Generalized Flows, Intrinsic Stochasticity, and Turbulent Transport

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ABSTRACT **The study of passive scalar transport in a turbulent velocity field leads naturally to the notion of generalized flows which are families of probability distributions on the space of solutions to the associated ODEs, which no longer admit unique solutions in general. Two most natural regularizations of this problem, namely the regularization via adding small molecular diffusion and the regularization via smoothing out the velocity field are considered. White-in-time random velocity fields are used as an example to examine the variety of phenomena that take place when the velocity field is not spatially regular. Three different regimes characterized by their degrees of compressibility are isolated in the parameter space. In the regime of intermediate compressibility, the two different regularizations give rise to two different generalized flows and consequently two different scaling behavior for the structure functions of the passive scalar. Physically this means that the scaling depends on Prandtl number. Surprisingly the two different regularizations give rise to the same generalized flows in the other two regimes even though the sense of convergence can be very different. The "one force, one solution" principle is established for the scalar field in the weakly compressible regime, and for the difference of the scalar in the strongly compressible regime which is the regime of inverse cascade. Existence and uniqueness of an invariant measure is also proved in these regimes when the transport equation is suitably forced. Finally incomplete self-similarity in the spirit of Barenblatt-Chorin is established.**

Introduction

Recent efforts on the understanding of the fundamental physics of hydrodynamic turbulence have concentrated on the explanation of the observed violations of Kolmogorov's scaling. These violations reflect the occurrence of large fluctuations in the velocity field on the small scales, a phenomenon referred to as intermittency. Some progress in the understanding of intermittency has been achieved recently through the study of simple model problems that include Burgers equation [1, 2] and the passive advection of a scalar by a velocity field of known statistics [3, 4, 5, 6]. This paper is a summary of the many interesting mathematical issues that arise in the problem of passive scalar advection together with our understanding of these issues. We put some of our results in the perspective of a new phenomenological model proposed recently by Barenblatt and Chorin [7, 8] using the formalism of incomplete self-similarity.

Generalized Flows

Consider the transport equation for the scalar field $\theta^\kappa(x,t)$ in $\mathbb{R}^d$:

$$\frac{\partial \theta^\kappa}{\partial t} + (u(x,t) \cdot \nabla)\theta^\kappa = \kappa \Delta \theta^\kappa. \tag{1}$$

We will be interested in θ^κ in the limit as $\kappa \to 0$. It is known from classical results that if u is Lipschitz continuous in x, then as $\kappa \to 0$, θ^κ converges to θ, the solution of

$$\frac{\partial \theta}{\partial t} + (u(x,t) \cdot \nabla)\theta = 0. \tag{2}$$

Furthermore, if we define $\{\varphi_{s,t}(x)\}$ to be the flow generated by the velocity field u, satisfying the ordinary differential equations (ODEs)

$$\frac{d\varphi_{s,t}(x)}{dt} = u(\varphi_{s,t}(x), t), \qquad \varphi_{s,s}(x) = x, \tag{3}$$

for $s < t$, then the solution of the transport equation in 2 for the initial condition $\theta^\kappa(x,0) = \theta_0(x)$ is given by

$$\theta(x,t) = \theta_0(\varphi_{0,t}^{-1}(x)) = \theta_0(\varphi_{t,0}(x)). \tag{4}$$

This classical scenario breaks down when u fails to be Lipschitz continuous in x, which is precisely the case for fully developed turbulent velocity fields. In this case Kolmogorov's theory of turbulent flows suggests that u will only be Hölder continuous with an exponent roughly equal to $\frac{1}{3}$. In such situations the solution of the ODEs in 3 may fail to be unique [9], and we then have to consider probability distributions on the set of solutions in order to solve the transport equation in 2. This is the essence of the notion of generalized flows proposed by Brenier [10, 11] (see also [12]).

There are two ways to think about probability distributions on the solutions of the ODEs in 3. We can either think of it as probability measures on the path-space (functions of t) supported by paths which are solutions of 3, or we can think of it as transition probability at time t if the starting position at time s is x. In the classical situation when u is Lipschitz continuous, this transition probability degenerates to a point mass centered at the unique solution of 3. When Lipschitz condition fails, this transition probability may be non-degenerate and the system in 3 is intrinsically stochastic.

There is a parallel story for the case when u is a white-in-time random process defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We will denote the elements in Ω by ω and indicate the dependence on realization of the random velocity field by a super- or a subscript ω. In connection with the transport equation in 2, it is most natural to consider the stochastic ODEs

$$d\varphi^{\omega}_{s,t}(x) = u(\varphi^{\omega}_{s,t}(x), t)dt, \qquad \varphi^{\omega}_{s,s}(x) = x, \tag{5}$$

in Stratonovich sense. In this case, it is shown [13] that if the local characteristic of u is spatially twice continuously differentiable, then the system in 5 has a unique solution. Such conditions are not satisfied by typical turbulent velocity fields on the scale of interest. When the regularity condition on u fails, there are at least two natural ways to regularize 3 or 5. The first is to add diffusion:

$$d\varphi^{\omega,\kappa}_{s,t}(x) = u(\varphi^{\omega,\kappa}_{s,t}(x), t)dt + \sqrt{2\kappa}d\beta(t), \tag{6}$$

and consider the limit as $\kappa \to 0$. We will call this the κ-limit. The second is to smooth out the velocity field. Let ψ_ε be defined as

$$\psi_\varepsilon(x) = \frac{1}{\varepsilon^d}\psi\left(\frac{x}{\varepsilon}\right),$$

where ψ is a standard mollifier: $\psi \geq 0$, $\int_{\mathbb{R}^d} \psi dx = 1$, ψ decays fast at infinity. Let $u^\varepsilon = u \star \psi_\varepsilon$ and consider

$$d\varphi^{\omega,\varepsilon}_{s,t}(x) = u^\varepsilon(\varphi^{\omega,\varepsilon}_{s,t}(x), t)dt, \tag{7}$$

in the limit as $\varepsilon \to 0$. We will call this the ε-limit. Physically κ plays the role of molecular diffusivity, ε can be thought of as a crude model of the viscous cut-off scale. The κ-limit corresponds to the situation when the Prandtl number tends to zero, $Pr \to 0$, whereas the ε-limit corresponds to the situation when the Prandtl number diverges, $Pr \to \infty$. The following questions naturally arise:

