ANALYSIS OF THE HETEROGENEOUS MULTISCALE METHOD FOR ELLIPTIC HOMOGENIZATION PROBLEMS

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Abstract. A comprehensive analysis is presented for the heterogeneous multiscale method (HMM for short) applied to various elliptic homogenization problems. These problems can be either linear or nonlinear, with deterministic or random coefficients. In most cases considered, optimal estimates are proved for the error between the HMM solutions and the homogenized solutions. Strategies for retrieving the microstructural information from the HMM solutions are discussed and analyzed.

Contents

1. Introduction and the main results 2
1.1. General methodology 2
1.2. Heterogeneous multiscale methods 3
1.3. Main results 5
1.4. Recovering the microstructural information 6
2. Generalities 7
3. Estimating $e(\text{HMM})$ 12
3.1. Problems with locally periodic coefficients 12
3.2. Problems with random coefficients 16
4. Reconstruction and compression 22
4.1. Reconstruction procedure 22
4.2. Compression operator 24
5. Nonlinear homogenization problems 25
5.1. Algorithms and main results 25
5.2. Estimating $e(\text{HMM})$ 30
References 37

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1. Introduction and the main results

1.1. General methodology. Consider the classical elliptic problem

\[
\begin{align*}
-\text{div} \left( a^\varepsilon(x) \nabla u^\varepsilon(x) \right) &= f(x), & x \in D \subset \mathbb{R}^d, \\
u^\varepsilon(x) &= 0, & x \in \partial D.
\end{align*}
\]

Here \( \varepsilon \) is a small parameter that signifies explicitly the multiscale nature of the coefficients \( a^\varepsilon(x) \). There already exist several classical multiscale methodologies for the numerical solution of this elliptic problem, the most well-known among which is the multigrid technique [9]. These classical multiscale methods are designed to resolve the details of fine scale problem (1.1) and are applicable for general problems, i.e., no special assumptions are required for the coefficient \( a^\varepsilon(x) \). In contrast, modern multiscale methods are designed specifically for recovering partial information about \( u^\varepsilon \) at a sublinear cost, i.e., the total cost grows sublinearly with the cost of solving the fine scale problem [19]. This is only possible by exploring the special features that \( a^\varepsilon(x) \) might have, such as scale separation. The simplest example is when

\[
a^\varepsilon(x) = a \left( x, \frac{x}{\varepsilon} \right),
\]

where \( a(x, y) \) can either be periodic in \( y \), in which case we assume the period to be \( I = [-\frac{1}{2}, \frac{1}{2}]^d \), or random but stationary under shifts in \( y \), for each fixed \( x \in D \). In both cases, it has been shown that

\[
\| u^\varepsilon(x) - U(x) \|_{L^2(D)} \to 0,
\]

where \( U(x) \) is the solution of a homogenized equation:

\[
\begin{align*}
-\text{div} \left( \mathcal{A}(x) \nabla U(x) \right) &= f(x), & x \in D, \\
U(x) &= 0, & x \in \partial D.
\end{align*}
\]

The homogenized coefficient \( \mathcal{A}(x) \) can be obtained from the solutions of the so-called cell problem. In general, there are no explicit formulas for \( \mathcal{A}(x) \), except in one dimension.

Several numerical methods have been developed to deal specifically with the case when \( a(x, y) \) is periodic in \( y \). [3, 8, 14] propose to solve the homogenized equations as well as the equations for the cell problem. Schwab et al. [26, 35] use multiscale test functions of the form \( \varphi(x, x/\varepsilon) \) where \( \varphi(x, y) \) is periodic in \( y \) to extract the leading order behavior of \( u^\varepsilon(x) \), extending an idea that was used analytically in the work of [2, 16, 30, 40] for the homogenization problems. These methods have the feature that their cost is independent of \( \varepsilon \), hence sublinear as \( \varepsilon \to 0 \), but so far they are restricted to the periodic homogenization problem.
1.2. **Heterogeneous multiscale methods.** HMM [17, 18, 19] is a general methodology for designing sublinear algorithms by exploiting scale separation and other special features of the problem. It consists of two components: selection of a macroscopic solver, and estimating the missing macroscale data by solving locally the fine scale problem.

For (1.1) the macroscopic solver can be chosen as a conventional $P_k$ finite element method on a triangulation of element size $H$ which should resolve the macroscale features of $a^\varepsilon(x)$. The missing data is the effective stiffness matrix at this scale. This is obtained as follows. Assuming that the effective coefficient at this scale is $A^H(x)$, we could evaluate the quadratic form

$$A^H(V,V) = \int_D \nabla V(x) \cdot A^H(x) \nabla V(x) \, dx$$

by numerical quadrature if we know $A^H(x)$ explicitly: For any $V \in X^H$, the finite element space,

$$A^H(V,V) \simeq \sum_{K \in T_H} |K| \sum_{x_\ell \in K} \omega_\ell \left( \nabla V \cdot A^H \nabla V \right)(x_\ell),$$

where $x_\ell$ and $\omega_\ell$ are the quadrature points and weights in $K$. In the absence of explicit knowledge of $A^H(x)$, we approximate $\left( \nabla V \cdot A^H \nabla V \right)(x_\ell)$ by solving the problem:

$$\begin{cases}
- \text{div} \left( a^\varepsilon(x) \nabla v^\varepsilon_\ell(x) \right) = 0, & \text{in } I_\delta(x_\ell), \\
 v^\varepsilon_\ell(x) = V_\ell(x), & \text{on } \partial I_\delta(x_\ell),
\end{cases}$$

where $I_\delta(x_\ell)$ is a cube of size $\delta$ centered at $x_\ell$, and $V_\ell$ is the linear approximation of $V$ at $x_\ell$. We then let

$$\left( \nabla V \cdot A^H \nabla V \right)(x_\ell) = \frac{1}{\delta^d} \int_{I_\delta(x_\ell)} \nabla v^\varepsilon_\ell(x) \cdot a^\varepsilon(x) \nabla v^\varepsilon_\ell(x) \, dx.$$  

In order to reduce the effect of the imposed boundary condition on $\partial I_\delta(x_\ell)$, we may replace (1.8) with

$$\left( \nabla V \cdot A^H \nabla V \right)(x_\ell) = \frac{1}{(\delta')^d} \int_{I_{\delta'}(x_\ell)} \nabla v^\varepsilon_\ell(x) \cdot a^\varepsilon(x) \nabla v^\varepsilon_\ell(x) \, dx,$$

where $\delta' < \delta$. For example, we may choose $\delta' = \delta/2$. In (1.7), we used Dirichlet boundary condition. Other boundary conditions are possible. In the case when $a^\varepsilon(x) = a(x, x/\varepsilon)$ and $a(x, y)$ is periodic in $y$, one can take $I_\delta(x_\ell)$ to be $x_\ell + \varepsilon I$, i.e., $\delta = \varepsilon$ and use the boundary condition that $v^\varepsilon_\ell(x) - V_\ell(x)$ is periodic on $I_\delta$. 

So far the algorithm is completely general. The savings compared with solving the full fine scale problem comes from the fact that we can choose $I_\delta(x_\ell)$ to be smaller than $K$. The size of $I_\delta(x_\ell)$ is determined by many factors, including the accuracy and cost requirement, the degree of scale separation, and the microstructure in $a^\varepsilon(x)$. One purpose for the error estimates that we present below is to give guidelines on how to select $I_\delta(x_\ell)$. In the special case when $a^\varepsilon(x) = a(x, x/\varepsilon)$ and $a(x, y)$ is periodic in $y$, we can simply choose $I_\delta(x_\ell)$ to be $x_\ell + \varepsilon I$, i.e., $\delta = \varepsilon$. If $a(x, y)$ is random, then $\delta$ should be a few times larger than the local correlation length. In the former case, the total cost is independent of $\varepsilon$. In the latter case, the total cost depends only weakly on $\varepsilon$ (see [27]).

The final problem is to solve

$$
\min_{V \in X_H} \frac{1}{2} A_H(V, V) - (f, V).
$$

![Figure 1. Illustration of HMM for solving (1.1). The dots are the quadrature points. The little squares are the microcell $I_\delta(x_\ell)$.](image)

To summarize, HMM has the following features:

1. It gives a framework that allows us to maximally take advantage of the scale separation. The coefficient $a^\varepsilon$ need not to be the form $a(x, x/\varepsilon)$. For periodic homogenization problems, the cost of HMM is comparable to the special techniques discussed in [3, 8, 14, 26, 31]. However HMM is also applicable for random problems and for problems whose coefficients $a^\varepsilon(x)$ does not has the structure of $a(x, x/\varepsilon)$.

2. For problems without scale separation, HMM becomes a fine scale solver by choosing $H$ that resolves the fine scales and letting $A_H(x) = a^\varepsilon(x)$. 


Some related ideas exist in the literature. Durlofsky [15] proposed an up-scaling method, which solves directly some local problems to obtain the effective coefficients [37]. Oden et al [31] proposed a method that aims at recovering the oscillation in $\nabla u^\varepsilon$ locally by solving a local problem with some given approximation to the macroscopic state $U_0$ as the boundary condition. Their method is sometimes used in HMM to recover the microstructural information.

The numerical performance of HMM including comparison with other methods is discussed in [27].

The analysis in this paper is restricted to the case of periodic and random homogenization problems. This is mainly due to the fact that the analytical behavior of $u^\varepsilon(x)$ is best understood in these cases. The strategy of the analysis is however not restricted to these cases, and can be extended to other cases as long as some analytical results on $u^\varepsilon(x)$ are available.

We will always assume that $a^\varepsilon(x)$ is smooth, symmetric and uniformly elliptic:

$$\lambda I \leq a^\varepsilon(x) \leq \Lambda I$$

for some $\lambda, \Lambda > 0$. We will use summation conventional rule. Standard notations for Sobolev spaces are used (see [1]). For the quadrature formula, we will assume the following accuracy conditions: $k$th-order numerical quadrature scheme [11]:

$$\int_K p(x) = \sum_{\ell=1}^{L} \omega_\ell p(x_\ell) \quad \forall p(x) \in P_{2k-2}.$$  

Here $\omega_\ell > 0$, for $\ell = 1, \cdots, L$.

1.3. Main results. Our main results for the linear problem are as follows.

**Theorem 1.1.** Denote by $U_0$ and $U_{\text{HMM}}$ the solutions of (1.4) and the HMM solution, respectively. Let

$$e(\text{HMM}) = \max_{x_\ell \in T_H} \| A(x_\ell) - A_H(x_\ell) \|,$$

and $\| \cdot \|$ is the Euclidean norm. If (1.10) holds, then there exists a constant $C$ independent of $\varepsilon, \delta$ and $H$, such that

$$\| U_0 - U_{\text{HMM}} \|_1 \leq C \left( H^k + e(\text{HMM}) \right).$$

If (1.10) holds for $k \geq 2$ and the quadrature scheme (1.10) is exact for $P_1$ if $k = 1$, then

$$\| U_0 - U_{\text{HMM}} \|_0 \leq C \left( H^{k+1} + e(\text{HMM}) \right).$$
If there exists a constant $C_0$ such that $e(\text{HMM}) < C_0$, then there exists a constant $H_0$ such that for all $H \leq H_0$, we have

$$\|U_0 - U_{\text{HMM}}\|_{1,\infty} \leq C\left(H^k + e(\text{HMM})\right)|\ln H|.$$  

At this stage, no assumption on the form of $a_\varepsilon(x)$ is necessary. $U_0$ can be the solution of an arbitrary macroscopic equation with the same right-hand side. Of course for $U_{\text{HMM}}$ to converge to $U_0$, i.e., $e(\text{HMM}) \to 0$, $U_0$ must be chosen on the solution of an effective homogenized equation, which we now assume exist. To obtain quantitative estimates on $e(\text{HMM})$, we must make more specific assumptions on the form of $a_\varepsilon(x)$.

In the special case of homogenization problems, $e(\text{HMM})$ can be estimated as follows.

**Theorem 1.2.** For the periodic homogenization problem, we have

$$e(\text{HMM}) \leq \begin{cases} C\delta & \text{if } \delta \text{ is an integer multiple of } \varepsilon, \\ C\left(\frac{\varepsilon}{\delta} + \delta\right) & \text{if } \delta \text{ is not an integer multiple of } \varepsilon. \end{cases}$$

In the first case, we replace the boundary condition in (1.7) by a periodic boundary condition: $v_\varepsilon^\ell - V_\ell$ is periodic with period $I_\delta$.

**Theorem 1.3.** For the random homogenization problem, assuming that (A) in §3.2 holds (see [39]), then we have

$$\mathbb{E}e(\text{HMM}) \leq \begin{cases} C\left(\frac{\varepsilon}{\delta}\right)^\kappa, & d = 3 \\ \text{remains open}, & d = 2 \\ C\left(\frac{\varepsilon}{\delta}\right)^{1/2}, & d = 1 \end{cases}$$

where

$$\kappa = \frac{6 - 12\gamma}{25 - 8\gamma}$$

for any $0 < \gamma < 1/2$.

The probabilistic set-up will be given in §3.2. To prove this result, we assume that (1.8') is used with $\delta' = \delta/2$.

1.4. **Recovering the microstructural information.** In many cases, the microstructure information in $u_\varepsilon(x)$ is very important. $U_{\text{HMM}}$ by itself does not give this information. However, this information can be recovered using a simple post-processing technique. For the general case, having $U_{\text{HMM}}$, one can obtain locally the microstructural information using an idea in [31]. Assume that we are interested in recovering $u_\varepsilon$ and $\nabla u_\varepsilon$ only in the subdomain $\Omega \subset D$. Consider the
following auxiliary problem:

\[
\begin{cases}
- \text{div} \left( a^\varepsilon(x) \nabla u^\varepsilon(x) \right) = f(x), & x \in \Omega_\eta, \\
u^\varepsilon(x) = U_{\text{HMM}}(x), & x \in \partial \Omega_\eta,
\end{cases}
\]

where \( \Omega_\eta \) satisfies \( \Omega \subset \Omega_\eta \subset D \) and \( \text{dist}(\partial \Omega, \partial \Omega_\eta) = \eta \). We then have

**Theorem 1.4.** We have

\[
(1.15) \quad \left( \int_\Omega |\nabla (u^\varepsilon - \overline{u}^\varepsilon)|^2 \, dx \right)^{1/2} \leq C \left( \|U_0 - U_{\text{HMM}}\|_{L^\infty(\Omega_\eta)} + \|u^\varepsilon - U_0\|_{L^\infty(\Omega_\eta)} \right).
\]

For the random problem, the last term was estimated in [39].

For the periodic homogenization problem, in particular, consider the case when \( k = 1 \). Let \( \overline{u}^\varepsilon \) be defined as follows:

1. \( \left( \overline{u}^\varepsilon - U_{\text{HMM}} \right)|_K \) is periodic with period \( \varepsilon I \).
2. \( \overline{u}^\varepsilon(x) = v^\varepsilon(x) \) on \( I_{\delta/2} \), where \( v^\varepsilon(x) \) is the solution of (1.7) with \( V_l(x) = U_{\text{HMM}}(x) \).

