit machine zeros with ansatz (32). The functions $C(k,t)$ and $\beta(k,t)$ exhibit the same feature.
FIG. 13. $\alpha(k,t)$ from $t=0.4$ to 1, with constant time increment $\Delta t=0.02$ between consecutive curves. Time goes downward. The numerical results are computed on a 256 grid. For fixed $t$, the function $\alpha(\cdot,t)$ is basically constant in the range of wave numbers displayed. This confirms the validity of the ansatz (32) for the energy spectrum.

Now we can average the values of $\alpha$ over the intermediate wave numbers and safely regard the averaged value (as a function of $t$) as the width of the analyticity strip of the solution at that time. This function will be denoted by $\tilde{\alpha}(t)$. The corresponding function for $\beta$ will be denoted by $\tilde{\beta}(t)$. Displayed in Fig. 15 is the function $\alpha(t)$ for two different numerical resolutions: 256 and 400. Results obtained on both grids indicate that the width of the analyticity strip for the solution vanishes at $t^*\approx 0.9$, suggesting a singularity formation at this time. The corresponding results for $\beta$ are displayed in Fig. 16. As $t\to t^*$, $\tilde{\beta}(t)$ converges to 4.

The other numerical parameter in our method is the order of the filter. Figures 17 and 18 display the results for $\tilde{\alpha}$, and $\tilde{\beta}$ as we vary the order of the filter. We see that the approximated values of $\alpha$ and $\beta$ are not sensitive to the change of the filters. However, when the filter is too weak ($m_f=15$), the numerical solution deteriorates drastically at a time before $t^*$, since it can no longer handle the large gradients developed in the vorticity field. The computed value of $\tilde{\beta}$ fluctuates much more, therefore the result for $t>0.7$ is not shown in Fig. 18. This is not unexpected since in general $\beta$ is much more sensitive than $\alpha$.

To get an idea about the nature of the singularity, we display in Fig. 19 the time evolution of $\omega$ at $x_2=2\pi/3$ with $t=0.3, 0.6, and 0.9$. At $t=0.9$, a cusp seems to have formed on the profile of $\omega$. Figure 20 is the contour plot of $\omega$ at $t=0.9$. We observe that the cusp structures occur on isolated lines, consistent with the fact that the spectrum decays like $k^{-4}$. Notice also the remarkable similarity between Figs. 3 and 20. This is not unexpected since at this time, $\omega_A$ should be close to $\omega$. The vorticity itself, on the other hand, is not blowing up. This can be seen from Fig. 21 which displays the time history of the maximum of the computed vorticity on different grids: 128, 256, and 400.

The fact that vorticity remains bounded at the point of singularity formation does not contradict the results of Beale et al., which assert that if higher derivatives of the solutions of Euler's equations blow up, then vorticity must blow up. The reason is that the effective equation is not a
symmetric hyperbolic system. In fact, it is hardly hyperbolic since its linearization contains a nontrivial Jordan block.

We mention briefly the interesting behavior of $\omega$ at later times, after the singularity. The Fourier collocation method failed for this purpose since it cannot handle the large gradients of $\omega$. Instead we designed a finite difference method with nonlinear numerical viscosity built-in, in the spirit of the ENO schemes. This method not only enabled us to confirm the results reported above, it also allowed us to continue the calculation beyond the formation of singularities. The numerical results displayed in Figs. 22 and 23 suggest that the spectrum saturates again at about $t=2$ with an asymptotic form $k^{-2}$. Furthermore, the flow field develops features that are remarkably similar to the ones seen in Fig. 5.

The behavior of the solutions reported above seems to be generic. It occurs in other calculations we did using different initial data.

V. CONCLUDING REMARKS

We have displayed the remarkable structure exhibited in the inverse cascade process in Kolmogorov flows at early and intermediate times. We have also put forward a rather satisfactory theory to explain these phenomena. In the context of two-dimensional turbulence, our results demonstrate quite convincingly the existence of a $k^{-2}$ inertial range for the Kolmogorov flows. While there has been plenty of two-dimensional turbulence theories proposed, each of these theories requires spontaneous singularities of some sort in the Euler flows. On the other hand, it is well known that in two dimensions, Euler flows do not form spontaneous singularities. This dilemma is a major obstacle in the understanding of two-dimensional
turbulence. Our results suggest that if the system is forced at very high wave numbers, then the Euler equations are replaced by the effective equations derived in Sec. III, and the dilemma mentioned above can then be resolved in this case.

One naturally asks what happens at much later times. Without the forcing, this has been studied in depth by a number of authors. In this case, inverse cascade is mainly reflected in the merger of vortices. In a periodic geometry, two coherent vortices are formed at very long times which capture essentially all the energy in the system. It is not at all clear that this will happen in a forced flow, particularly when the forcing wave number is larger than the usual viscous cutoff. She and Platt et al. studied the long time behavior when the forcing wave number is small. While some observations on the formation of large-scale structures were made, their main interest is to study the bifurcation in the phase space as the Reynolds number varies while the forcing wave number is kept fixed. A detailed study of the inverse cascade process at long times has yet to be carried out.

As we mentioned in the Introduction, the Kolmogorov flow belongs to a large class of problems which exhibit large-scale instability. A typical example in three dimensions is the Arnold-Beltrami-Childress (ABC) flow which is known to be unstable to large-scale perturbations. It would be interesting to study the resulting inverse cascade process when the flow is forced by high wave-number ABC flows. In principle, the ideas and methods presented above should apply to these problems also, although to actually carry out the program is difficult mainly because the computational cost increases drastically for three-dimensional problems. However, it is clear that such instabilities will trigger the transport of energy from small scales to large ones, thereby effecting the mean flow quantities.

One can also attempt to study the interaction of different scales by studying solutions with two-scale initial data: \( u_0 = u_0(x, \theta(x)/\varepsilon) \). Formally one can use multiple scale asymptotics to derive effective equations for the large scales. This is done in Refs. 22 and 23. However, a closer look at these effective equations reveals that the linearized equations are exponentially ill-posed: high wave-number perturbations grow at a rate that is exponential in the wave number. The reason for this is very simple. The incompressibility condition forces the microstructure to be shear flows, i.e., after a coordinate transformation, one can assume without loss of generality that \( u_0 = u_0(x, x_2/\varepsilon, u_0^2(x)) \). Therefore the microstructure appears like stacks of vortex sheets when viewed at the large scale. The linear ill-posedness of the effective equation is simply a manifestation of the Kelvin–Helmholtz instability of these vortex sheets. In such situations multiple scale asymptotics does not provide useful information since it is
based on the ansatz of scale separation, an assumption which breaks down catastrophically as time evolves. For the Kolmogorov flow, we are saved since there is a fixed time interval in which the scales are separated and we can get useful information from two-scale expansions. The information can then be used as guidance to study the process according to which a continuum of scales emerge in the flow.

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