(Q1) Do the regularized equations in 6 and 7 converge to the same limit?

(Q2) If not, what characterizes these limits?

(Q3) Does there exist a unique statistical steady state when the transport equation in 1 is suitably forced?

(Q4) What are the statistical and geometrical properties of solutions in the statistical steady state?

Below we address questions Q1, Q3, Q4 on a specific model introduced by Kraichnan [14]. Question Q2 is an extremely interesting one which we intend to consider in a future publication.

Before proceeding further, we relate the regularized flows in 6, 7 to the solutions of the transport equations. Consider the κ-regularization first. It is convenient to introduce the backward transition probability

$$g^{\kappa}_{\omega}(x,t|dy,s) = \mathbf{E}_{\beta}\delta(y - \varphi^{\omega,\kappa}_{t,s}(x))dy, \quad s < t, \tag{8}$$

where the expectation is taken with respect to $\beta(t)$, and $\varphi^{\omega,\kappa}_{t,s}(x)$ is the flow inverse to $\varphi^{\omega,\kappa}_{s,t}(x)$ defined in 6 (i.e. $\varphi^{\omega,\kappa}_{s,t}(x)$ is the forward flow and $\varphi^{\omega,\kappa}_{t,s}(x)$ is the backward flow). The action of g^{κ}_{ω} generates a semi-group of transformation

$$S^{\omega,\kappa}_{t,s}\psi(x) = \int_{\mathbb{R}^d} \psi(y) g^{\kappa}_{\omega}(x,t|dy,s), \tag{9}$$

for all test functions ψ. $\theta^{\kappa}_{\omega}(x,t) = S^{\omega,\kappa}_{t,s}\psi(x)$ solve the transport equation in 1 for the initial condition $\theta^{\kappa}_{\omega}(x,s) = \psi(x)$. Similarly, for the flow in 7, define

$$S^{\omega,\varepsilon}_{t,s}\psi(x) = \psi(\varphi^{\omega,\varepsilon}_{t,s}(x)), \qquad s < t. \tag{10}$$

$\theta^{\varepsilon}_{\omega}(x,t) = S^{\omega,\varepsilon}_{t,s}\psi(x)$ solves the transport equation

$$\frac{\partial \theta^{\varepsilon}}{\partial t} + (u^{\varepsilon}(x,t)\cdot\nabla)\theta^{\varepsilon} = 0, \tag{11}$$

with initial condition $\theta(x,s) = \psi(x)$. Similar definitions can be given for forward flows but we will restrict attention to the backward ones since we are primarily interested in scalar transport. The results given below generalize trivially for forward flows.

Kraichnan Model

In [14] Kraichnan introduced one of the simplest model of passive scalar by considering the advection by a Gaussian, spatially non-smooth and white-in-time velocity field. The fact that white-in-time velocity fields may exhibit intermittency was first recognized by Majda [15]. Definitive work on Kraichnan model has been done afterwards in [3, 4, 5, 6].

We will consider a generalization of Kraichnan model introduced in [16] (see also [17]). The velocity u is assumed to be a statistically homogeneous, isotropic and stationary Gaussian field with mean zero and covariance

$$\mathbf{E}\, u_\alpha(x,t) u_\beta(y,s) = (C_0 \delta_{\alpha\beta} - c_{\alpha\beta}(x-y))\delta(t-s). \quad (12)$$

We assume that u has a correlation length ℓ_0, i.e. the covariance in 12 decays fast for $|x-y| > \ell_0$. Consequently $c_{\alpha\beta}(x) \to C_0\delta_{\alpha\beta}$ as $|x|/\ell_0 \to \infty$. On the other hand, we will be mainly interested in small scale phenomena for which $|x| \ll \ell_0$. In this range, we take $c_{\alpha\beta}(x) = d_{\alpha\beta}(x) + O(|x|^2/\ell_0^2)$ with

$$d_{\alpha\beta}(x) = A d^P_{\alpha\beta}(x) + B d^S_{\alpha\beta}(x), \quad (13)$$

and

$$\begin{aligned} d^P_{\alpha\beta}(x) &= D\left(\delta_{\alpha\beta} + \xi \frac{x_\alpha x_\beta}{|x|^2}\right)|x|^\xi, \\ d^S_{\alpha\beta}(x) &= D\left((d+\xi-1)\delta_{\alpha\beta} - \xi \frac{x_\alpha x_\beta}{|x|^2}\right)|x|^\xi. \end{aligned} \quad (14)$$

D is a parameter with dimension $[\text{length}]^{2-\xi}[\text{time}]^{-1}$. The dimensionless parameters A and B measure the divergence and rotation of the field u. $A=0$ corresponds to incompressible fields with $\nabla \cdot u = 0$. $B = 0$ corresponds to irrotational fields with $\nabla \times u = 0$. The parameter ξ measures the spatial regularity of u. For $\xi \in (0,2)$, the local characteristic of u fails to be twice differentiable and this fact has important consequences on both the transport equation in 2 and the systems of ODEs in 3 or 5.