If the period is known a-priori, the period extension in the second step can be made over \( I_\varepsilon \). For this case, we can prove

**Theorem 1.5.** Let \( \overline{u}^\varepsilon \) be defined as above, then

\[
(1.16) \quad \left( \sum_{K \in T_H} \|u^\varepsilon - \overline{u}^\varepsilon\|_{1,K}^2 \right)^{1/2} \leq C \left( \sqrt{\varepsilon + H} \right).
\]

For the general case, we have

\[
\left( \sum_{K \in T_H} \|u^\varepsilon - \overline{u}^\varepsilon\|_{1,K}^2 \right)^{1/2} \leq C \left( \sqrt{\varepsilon + \frac{\varepsilon}{\delta} + \delta + H} \right).
\]

This can be proved by combining the proofs of Lemma 3.2 and Theorem 1.5.

We leave the details to the interested readers.

Similar results with some modification hold for nonlinear problems. The details are given in §5.

## 2. Generalities

We prove Theorem 1.1. We will let \( U_H = U_{\text{HMM}} \) for short.

Obviously, \( U_H \) is the numerical solution associated with the quadratic form \( A_H \), and \( U_0 \) is the exact solution associated with the quadratic form \( A \), which is defined for any \( V \in H_0^1(D) \) as

\[
A(V,V) = \int_D \nabla V(x) \cdot \mathcal{A}(x) \nabla V(x) \, dx.
\]
To estimate $U_0 - U_H$, we view $A_H(V, V)$ as an approximation to $A(V, V)$, and use Strang’s first lemma [10]. We begin with the following observation, which gives stability.

On $I_\delta(x_\ell)$, notice that $v_\varepsilon^\ell = V_\ell$ on the edges of $I_\delta(x_\ell)$, using the fact that $\nabla V_\ell$ is a constant in $I_\delta(x_\ell)$, an integration by parts leads to

$$\int_{I_\delta(x_\ell)} \nabla (v_\varepsilon^\ell - V_\ell)(x) \nabla V_\ell(x) \, dx = 0,$$

which implies

$$\int_{I_\delta(x_\ell)} |\nabla v_\varepsilon^\ell(x)|^2 \, dx = \int_{I_\delta(x_\ell)} |\nabla V_\ell(x)|^2 \, dx + \int_{I_\delta(x_\ell)} |\nabla (v_\varepsilon^\ell - V_\ell)(x)|^2 \, dx. \tag{2.1}$$

Multiplying (1.7) by $v_\varepsilon^\ell(x) - V_\ell(x)$ and integrating by parts we obtain

$$\int_{I_\delta(x_\ell)} \nabla v_\varepsilon^\ell(x) \cdot a_\varepsilon^\ell(x) \nabla v_\varepsilon^\ell(x) \, dx + \int_{I_\delta(x_\ell)} \nabla (v_\varepsilon^\ell - V_\ell)(x) \cdot a_\varepsilon^\ell(x) \nabla (v_\varepsilon^\ell - V_\ell)(x) \, dx
\tag{2.2}
= \int_{I_\delta(x_\ell)} \nabla V_\ell(x) \cdot a_\varepsilon^\ell(x) \nabla V_\ell(x) \, dx.$$

Using (2.1) and (1.10), for any $V \in X_H$, we have

$$A_H(V, V) \geq \lambda \sum_{K \in T_H} |K| \sum_{x_\ell \in K} \omega_\ell \int_{I_\delta(x_\ell)} |\nabla V_\ell(x)|^2 \, dx
\tag{2.3}
= \lambda \|\nabla V\|_0^2.$$  

Similarly, for any $V, W \in X_H$, in light of (2.2) and (1.10), we obtain

$$|A_H(V, W)| \leq A \sum_{K \in T_H} \sum_{x_\ell \in K} \frac{|K|}{|I_\delta(x_\ell)|} \omega_\ell \|\nabla v_\varepsilon^\ell\|_{0,I_\delta(x_\ell)} \|\nabla w_\varepsilon^\ell\|_{0,I_\delta(x_\ell)}
\leq \frac{A^2}{\lambda} \sum_{K \in T_H} \sum_{x_\ell \in K} \frac{|K|}{|I_\delta(x_\ell)|} \omega_\ell \|\nabla V_\ell\|_{0,I_\delta(x_\ell)} \|\nabla W_\ell\|_{0,I_\delta(x_\ell)}
\tag{2.4}
= \frac{A^2}{\lambda} \sum_{K \in T_H} |K| \int_{I_\delta(x_\ell)} |\nabla V(x) \nabla W(x)| \, dx.$$

The existence and the uniqueness of the solutions to (1.9) follows from (2.3) and (2.4) via Lax-Milgram lemma.
Remark 2.1. In view of (2.3) and (2.4), we have
\[ \lambda I \leq A_H(x) \leq \frac{A^2}{\lambda} I. \]
This is consistent with the bounds on the effective tensor for general elliptic problem (1.1) in the sense of $H-$convergence [29].

To streamline the proof of Theorem 1.1, we introduce the following auxiliary bilinear form $\hat{A}_H$.
\[ \hat{A}_H(V, W) = \sum_{K \in T_H} \hat{A}_K(V, W) \quad \text{with} \quad \hat{A}_K(V, W) = |K| \sum_{x_\ell \in K} \omega_\ell (\nabla W \cdot A \nabla V)(x_\ell). \]

Classical results on numerical integration [11, Theorem 7] yields that for any $V, W \in X_H$,
\[ |\hat{A}_K(V, W) - \int_K \nabla W \cdot A \nabla V \, dx| \leq CH^m \|V\|_{m,K} \|\nabla W\|_{0,K} \quad 1 \leq m \leq k. \]

Moreover, for any $V, W \in X_H$, if $\|V\|_{k+1}$ and $\|W\|_2$ are bounded, we have [11, Theorem 8],
\[ |\hat{A}_H(V, W) - A(V, W)| \leq CH^{k+1} \|V\|_{k+1} \|W\|_2. \]

**Proof for Theorem 1.1.** Invoking the first Strang lemma [10, Theorem 4.1.1], we have
\[ \|U_0 - U_H\|_1 \leq C \inf_{V \in X_H} \left( \|U_0 - V\|_1 + \sup_{W \in X_H} \frac{|A_H(V, W) - A(W, W)|}{\|W\|_1} \right). \]
Let $\Pi$ be the standard Lagrange interpolate operator [10], $V = \Pi U_0$, we have
\[ \inf_{V \in X_H} \|U_0 - V\|_1 \leq \|U_0 - \Pi U_0\|_1 \leq CH^k. \]

It remains to estimate $|A_H(V, W) - A(V, W)|$ for $V = \Pi U_0$ and $W \in X_H$. In view of (2.5), we get
\[ |A_H(V, W) - A(V, W)| \leq |A_H(V, W) - \hat{A}_H(V, W)| + |\hat{A}_H(V, W) - A(V, W)| \leq \left( e(HMM) \|\nabla V\|_0 + CH^k \|V\|_k \right) \|\nabla W\|_0. \]
This gives (1.11)

To get the $L^2$ estimate, we use the Aubin-Nitsche argument [10]. To this end, consider the following auxiliary problem: Find $w \in H_0^1(D)$ such that
\[ A(v, w) = (U_0 - U_H, v) \quad \forall v \in H_0^1(D). \]
Standard regularity result reads [23]
\[ \|w\|_2 \leq C \|U_0 - U_H\|_0. \]
Putting $v = U_0 - U_H$ into the right-hand side of (2.9), we obtain

\[
\|U_0 - U_H\|^2_0 = A(U_0 - U_H, w - \Pi w) + \left( A_H(U_H, \Pi w) - A(U_H, \Pi w) \right)
+ \left( A_H(U_H - \Pi U_0, \Pi w) - A(U_H - \Pi U_0, \Pi w) \right)
\]

(2.11)

\[
+ \left( A_H(\Pi U_0, \Pi w) - A(\Pi U_0, \Pi w) \right).
\]

Using (2.8) with $k = 1$, we bound the first two terms at the right-hand side of the above identities as

\[
|A(U_0 - U_H, w - \Pi w)| \leq C\|U_0 - U_H\|_1 \|w - \Pi w\|_1 \leq CH\|U_0 - U_H\|_1 \|w\|_2,
\]

and

\[
|A_H(U_H - \Pi U_0, \Pi w) - A(U_H - \Pi U_0, \Pi w)| \leq \left( e(\text{HMM}) + CH \right) \|U_0 - U_H\|_1 \|\Pi w\|_1.
\]

The last term at the right-hand side of (2.11) may be decomposed into

\[
A_H(\Pi U_0, \Pi w) - A(\Pi U_0, \Pi w) = \left( A_H(\Pi U_0, \Pi w) - \hat{A}_H(\Pi U_0, \Pi w) \right)
+ \left( \hat{A}_H(\Pi U_0, \Pi w) - A(\Pi U_0, \Pi w) \right).
\]

In view of (2.6), it follows that

\[
|\hat{A}_H(\Pi U_0, \Pi w) - A(\Pi U_0, \Pi w)| \leq CH^{k+1}\|U_0\|_{k+1} \|w\|_2.
\]

By definition, we get

\[
|A_H(\Pi U_0, \Pi w) - \hat{A}_H(\Pi U_0, \Pi w)| \leq e(\text{HMM})\|\nabla \Pi U_0\|_0 \|w\|_2.
\]

Combining the above estimates and using (2.10) leads to (4.1).

It remains to prove (1.13). For a given $z \in D$, we define the Green’s function $G^z \in H^1_0(D)$ and the discrete Green’s function $G_H^z \in X_H$ as

\[
A(G^z, V) = \partial_V(z) \quad \forall V \in H^1_0(D),
A(G_H^z, V) = \partial_V(z) \quad \forall V \in X_H.
\]

(2.12)

Here $\partial_V(z)$ is a generic notation for $\partial_z V(z)$. It is well-known [33]

\[
\|G^z - G_H^z\|_{1,1} \leq C \quad \text{and} \quad \|G_H^z\|_{1,1} \leq C|\ln H|.
\]

(2.13)
Using the definition of $G_z$ and $G_z^H$, a simple manipulation gives
\[
\partial(U_0 - U_H)(z) = A(G_z, U_0 - PU_0) + A(G_z, PU_0 - U_H)
\]
\[
= A(G_z - G_z^H, U_0 - PU_0) + A(G_z^H, U_0 - U_H)
\]
\[
= A(G_z - G_z^H, U_0 - PU_0) + A_H(U_H, G_z^H) - A(U_H, G_z^H)
\]
\[
= A(G_z - G_z^H, U_0 - PU_0) + A_H(IIU_0, G_z^H) - A_H(U_H, G_z^H)
\]
\[
+ \left(A_H(U_H - IIU_0, G_z^H) - A(U_H - IIU_0, G_z^H)\right).
\]

Using (2.13), we obtain
\[
\|U_0 - U_H\|_{1,\infty} \leq C\|U_0 - PU_0\|_{1,\infty} + |A(IIU_0, G_z^H) - A_H(IIU_0, G_z^H)|
\]
\[
+ |A(U_H - IIU_0, G_z^H) - A_H(U_H - IIU_0, G_z^H)|.
\]

Using (2.8) once again, we get
\[
|A(IIU_0, G_z^H) - A_H(IIU_0, G_z^H)| \leq \left(e(\text{HMM}) + CH^k\right) \sum_{K \in T_H} \|IIU_0\|_{k,K}\|\nabla G_z^H\|_{0,K}
\]
\[
\leq C\left(e(\text{HMM}) + H^k\right) \sum_{K \in T_H} \|IIU_0\|_{k,\infty,K}\|\nabla G_z^H\|_{0,1,K}
\]
\[
\leq C\left(e(\text{HMM}) + H^k\right) \ln H \|U_0\|_{k+1,\infty},
\]
where we have used the inverse inequality [10] on each element.

Similarly, we have
\[
|A(U_H - IIU_0, G_z^H) - A_H(U_H - IIU_0, G_z^H)|
\]
\[
\leq \left(e(\text{HMM}) + CH\right) \sum_{K \in T_H} \|U_H - II\|_{1,K}\|\nabla G_z^H\|_{0,K}
\]
\[
\leq C\left(e(\text{HMM}) + H\right) \sum_{K \in T_H} \|U_H - IIU_0\|_{1,\infty,K}\|\nabla G_z^H\|_{0,1,K}
\]
\[
\leq C\left(e(\text{HMM}) + H\right) \ln H \|U_0 - U_H\|_{1,\infty}
\]
\[
+ C\left(e(\text{HMM}) + H^k\right) \ln H \|U_0\|_{k+1,\infty}.
\]

A combination of the above three yields
\[
\|U_0 - U_H\|_{1,\infty} \leq CH^k + C\left(e(\text{HMM}) + H\right) \ln H \|U_0 - U_H\|_{1,\infty}
\]
\[
+ C\left(e(\text{HMM}) + H^k\right) \ln H \|U_0\|_{k+1,\infty}.
\]

Let $C_0 = \frac{1}{2C}$, then if $e(\text{HMM}) < C_0$, there exits a constant $H_0$ such that for all $H \leq H_0$, $C\left(e(\text{HMM}) + H\right) \ln H \leq 2Ce(\text{HMM}) < 1$. 

we thus obtain (1.13) and completes the proof.

Combining the proof for $L^2$ and $W^{1,\infty}$ estimates in the above lemma, using the Green’s function defined in [36], we obtain

**Remark 2.2.** Under the same condition for $W^{1,\infty}$ estimate in Theorem 1.1, we have

$$
\|U_0 - U_H\|_{L^\infty} \leq C \left( e(H^{\text{MM}}) + H^{k+1} \right) |\ln H|^2.
$$

3. Estimating $e(H^{\text{MM}})$

In this section, we estimate $e(H^{\text{MM}})$ for various cases.

3.1. Problems with locally periodic coefficients. We assume that $a^\varepsilon(x) = a(x, x/\varepsilon)$, $a^\varepsilon$ is smooth in $x$ and periodic in $y$ with period $I$. Define $\kappa = \lfloor \delta/\varepsilon \rfloor$, and we introduce $\hat{V}_\ell$ as

$$
\hat{V}_\ell(x) = V_\ell(x) + \varepsilon \chi^k \left( x, \frac{x}{\varepsilon} \right) \frac{\partial V_\ell}{\partial x_k}(x),
$$

where $\{\chi^j\}_{j=1}^d$ is defined as: For $j = 1, \cdots, d$, $\chi^j$ is periodic in $y$ with period $I$ and satisfy

$$
- \frac{\partial}{\partial y_i} \left( a_{ik} \frac{\partial \chi^j}{\partial y_k} \right)(x,y) = \frac{\partial}{\partial y_i} a_{ij}(x,y) \quad \text{in} \ I, \quad \int_I \chi^j(x,y) \, dy = 0.
$$

A direct calculation leads to

$$
\nabla W(x_\ell) \cdot A(x_\ell) \nabla V(x_\ell) = \nabla W_\ell(x) \cdot A(x_\ell) \nabla \hat{V}_\ell(x)
$$

$$
= \int_{I_\kappa(x_\ell)} \nabla W_\ell \cdot a \left( x_\ell, \frac{x}{\varepsilon} \right) \nabla \hat{V}_\ell \, dx.
$$

We first establish some estimates on the solution of the cell problem (1.7). $I_\delta(x_\ell)$ will be replaced by $I_\delta$ if there is no risk of confusion.