Existing physics literature concentrates on the κ-limit for Kraichnan model. Let $\mathcal{S}^2 = A + (d-1)B$, $\mathcal{C}^2 = A$, $\mathcal{P} = \mathcal{C}^2/\mathcal{S}^2$. $\mathcal{P} \in [0,1]$ is a measure of the degree of compressibility of u. The pioneering work of Gawędzki and Vergassola [16] (see also [17]) identifies two different regimes for the κ-limit:

1. The strongly compressible regime when $\mathcal{P} \geq d/\xi^2$. In this regime g^κ_ω converges to a flow of maps, i.e. there exists a two-parameter family of maps $\{\varphi^\omega_{t,s}(x)\}$ such that

$$g^\kappa_\omega(x,t|dy,s) \to \delta(y - \varphi^\omega_{t,s}(x))dy. \quad (15)$$

Moreover particles have finite probability to coalesce under the flow of $\{\varphi^\omega_{t,s}(x)\}$.

2. When $\mathcal{P} < d/\xi^2$, g^κ_ω converges to a "generalized stochastic flow"

$$g^\kappa_\omega(x,t|dy,s) \to g_\omega(x,t|dy,s), \tag{16}$$

and the limit g_ω is a nontrivial probability distribution in y. This means that the image of a particle under the flow defined by the velocity field u is non-unique and has a non-trivial distribution. In other words, particle trajectories branch.

The following result provides rigorous justification of these predictions and also answer the question Q1. In particular, it points out that there are three different regimes if both the κ and the ε-limits are considered.

Theorem 1 *In the strongly compressible regime when*

$$\mathcal{P} \geq \frac{d}{\xi^2}, \tag{17}$$

there exists a two-parameter family of random maps $\{\varphi^\omega_{t,s}(x)\}$*, such that for all smooth test functions* ψ *and for all* (s,t,x)*,* $s < t$*,*

$$\mathbf{E}\left(S^{\omega;\kappa}_{t,s}\psi(x) - \psi(\varphi^\omega_{t,s}(x))\right)^2 \to 0, \tag{18}$$

as $\kappa \to 0$*, and*

$$\mathbf{E}\left(\psi(\varphi^{\omega,\varepsilon}_{t,s}(x)) - \psi(\varphi^\omega_{t,s}(x))\right)^2 \to 0, \tag{19}$$

as $\varepsilon \to 0$*. Moreover, the limiting flow* $\{\varphi^\omega_{t,s}(x)\}$ *coalesces in the sense that for almost all* (t,x,y)*,* $x \neq y$*, we can define a time* τ *such that* $-\infty < \tau < t$ *a.s. and*

$$\varphi^\omega_{t,s}(x) = \varphi^\omega_{t,s}(y), \tag{20}$$

for $s \leq \tau$*.*

In the weakly compressible regime when

$$\mathcal{P} \leq \frac{d+\xi-2}{2\xi}, \tag{21}$$

there exists a random family of generalized flows $g_\omega(x,t|dy,s)$*, such that for all test function* ψ*,*

$$S^\omega_{t,s}\psi(x) = \int_{\mathbb{R}^d} \psi(y) g_\omega(x,t|dy,s), \tag{22}$$

satisfies

$$\mathbf{E}\left(S^{\omega,\kappa}_{t,s}\psi(x) - S^\omega_{t,s}\psi(x)\right)^2 \to 0, \tag{23}$$

as $\kappa \to 0$ *for all* (s,t,x)*,* $s < t$*, and*

$$\mathbf{E}\Big(\int_{\mathbb{R}^d} \eta(x)\left(\psi(\varphi^{\omega,\varepsilon}_{t,s}(x)) - S^\omega_{t,s}\psi(x)\right) dx\Big)^2 \to 0, \tag{24}$$

as $\varepsilon \to 0$ for all (s,t), $s < t$, and for all test functions η. Moreover, $g_\omega(x,t|dy,s)$ is non-degenerate in the sense that

$$S^\omega_{t,s}\psi^2(x) - \left(S^\omega_{t,s}\psi(x)\right)^2 > 0 \quad a.s. \tag{25}$$

In the intermediate regime when

$$\frac{d+\xi-2}{2\xi} < \mathcal{P} < \frac{d}{\xi^2}, \tag{26}$$

there exists a random family of maps $\{\varphi^\omega_{t,s}(x)\}$, and a random family of generalized flows $g_\omega(x,t|dy,s)$, such that for all test function ψ and for all (s,t,x), $s<t$,

$$\mathbf{E}\left(S^{\omega;\kappa}_{t,s}\psi(x) - S^\omega_{s,t}\psi(x)\right)^2 \to 0 \tag{27}$$

as $\kappa \to 0$, and

$$\mathbf{E}\left(\psi(\varphi^{\omega,\varepsilon}_{t,s}(x)) - \psi(\varphi^\omega_{t,s}(x))\right)^2 \to 0 \tag{28}$$

as $\varepsilon \to 0$. Furthermore, the limit $\{\varphi^\omega_{t,s}(x)\}$ coalesces in the sense of 20, and the limit g_ω is non-degenerate in the sense of 25.

Rephrasing the content of this result, we have strong convergence to a family of flow maps in the strongly compressible regime for both the κ-limit and the ε-limit. In the weakly compressible regime, we have strong convergence to a family of generalized flows for the κ-limit, but weak convergence to the same limit for the ε-regularization. In fact, using the terminology of Young measures [18], the limiting generalized flow $\{g_\omega(x,t|dy,s)\}$ is nothing but the Young measure for the sequence of flow maps $\{\varphi^{\omega,\varepsilon}_{s,t}(x)\}$. Finally, in contrast to what is observed in the other two regimes, the ε-limit and κ-limit are not the same in the intermediate regime. It would be interesting to characterize all the generalized flows in this case.

From Theorem 1, it is natural to define the solution of the transport equation in 2 for the initial condition $\theta_\omega(x,s) = \theta_0(x)$ as

$$\theta_\omega(x,t) = S_{t,s}\theta_0(x) = \int_{\mathbb{R}^d} \theta_0(y) g_\omega(x,t|dy,s), \tag{29}$$

for the non-degenerate case, and as

$$\theta_\omega(x,t) = \theta_0(\varphi^\omega_{t,s}(x)), \tag{30}$$

for the coalescence case. Some consequences of Theorem 1 on θ_ω are investigated in the next section.

The proof of this result is given in [19]. Crucial to the proof is the study of $P(\rho|r,s)$ defined for all test function η as

$$\int_0^\infty \eta(r)P(\rho|r,s-t)dr$$
$$= \int\limits_{\mathbb{R}^d\times\mathbb{R}^d} \eta(|x-x'|)\mathbf{E}\left(g_\omega(y,t|x,s)g_\omega(z,t|x',s)\right)dxdx', \tag{31}$$

in the non-degenerate case, and as

$$\int_0^\infty \eta(r)P(\rho|r,s-t)dr = \mathbf{E}\,\eta(|\varphi_{t,s}(y)-\varphi_{t,s}(z)|), \tag{32}$$

in the coalescence case. Here $\rho = |y-z|$ and $s<t$. $P(\rho|r,s)$ can be thought of as the probability density that two particles have distance r at time $s<t$ if their final distance is ρ at time t. For Kraichnan model, P satisfies the backward equation

$$-\frac{\partial P}{\partial s} = -\frac{\partial}{\partial r}\left(b(r)P\right) + \frac{\partial^2}{\partial r^2}\left(a(r)P\right), \tag{33}$$

for the final condition $\lim_{s\to 0-} P(\rho|r,s) = \delta(r-\rho)$, and with $a(r)$, $b(r)$ such that

$$\begin{aligned} a(r) &= D(\mathcal{S}^2+\xi\mathcal{C}^2)r^\xi + O(r^2/\ell_0^2), \\ b(r) &= D((d-1+\xi)\mathcal{S}^2-\xi\mathcal{C}^2)r^{\xi-1} + O(r/\ell_0^2). \end{aligned} \tag{34}$$

For $r \gg \ell_0$, $a(r)$ tends to C_0, $b(r)$ to $C_0(d-1)/r$, and the equation in 33 reduces to a diffusion equation with constant coefficient. The equation in 33 is singular at $r=0$. The proof of Theorem 1 is essentially reduced to the study of this singular diffusion equation. This is also the main step for which the white-in-time nature of the velocity field is crucial.

Dissipative Anomaly and Structure Functions

We now study some consequences of Theorem 1 for the passive scalar θ_ω defined in 29 or 30. A first key observation is that when g_ω is non-degenerate, there exists an anomalous dissipation mechanism for the scalar, whereas no such anomalous dissipation is present in the coalescence case [16]. The presence of anomalous dissipation is the primary reason why the transport equation in 1 has a statistical steady state (invariant measure) if it is appropriately forced, as we will show later. Notice that since u is compressible, the transport equation in 1 does not conserve $\int_{\mathbb{R}^d} \theta_\omega^2 dx$. To isolate anomalous dissipation, we assume that θ_0 is an homogeneous random process, i.e. such that for all x

$$\theta_0(\cdot) \stackrel{D}{=} \theta_0(\cdot + x), \tag{35}$$

and consider the average energy

$$\mathbf{E}\, \theta_\omega^2(x,t), \tag{36}$$

where the expectation is taken with respect to the statistics of both u and θ_0. If a (generalized) flow of maps can be associated with the dynamics in 1, $\mathbf{E}\,\theta_\omega^2$ is conserved by the dynamics in Kraichnan model since, using the statistical homogeneity of θ_0, we have

$$\mathbf{E}\, \theta_\omega^2(x,t) = \mathbf{E}\, \theta_0^2(\varphi_{t,s}^\omega(x)) = \mathbf{E}\, \theta_0^2(x). \tag{37}$$

Equation 37 holds in the strongly compressible regime and in the intermediate regime under the ε-limit. In these cases, it follows from 37 and the moment inequality $\mathbf{E}\,\theta_\omega^{2n} \geq (\mathbf{E}\,\theta_\omega^n)^2$, $n \in \mathbb{N}$, that the higher order moments are conserved as well

$$\mathbf{E}\, \theta_\omega^{2n}(x,t) = \mathbf{E}\, \theta_0^{2n}(x). \tag{38}$$

In contrast, when the generalized flow is non-degenerate (i.e. in the weakly compressible regime or in the intermediate regime in the κ-limit) the energy is not conserved:

$$\mathbf{E}\, \theta_\omega^2(x,t) < \mathbf{E}\, \theta_0^2(x). \tag{39}$$

Indeed 39 is equivalent to

$$\begin{aligned} \int\limits_{\mathbb{R}^d \times \mathbb{R}^d} & \mathbf{E}\theta_0(y)\theta_0(z)\, \mathbf{E} g_\omega(x,t|dy,s) g_\omega(x,t|dz,s) \\ & < \int_{\mathbb{R}^d} \mathbf{E}\theta_0^2(y) \mathbf{E} g_\omega(x,t|dy,s) = \mathbf{E}\theta_0^2(x), \end{aligned} \tag{40}$$

which follows by the non-degeneracy condition in 25.