**Lemma 3.1.** There exists a constant $C$ independent of $\varepsilon$ and $\delta$ such that for each $\ell$,

$$
\| \nabla (v^\varepsilon_\ell - \hat{V}_\ell) \|_{0,I_\delta} \leq C \left( \left( \frac{\varepsilon}{\delta} \right)^{1/2} + \delta \right) \| \nabla \hat{V}_\ell \|_{0,I_\delta},
$$

and

$$
\| \nabla v^\varepsilon_\ell \|_{0,I_\delta \setminus I_{\kappa \ell}} \leq C \left( \left( \frac{\varepsilon}{\delta} \right)^{1/2} + \delta \right) \| \nabla V_\ell \|_{0,I_\delta}.
$$

**Proof.** Denote by $\hat{v}^\varepsilon_\ell$ the solution of (1.7) with the coefficients $a^\varepsilon(x)$ replaced by $a(x, x/\varepsilon)$. Obviously, we still have

$$
\| \nabla V_\ell \|_{0,I_\delta} \leq \| \nabla \hat{v}^\varepsilon_\ell \|_{0,I_\delta} \leq \sqrt{\frac{A}{\lambda}} \| \nabla \hat{V}_\ell \|_{0,I_\delta}.
$$
In view of the definitions of \( v^{\ell}_{\epsilon} \) and \( \hat{v}^{\ell}_{\epsilon} \), we get

\[
\lambda \| \nabla (v^{\ell}_{\epsilon} - \hat{v}^{\ell}_{\epsilon}) \|_{0, I_{0\delta}} \leq \int_{I_{0\delta}} \nabla (v^{\ell}_{\epsilon} - \hat{v}^{\ell}_{\epsilon}) \cdot a\left(x, \frac{x}{\epsilon}\right) \nabla (v^{\ell}_{\epsilon} - \hat{v}^{\ell}_{\epsilon}) \, dx
\]

\[
= \int_{I_{0\delta}} \nabla (v^{\ell}_{\epsilon} - \hat{v}^{\ell}_{\epsilon}) \cdot \left[ a\left(x_{\ell}, \frac{x}{\epsilon}\right) - a\left(x, \frac{x}{\epsilon}\right) \right] \nabla \hat{v}^{\ell}_{\epsilon} \, dx
\]

\[
\leq C\delta \| \nabla \hat{v}^{\ell}_{\epsilon} \|_{0, I_{0\delta}} \| \nabla (v^{\ell}_{\epsilon} - \hat{v}^{\ell}_{\epsilon}) \|_{0, I_{0\delta}},
\]

which together with (3.6) gives

\[
\| \nabla (v^{\ell}_{\epsilon} - \hat{v}^{\ell}_{\epsilon}) \|_{0, I_{0\delta}} \leq C\delta \| \nabla V^{\ell}_{\epsilon} \|_{0, I_{0\delta}}.
\]

Define \( \theta^{\ell}_{\epsilon} = \hat{v}^{\ell}_{\epsilon} - \hat{V}^{\ell} \), which obviously satisfies

\[
\begin{cases}
- \text{div} \left( a\left(x_{\ell}, \frac{x}{\epsilon}\right) \nabla \theta^{\ell}_{\epsilon}(x) \right) = 0, & \text{in } I_{\delta}(x_{\ell}), \\
\theta^{\ell}_{\epsilon}(x) = -\epsilon \chi^{k}\left(x_{\ell}, \frac{x}{\epsilon}\right) \frac{\partial \hat{V}^{\ell}}{\partial x_{k}}(x), & \text{on } \partial I_{\delta}(x_{\ell}).
\end{cases}
\]

It is seen that \( \theta^{\ell}_{\epsilon} \) is just the ”boundary layer correction” for the cell problem (1.7) [6]. Rescaling the problem by introducing \( x' = x/\delta \), we get a problem over \( I \) with \( \epsilon' = \epsilon/\delta \). Invoking a well-known result for this rescaled homogenization problem [41, (1.51) in §1.4], we obtain

\[
\| \nabla \theta^{\ell}_{\epsilon} \|_{0, I_{\delta}} \leq C \left( \frac{\epsilon}{\delta} \right)^{1/2} \| \nabla V^{\ell}_{\epsilon} \|_{0, I_{\delta}},
\]

which together with (5.24) leads to (3.4).

A straightforward calculation gives

\[
\| \nabla \hat{V}^{\ell} \|_{0, I_{\delta}\setminus I_{\delta\epsilon}} \leq C \left( \frac{\epsilon}{\delta} \right)^{1/2} \| \nabla V^{\ell}_{\epsilon} \|_{0, I_{\delta}}.
\]

Using (3.4) we have

\[
\| \nabla v^{\ell}_{\epsilon} \|_{0, I_{\delta}\setminus I_{\delta\epsilon}} \leq \| \nabla \hat{V}^{\ell} \|_{0, I_{\delta}\setminus I_{\delta\epsilon}} + \| \nabla (v^{\ell}_{\epsilon} - \hat{V}^{\ell}) \|_{0, I_{\delta\epsilon}}
\]

\[
\leq \| \nabla \hat{V}^{\ell} \|_{0, I_{\delta}\setminus I_{\delta\epsilon}} + \| \nabla (v^{\ell}_{\epsilon} - \hat{V}^{\ell}) \|_{0, I_{\delta}}
\]

\[
\leq C \left( \left( \frac{\epsilon}{\delta} \right)^{1/2} + \delta \right) \| \nabla V^{\ell}_{\epsilon} \|_{0, I_{\delta}}.
\]

This gives (3.5).

□

**Theorem 3.2.**

\[
e(HMM) \leq C \left( \frac{\epsilon}{\delta} + \delta \right).
\]
Proof. In view of (3.3), we have
\[
\nabla W(x_\varepsilon) \cdot (A_H - A)(x_\varepsilon) \nabla V(x_\varepsilon) = \int_{I_s} \nabla w_\varepsilon \cdot a\left( x, \frac{x}{\varepsilon} \right) \nabla v_\varepsilon \, dx
\]
\[
- \int_{I_{ne}} \nabla W_\varepsilon \cdot a\left( x, \frac{x}{\varepsilon} \right) \nabla \hat{V}_\varepsilon \, dx,
\]
with
\[
\int_{I_s} \nabla w_\varepsilon \cdot a\left( x, \frac{x}{\varepsilon} \right) \nabla v_\varepsilon \, dx = I_1 + \cdots + I_4,
\]
where
\[
I_1 = \int_{I_s} \nabla w_\varepsilon \cdot \left[ a\left( x, \frac{x}{\varepsilon} \right) - a\left( x, \frac{x}{\varepsilon} \right) \right] \nabla v_\varepsilon \, dx,
\]
\[
I_2 = \int_{I_s} \nabla w_\varepsilon \cdot a\left( x, \frac{x}{\varepsilon} \right) \nabla (v_\varepsilon - \hat{v}_\varepsilon) \, dx,
\]
\[
I_3 = \int_{I_s} \nabla w_\varepsilon \cdot a\left( x, \frac{x}{\varepsilon} \right) \nabla (\hat{v}_\varepsilon - \hat{V}_\varepsilon) \, dx,
\]
\[
I_4 = \int_{I_s} \nabla w_\varepsilon \cdot a\left( x, \frac{x}{\varepsilon} \right) \nabla \hat{V}_\varepsilon \, dx.
\]

Using (2.4), we have
\[
|I_1| \leq C \delta^{1-d} \| \nabla v_\varepsilon \|_{0, I_s} \| \nabla w_\varepsilon \|_{0, I_s} \leq C \delta^{1-d} \| \nabla V_\varepsilon \|_{0, I_s} \| \nabla W_\varepsilon \|_{0, I_s} = C \delta |\nabla V_\varepsilon| |\nabla W_\varepsilon|.
\]

Using (5.24) and (2.4), we bound $I_2$ as
\[
|I_2| \leq \delta^{-d} \| \nabla (v_\varepsilon - \hat{v}_\varepsilon) \|_{0, I_s} \| \nabla w_\varepsilon \|_{0, I_s} \leq C \delta^{1-d} \| \nabla V_\varepsilon \|_{0, I_s} \| \nabla W_\varepsilon \|_{0, I_s} = C \delta |\nabla V_\varepsilon| |\nabla W_\varepsilon|.
\]

In view of the symmetry of $a^\varepsilon$, $I_3$ can be rewritten as
\[
I_3 = \int_{I_s} \nabla \theta_\varepsilon \cdot a\left( x, \frac{x}{\varepsilon} \right) \nabla w_\varepsilon \, dx + \int_{I_s} \nabla \theta_\varepsilon \cdot \left[ a\left( x, \frac{x}{\varepsilon} \right) - a\left( x, \frac{x}{\varepsilon} \right) \right] \nabla w_\varepsilon \, dx.
\]
The first term at the right-hand side of the above identity can be further decomposed into
\[
\int_{I_s} \nabla \theta_\varepsilon \cdot a\left( x, \frac{x}{\varepsilon} \right) \nabla \theta_\varepsilon \, dx = \int_{I_s} \nabla \left( \theta_\varepsilon^\varepsilon \varepsilon k \right) \left( x, x, \frac{x}{\varepsilon} \right) \frac{\partial V_\varepsilon}{\partial x_k} \left( 1 - \rho_\varepsilon \right) \cdot a\left( x, \frac{x}{\varepsilon} \right) \nabla w_\varepsilon \, dx
\]
\[
- \int_{I_s} \nabla \left( \varepsilon k \left( x, x, \frac{x}{\varepsilon} \right) \frac{\partial V_\varepsilon}{\partial x_k} \left( 1 - \rho_\varepsilon \right) \cdot a\left( x, \frac{x}{\varepsilon} \right) \nabla w_\varepsilon \, dx.
\]
where $\rho_\varepsilon(x) \in C^\infty_0(I_s)$, $|\nabla \rho_\varepsilon| \leq C/\varepsilon$, and
\[
(3.12) \quad \rho_\varepsilon(x) = \begin{cases} 1 & \text{dist}(x, \partial I_s) \geq 2\varepsilon, \\ 0 & \text{dist}(x, \partial I_s) \leq \varepsilon. \end{cases}
\]
An integration by parts makes the first term on the right-hand side of $I_3$ vanish due to (1.7) and $\theta^\varepsilon_j + \varepsilon \chi (x, \varepsilon^{(1 - \rho^\varepsilon)}(1 - \rho^\varepsilon) \in H_0^1(I_\delta)$. We expand $I_3$ into

$$I_3 = - \int_{I_\delta} a_{ij} \left( x, \frac{x}{\varepsilon} \right) \frac{\partial w^\varepsilon_i}{\partial x_j} \frac{\partial w^\varepsilon_k}{\partial x_i} \frac{\partial V_\ell}{\partial x_k} (1 - \rho^\varepsilon) \, dx + \varepsilon \int_{I_\delta} a_{ij} \left( x, \frac{x}{\varepsilon} \right) \frac{\partial w^\varepsilon_i}{\partial x_j} \varepsilon^k \frac{\partial V_\ell}{\partial x_k} \frac{\partial \rho^\varepsilon}{\partial x_j} \, dx$$

$$+ \int_{I_\delta} \nabla \theta^\varepsilon_j \cdot \left[ a \left( x, \frac{x}{\varepsilon} \right) - a \left( x, \frac{x}{\varepsilon} \right) \right] \nabla w^\varepsilon_i \, dx.$$

Invoking (3.5), we bound $I_3$ as

$$|I_3| \leq C \left( \frac{\varepsilon}{\delta} \right)^{1/2} \delta^{1-d} \| \nabla V_\ell \|_{0,I_\delta} \| \nabla W_\ell \|_{0,I_\delta} + C \left( \frac{\varepsilon}{\delta} \right)^{1/2} \delta^{1-d} \| \nabla V_\ell \|_{0,I_\delta} \| \nabla W_\ell \|_{0,I_\delta}$$

$$\leq C \left( \frac{\varepsilon}{\delta} + \delta^2 + (\varepsilon \delta)^{1/2} \right) \| \nabla W_\ell \| \| \nabla V_\ell \|.$$

Using (3.2) and integrating by parts give

$$I_4 = \int_{I_\delta} \nabla W_\ell \cdot a \left( x, \frac{x}{\varepsilon} \right) \nabla \hat{V}_\ell \, dx.$$

So we have

$$I_4 - \int_{I_{ns}} \nabla W_\ell \cdot a \left( x, \frac{x}{\varepsilon} \right) \nabla \hat{V}_\ell \, dx = \left( \frac{1}{\delta^d} - \frac{1}{| \kappa \varepsilon |^d} \right) \int_{I_{ns}} \nabla W_\ell \cdot a \left( x, \frac{x}{\varepsilon} \right) \nabla \hat{V}_\ell \, dx$$

$$+ \frac{1}{\delta^d} \int_{I_\delta \setminus I_{ns}} \nabla W_\ell \cdot a \left( x, \frac{x}{\varepsilon} \right) \nabla \hat{V}_\ell \, dx.$$

Invoking (3.10), we get

$$\left| \frac{1}{\delta^d} - \frac{1}{| \kappa \varepsilon |^d} \right| \int_{I_{ns}} \nabla W_\ell \cdot a \left( x, \frac{x}{\varepsilon} \right) \nabla \hat{V}_\ell \, dx \leq C \left( \frac{\varepsilon}{\delta} \right)^{d-1} \| \nabla \hat{V}_\ell \|_{0,I_{ns}} \| \nabla W_\ell \|_{0,I_{ns}}$$

$$\leq C \left( \frac{\varepsilon}{\delta} \right)^{d-1} \| \nabla \hat{V}_\ell \|_{0,I_\delta} \| \nabla W_\ell \|_{0,I_\delta}$$

$$\leq C \left( \frac{\varepsilon}{\delta} \right)^{d-1} \| \nabla \hat{V}_\ell \| \| \nabla W_\ell \|,$$

and

$$\delta^{-d} \int_{I_\delta \setminus I_{ns}} \nabla W_\ell \cdot a \left( x, \frac{x}{\varepsilon} \right) \nabla \hat{V}_\ell \, dx \leq C \left( \frac{1}{\delta^d} \right) \| \nabla \hat{V}_\ell \|_{0,I_\delta \setminus I_{ns}} \| \nabla W_\ell \|_{0,I_\delta \setminus I_{ns}}$$

$$\leq C \left( \frac{\varepsilon}{\delta} + \delta^2 \right) \| \nabla \hat{V}_\ell \| \| \nabla W_\ell \|,$$

Combining the estimates for $I_1, I_2, I_3$ and the above two estimates give the desired result (3.11).

□
If we know the exact local period of the coefficients a-priori, we do not need to solve the problem over \( I_\delta \). It is enough to solve the cell problem (1.7) over one period \( I_\epsilon \) with periodic boundary condition. In this case (5.24) is valid with \( \delta \) replaced by \( \epsilon \) and \( \hat{v}_\ell^{\epsilon}(x) = \hat{V}_\ell(x) \) for \( 1 \leq \ell \leq L \), so a direct calculation yields

\[
\nabla W(x_\ell) \cdot (A_H - A)(x_\ell) \nabla V(x_\ell) = \int_{I_\epsilon} \nabla w_\ell^{\epsilon} \cdot \left[ a \left( x, \frac{x}{\epsilon} \right) - a \left( x_\ell, \frac{x}{\epsilon} \right) \right] \nabla v_\ell^{\epsilon} \, dx
\]

\[
+ \int_{I_\epsilon} \nabla w_\ell^{\epsilon} \cdot a \left( x_\ell, \frac{x}{\epsilon} \right) \nabla (v_\ell^{\epsilon} - \hat{v}_\ell^{\epsilon}) \, dx.
\]

Invoking (3.3) once again, we get

\[
e(HMM) \leq C\epsilon.
\]

Similarly, if \( \delta \) is an integer multiple of \( \epsilon \), and the periodic boundary condition is used, then we have

\[
e(HMM) \leq C\delta.
\]

**Remark 3.3.** An explicit expression for \( v_\ell^{\epsilon} \) is available in one dimension, from which we may show that the bound for \( e(HMM) \) is sharp.

### 3.2. Problems with random coefficients.

Here we estimate \( e(HMM) \) for the random case. Our basic strategy follows that of [39].

Denote by \((\Omega, \mathcal{F}, P)\) a probability space and let \( a(y, \omega) = (a_{ij}(y, \omega)) \) be a random field, whose statistics is invariant under integer shifts, and satisfies a uniform ellipticity condition that there exist constants \( \lambda \) and \( \Lambda \) such that

\[
\lambda |\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij}(y, \omega) \xi_i \xi_j \leq \Lambda |\xi|^2,
\]

for almost all \( y \in \mathbb{R}^d \) and \( \omega \in \Omega \). For \( j = 1, \cdots, d \), denote by \( \varphi_j(y, \omega) \) the solutions of the cell problem:

\[
(3.13) \quad L_y \varphi_j: = - \text{div}_y \left( a_{ij}(y, \omega) \nabla_y \varphi_j \right) = \text{div}_y (a_{ij} \cdot e_j).
\]

where \( \{e_j\}_{j=1}^{d} \) are the canonic basis in \( \mathbb{R}^d \). \( \nabla \varphi_j \) is required to be stationary under integer shift. The existence of \( \varphi_j \) is proved in [32, 25]. In general \( \varphi_j \) is not stationary. The homogenization coefficient \( A \) [32, 25] is given by

\[
A = \langle a(I + \nabla \varphi) \rangle.
\]

Here and in the following, we use the notation

\[
\langle f \rangle = \mathbb{E} \int_{[0,1]^d} f(y) \, dy,
\]
and
\[ [f; m] = \frac{1}{m^d} \int_{[0, m]^d} f(y) \, dy. \]

As in [39], we will consider the following auxiliary problem:
\[ (3.14) \quad \mathcal{L}u + \rho u = f, \]
for any \( \rho > 0 \), where \( f \) is of the form
\[ f = \sum_{j=1}^{d} D_j g_j + h, \]
with \( g_j, h \in \rho G \) which is defined as
\[ \rho G := \{ \psi \mid \langle |\psi|^2 \rangle \leq G^2 \}, \]
and \( \psi \) a stationary random field, whose statistics is stationary with respect to integer shifts.

The solution of (3.14) can be expressed with the help of a diffusion process \( \eta \) generated by the operator \( \mathcal{L} \).

For each fixed realization of \( \{ a(y, \cdot) \} \), denote by \( \eta_x \) the diffusion process generated by \( \mathcal{L}_1 \), and \( M_x \) the expectation with respect to \( \eta_x \). Let
\[ \Gamma(\tau) = \int_0^\tau f(\eta(t))e^{-\rho t} \, dt, \]
then it is well-known [21] that the solution of (3.14) is given by
\[ u(x) = M_x \Gamma(\infty). \]
The following results are either standard or proved in [39, 25].

**Lemma 3.4.** If \( u \) is the solution of (3.14), then there exists constants independent of \( \rho \) such that
\[ (3.15) \quad \langle |\nabla u|^2 \rangle + \rho \langle u^2 \rangle \leq C \left( \langle g^2 \rangle + \frac{1}{\rho} \langle h^2 \rangle \right). \]
\[ (3.16) \quad \langle (M_x \Gamma(\infty))^2 \rangle^{1/2} \leq C \frac{G^2}{\rho}. \]
\[ (3.17) \quad \langle M_x(\Gamma(\infty) - \Gamma(s))^2 \rangle \leq C \frac{G^2}{\rho} e^{-2s\rho}. \]

Because of the lowest order term \( \rho u \), the Green’s function associated with the operator \( \mathcal{L}_1 + \rho I \) decays exponentially with rate \( \mathcal{O}(\sqrt{\rho}) \). To make this statement precise, we define the norm \( \|x\| := \max_i |x_i|, \) and
\[ Q_\rho := \left\{ x \in \mathbb{R}^d \mid \|x\| \leq \left( \frac{1}{\rho} \right)^{\frac{1}{2}(\ln \frac{1}{\rho})^{1/2}} \right\}. \]

Let \( \tau \) be the first exit time of \( Q_\rho \) starting at \( x \in Q_\rho \). Denote by \( \hat{\varphi}_\rho(x) = M_x \Gamma(\tau) \).
Lemma 3.5. [39] If $\rho$ is sufficiently small, then
\begin{equation}
E \int_{\|x\| \leq 10} |\varphi_\rho(x) - \hat{\varphi}_\rho(x)|^2 \, dx \leq C G^2 e^{-C \ln(1/\rho)^2}.
\end{equation}
\begin{equation}
E \int_{\|x\| \leq 1} |\nabla \varphi_\rho(x) - \nabla \hat{\varphi}_\rho(x)|^2 \, dx \leq C G^2 e^{-C \ln(1/\rho)^2}.
\end{equation}

To prepare for the discussion on the consequence of the mixing condition, we mention

Lemma 3.6. [39] Let $\{a_{ij}, g_j\}$ and $\{\tilde{a}_{ij}, \tilde{g}_j\}$ be two sets of data such that $\{a_{ij}(y), g_j(y)\} = \{\tilde{a}_{ij}(y), \tilde{g}_j(y)\}$ for $y \notin B$, where $B$ is a domain in $\mathbb{R}^d$, and let $\varphi_\rho$ and $\hat{\varphi}_\rho$ be the solutions of (3.14) associated with $\{a_{ij}, g_j\}$ and $\{\tilde{a}_{ij}, \tilde{g}_j\}$, respectively (with $h = 0$), then
\begin{equation}
\int_{\mathbb{R}^d} |\varphi_\rho(x) - \hat{\varphi}_\rho(x)|^2 \, dx \leq \frac{C}{\rho} \int_{\mathbb{R}^d} (G^2 + |\nabla \varphi_\rho(x)|^2) I_B(x) \, dx,
\end{equation}
where $I_B$ is the indicator function of the domain $B$.

Now we introduce the crucial mixing condition. Let $B$ be a domain in $\mathbb{R}^d$. Denote by $\mathcal{F}(B)$ the $\sigma$-algebra generated by $\{a_{ij}(y, \omega), y \in B\}$. Let $\xi, \eta$ be two random variables that are measurable with respect to $\mathcal{F}(B_1)$ and $\mathcal{F}(B_2)$, respectively, then

\begin{equation}
|\mathbb{E}[\xi \eta] - \mathbb{E}[\xi] \mathbb{E}[\eta]| \leq \lambda q,
\end{equation}
where $q = \text{dist}(B_1, B_2)$.

Lemma 3.7. Under the condition (A), we have
\begin{equation}
E[\varphi_\rho; m] \leq C \left( \frac{G^2}{\rho} \left( \frac{\ln(1/\rho)^2}{\rho^{1/2} m} \right)^d + e^{-c(\ln(1/\rho)^2)} \right).
\end{equation}

Proof. For $\ell = (\ell_1, \cdots, \ell_d) \in \mathbb{Z}^d \cap [0, m]^d = \mathbb{Z}^d_m$. Denote by $I_\ell$ the cube of size 1 centered at $\ell + \frac{1}{2} = (\ell_1 + \frac{1}{2}, \cdots, \ell_d + \frac{1}{2})$, and let $\varphi^\ell = \int_{I_\ell} \varphi(x) \, dx$, then

\begin{equation}
[\varphi; m] = \frac{1}{m^d} \sum_{\ell \in \mathbb{Z}^d_m} \varphi^\ell.
\end{equation}

We first estimate $E\varphi^\ell \varphi^k$. If $|\ell - k| \leq C\rho^{-1/2}|\ln(1/\rho)|^2$, then
\begin{equation}
E\varphi^\ell \varphi^k \leq (\varphi_\rho^2(x))^{1/2} (\varphi_\rho^2(x))^{1/2} \leq C \frac{G^2}{\rho}.
\end{equation}

If $|\ell - k| \geq C\rho^{-1/2}|\ln(1/\rho)|^2$, then let $B_1 = \ell + B(\rho^{-1/2}|\ln(1/\rho)|^2)$ and $B_2 = k + B(\rho^{-1/2}|\ln(1/\rho)|^2)$, where $B(s)$ is a ball of size $s$ in the norm $\| \cdot \|$. Denote
by \( \tilde{\varphi}_1(x) \) the solution of (3.14) in which the coefficient \( \left( a_{ij}(y, \omega) \right) \) is modified in \( \mathbb{R}^d \setminus B_1 \) such that it is measurable with respect to \( \mathcal{F}(B_1) \), and similarly \( \tilde{\varphi}_2(x) \) the solution of (3.14) in which the coefficient \( \left( a_{ij}(y, \omega) \right) \) is modified in \( \mathbb{R}^d \setminus B_2 \) such that it is measurable with respect to \( \mathcal{F}(B_2) \). The modified coefficients \( \left( \tilde{a}_{ij}(y, \omega) \right) \) should still satisfy the condition on \( a_{ij} \) listed in the beginning of this subsection. Using \( \tilde{\varphi}_1 \) and \( \tilde{\varphi}_2 \), we can similarly define \( \tilde{\varphi}_1^\ell \) and \( \tilde{\varphi}_2^\ell \). Using (3.18), we have

\[
\mathbb{E}(\tilde{\varphi}_1^\ell - \tilde{\varphi}^\ell)^2 \leq CG^2 e^{-C|\ln(1/\rho)|^2},
\]

\[
\mathbb{E}(\tilde{\varphi}_2^\ell - \tilde{\varphi}^\ell)^2 \leq CG^2 e^{-C|\ln(1/\rho)|^2}.
\]

Since

\[
\mathbb{E}(\varphi^\ell \tilde{\varphi}^k) = \mathbb{E}(\varphi^\ell \varphi^k) + \mathbb{E}(\varphi^\ell - \varphi^k)\tilde{\varphi}^k + \mathbb{E}\tilde{\varphi}^\ell(\varphi^k - \tilde{\varphi}^k) + \mathbb{E}(\varphi^\ell - \varphi^k)(\varphi^k - \tilde{\varphi}^k),
\]

and

\[
|\mathbb{E}\tilde{\varphi}^\ell_1|, |\mathbb{E}\tilde{\varphi}^k_2| \leq CGe^{-C|\ln \frac{1}{\rho}|^2},
\]

\[
|\mathbb{E}\tilde{\varphi}^\ell_1\tilde{\varphi}^k_2| \leq C\left( \frac{G^2}{\rho} e^{-C|\ell-k|} + G^2 e^{-C|\ln \frac{1}{\rho}|^2} \right).
\]

We thus have

\[
|\mathbb{E}\varphi^\ell \tilde{\varphi}^k| \leq C\left( \frac{G^2}{\rho} e^{-C|\ell-k|} + G^2 e^{-C|\ln \frac{1}{\rho}|^2} \right).
\]

Hence

\[
\mathbb{E}[\varphi^\ell; m]^2 = \frac{1}{m^{2d}} \sum_{\ell, k \in \mathbb{Z}^d_m} \mathbb{E}\varphi^\ell \varphi^k
\]

\[
= \frac{1}{m^{2d}} \left( \sum_{|\ell-k| \leq \rho^{-1/2}|\ln(1/\rho)|^2} \mathbb{E}\varphi^\ell \varphi^k + \sum_{|\ell-k| > \rho^{-1/2}|\ln(1/\rho)|^2} |\ln(1/\rho)|^2 \right) \mathbb{E}\varphi^\ell \varphi^k
\]

\[
\leq \frac{1}{m^{2d}} \left( \frac{G^2}{\rho} m^d \left( \rho^{-1/2}|\ln(1/\rho)|^2 \right)^d + m^{2d} G^2 e^{-C|\ln(1/\rho)|^2} \right.
\]

\[
\times \frac{G^2}{\rho} \sum_{|\ell-k| > \rho^{-1/2}|\ln(1/\rho)|^2} e^{-C|\ell-k|}
\]

\[
\leq \frac{CG^2}{\rho} \left( \frac{\ln(1/\rho)}{\rho^{1/2} m} \right)^d + e^{-C|\ln(1/\rho)|^2}.
\]

\[\square\]

Proceeding along the same line as in [39, Theorem 2.1], using condition (A), we have
Lemma 3.8. For any $0 < \gamma < 1/2$, under condition (A), there exists a constant $C$ such that
\begin{equation}
|A - \langle a(I + \nabla \chi_\rho) \rangle| \leq C \rho^{\frac{\gamma - 2\gamma_s}{4 + \delta}},
\end{equation}
where $\chi_\rho = (\chi_{1,\rho}, \ldots, \chi_{d,\rho})$, and $\chi_{k,\rho}$ is the solution of (3.13) with $g = (a_k, \ldots, a_k)$.

Now we are ready to estimate $e(\text{HMM})$. Define $m = \frac{\delta}{2\eta}$ and denote by $\varphi_j^m$ the solution of (3.13) on $I_m = [0, m]^d$ with the boundary condition that $\varphi_j^m(y) = 0$ on $\partial I_m$, and let $\varphi_j^m = (\varphi_1^m, \ldots, \varphi_d^m)$, then
\[ e(\text{HMM}) = |A - [a(I + \nabla \varphi^m)]_m|. \]
Define $\varphi_\rho, \varphi_\rho^m$ similarly. We have
\[ e(\text{HMM}) \leq E_1 + E_2 + E_3, \]
where
\begin{align*}
E_1 &= |A - [a(I + \nabla \varphi_\rho)]_m|, \\
E_2 &= |[a(I + \nabla \varphi_\rho) - a(I + \nabla \varphi_\rho^m)]_m|, \\
E_3 &= |[a \nabla (\varphi_\rho^m - \varphi^m)]_m|. \\
\end{align*}

Obviously,
\[ E_1 = |A - \langle a(I + \nabla \varphi_\rho) \rangle + [\tilde{\psi}]_m|, \]
with $\tilde{\psi} = \langle a(I + \nabla \varphi_\rho) \rangle - a(I + \nabla \varphi_\rho)$. It was proved in [39, Lemma 2.5] that
\[ \mathbb{E} |\tilde{\psi}|_m \leq \left( \mathbb{E} |\tilde{\psi}|_m^2 \right)^{1/2} \leq C \left( \frac{\ln \frac{1}{\rho}}{\rho^{1/2}} \right)^{d/2}. \]
The above inequality together with Lemma 3.8 gives
\[ \mathbb{E} E_1 \leq C \left( G \rho^{\frac{\gamma - 2\gamma_s}{4 + \delta}} + \left( \frac{\ln \frac{1}{\rho}}{\rho^{1/2}} \right)^{d/2} \right). \]
To estimate $E_2$, denote by $\tau_m$ the first exist time for the domain $I_{2m}$, then $\varphi_\rho^{2m} = Mx \Gamma_\rho(\tau_m)$. For any $s > 0$,
\begin{align*}
|\varphi_\rho - \varphi_\rho^{2m}| &= |Mx (\Gamma(\infty) - \Gamma(\tau_m))| \\
&\leq Mx \left\{ |\Gamma(\infty)| + |\Gamma(\tau_m)|; \tau_m \leq s \right\} + Mx \{ e^{-s \rho_\rho} M_{\rho(s)} |\Gamma(\infty) - \Gamma(\tau_m)|; \tau_m > s \} \\
&\leq C \left( Mx (|\Gamma(\infty)|^2 + |\Gamma(\tau_m)|^2) \right)^{1/2} \{ P_x (\tau_m \leq s)^{1/2} + e^{-s \rho} \},
\end{align*}

since
\[ P \{ \tau_m \leq s \} \leq e^{-Cm^2/s}, \]
we get
\[ \mathbb{E} [|\varphi_\rho - \varphi_\rho^{2m}|_m^2]_{2m} \leq C \frac{G^2}{\rho} \left( e^{-Cm^2/s} + e^{-s \rho} \right)^2. \]
Optimizing in \( s \), we get
\[
\mathbb{E}[|\varphi_\rho - \varphi_\rho^{2m}|^2] \leq C \frac{G^2}{\rho} e^{-C\rho^2}.
\]