Another interesting consequence of Theorem 1 is that the scaling of the second-order structure function is the same for the κ- and the ε-limits in the strongly and the weakly compressible cases [20], but it differs in the intermediate regime as a result of the difference between the limits in 27 and 28. For simplicity of presentation, we assume that θ_0 is isotropic and Gaussian, with covariance

$$\mathbf{E}\, \theta_0(x)\theta_0(y) = D_0 - D(|x-y|), \tag{41}$$

where $D(r)$ is $O(r^2)$ for small r and tends rapidly to D_0 for $r > \lambda$ (i.e. λ is the correlation length for θ_0). Denote ($n \in \mathbb{N}$)

$$S_{2n}(|x-y|,t) = \mathbf{E}(\theta_\omega(x,t) - \theta_\omega(y,t))^{2n}. \tag{42}$$

In the strongly compressible case, we have for both the κ- and the ε-limits

$$S_2(r,t) = O(r^\zeta), \tag{43}$$

with

$$\zeta = \frac{2 - d - \xi + 2\xi\mathcal{P}}{1 + \xi\mathcal{P}}. \tag{44}$$

In the weakly compressible case, we have for both the κ- and the ε-limits

$$S_2(r) = O(r^{2-\xi}). \tag{45}$$

In the intermediate regime, the limits differ, and the κ-limit scales as in 45, whereas the ε-limit scales as in 43. The equations in 43 and 45 can be derived from

$$S_2(r,t) = 2\int_0^\infty (D_0 - D(\rho))(P(0|\rho,-t) - P(r|\rho,-t))d\rho, \tag{46}$$

where P satisfies the equation in 33. In the coalescence case, because P develops a delta peak with mass $1 - \int_{0+}^\infty P d\rho$ at $r = 0$, the equation in 46 can be further simplified to

$$S_2(r,t) = 2\int_0^\infty D(\rho)P(r|\rho,-t)d\rho. \tag{47}$$

It is interesting to discuss the higher order structure functions both in the non-degenerate and in the coalescence cases in 43 and 45 since their scalings highlight very different behavior of the scalar. We consider first the coalescence case which is simpler. In this case, because of the very existence of a flow of maps and the absence of dissipative anomaly, we have

$$S_{2n}(r,t) = \frac{(2n)!}{n!}\int_0^\infty (D(\rho))^n P(r|\rho,-t)d\rho. \tag{48}$$

This gives

$$S_{2n}(r,t) = O(r^\zeta), \tag{49}$$

with ζ given by 44 for all $n \geq 2$. In fact, coalescence implies that the temperature field θ_ω tends to become flat except possibly on a zero-measure set where it presents shock-like discontinuities. Such a situation with two kinds of spatial structures for θ_ω is usually refered to as bi-fractal, and, in simple cases, one may identify ζ with the codimension of the set supporting the discontinuities of θ_ω [21, 22].

The non-degenerate case is more complicated. In this case, one expects that θ_ω presents a spatial behavior much richer than the coalescence case, with all kinds of scalings present. This is the multi-fractal situation for which the higher order structure functions behave as

$$S_{2n}(r,t) = O(r^{\zeta_{2n}}), \tag{50}$$

with $\zeta_{2n} < n(2-\xi)$ for $2n > 2$. The actual value of the ζ_n's cannot be obtained by dimensional analysis, and one has to resort to various sophisticated perturbation techniques (see [3, 4, 5, 6]). We will consider again the scaling of the structure functions at statistical steady state in the section on incomplete self-similarity.

One Force, One Solution Principle

We now turn to question Q3, and first restrict attention to the non-degenerate case. This includes the weakly compressible regime and the intermediate regime in the κ-limit. As we already know from the equation 39, the non-degeneracy of $g_\omega(x,t|dy,s)$ as a probability distribution in y implies dissipation of energy or, phrased differently, decay in memory in the semi-group $S_{t,s}$ generated by $\{g_\omega\}$. This is the main reason that the forced transport equation has a unique invariant measure, as we explain now.

We will consider (compare with 1)

$$\frac{\partial\theta}{\partial t} + (u(x,t)\cdot\nabla)\theta = b(x,t). \tag{51}$$

where b is a white-noise forcing such that

$$\mathbf{E}\, b(x,t)b(y,s) = B(|x-y|)\delta(t-s). \tag{52}$$

$B(r)$ is assumed to be smooth and rapidly decaying to zero for $r > L$; L will be referred to as the forcing scale. The solution of 51 for the initial condition $\theta_\omega(x,s) = \theta_0(x)$ is understood as

$$\theta_\omega(x,t) = S_{t,s}\theta_0(x) + \int_s^t S_{t,\tau}b(x,\tau)d\tau. \tag{53}$$

Define the product probability space $(\Omega_u\times\Omega_b, \mathcal{F}_u\times\mathcal{F}_b, \mathcal{P}_u\times\mathcal{P}_b)$, and the shift operator

$$T_\tau\omega(t) = \omega(t+\tau), \tag{54}$$

with $\omega = (\omega_u, \omega_b)$.