Using standard interior estimates, we have
\[
\mathbb{E}E_2 \leq C \mathbb{E}[|\nabla(\varphi_\rho - \varphi_\rho^m)|^2] \leq \frac{C}{m} \mathbb{E}[|\varphi_\rho - \varphi_\rho^{2m}|^2] \leq \frac{CG^2}{m\rho} e^{-C\rho^2}.
\]

As for \( E_3 \), proceeding along the same line as in the estimate of \( E_1 \), we get
\[
\mathbb{E}E_3 \leq C \left( G^2 \rho^{\frac{d-2}{4+\gamma}} + \left( \frac{|\ln \frac{1}{\rho}|^2}{\rho^{1/2} m} \right)^{d/2} \right).
\]

To sum up, we have
\[
\mathbb{E}e(HMM) \leq C \left( G^2 \rho^{\frac{d-2}{4+\gamma}} + \left( \frac{|\ln \frac{1}{\rho}|^2}{\rho^{1/2} m} \right)^{d/2} + \frac{CG^2}{m\rho} e^{-C\rho^2} \right).
\]

Optimizing in \( \rho \) with respect to the first two terms, we get \( \rho_0 = m^{-\frac{2d}{d+4\gamma}} \). Hence
\[
(3.23) \quad \mathbb{E}e(HMM) \leq C \left( \frac{|\ln m|^d}{m^\kappa} + \frac{CG^2}{m\rho_0} e^{-C\rho_0^2} \right) \leq C \frac{|\ln m|^d}{m^\kappa},
\]

with
\[
\kappa = \frac{d/2}{1 + \frac{d(d + 4)/4}{d - 2 - 2\gamma}}, \quad 0 < \gamma < \frac{1}{2}.
\]

Obviously, the \(|\ln m|^d\) factor in (3.23) can be absorbed into the factor \( m^{-\kappa} \). This inequality together with Theorem 1.2 completes the proof of Theorem 1.1 for the random case.

**Remark 3.9.** The estimate (3.23) is unlikely to be optimal. In \( 1 - d \), a direct calculation gives:
\[
\mathbb{E}e(HMM) \leq C \left( \frac{\epsilon}{\delta} \right)^{1/2},
\]

whereas the estimate (3.23) does not apply for \( d = 1 \) and \( d = 2 \). We may use the techniques in [13] to derive improved bounds for \( e(HMM) \) if the magnitude of the oscillation in the coefficients \( (a_0^\alpha) \) is sufficiently small.

4. **RECONSTRUCTION AND COMPRESSION**

4.1. **Reconstruction procedure.** It is seen from Theorem 1.1 that the HMM solution \( U_H \) is indeed a good approximation of the macroscopic solution \( U_0 \). Next we consider how \( u^\varepsilon \) can be approximated. We will restrict to \( k = 1 \).
Proof of Theorem 1.4. Subtracting (1.1) from (1.14), we obtain
\[
- \text{div} \left( \frac{a_\varepsilon(x)}{\varepsilon} \nabla (\overline{u}^\varepsilon - u^\varepsilon(x)) \right) = 0, \quad x \in \Omega_\eta, \\
\overline{u}^\varepsilon(x) - u^\varepsilon(x) = U_H(x) - u^\varepsilon(x), \quad x \in \partial \Omega_\eta.
\]
In view of the classic interior estimate for elliptic equations [23], we have
\[
\| \nabla (\overline{u}^\varepsilon - u^\varepsilon) \|_{0, \Omega} \leq \frac{C}{\eta} \| \overline{u}^\varepsilon - u^\varepsilon \|_{0, \Omega_\eta}.
\]
By Hopf maximum principle, we get
\[
\frac{1}{\eta^2} \int_{\Omega_\eta} |(\overline{u}^\varepsilon - u^\varepsilon)(x)|^2 \, dx \leq \frac{C}{\eta^2} \| \overline{u}^\varepsilon - u^\varepsilon \|_{L^\infty(\Omega_\eta)} \leq \frac{C}{\eta^2} \| U_H \|_{L^\infty(\partial \Omega_\eta)} \\
\leq \frac{C}{\eta^2} \left( \| U_0 - U_H \|_{L^\infty(\Omega_\eta)} + \| u^\varepsilon - U_0 \|_{L^\infty(\Omega_\eta)} \right).
\]
A combination of the above two implies (1.4). \qed

Proof of Theorem 1.5. Denote by \( \hat{u}^\varepsilon \) the solution of (1.7) with \( V_\ell = U_H \) and the coefficients \( a^\varepsilon(x) \) replaced by \( a(x_K, x/\varepsilon) \), where \( x_K \) is the barycenter of \( K \). In light of (5.24), we have
\[
\| \nabla (\overline{u}^\varepsilon - \hat{u}^\varepsilon) \|_{0, I_\varepsilon(x_K)} \leq C \varepsilon \| \nabla U_H \|_{0, I_\varepsilon(x_K)}.
\]
Due to the construction of \( \overline{u}^\varepsilon \) and the expression of \( \hat{u}^\varepsilon \) (3.1), we have for any cell \( I_\varepsilon(x) \), where \( x_1 \) is an arbitrary point in \( K \),
\[
\| \nabla (\overline{u}^\varepsilon - \hat{u}^\varepsilon) \|_{0, I_\varepsilon(x_1)} = \| \nabla (\overline{u}^\varepsilon - \hat{u}^\varepsilon) \|_{0, I_\varepsilon(x_K)}.
\]
Since \( K \) is compact, we may find a finite covering such that \( K \subset \bigcup_{x \in K} I_\varepsilon(x) \), so combining the above two estimates and notice that \( | \bigcup_{x \in K} I_\varepsilon(x) | \leq C |K| \), we get
\[
(4.1) \quad \| \nabla (\overline{u}^\varepsilon - \hat{u}^\varepsilon) \|_{0, K} \leq C \varepsilon \| \nabla U_H \|_{0, K}.
\]
Invoking (3.1) once again, we have
\[
\frac{\partial \hat{u}^\varepsilon}{\partial x_i} = \frac{\partial U_H}{\partial x_i} + \frac{\partial \chi^k}{\partial y_i} \left( x_K, \frac{x}{\varepsilon} \right) \frac{\partial U_H}{\partial x_k},
\]
where \( x_K \) is the barycenter of \( K \). Define the 1-order approximation of \( u^\varepsilon \) as
\[
u^\varepsilon_1(x) = U_0 + \varepsilon \chi^k \left( x, \frac{x}{\varepsilon} \right) \frac{\partial U_0}{\partial x_k},
\]
where \( \{ \chi^k \}_{k=1}^d \) is the solutions of (3.2). Obviously,
\[
\frac{\partial u^\varepsilon_1}{\partial x_i} = \frac{\partial U_0}{\partial x_i} + \left( \varepsilon \frac{\partial \chi^k}{\partial x_i} + \frac{\partial \chi^k}{\partial y_i} \right) \left( x, \frac{x}{\varepsilon} \right) \frac{\partial U_0}{\partial x_k} + \varepsilon \chi^k \left( x, \frac{x}{\varepsilon} \right) \frac{\partial^2 U_0}{\partial x_i \partial x_k}.
\]
A combination of the above leads to
\[
\|\nabla (\hat{u}^\varepsilon - u_1^\varepsilon)\|_{0,K} \leq C\|\nabla (U_H - U_0)\|_{0,K} + C\varepsilon|U_0|_{1,K}
\]
\[
+ \left\| \left( \frac{\partial \chi^k}{\partial y_1}(x, \frac{x}{\varepsilon}) - \frac{\partial \chi^k}{\partial y_1}(x_K, \frac{x}{\varepsilon}) \right) \frac{\partial U_0}{\partial x_K} \right\|_{0,K} + C\varepsilon|U_0|_{2,K}.
\]
\[
\leq C(|U_0 - U_H|_{1,K} + C(\varepsilon + H)\|U_0\|_{2,K}.
\]

Summing up for all \(K \in T_H\) and using Theorem 1.1 for \(k = 1\) and the fact that \(I_\delta = I_\varepsilon\), we get
\[
\left( \sum_{K \in T_H} \|\hat{u}^\varepsilon - u_1^\varepsilon\|_{l,K}^2 \right)^{1/2} \leq C(\varepsilon + H),
\]
which together with the classical estimate for \(u^\varepsilon - u_1^\varepsilon\) [41, 28, 6], i.e.,
\[
\|u^\varepsilon - u_1^\varepsilon\|_1 \leq C\sqrt{\varepsilon},
\]
gives
\[
\left( \sum_{K \in T_H} \|\hat{u}^\varepsilon - u_1^\varepsilon\|_{l,K}^2 \right)^{1/2} \leq \|u^\varepsilon - u_1^\varepsilon\|_1 + \left( \sum_{K \in T_H} \|\hat{u}^\varepsilon - u_1^\varepsilon\|_{l,K}^2 \right)^{1/2}
\]
\[
\leq C(\sqrt{\varepsilon} + H).
\]
A combination of this estimate and (4.1) gives the desired result. \(\square\)

**Corollary 4.1.**

\[(4.2) \quad \|\pi^\varepsilon - u^\varepsilon\|_0 \leq C(\varepsilon + H^2).\]

**Proof.** Since \(\int_{I_\delta} (\pi^\varepsilon - \hat{u}^\varepsilon)(x) \, dx = 0\), so application of the Poincaré inequality gives
\[
\|\pi^\varepsilon - \hat{u}^\varepsilon\|_{0,I_\varepsilon(x_1)} \leq C\varepsilon\|\nabla(\pi^\varepsilon - \hat{u}^\varepsilon)\|_{0,I_\varepsilon(x_K)} \leq C\varepsilon^2\|\nabla U_H\|_{0,I_\varepsilon(x_K)}.
\]
As before for any \(I_\varepsilon(x_1)\) we have
\[
\|\pi^\varepsilon - \hat{u}^\varepsilon\|_{0,I_\varepsilon(x_1)} = \|\pi^\varepsilon - \hat{u}^\varepsilon\|_{0,I_\varepsilon(x_K)}.
\]
So
\[(4.3) \quad \|\pi^\varepsilon - \hat{u}^\varepsilon\|_{0,K} \leq C\varepsilon^2\|\nabla U_H\|_{0,K}.
\]
On each element \(K\), we have
\[
\|\hat{u}^\varepsilon - U_H\|_{0,K} \leq C\varepsilon\|\nabla U_H\|_{0,K}.
\]
A combination of the above two and summing up for all \(K \in T_H\), we get
\[
\|\pi^\varepsilon - U_H\|_0 \leq C\varepsilon\|\nabla U_H\|_0 \leq C\varepsilon,
\]
which together with
\[
\|u^\varepsilon - U_H\|_0 \leq \|u^\varepsilon - U_0\| + \|U_0 - U_H\|_0 \leq C(\varepsilon + H^2).
\]
leads to (4.2), where we have used the estimate for $U_0$ [41, 28, 6], i.e.,
\[ \|u^\varepsilon - U_0\|_0 \leq C\varepsilon. \]

\[ \Box \]

4.2. **Compression operator.** Compression operator (denoted by $Q$) maps the micro variables to the macro variables [17]. It plays an important role in the general framework of HMM, even though for the present problem, HMM can be formulated without explicitly specifying the compression operator beforehand. Typically the compression operator is some spatial/temporal averaging, or projection to some slow manifolds. It is of interest to consider the error bound for $Qu^\varepsilon - U_H$. We first list some natural properties of the compression operator.

- For any $\phi \in X$, $Q\phi \in X_H$.
- There exists a constant $C$ such that
  \[ \|Q\phi\|_0 \leq C\|\phi\|_0. \]
- If $\phi \in H^2(\Omega) \cap H^1_0(\Omega)$, then
  \[ \|Q\phi - \phi\|_0 \leq CH^2|\phi|_2. \]

**Theorem 4.2.** Assuming that $Q$ satisfies all three requirements, then
\[ (4.4) \quad \|Qu^\varepsilon - U_H\|_0 \leq C(\varepsilon + H^2). \]

Moreover, if $\mathcal{T}_H$ is quasi-uniform, then
\[ (4.5) \quad \|Qu^\varepsilon - U_H\|_1 \leq C\left(\frac{\varepsilon}{H} + H\right). \]

**Proof.** We decompose $Qu^\varepsilon - U_H$ into
\[ (4.6) \quad Qu^\varepsilon - U_H = Q(u^\varepsilon - U_0) + (QU_0 - U_0) + (U_0 - U_H). \]

Using the fact that $Q$ is $L^2$ bounded, we obtain
\[ \|Q(u^\varepsilon - U_0)\|_0 \leq C\|u^\varepsilon - U_0\|_0 \leq C\varepsilon. \]

In view of the second property of $Q$, we have
\[ \|QU_0 - U_0\|_0 \leq CH^2. \]

A combination of (4.1) and the first estimate in Theorem 1.2 gives
\[ \|U_0 - U_H\|_0 \leq C(\varepsilon + H^2). \]

A combination of these three implies (4.4), which together with the inverse inequality [10, 3.2.28] leads to (4.5).

\[ \Box \]

It remains to gives some examples of the compression operator $Q$. The following two types operators meet all three requirements.

- The $L^2$-projection operator onto $X_H$. 
Remark 4.3. Notice that in one-dimension, the standard Lagrange interpolant does not meet the second requirement. However, it is still possible to derive (4.5) via another approach. Moreover, a careful study of one-dimensional examples shows that the term $\varepsilon/H$ in (4.5) is sharp.

5. Nonlinear homogenization problems

5.1. Algorithms and main results. We consider the following nonlinear problem which has been discussed in [7].

\begin{equation}
\begin{aligned}
-\text{div} \left( a^\varepsilon(x, u^\varepsilon(x)) \nabla u^\varepsilon(x) \right) &= f(x) \quad \text{in } D, \\
 u^\varepsilon(x) &= 0 \quad \text{on } \partial D.
\end{aligned}
\end{equation}

In this section, we define $X := W^{1,p}_0(D)$ with $p > 1$ and $X_H$ is defined as the $P_k$ finite element subspace of $X$.