Theorem 2 (One force–one solution I) *For $d > 2$, for almost all ω, there exists a unique solution of 51 defined on $\mathbb{R}^d \times (-\infty, \infty)$. This solution can be expressed as*

$$\theta^\star_\omega(x,t) = \int_{-\infty}^t S_{t,s} b(x,s) ds. \tag{55}$$

Furthermore the map $\omega \to \theta^\star_\omega$ satisfies the invariance property

$$\theta^\star_{T_\tau \omega}(x,t) = \theta^\star_\omega(x, t+\tau). \tag{56}$$

Theorem 2 is the "one force, one solution" principle articulated in [23]. Because of the invariance property 56, the map in 55 leads to a natural invariant measure. As a consequence we have

Theorem 3 *For $d > 2$, there exists a unique invariant measure on $L^2_{loc}(\mathbb{R}^d \times \Omega)$ for the dynamics defined by 51.*

The connection between the map 55 and the invariant measure, together with uniqueness, is explained in [23]. The restriction on the dimensionality in Theorems 2 and 3 arises because the velocity field has finite correlation length ℓ_0, and can be relaxed upon considering the limit as $\ell_0 \to \infty$ after appropriate redefinition of the velocity field as in (63) below.

We sketch the proof of Theorem 2. Basically, it amounts to verifying that the dissipation in the system is strong enough in the sense that

$$\mathbf{E}\Big(\int_{T_1}^{T_2} \int_{\mathbb{R}^d} b(y,s) g(x,t|dy, t+s) ds\Big)^2 \to 0, \tag{57}$$

as $T_1, T_2 \to -\infty$ for fixed x and t. The average in 57 is given explicitly by

$$\int_{T_1}^{T_2} \int_0^\infty B(r) P(0|r,s) dr ds, \tag{58}$$

where P satisfies 33. The convergence of the integral in 57 depends on the rate of decay in $|s|$ of $P(0|r,s)$. Because of the integral in r in 58 has a cut-off at the forcing scale L due to $B(r)$, we can restrict attention to the behavior of $P(0|r,s)$ for $r < L$. For large $|s|$, it follows from the equation in 33 that P can be approximated by

$$P(0|r,s) = \frac{C}{a(r)|s|^\nu} \exp\left(\int_0^r \frac{b(\rho)}{a(\rho)} d\rho\right) + o(|s|^{-a}), \tag{59}$$

where C is a constant and the exponent ν is yet to be determined. The range of value for r in which the approximation in 59 is valid increases with $|s|$. For $|s|$ large enough, most of the mass of $P(0|r,s)$ is in the range $r \gg \ell_0$, where P satisfies a diffusion equation with constant diffusion coefficient C_0 whose exact solution is known. A standard matching argument between this solution and the approximation in 59 can be used to estimate $\nu = d/2$. Thus, using (34) to evaluate the integral in (59), we obtain for $r < L \ll \ell_0$

$$P(0|r,s) = \frac{Cr^\alpha}{|s|^{d/2}} + o(|s|^{-d/2}), \tag{60}$$

where $\alpha = (d-1-\xi(\xi+1)\mathcal{P})/(1+\xi\mathcal{P})$. Using 60 gives the following leading order estimate for the average in 57

$$C\int_0^\infty B(r)r^\alpha dr \int_{T_1}^{T_2} |s|^{-d/2} ds. \tag{61}$$

The integral in s in this expression tends to zero as $T_1, T_2 \to -\infty$ if $d > 2$. It follows that the invariant measure in 55 exists provided $d > 2$.

Consider now the coalescence case, i.e the strongly compressible regime and the intermediate regime in the ε-limit. Since no anomalous dissipation is present in this case, no invariant measure for the temperature field as the one in (55) exists. It makes sense, however, to ask about the existence of an invariant measure for the temperature difference, i.e. to consider

$$\delta\theta^\star_\omega(x,y,t) = \int_{-\infty}^t S_{t,s}(b(x,s)-b(y,s))ds. \tag{62}$$

When $\theta^\star_\omega$ exists, one has $\delta\theta^\star_\omega(x,y,t) = \theta^\star_\omega(x,t) - \theta^\star_\omega(y,t)$, but it is conceivable that $\delta\theta^\star_\omega$ exists in the coalescence case even though $\theta^\star_\omega$ is not defined. The reason is that coalescence of the generalized flow implies that the temperature field flattens with time, which is a dissipation mechanism as far as the temperature difference is concerned. Of course, this effect has to overcome the fluctuations produced by the forcing, and the existence of an invariant measure such as 62 will depend on how fast particles coalesce under the flow.

At this point a difficulty arises. If we were to consider two particles separated by much more than the correlation length ℓ_0, the dynamics of their distance under the flow is governed by the equation in 33 for $r \gg \ell_0$, i.e. by a diffusion equation with constant diffusion coefficient on the scale of interest. It follows that no tendency of coalescence is observed before the distance becomes smaller than ℓ_0, which, as shown below, does not happen fast enough in order to overcome the the fluctuations produced by the forcing. In other words,

Lemma 4 *In the coalescence case, for finite ℓ_0, there is no invariant measure with finite energy for the temperature difference.*

The obvious question to ask next is what happens if we let $\ell_0 \to \infty$? This question, however, has to be considered carefully because the velocity field with the covariance in 12 diverges as $\ell_0 \to \infty$. The right way to proceed is to consider an alternative velocity v, taken to be Gaussian, white-in-time, but *non-homogeneous*, with covariance

$$\begin{aligned} &\mathbf{E}\, v_\alpha(x,t) v_\beta(y,s) \\ &= (c_{\alpha\beta}(x) + c_{\alpha\beta}(y) - c_{\alpha\beta}(x-y))\delta(t-s). \end{aligned} \tag{63}$$