We assume that $a^\varepsilon(x, u^\varepsilon)$ satisfies

$$
\lambda |\xi|^2 \leq a^\varepsilon_{ij}\xi_i\xi_j \leq A|\xi|^2, \quad \forall \xi \in \mathbb{R}^d,
$$

with $0 < \lambda \leq A$. Moreover, we assume that $a^\varepsilon(x, z)$ is equi-continuous in $z$ uniformly with respect to $x$ and $\varepsilon$. Under these conditions, it is proved in [7] that $u^\varepsilon$ converges weakly in $X$ to the solution of the following problem:

\begin{equation}
\begin{aligned}
\mathcal{L}U_0: = -\text{div} \left( A(x, U_0(x)) \nabla U_0(x) \right) &= f(x) \quad \text{in } D, \\
 U_0(x) &= 0 \quad \text{on } \partial D.
\end{aligned}
\end{equation}

If we let

$$
A(v, w) = (A(x, v)\nabla v, \nabla w) \quad \forall v, w \in X,
$$

then

\begin{equation}
A(U_0, v) = (f, v) \quad \forall v \in X',
\end{equation}

where $X'$ is the dual space of $X$. The tensor $A(x, z)$ is also equi-continuous in $z$ uniformly with respect to $x$ and satisfies

$$
\lambda |\xi|^2 \leq A^\varepsilon_{ij}\xi_i\xi_j \leq \frac{A^2}{\lambda} |\xi|^2, \quad \forall \xi \in \mathbb{R}^d.
$$

The linearized operator of $\mathcal{L}$ at $U_0$ is defined for any $v \in H^1_0(D)$ by

$$
\mathcal{L}_{\text{lin}}(U_0)v = -\text{div} \left( A(x, U_0)\nabla v + A_p(x, U_0)\nabla U_0v \right),
$$

where $A_p(x, z) := \nabla_z A(x, z)$. $\mathcal{L}_{\text{lin}}$ induces a bilinear form through

$$
\hat{A}(u; v, w) = (A(x, u)\nabla v, \nabla w) + (A_p(x, u)\nabla u v, \nabla w) \quad \forall (v, w) \in H^1_0(D) \times H^1_0(D).
$$
Our basic assumption is that the linearized operator $L_{\text{lin}}$ is an isomorphism from $H^1_0(D)$ to $H^{-1}(D)$, so $U_0$ must be an isolated solution of (5.2).

To formulate HMM, for each quadrature points $x_\ell$, define $v^\ell_\varepsilon$ to be the solutions of:

\begin{equation}
\begin{cases}
- \text{div} \left( a^\varepsilon(x, v^\ell_\varepsilon) \nabla v^\ell_\varepsilon(x) \right) = 0 & x \in I_\delta(x_\ell), \\
v^\ell_\varepsilon(x) = V_\ell(x) & x \in \partial I_\delta(x_\ell).
\end{cases}
\end{equation}

$v^\ell_\varepsilon$ can be defined similarly.

For any $V, W \in X_H$, define

$$\nabla W(x_\ell) \cdot A_H(x_\ell, V(x_\ell)) \nabla V(x_\ell) = \frac{1}{\delta^d} \int_{I_\delta(x_\ell)} \nabla w^\ell_\varepsilon(x) \cdot a^\varepsilon(x, v^\ell_\varepsilon(x)) \nabla v^\ell_\varepsilon(x) \, dx,$$

and

$$A_H(V, W) := \sum_{K \in T_H} |K| \sum_{\tau \in K} \omega_{\tau} \nabla W(x_\tau) \cdot A_H(x_\tau, V(x_\tau)) \nabla V(x_\tau).$$

The HMM solution is given by the problem:

**Problem 5.1.** Find $U_H \in X_H$ such that

\begin{equation}
A_H(U_H, V) = (f, V) \quad \forall V \in X_H.
\end{equation}

It is observed that the solution of (5.3) still satisfies (2.3), (2.4) and

\begin{equation}
\|\nabla V_\ell\|_{0, I_\delta} \leq \|\nabla v^\ell_\varepsilon\|_{0, I_\delta} \leq \sqrt{\frac{\Lambda}{\chi}} \|\nabla V_\ell\|_{0, I_\delta}.
\end{equation}

For any $v, v_H, w \in X$, define

\begin{equation}
R(v, v_H, w) := A(v_H, w) - A(v, w) - \hat{A}(v; v_H - v, w).
\end{equation}

Here $R$ satisfies for $e_H := v - v_H$ and $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \geq 1$,

\begin{equation}
|R(v, v_H, w)| \leq C(M)(\|e_H\|_{0, 2p} + \|e_H \nabla e_H\|_{0, p}) \|\nabla w\|_{0, q}
\end{equation}

with any $v$ and $v_H$ satisfying $\|v\|_{1, \infty} + \|v_H\|_{1, \infty} \leq M$ (see [38, Lemma 3.1] for a similar result). Therefore we have

**Lemma 5.2.** $U_H \in X_H$ is the solution of Problem 5.1 if and only if

\begin{equation}
\hat{A}(U_0; U_0 - U_H, V) = R(U_0, U_H, V) + A_H(U_H, V) - A(U_H, V) \quad \forall V \in X_H.
\end{equation}

For any $V, W \in X_H$, define

\begin{equation}
E(V, W) := \nabla W(x_\ell) \cdot (A_H - A)(x_\ell, V(x_\ell)) \nabla V(x_\ell).
\end{equation}

Define $e(\text{HMM})$ as

\begin{equation}
e(\text{HMM}) = \max_{x_\ell \in K, K \in T_H, V, W \in X_H \cap W^{1, \infty}(D)} \frac{|E(V, W)|}{\|\nabla V_\ell\| \|\nabla W_\ell\|}.
\end{equation}
The existence and uniqueness of the solution of Problem 5.1 are proved in the following lemma.

**Lemma 5.3.** Assuming that $U_0 \in W^{2,p}(D)$ with $p > d$, then there exists a constant $H_0 > 0$ such that for $0 < H \leq H_0$, Problem 5.1 has a solution $U_H$ satisfying

\begin{align}
\|U_H - P_H U_0\|_{1,\infty} &\leq e(H\text{MM})^{1/2} + H^{1-d/p}, \\
\|U_0 - U_H\|_{1,\infty} &\leq C\left(e(H\text{MM})^{1/2} + H^{1-d/p}\right),
\end{align}

where $P_H U_0$ is defined as

\begin{align}
\hat{A}(U_0; P_H U_0, V) = \hat{A}(U_0; U_0, V) &\forall V \in X_H.
\end{align}

Moreover, if there exists a constant $\eta$ with $0 < \eta < 1$ such that

\begin{align}
\sum_{K \in T_H} \sum_{x \in K} \omega_K |E(V_1, W) - E(V_2, W)| &\leq \eta \|V_1 - V_2\|_1 \|W\|_1
\end{align}

for any $V_1, V_2$ and $W \in X_H \cap W^{1,\infty}(D)$, then there exists a constant $H_0 > 0$ such that for $0 < H \leq H_0$, the HMM solution $U_H$ is unique.

**Proof.** Notice that $L_{\text{lin}}$ is an isomorphism from $H^{1,0}(D)$ to $H^{1,0}(D)$ and using Schatz’s argument [34], we infer that there exists a constant $H_0 > 0$ such that for $0 < H \leq H_0$,

\begin{align}
\sup_{W \in X_H} \frac{\hat{A}(U_0; V, W)}{\|W\|_1} &\geq C \|V\|_1 &\forall V \in X_H.
\end{align}

So there is a unique solution $P_H U_0$ satisfying (5.12) and

\begin{align}
\|U_0 - P_H U_0\|_{1,\infty} &\leq C H^{1-d/p}.
\end{align}

Define a nonlinear mapping $T \colon X_H \to X_H$ by

\begin{align}
\hat{A}(U_0; T(V), W) = \hat{A}(U_0; U_0, W) - R(U_0, V, W) + A(V, W) - A_H(V, W),
\end{align}

for any $W \in X_H$. Obviously $T$ is continuous due to (5.14) and (5.7).

Define the set

\begin{align}
B = \{ V \in X_H \mid \|V - P_H U_0\|_{1,\infty} \leq e(H\text{MM})^{1/2} + H^{1-d/p} \}.
\end{align}

We claim that there exists a constant $H_0 > 0$ such that for all $0 < H \leq H_0$, $T(B) \subset B$. Notice that

\begin{align}
\hat{A}(U_0; T(V) - P_H U_0, W) = -R(U_0, V, W) + A(V, W) - A_H(V, W).
\end{align}
Taking $W = G^*_H$ as defined in (2.12)$_2$, we obtain

$$
\|T(V) - P_H U_0\|_{1, \infty} \leq C |\ln H| \|U_0 - V\|_{1, \infty}^2 + C \left( e(\text{HMM}) + H \right) |\ln H|
$$

$$
\leq C \left( \|U_0 - P_H U_0\|_{1, \infty}^2 + \|P_H U_0 - V\|_{1, \infty}^2 + e(\text{HMM}) + H \right) |\ln H|
$$

$$
\leq C \left( e(\text{HMM}) + H^{2-2d/p} + H \right) |\ln H|
$$

$$
\leq e(\text{HMM})^{1/2} + H^{1-d/p}.
$$

An application of Brouwer’s fixed point theorem gives the existence of a $U_H \in B$ such that $T(U_H) = U_H$. By definition, $U_H$ satisfies (5.11)$_1$. An application of the triangle inequality and (5.19) yield (5.11)$_2$.

Suppose that $U_H$ and $\hat{U}_H$ are both solutions of (5.4). Using (5.14) we obtain

$$
C \|U_H - \hat{U}_H\|_1 \leq \sup_{W \in X} \int_0^1 \frac{\hat{A}(U_H; U_H - \hat{U}_H, W)}{\|W\|_1} dt
$$

$$
\leq \sup_{W \in X} \frac{|A(U_H, W) - A(\hat{U}_H, W)|}{\|W\|_1},
$$

where $U_H^t = (1-t)\hat{U}_H + tU_H$. Notice that

$$
A(U_H, W) - A(\hat{U}_H, W) = \left( A(U_H, W) - A_H(U_H, W) \right)
$$

$$
- \left( A(\hat{U}_H, W) - A_H(\hat{U}_H, W) \right).
$$

In view of (5.13), we finally have

$$
\|U_H - \hat{U}_H\|_1 \leq (\eta + CH) \|U_H - \hat{U}_H\|_1.
$$

Hence, for $\eta < 1$, there exists a constant $H_0 > 0$ such that for $0 < H \leq H_0$,

$$
\eta + CH \leq \eta + CH_0 \leq \frac{1}{2} (1 + \eta) < 1,
$$

this implies $U_H = \hat{U}_H$. Therefore the HMM solution is unique. \qed

Based on the above lemma, we can prove a nonlinear analog of Theorem 1.1.

**Theorem 5.4.** Let $U_0$ and $U_H$ be solutions of (5.2) and (5.4), respectively. Assuming that $U_0 \in W^{k+1, \infty}$, then there exist $C_0$ ($0 < C_0 < 1$) and $\eta_0$ such that if

$$
e(\text{HMM}) < C_0, \quad \|U_0 - U_H\|_{1, \infty} \leq \eta_0,
$$

then there exists a constant $H_0$ such that for $0 < H < H_0$, we have

$$
\|U_0 - U_H\|_1 \leq C \left( H^k + e(\text{HMM}) \right),
$$

$$
\|U_0 - U_H\|_{1, \infty} \leq C \left( H^k + e(\text{HMM}) \right) |\ln H|.
$$
Proof. Note that $U_0 \in W^{k+1,\infty}$ and from (5.14), we have
\begin{equation}
(5.19) \quad \|U_0 - P_H U_0\|_1 \leq C H^k, \quad \|U_0 - P_H U_0\|_{1,\infty} \leq C H^k.
\end{equation}
Using (5.8) with $V = P_H U_0 - U_H$, invoking (5.14) and (2.5) we obtain
\[
\|P_H U_0 - U_H\|_1 \leq C \|U_0 - U_H\|^2_{1,4} + \left( e(\text{HMM}) + C H \right) \|P_H U_0 - U_H\|_1 \\
+ C \left( H^k + e(\text{HMM}) \right).
\]
If $e(\text{HMM}) < C_1 < 1$, then there exists a constant $H_0$ such that for $0 < H < H_0$,
\[
e(\text{HMM}) + C H \leq C_1 + C H_0 \leq \frac{1}{2} (1 + C_1) < 1,
\]
which implies
\[
\|P_H U_0 - U_H\|_1 \leq C \|U_0 - U_H\|^2_{1,4} + C \left( H^k + e(\text{HMM}) \right).
\]
Using the interpolation inequality we have
\[
\|U_0 - U_H\|^2_{1,4} \leq \|U_0 - U_H\|_1 \|U_0 - U_H\|_{1,\infty}.
\]
If $\|U_0 - U_H\|_{1,\infty} \leq \eta_1 = \frac{1}{2C}$, then combining the above two inequalities leads to (5.17).

Let $V = G^2_H$ in (5.8) and using (2.13) we obtain
\[
\|P_H U_0 - U_H\|_{1,\infty} \leq C \left( e(\text{HMM}) + H \right) \|P_H U_0 - U_H\|_{1,\infty} \\
+ \left( \|U_0 - U_H\|^2_{1,\infty} + e(\text{HMM}) + H^k \right) |\ln H|.
\]
If $e(\text{HMM}) < \frac{1}{2C}$, then there exists a constant $H_0$ such that for $0 < H < H_0$,
\[
C \left( e(\text{HMM}) + H \right) < \frac{1}{2} + CH_0 < 1,
\]
we thus have
\[
\|P_H U_0 - U_H\|_{1,\infty} \leq C \left( \|U_0 - U_H\|^2_{1,\infty} + e(\text{HMM}) + H^k \right) |\ln H|.
\]
If $\|U_0 - U_H\|_{1,\infty} \leq \eta_2 = \frac{1}{2C |\ln H|}$, then we have
\[
\|U_0 - U_H\|_{1,\infty} \leq \|U_0 - P_H U_0\|_{1,\infty} + C \eta_2 |\ln H| \|U_0 - U_H\|_{1,\infty} \\
+ C \left( H^k + e(\text{HMM}) \right) |\ln H|,
\]
which implies
\[
\|U_0 - U_H\|_{1,\infty} \leq \|U_0 - P_H U_0\|_{1,\infty} + C \left( H^k + e(\text{HMM}) \right) |\ln H|,
\]
this inequality together with the standard interpolation results gives (5.18).

To sum up, define $C_0 = \min(C_1, 1/(2C))$ and $\eta_0 = \min(\eta_1, \eta_2)$, and if (5.16) holds for such $\eta_0$, we get (5.17) and (5.18). □
5.2. Estimating \( e(\text{HMM}) \). It remains to estimate \( e(\text{HMM}) \) and verify the assumptions (5.13) and (5.16). We assume that \( a^\varepsilon(x, u^\varepsilon) = (a_{ij}(x, x/\varepsilon, u^\varepsilon)) \), and for \( 1 \leq i, j \leq d \), the coefficients \( a_{ij}^\varepsilon(x, y, z) \) are smooth in \( x, z \) and periodic in \( y \) with period \( I \). These types of problems have been considered in [6, 7, 22]; among others. The homogenized coefficient \( A = (A_{ij}(x, p)) \) is given for any \( p \in \mathbb{R} \) by

\[
A_{ij}(x, p) = \int_I \left(a_{ij} + a_{ik} \frac{\partial \chi^k}{\partial y_j}\right)(x, y, p) \, dy,
\]

where \( \{\chi^k\}_{k=1}^d \) is defined for any \( p \in \mathbb{R} \) by

\[
-\frac{\partial}{\partial y_k} \left(a_{ij} \frac{\partial \chi^k}{\partial y_j}\right)(x, y, p) = \frac{\partial}{\partial y_i} a_{ij}(x, y, p),
\]

with periodic boundary condition in \( y \) and \( \int_I \chi^k(x, y, p) \, dy = 0 \). It is clear that \( A(x, p) \) is also smooth in \( x \) and \( p \).