For finite ℓ_0, one has $v(x,t) = u(x,t) - u(a,t)$, where a is arbitrary but fixed. However, v makes sense in the limit as $\ell_0 \to \infty$. Denote by $\vartheta_\omega(x,t)$ the temperature field advected by v, i.e. the solution of the transport equation 51 with u replaced by v. Restricting to zero initial condition, it follows from the homogeneity of the forcing that the single-time moments of θ_ω and ϑ_ω coincide for finite ℓ_0, but in contrast to θ_ω, ϑ_ω makes sense as $\ell_0 \to \infty$. Thus, ϑ_ω is a natural process to study the limit as $\ell_0 \to \infty$, and from now on we restrict attention to this case. Let $\delta\vartheta_\omega(x,y,t) = \vartheta_\omega(x,t) - \vartheta_\omega(y,t)$. The temperature difference $\delta\vartheta_\omega$ satisfies the transport equation

$$\frac{\partial \delta\vartheta}{\partial t} + (v(x,t)\cdot\nabla_x + v(y,t)\cdot\nabla_y)\delta\vartheta = b(x,t) - b(y,t). \tag{64}$$

We have

Theorem 5 (One force–one solution II) *For almost all ω, in the strongly and the weakly compressible regimes, as well as in the intermediate regime if the flow is non-degenerate, there exists a unique solution of 64 defined on $\mathbb{R}^d \times (-\infty,\infty)$. This solution can be expressed as*

$$\delta\vartheta^\star_\omega(x,y,t) = \int_{-\infty}^t S_{t,s}(b(x,s) - b(y,s))ds. \tag{65}$$

Furthermore the map $\omega \to \delta\vartheta^\star_\omega$ satisfies the invariance property

$$\delta\vartheta^\star_{T_\tau\omega}(x,y,t) = \delta\vartheta^\star_\omega(x,y,t+\tau). \tag{66}$$

In contrast, in the intermediate regime if the flow coalesces (ε-limit) there is no such solution with finite covariance.

An immediate consequence of this theorem is

Theorem 6 *In the strongly and the weakly compressible regimes, as well as in the intermediate regime if the flow is non-degenerate, there exists a unique invariant measure on* $L^2_{loc}(\mathbb{R}^d \times \Omega)$ *for the dynamics defined by 64. In the intermediate regime if the generalized flow coalesces, there is no invariant measure for the equation in 64 with finite energy.*

In regimes for which the generalized flow is non-degenerate, Theorem 5 follows from Theorem 2. In the coalescence cases one proceeds similarly as in the proof of Theorem 2 and study the convergence as $T_1, T_2 \to -\infty$ of

$$\mathbf{E}\Big(\int_{T_1}^{T_2} (b(\varphi^{\omega}_{t,s}(x,s) - b(\varphi^{\omega}_{t,s}(y,s))ds\Big)^2. \qquad (67)$$

The average in 57 is given explicitly by

$$2\int_{T_1}^{T_2}\int_0^{\infty}(B(0)-B(\rho))P(r|\rho,s)drds, \qquad (68)$$

where $r = |x-y|$, P satisfies 33 for $\ell_0 = \infty$. Since in the coalescence case $r = 0$ is an exit boundary, it follows that P looses mass at $r = 0+$. The convergence of the time integral in 68 depends on the rate at which mass is lost (i.e. the rate at which particles coalesce). The analysis of the equation in 33 shows that the process is fast enough in order that the integral over s in 68 tends to zero as $T_1, T_2 \to -\infty$ in the strongly compressible regime. In contrast, the integral diverges in the weakly compressible regime in the coalescence case.

Incomplete self-similarity

We finally turn to question Q4 and consider the scaling of the structure functions based on the invariant measure $\delta\vartheta^\star$ defined in 65. Denote

$$\mathcal{S}_n(|x-y|) = \mathbf{E}\,|\delta\vartheta^\star(x,y,t)|^n. \qquad (69)$$

The dimensional parameters are $B_0 = B(0)$ ([temperature]^2[time]$^{-1}$), D ([length]$^{2-\xi}$[time]$^{-1}$), and L ([length]). It follows that

$$\mathcal{S}_n(r) = \left(\frac{B_0 r^{2-\xi}}{D}\right)^{n/2} f_n\left(\frac{r}{L}\right), \qquad (70)$$

where the f_n's are dimensionless functions which cannot be obtained by dimensional arguments. For instance, the scalings in 49, 50 correspond to different f_n. It is however obvious from the equation 70 that, provided the limit exists and is non-zero

$$\lim_{L\to\infty} \mathcal{S}_n(r) = C_n \left(\frac{B_0 r^{2-\xi}}{D}\right)^{n/2} = O(r^{n(2-\xi)/2}). \qquad (71)$$

where $C_n = \lim_{r\to\infty} f_n(r/L)$ are numerical constants. The scaling in 71 is usually referred to as the normal scaling since, consistent with Kolmogorov's picture, it is independent of the forcing or the dissipation scales. In contrast, anomalous scaling is a statement that the structure functions diverge in the limit of infinite forcing scale, $L \to \infty$. In the spirit of Barenblatt-Chorin [7, 8], we may say that normal scaling holds in case of complete self-similarity, whereas anomalous scaling is equivalent to incomplete self-similarity.