To simplify the presentation, we will show how to estimate \( e(\text{HMM}) \) when (5.3) is slightly changed into

\[
\begin{align*}
-\text{div} \left(a^\varepsilon(x, V(x_\ell)) \nabla v_\ell^\varepsilon(x)\right) &= 0 \quad x \in I_\delta(x_\ell), \\
v_\ell^\varepsilon(x) &= V_\ell(x) \quad x \in \partial I_\delta(x_\ell).
\end{align*}
\]  

(5.21)

If \( \delta = \varepsilon \), we replace the Dirichlet boundary condition in (5.21) by the periodic boundary condition, i.e., \( v_\ell^\varepsilon(x) - V_\ell(x) \) is periodic on \( \partial I_\varepsilon(x_\ell) \).

**Theorem 5.5.** In general, (5.13) and (5.16) hold if \( \delta \) and \( \varepsilon/\delta \) are sufficiently small, and

\[
e(\text{HMM}) \leq C \left(\left(\frac{\varepsilon}{\delta}\right)^{1/2} + \delta\right).
\]

(5.22)

If \( \delta = \varepsilon \), then (5.13) and (5.16) hold if \( \varepsilon \) is sufficiently small,

\[
e(\text{HMM}) \leq C \varepsilon.
\]

(5.23)

In what follows, we concentrate on the case when \( \delta \) is not an integer multiple of \( \varepsilon \). The other case will be commented on.

Let us first fix more notations. Denote by \( \hat{v}_\ell^\varepsilon \) the solutions of (5.21) with the coefficients \( a(x, x/\varepsilon, V(x_\ell)) \) replaced by \( a(x_\ell, x/\varepsilon, V(x_\ell)) \). Similarly we define \( \hat{w}_\ell^\varepsilon \) to be the solutions of (5.21) with \( V \) replaced by \( W \in X_H \). \( \hat{v}_\ell^\varepsilon \) can be defined in the same way. \( v_\ell^\varepsilon \) and \( \hat{v}_\ell^\varepsilon \) can be viewed as the perturbation of \( v_\ell^\varepsilon \) and \( \hat{v}_\ell^\varepsilon \), respectively.

Observe that \( \hat{v}_\ell^\varepsilon \) and \( \hat{w}_\ell^\varepsilon \) also satisfy (5.5), and

\[
\|\nabla(v_\ell^\varepsilon - \hat{v}_\ell^\varepsilon)\|_{0, I_\delta} \leq C \delta \|\nabla V_\ell\|_{0, I_\delta}, \quad \|\nabla(w_\ell^\varepsilon - \hat{w}_\ell^\varepsilon)\|_{0, I_\delta} \leq C \delta \|\nabla W_\ell\|_{0, I_\delta}.
\]

(5.24)
Lemma 5.6. We have

\[
\|\nabla (\hat{v}_\e^\ell - \hat{w}_\e^\ell)\|_{0,I_\delta} \leq C \left( |(V - W)(x_\ell)| \|\nabla V\|_{0,I_\delta} + \|\nabla W\|_{0,I_\delta} \right) \\
+ \|\nabla (V_\ell - W_\ell)\|_{0,I_\delta}.
\]  
(5.25)

Proof. In view of the definitions of \( v_\e^\ell \) and \( w_\e^\ell \), we have

\[
\lambda \|\nabla (\hat{v}_\e^\ell - \hat{w}_\e^\ell)\|^2_{0,I_\delta} \leq \int_{I_\delta} \nabla (\hat{v}_\e^\ell - \hat{w}_\e^\ell) \cdot \left[ a\left(x_\ell, \frac{x}{\e}, V(x_\ell)\right) \nabla \hat{v}_\e^\ell - a\left(x_\ell, \frac{x}{\e}, W(x_\ell)\right) \nabla \hat{w}_\e^\ell \right] dx
\]
(5.26)

where

\[
I_1 = \int_{I_\delta} \nabla (V_\ell - W_\ell) \cdot \left[ a\left(x_\ell, \frac{x}{\e}, V(x_\ell)\right) \nabla \hat{v}_\e^\ell - a\left(x_\ell, \frac{x}{\e}, W(x_\ell)\right) \nabla \hat{w}_\e^\ell \right] dx
\]

\[
I_2 = \int_{I_\delta} \nabla (\hat{v}_\e^\ell - \hat{w}_\e^\ell) \cdot \left[ a\left(x_\ell, \frac{x}{\e}, V(x_\ell)\right) - a\left(x_\ell, \frac{x}{\e}, W(x_\ell)\right) \right] \nabla w_\e^\ell dx.
\]

Obviously, \( I_1 \) can be decomposed into

\[
I_1 = \int_{I_\delta} \nabla (V_\ell - W_\ell) \cdot \left[ a\left(x_\ell, \frac{x}{\e}, V(x_\ell)\right) - a\left(x_\ell, \frac{x}{\e}, W(x_\ell)\right) \right] \nabla v_\e^\ell dx
\]

\[
+ \int_{I_\delta} \nabla (V_\ell - W_\ell) \cdot a\left(x_\ell, \frac{x}{\e}, W(x_\ell)\right) \nabla (\hat{v}_\e^\ell - \hat{w}_\e^\ell) dx.
\]

Therefore we bound \( I_1 \) and \( I_2 \) as

\[
|I_1| \leq C |(V - W)(x_\ell)| \|\nabla \hat{v}_\e^\ell\|_{0,I_\delta} \|\nabla (V_\ell - W_\ell)\|_{0,I_\delta}
+ A \|\nabla (\hat{v}_\e^\ell - \hat{w}_\e^\ell)\|_{0,I_\delta} \|\nabla (V_\ell - W_\ell)\|_{0,I_\delta},
\]

and

\[
|I_2| \leq C |(V - W)(x_\ell)| \|\nabla \hat{w}_\e^\ell\|_{0,I_\delta} \|\nabla (\hat{v}_\e^\ell - \hat{w}_\e^\ell)\|_{0,I_\delta}
\]

Putting the estimates for \( I_1 \) and \( I_2 \) into (5.26), and using (5.5) we come to (5.25). \( \square \)
Next we establish the estimate for \((\psi^\varepsilon_\ell - \tilde{\psi}^\varepsilon_\ell) - (\hat{w}^\varepsilon_\ell - \tilde{w}^\varepsilon_\ell)\). Let \(\psi^\varepsilon_\ell = v^\varepsilon_\ell - \hat{v}^\varepsilon_\ell\) and \(\tilde{\psi}^\varepsilon_\ell = w^\varepsilon_\ell - \tilde{w}^\varepsilon_\ell\). Clearly, \(\psi^\varepsilon_\ell, \tilde{\psi}^\varepsilon_\ell\) vanish on \(\partial I_\delta(x_\ell)\) and satisfy:

\[-\text{div} \left( a \left( x, \frac{x}{\varepsilon}, V(x_\ell) \right) \nabla \psi^\varepsilon_\ell \right) = \text{div} \left( \left[ a \left( x, \frac{x}{\varepsilon}, V(x_\ell) \right) - a \left( x_\ell, \frac{x}{\varepsilon}, V(x_\ell) \right) \right] \nabla \hat{v}^\varepsilon_\ell \right),\]

\[-\text{div} \left( a \left( x, \frac{x}{\varepsilon}, W(x_\ell) \right) \nabla \tilde{\psi}^\varepsilon_\ell \right) = \text{div} \left( \left[ a \left( x, \frac{x}{\varepsilon}, W(x_\ell) \right) - a \left( x_\ell, \frac{x}{\varepsilon}, W(x_\ell) \right) \right] \nabla \tilde{w}^\varepsilon_\ell \right).\]

**Lemma 5.7.** We have

\[\|\nabla(\psi^\varepsilon_\ell - \tilde{\psi}^\varepsilon_\ell)\|_{0,I_\delta} \leq C \delta \left( \| (V - W)(x_\ell) \| + \| \nabla V \|_{0,I_\delta} + \| \nabla W \|_{0,I_\delta} \right) \]

\[\text{div} (V_\ell - W_\ell)\|_{0,I_\delta}.\]

**Proof.** Proceeding along the same line as in Lemma 5.6, using \((5.27)\) and \((5.28)\), we have

\[\lambda \| \nabla(\psi^\varepsilon_\ell - \tilde{\psi}^\varepsilon_\ell) \|^2_{0,I_\delta} \leq \int_{I_\delta} \nabla(\psi^\varepsilon_\ell - \tilde{\psi}^\varepsilon_\ell) \cdot \left( a \left( x_\ell, \frac{x}{\varepsilon}, V(x_\ell) \right) - a \left( x, \frac{x}{\varepsilon}, V(x_\ell) \right) \right) \nabla \hat{v}^\varepsilon_\ell \, dx\]

\[\quad + \int_{I_\delta} \nabla(\psi^\varepsilon_\ell - \tilde{\psi}^\varepsilon_\ell) \cdot \left( a \left( x_\ell, \frac{x}{\varepsilon}, W(x_\ell) \right) - a \left( x, \frac{x}{\varepsilon}, W(x_\ell) \right) \right) \nabla \tilde{w}^\varepsilon_\ell \, dx\]

\[\quad + \int_{I_\delta} \nabla(\psi^\varepsilon_\ell - \tilde{\psi}^\varepsilon_\ell) \cdot \left( a \left( x_\ell, \frac{x}{\varepsilon}, W(x_\ell) \right) - a \left( x, \frac{x}{\varepsilon}, V(x_\ell) \right) \right) \, dx\]

\[\quad = I_1 + I_2.\]

We decompose \(I_1\) into

\[I_1 = \int_{I_\delta} \nabla\left( \psi^\varepsilon_\ell - \tilde{\psi}^\varepsilon_\ell \right) \cdot \left[ a \left( x_\ell, \frac{x}{\varepsilon}, V(x_\ell) \right) - a \left( x, \frac{x}{\varepsilon}, V(x_\ell) \right) \right] \nabla(\hat{v}^\varepsilon_\ell - \tilde{w}^\varepsilon_\ell) \, dx\]

\[\quad + \int_{I_\delta} \nabla\left( \psi^\varepsilon_\ell - \tilde{\psi}^\varepsilon_\ell \right) \cdot \left[ a \left( x_\ell, \frac{x}{\varepsilon}, V(x_\ell) \right) - a \left( x, \frac{x}{\varepsilon}, V(x_\ell) \right) \right] \nabla \hat{v}^\varepsilon_\ell \, dx\]

\[\quad + a \left( x_\ell, \frac{x}{\varepsilon}, W(x_\ell) \right) - a \left( x, \frac{x}{\varepsilon}, W(x_\ell) \right) \right] \nabla \tilde{w}^\varepsilon_\ell \, dx.\]

Similar to \((5.25)\), we have

\[\|\nabla(\hat{v}^\varepsilon_\ell - \tilde{w}^\varepsilon_\ell)\|_{0,I_\delta} \leq C \delta \left( \| (V - W)(x_\ell) \|_{0,I_\delta} + \| \nabla V \|_{0,I_\delta} + \| \nabla W \|_{0,I_\delta} \right) \]

\[\text{div} (V_\ell - W_\ell)\|_{0,I_\delta}.\]

We thus bound \(I_1\) as

\[|I_1| \leq C \delta \left( \| (V - W)(x_\ell) \|_{0,I_\delta} + \| \nabla V \|_{0,I_\delta} + \| \nabla W \|_{0,I_\delta} \right) \|\nabla(\psi^\varepsilon_\ell - \tilde{\psi}^\varepsilon_\ell)\|_{0,I_\delta}.\]
\( \hat{w}_\varepsilon^{\ell} \) can be viewed as a perturbation of \( w_\varepsilon^{\ell} \), it follows from (5.24) that

\[
\| \nabla \hat{\psi}^{\ell}_\varepsilon \|_{0,t} \leq C \delta \| \nabla W_\varepsilon \|_{0,t}.
\]

So \( I_2 \) can be bounded as

\[
|I_2| \leq C \delta \|(V - W)(x_\ell)\| \| \nabla \hat{\psi}^{\ell}_\varepsilon \|_{0,t} \|
abla (\psi_\varepsilon^{\ell} - \hat{\psi}^{\ell}_\varepsilon)\|_{0,t} \\
\leq C \delta \|(V - W)(x_\ell)\| \| \nabla W_\varepsilon \|_{0,t} \|
abla (\psi_\varepsilon^{\ell} - \hat{\psi}^{\ell}_\varepsilon)\|_{0,t}.
\]

Putting the estimates for \( I_1 \) and \( I_2 \) into (5.30), we get (5.7). \( \square \)

Define

\[
\hat{V}_\ell(x) = V_\ell(x) + \varepsilon \chi^k \left( x_\ell, \frac{x}{\varepsilon}, V(x_\ell) \right) \frac{\partial V_\ell}{\partial x_k}(x), \\
\hat{W}_\ell(x) = W_\ell(x) + \varepsilon \chi^k \left( x_\ell, \frac{x}{\varepsilon}, W(x_\ell) \right) \frac{\partial W_\ell}{\partial x_k}(x),
\]

where \( \{ \chi^k \}_{k=1}^d \) are the solutions of (3.2) with coefficients replaced by \( a_{ij}(x_\ell, y, V(x_\ell)) \).

Denote \( \theta_\varepsilon^{\ell} = \hat{v}_\varepsilon^{\ell} - \hat{V}_\ell \) and \( \hat{\theta}_\varepsilon^{\ell} = \hat{w}_\varepsilon^{\ell} - \hat{W}_\ell \). Observe that

\[
\begin{cases}
- \text{div} \left( a \left( x_\ell, \frac{x}{\varepsilon}, V(x_\ell) \right) \theta_\varepsilon^{\ell}(x) \right) = 0 & x \in I_\delta(x_\ell), \\
\theta_\varepsilon^{\ell}(x) = -\varepsilon \chi^k \left( x_\ell, \frac{x}{\varepsilon}, V(x_\ell) \right) \frac{\partial V_\ell}{\partial x_k}(x) & x \in \partial I_\delta(x_\ell).
\end{cases}
\]

and

\[
\begin{cases}
- \text{div} \left( a \left( x_\ell, \frac{x}{\varepsilon}, W(x_\ell) \right) \hat{\theta}_\varepsilon^{\ell}(x) \right) = 0 & x \in I_\delta(x_\ell), \\
\hat{\theta}_\varepsilon^{\ell}(x) = -\varepsilon \chi^k \left( x_\ell, \frac{x}{\varepsilon}, W(x_\ell) \right) \frac{\partial W_\ell}{\partial x_k}(x) & x \in \partial I_\delta(x_\ell).
\end{cases}
\]

Similar to (3.4), we have

\[
\begin{align*}
\| \nabla \theta_\varepsilon^{\ell} \|_{0,t} & \leq C \left( \frac{\varepsilon}{\delta} \right)^{1/2} \| \nabla V_\ell \|_{0,t}, \\
\| \nabla \hat{\theta}_\varepsilon^{\ell} \|_{0,t} & \leq C \left( \frac{\varepsilon}{\delta} \right)^{1/2} \| \nabla W_\ell \|_{0,t}.
\end{align*}
\]

**Lemma 5.8.** We have

\[
\| \nabla (\theta_\varepsilon^{\ell} - \hat{\theta}_\varepsilon^{\ell}) \|_{0,t} \leq C \left( \frac{\varepsilon}{\delta} \right)^{1/2} \left( \| \nabla (V_\ell - W_\ell) \|_{0,t} \right) \\
+ \|(V - W)(x_\ell)\| \| \nabla V_\ell \|_{0,t} + \| \nabla W_\ell \|_{0,t}.
\]
Proof. Obviously we have
\[
\lambda \| \nabla (\theta_\varepsilon - \tilde{\theta}_\varepsilon) \|^2_{0, I_s} \leq \int_{I_s} \nabla (\theta_\varepsilon - \tilde{\theta}_\varepsilon) \cdot \left[ a \left( x_\varepsilon, \frac{x}{\varepsilon}, W(x_\varepsilon) \right) - a \left( x_\varepsilon, \frac{x}{\varepsilon}, V(x_\varepsilon) \right) \right] \nabla \tilde{\theta}_\varepsilon \, dx \\
- \int_{I_s} \nabla (\theta_\varepsilon - \tilde{\theta}_\varepsilon) \cdot \left( a \left( x_\varepsilon, \frac{x}{\varepsilon}, V(x_\varepsilon) \right) \nabla \theta_\varepsilon - a \left( x_\varepsilon, \frac{x}{\varepsilon}, W(x_\varepsilon) \right) \nabla \tilde{\theta}_\varepsilon \right) \, dx \\
= : I_1 + I_2.
\]
Invoking (5.32), we bound \( I_1 \) as
\[
|I_1| \leq C \| (V - W)(x_\varepsilon) \| \| \nabla \tilde{\theta}_\varepsilon \|_{0, I_s} \| \nabla (\theta_\varepsilon - \tilde{\theta}_\varepsilon) \|_{0, I_s} \\
\leq C \left( \frac{\varepsilon}{\delta} \right)^{1/2} \| (V - W)(x_\varepsilon) \| \| \nabla W \|_{0, I_s} \| \nabla (\theta_\varepsilon - \tilde{\theta}_\varepsilon) \|_{0, I_s}.
\]
Denote by
\[
p_\varepsilon(x) = \chi^k \left( x_\varepsilon, \frac{x}{\varepsilon}, V(x_\varepsilon) \right) \frac{\partial V_\varepsilon}{\partial x_k} - \chi^k \left( x_\varepsilon, \frac{x}{\varepsilon}, W(x_\varepsilon) \right) \frac{\partial W_\varepsilon}{\partial x_k}.
\]
In light of the continuity of \( \{ \chi^k \}_{k=1} \), we have
\[
|p_\varepsilon|, \varepsilon |\nabla p_\varepsilon| \leq C |V(x_\varepsilon) - W(x_\varepsilon)| |\nabla V_\varepsilon| + C |\nabla (V_\varepsilon - W_\varepsilon)|.
\]
Obviously, \( \theta_\varepsilon - \tilde{\theta}_\varepsilon + p_\varepsilon (1 - \rho^\varepsilon) \in H^1_0(I_s) \), where \( \rho^\varepsilon \) is defined as in (3.12). Integrating by parts, we rewrite \( I_2 \) as
\[
I_2 = - \int_{I_s} \nabla \left( \varepsilon p_\varepsilon (1 - \rho^\varepsilon) \right) \cdot \left( a \left( x_\varepsilon, \frac{x}{\varepsilon}, V(x_\varepsilon) \right) - a \left( x_\varepsilon, \frac{x}{\varepsilon}, W(x_\varepsilon) \right) \right) \nabla \theta_\varepsilon \, dx \\
+ \int_{I_s} \nabla \left( \varepsilon p_\varepsilon (1 - \rho^\varepsilon) \right) \cdot a \left( x_\varepsilon, \frac{x}{\varepsilon}, W(x_\varepsilon) \right) \nabla \tilde{\theta}_\varepsilon \, dx.
\]
Invoking (5.32) and proceeding as in the estimate of \( I_3 \) in Lemma 3.2, we get
\[
|I_2| \leq C \left( \frac{\varepsilon}{\delta} \right)^{1/2} \left( |(V - W)(x_\varepsilon)||\nabla V_\varepsilon|_{0, I_s} + |\nabla (V_\varepsilon - W_\varepsilon)|_{0, I_s} \right) |(V - W)(x_\varepsilon)||\nabla \theta_\varepsilon|_{0, I_s} \\
+ C \left( \frac{\varepsilon}{\delta} \right)^{1/2} \left( |(V - W)(x_\varepsilon)||\nabla V_\varepsilon|_{0, I_s} + |\nabla (V_\varepsilon - W_\varepsilon)|_{0, I_s} \right) |\nabla (\theta_\varepsilon - \tilde{\theta}_\varepsilon)|_{0, I_s} \\
\leq C \left( \frac{\varepsilon}{\delta} \right)^{1/2} \left( |(V - W)(x_\varepsilon)|^2 |\nabla V_\varepsilon|^2_{0, I_s} + |\nabla (V_\varepsilon - W_\varepsilon)|^2_{0, I_s} \right) \\
+ C \left( \frac{\varepsilon}{\delta} \right)^{1/2} \left( |(V - W)(x_\varepsilon)| |\nabla V_\varepsilon|_{0, I_s} + |\nabla (V_\varepsilon - W_\varepsilon)|_{0, I_s} \right) |\nabla (\theta_\varepsilon - \tilde{\theta}_\varepsilon)|_{0, I_s}.
\]
A combination of the estimates for \( I_1 \) and \( I_2 \) gives (5.33). \( \square \)

In the next lemma, we shall prove that \( E(V, W) \) has certain continuity with respect to \( V \).
Lemma 5.9. For any $V, W, Z \in X_H$ satisfying $\|V\|_{1,\infty} + \|W\|_{1,\infty} + \|Z\|_{1,\infty} \leq M$, there exists a constant $C(M)$ such that

$$\left| \mathcal{K} \sum_{x_\ell \in \mathcal{K}} \omega_{x_\ell} E(V, Z) - E(W, Z) \right|$$

(5.34) \[ \leq C(M) \left( \left( \frac{\varepsilon}{\delta} \right)^{1/2} + \delta \right) \|V - W\|_{1,\kappa} \|\nabla Z\|_{0,\kappa} . \]

Proof. We decompose $E(V, Z) - E(W, Z)$ into the following terms:

$$E(V, Z) - E(W, Z) = : = I_1 + \cdots + I_6,$$

where

$$I_1 = \int_{I_\delta} \nabla z_\xi^\varepsilon \left[ a \left( x, \frac{x}{\varepsilon}, V(x_\ell) \right) - a \left( x, \frac{x}{\varepsilon}, W(x_\ell) \right) \right] \nabla (v_\ell^\varepsilon - \hat{v}_\ell^\varepsilon) \, dx,$$

$$I_2 = \int_{I_\delta} \nabla z_\xi^\varepsilon \cdot a \left( x, \frac{x}{\varepsilon}, W(x_\ell) \right) \nabla (v_\ell^\varepsilon - \hat{v}_\ell^\varepsilon - w_\ell^\varepsilon + \hat{w}_\ell^\varepsilon) \, dx.$$

$$I_3 = - \left( \frac{1}{|\kappa\varepsilon|} - \frac{1}{\delta^2} \right) \int_{I_\delta} \nabla Z_\ell \cdot \left( a \left( x_\ell, \frac{x}{\varepsilon}, V(x_\ell) \right) \nabla \hat{V}_\ell - a \left( x_\ell, \frac{x}{\varepsilon}, W(x_\ell) \right) \nabla \hat{W}_\ell \right) \, dx.$$

$$I_4 = \int_{|\kappa\varepsilon|^2} \int_{I_\delta \setminus I_{\kappa\varepsilon}} \nabla Z_\ell \cdot \left( a \left( x_\ell, \frac{x}{\varepsilon}, V(x_\ell) \right) \nabla \hat{V}_\ell - a \left( x_\ell, \frac{x}{\varepsilon}, W(x_\ell) \right) \nabla \hat{W}_\ell \right) \, dx$$

$$I_5 = \int_{I_\delta} \nabla Z_\ell \cdot \left( a \left( x_\ell, \frac{x}{\varepsilon}, V(x_\ell) \right) \nabla \theta_\ell^\varepsilon - a \left( x_\ell, \frac{x}{\varepsilon}, W(x_\ell) \right) \nabla \hat{\theta}_\ell^\varepsilon \right) \, dx$$

$$I_6 = - \left[ \int_{I_\delta} \nabla Z_\ell \cdot \left( a \left( x_\ell, \frac{x}{\varepsilon}, V(x_\ell) \right) \nabla v_\ell^\varepsilon - a \left( x_\ell, \frac{x}{\varepsilon}, W(x_\ell) \right) \nabla w_\ell^\varepsilon \right) \, dx \right. $$

$$\left. - \int_{I_\delta} \nabla Z_\ell \cdot \left( a \left( x_\ell, \frac{x}{\varepsilon}, V(x_\ell) \right) \nabla v_\ell^\varepsilon - a \left( x_\ell, \frac{x}{\varepsilon}, W(x_\ell) \right) \nabla w_\ell^\varepsilon \right) \, dx \right].$$

Using (5.24) to bound $I_1$, and (5.29) to bound $I_2$, we get

$$|I_1| + |I_2| \leq C\delta \left( |\nabla (V_\ell - W_\ell)| + |(V - W)(x_\ell)| (|\nabla V_\ell| + |\nabla W_\ell|) \right) |\nabla Z_\ell| .$$

$I_3$ and $I_4$ can be bounded as

$$|I_3|, |I_4| \leq C\frac{\varepsilon}{\delta} \left( |\nabla (V_\ell - W_\ell)| + |(V - W)(x_\ell)| (|\nabla V_\ell| + |\nabla W_\ell|) \right) |\nabla Z_\ell| .$$

Using Lemma 5.8 we bound $I_5$ as

$$|I_5| \leq C \left( \frac{\varepsilon}{\delta} \right)^{1/2} \left( |\nabla (V_\ell - W_\ell)| + |(V - W)(x_\ell)| (|\nabla V_\ell| + |\nabla W_\ell|) \right) |\nabla Z_\ell| .$$

Proceeding as $I_2$ in Lemma 5.7, we bound $I_6$ as

$$|I_6| \leq C\delta \left( |\nabla (V_\ell - W_\ell)| + |(V - W)(x_\ell)| (|\nabla V_\ell| + |\nabla W_\ell|) \right) |\nabla Z_\ell| .$$
Finally, we get
\[ |E(V, Z) - E(W, Z)| \leq C \left( \frac{\varepsilon}{\delta} + \delta + \left( \frac{\varepsilon}{\delta} \right)^{1/2} \right) \left( |\nabla(V_\ell - W_\ell)| + |(V - W)(x_\ell)| (|\nabla V_\ell| + |\nabla W_\ell|) \right) |\nabla Z_\ell|. \]

Summing up, we get
\[ |K| \sum_{x_\ell \in K} \omega_\ell |E(V, Z) - E(W, Z)| \leq C \delta \|\nabla (V - W)\|_{0,K} \|\nabla Z\|_{0,K} \]
\[ + C \delta \|V - W\|_{L^\infty(K)} (\|\nabla V\|_{0,K} + \|\nabla W\|_{0,K}) \|\nabla Z\|_{0,K}. \]

Using the inverse inequality on each element, we obtain
\[ \|V - W\|_{L^\infty(K)} \|\nabla V\|_{0,K} \leq C H_K^{-2/d} \|V - W\|_{0,K} H_K^{2/d} \|\nabla V\|_{L^\infty(K)} \]
\[ = C \|V - W\|_{0,K} \|\nabla V\|_{L^\infty(K)}. \]

Similarly
\[ \|V - W\|_{L^\infty(K)} \|\nabla W\|_{0,K} \leq C \|V - W\|_{0,K} \|\nabla W\|_{L^\infty(K)}. \]

A combination of the above estimates leads to (5.34).

Proof for the second case in Theorem 5.5. Let \( W = 0 \) in the above lemma, we obtain (5.22). Notice that for sufficiently small \( \varepsilon \) and \( \delta \), we have
\[ e(HMM) \leq C \left( \frac{\varepsilon}{\delta} \right)^{1/2} + \delta \right) < C_0. \]

Next let \( \eta = C \left( \frac{\varepsilon}{\delta} \right)^{1/2} + \delta \), then for sufficiently small \( \varepsilon \) and \( \delta \), we have \( \eta < 1 \), this verifies (5.13). Furthermore, let \( \eta = \delta + (\varepsilon/\delta)^{1/2} + H^k \), then for sufficiently small \( \varepsilon, \delta \) and \( H \), \( \eta \) is smaller than \( \eta_0 \), and (5.16) is verified.

Remark 5.10. Compared to the linear case, the upper bound for \( e(HMM) \) for the case when \( \delta/\varepsilon \not\in \mathbb{Z} \) degrades to \( \sqrt{\varepsilon/\delta} \). This is due to the fact that \( A_H \) is non-symmetric.

In case of \( \delta = \varepsilon \), note that \( \hat{v}_\ell^\varepsilon = \hat{V}_\ell \) and \( \hat{w}_\ell^\varepsilon = \hat{W}_\ell \), so a direct calculation gives Lemma 5.6 for this case. Lemma 5.7 is also valid with \( \delta \) replaced by \( \varepsilon \). We also have \( \theta_\ell^\varepsilon = 0 \) and \( \tilde{\theta}_\ell^\varepsilon = 0 \). Observing that for any \( V, Z \in X_H \),
\[ \int_{I_\ell(x_\ell)} \nabla z_\ell^\varepsilon \cdot a\left(x_\ell, \frac{x}{\varepsilon}, V(x_\ell)\right) \nabla \hat{v}_\ell^\varepsilon \, dx = \int_K \nabla Z_\ell \cdot A(x_\ell, V(x_\ell)) \nabla V_\ell \, dx, \]
we may rewrite $E(V, Z)$ as
\[
E(V, Z) = \int_{I_{\varepsilon}} \nabla z_{\varepsilon} \cdot a\left(x, \frac{x}{\varepsilon}, V(x_{\ell})\right) \nabla (v_{\ell} - \hat{v}_{\ell}) \, dx \\
+ \int_{I_{\varepsilon}} \nabla z_{\varepsilon} \cdot \left[ a\left(x, \frac{x}{\varepsilon}, V(x_{\ell})\right) - a\left(x_{\ell}, \frac{x_{\ell}}{\varepsilon}, V(x_{\ell})\right) \right] \nabla \hat{v}_{\ell} \, dx.
\]

Consequently, we have for any $V, W, Z \in X_H$,
\[
|K| \sum_{x_{\ell} \in K} \omega_{\ell} |E(V, Z) - E(W, Z)| \leq C(M)\varepsilon\|V - W\|_{1,K}\|\nabla Z\|_{0,K}.
\]

**Proof of Theorem 5.5.** Let $W = 0$ in the above lemma, we obtain (5.23). Notice that for sufficiently small $\varepsilon$, we have
\[
e(\text{HMM}) \leq C\varepsilon < C_0.
\]
Next let $\eta = C\varepsilon$, then for sufficiently small $\varepsilon$, we have $C\varepsilon < 1$, this proves (5.13). Furthermore, we let $\eta = \varepsilon^{1/2} + H^k$, so for sufficiently small $\varepsilon$ and $H$, $\eta$ is smaller than $\eta_0$. This proves (5.16). \(\square\)

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