It is interesting to discuss the existence or non-existence of the limit in 71 for both the coalescence and the non-degenerate cases. When the flow coalesces, because of the existence of a flow of maps and the absence of dissipative anomaly, the $\mathcal{S}_{2n}$'s of even order $2n \geq 2$ can be computed exactly from

$$\begin{aligned}\mathcal{S}_{2n}(r) = 2n(2n-2)\int_{-\infty}^{0}\int_{0}^{\infty} &(B_0 - B(\rho))^n \\ &\times \mathcal{S}_{2n-2}(r,s) P(r|\rho,s) d\rho ds.\end{aligned} \qquad (72)$$

Evaluation of the integrals in 72 shows that $\mathcal{S}_{2n}(r) = \infty$ for $n \geq \zeta/(2-\xi)$, whereas

$$\mathcal{S}_{2n}(r) = O(r^{\zeta}), \qquad \text{for} \ \ n < \frac{\zeta}{2-\xi}, \qquad (73)$$

where ζ is given in 44. Thus, for $n < \zeta/(2-\xi)$,

$$f_{2n}(r) = O\left((r/L)^{\zeta - n(2-\xi)}\right). \qquad (74)$$

It follows that f_{2n} and, hence, $\mathcal{S}_{2n}$ tend to zero as $L \to \infty$ for $2 \leq n < \zeta/(2-\xi)$, whereas they are infinite for all L for $n \geq \zeta/(2-\xi)$. In fact, in the coalescence case, it can be shown from the expressions in 72 that on scales much larger than the forcing scale L, the structure functions of order $n < \zeta/(2-\xi)$ behave as

$$\mathcal{S}_{2n}(r) \sim C_{2n} r^{n(2-\xi)} \qquad \text{as} \ \ r/L \to \infty. \qquad (75)$$

Thus in the coalescence case, it is more natural to consider the limit as $L \to 0$ of the structure functions, for which the expression in 75 shows the absence of intermittency corrections.

In the non-degenerate case, one has

$$\mathcal{S}_2(r) = O(r^{2-\xi}), \tag{76}$$

while perturbation analysis gives for the higher order structure functions [3, 4, 5, 6]

$$\mathcal{S}_{2n}(r) = O(r^{\zeta_{2n}}), \tag{77}$$

with $\zeta_{2n} < n(2-\xi)$ for $2n > 2$. It follows that $f_2(r) = O(1)$, while

$$f_{2n}(r) = O\left((r/L)^{\zeta_n - n(2-\xi)}\right), \qquad 2n > 2. \tag{78}$$

In other words, as $L \to \infty$, $\mathcal{S}_2$ has a limit which exhibits normal scaling, whereas the $\mathcal{S}_{2n}$'s, $2n > 2$, diverge. This may be closely related to the argument in [7, 8] that, in appropriate limits, intermittency corrections may disappear and higher than fourth order structure functions may not exist. We note, however, that Barenblatt and Chorin were discussing the case of infinite Reynolds number (here infinite Peclet number, $\kappa \to 0$) at finite L, whereas we require $L \to \infty$.

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References

[1] E, W., Khanin, K., Mazel, A., Sinai, Ya. G. *Phys. Rev. Lett.* **78** 1904–1907.

[2] E, W., Vanden Eijnden, E. (1999) *Phys. Rev. Lett.* **83** 2572–2575

[3] Shraiman, B., Siggia, E. (1994) *Phys. Rev. E* **49**, 2912–2927.

[4] Chertkov, M., Falkovich, G., Kolokolov, I., Lebedev, V. (1995) *Phys. Rev. E* **51**, 5609–5627.

[5] Gawędzki, K., Kupiainen, A. (1995) *Phys. Rev. Lett.* **75**, 3834–3837.

[6] Balkovsky, E., Lebedev, V. (1998) *Phys. Rev. E* **58**, 5776–5795.

[7] Barenblatt, G. I., Chorin, A. J. (1997) *Comm. Pure Appl. Math.* **50**, 381–398.

[8] Barenblatt, G. I., Chorin, A. J. (1998) *SIAM Rev.* **40**, 265–291.

[9] As long as u is continuous, Peano's theorem tells us that solutions to 3 do exist.

[10] Brenier, Y. (1989) *J. Amer. Math. Soc.* **2**, 225–255.

[11] Brenier, Y. (1993) *Arch. Rational Mech. Anal.* **122**, 323–351.

[12] Shnirelman, A. I. (1994) *Geom. Funct. Anal.* **4**, 586–620.

[13] Kunita, H. (1990) *Stochastic flows and stochastic differential equations* (Cambridge University Press, Cambridge).

[14] Kraichnan, R. H. (1968) *Phys. Fluids* **11**, 945–963.

[15] Majda, A. J. (1993) *J. Stat. Phys.* **73**, 515–542.

[16] Gawędzki, K., Vergassola, M. (1998) cond-mat/9811399.

[17] Le Jan, Y., Raimond, O. (1999) math.PR/9909147.

[18] Tartar, L. (1983) *Systems of nonlinear partial differential equations* (Oxford, 1982) 263–285, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci, 111 (Reidel, Dordrecht–Boston).

[19] E, W., Vanden Eijnden, E. (2000) in preparation, to be submitted to Physica D.

[20] For the weakly compressible case, because the ε-limit is a weak limit one has to consider the structure functions of the mollified temperature field $\psi_\varepsilon \star \theta^\varepsilon$ in order to obtain the limiting scaling as $\varepsilon \to 0$.

[21] Parisi, G., Frisch, U. (1985) *Turbulence and Predictability in Geophysical Fluid Dynamics* (Varenna, 1983) 83–87 (North-Holland, Amsterdam).

[22] Jaffard, S. (1997) *SIAM J. Math. Anal.* **28** 944–970, 971–998.

[23] E, W., Khanin, K., Mazel, A., Sinai, Ya. G. (2000) to appear in: *Ann. Math.*