Projection Method I :
Convergence and Numerical Boundary Layers

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Abstract: This is the first of a series of papers on the subject of projection methods for viscous incompressible flow calculations. The purpose of these papers is to provide a thorough understanding of the numerical phenomena involved in the projection methods, particularly when boundaries are present, and point to ways of designing more efficient, robust and accurate numerical methods based on the primitive variable formulation. The present paper contains the following topics:

1. Convergence and optimal error estimates for both velocity and pressure up to the boundary;

2. Explicit characterization of the numerical boundary layers in the pressure approximations and the intermediate velocity fields;

3. The effect of choosing different numerical boundary conditions at the projection step. We will show that different choice of boundary conditions gives rise to different boundary layer structures. In particular, the straightforward Dirichlet boundary condition for the pressure leads to O(1) numerical boundary layers in the pressure, and deteriorate the accuracy in the interior;

4. Post-processing the numerical solutions to get more accurate approximations for the pressure.

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§1. Introduction

Projection method was introduced years ago by Chorin [4] and independently by Temam [21] as a way of computing efficiently the solutions of incompressible Navier-Stokes equations (NSE). It is getting increasingly popular in applications to viscous incompressible flows at moderate Reynolds number. With periodic boundary conditions, the performance of the projection method is well-understood from the work of Chorin [5]. However, much less is known when physical boundary conditions such as the no-slip boundary conditions are used. It has been a mystery for twenty-five years that the projection method seems to perform better than expected. There are still controversies with regard to the optimal choice of boundary conditions at the projection step. Furthermore, although it is clear that numerical boundary layers must be present, little is known about their structures.

It is the purpose of this series of papers to fully clarify these issues. Besides being able to answer all these questions, we find that the effect of solid boundaries is not restricted to creating numerical boundary layers, they can also give rise to high frequency oscillations in the leading order error term, reducing the order of accuracy even in the interior of the domain. But when formulated appropriately, projection method is indeed an efficient numerical procedure for viscous incompressible flow calculations. Before our work, comparison of different formulations of projection method was only possible through careful numerical experiments. These numerical experiments are made difficult by the fact that in actual computations, the effect of time and space discretizations, as well as the numerical boundary conditions are all mixed together. Moreover, they usually involve a systematic study of 2-dimensional problems for which the resolution power of modern computers are still quite limited. In our work, we have developed procedures of studying separately the effect of different components of the projection method. In forthcoming papers, we will also make extensive use of 1-dimensional models which capture much of the computational difficulties for incompressible flow calculations.

The present paper is devoted to the explicit characterization of the numerical boundary layers. As a consequence, we also get optimal convergence and error estimates for both velocity and pressure up to the boundary. The boundary layer structure is strongly influenced by the boundary condition for pressure at the projection step. We will study different choices of the pressure boundary conditions and compare their performance in terms of the accuracy of the numerical solutions. Our analysis favors strongly the choice of Neumann boundary conditions.

Roughly speaking, projection method was based on the following philosophy: In incompressible flows, pressure does not carry any thermodynamic meaning and is present only as a Lagrange multiplier for the incompressibility constraint [4]. This observation motivated a time-splitting discretization scheme which decouples the computation of velocity and pressure, a key feature of the projection method. In the first step, an intermediate velocity field is computed using the momentum equation and ignoring the incompressibility constraint. In the second step, the intermediate velocity is projected to the space of divergence free vector fields to get the next update of velocity and pressure. This procedure is much more efficient than solving a coupled system of Stokes equations for velocity and pressure which would arise from a straightforward time discretization of the NSE (see §2). The price been paid, as we will see below, is that it introduces a numerical boundary layer on the pressure approximations and the intermediate velocity fields. This also signifies the main difficulty in the design and implementation of more efficient projection methods: treatment of the boundary conditions.

Over the years projection method has played a dominant role in the computation of viscous incompressible flows based on the primitive variable formulation . It has also acquired other names such as the splitting scheme, fractional step method, etc. Recently there has been a flourish of interest on the application of projection methods for the direct simulation of viscous incompressible flows at moderate Reynolds numbers [2, 3, 12, 13, 17, 24, etc]. Notable in these applications are the various spatial discretizations used, including flux (slope)-limited finite difference methods [2, 3], upwind differencing [17], and spectral-element methods [12].

The analysis of the projection methods was also initiated by Chorin and Temam. In the case of periodic boundary conditions Chorin proved the convergence of a projection method which uses backward Euler in time and centered-differencing in space. His analysis can be easily extended to other methods of a similar nature as long as the periodic boundary condition is retained. Chorin's analysis was facilitated by the fact that with periodic boundary conditions, the projection operator and the laplacian commute. This no longer holds for other types of boundary conditions. As a result, it is much more difficult to study the projection methods when the boundary condition is changed to more physical ones such as the no-slip condition, especially when it comes to the issue of accuracy. Indeed a crude analysis indicates that there is a real danger that the numerical boundary layer in pressure could pollute the numerical solutions in the interior and significantly reduce the overall accuracy (even for velocity) [2, 5], although numerical evidence seems to indicate otherwise.

Considerable progress was made in the recent papers of Shen [18, 19]. For no-slip boundary condition, Shen proved convergence together with some error estimates for various projection methods. Although the estimates obtained were not able to resolve the mystery mentioned earlier, Shen did realize that the key is to understand the time-discretization.

One result of the present paper is a proof that the numerical approximation of velocity indeed has the maximum accuracy. The proof is based on a systematic asymptotic analysis of the numerical solutions. The numerical method is viewed as a singular perturbation of the original NSE, and boundary layer analysis is used to construct approximate solutions which satisfy the numerical scheme to high order accuracy. This, plus the linear stability of the scheme implies the convergence results. This line of thought is often used in applied analysis and was first used by Strang [20] in the context of numerical analysis, although Strang only dealt with a regular perturbation problem. By using similar ideas, Michelson [14] extended Strang's argument to initial-boundary value problems for hyperbolic systems.

The advantage of this approach is that the numerical boundary layers are explicitly characterized. This enables us to propose simple ways of removing the numerical boundary layer by post-processing the numerical solutions. The disadvantage, however, is that it requires far more regularity of the exact solutions than necessary. This translates to a

Reynolds number dependence of the error estimates that are far from being optimal. This is an important issue since very often in actual computations, the smallest mesh size is set by the memory of the machine, and the issue is to resolve flows with the largest possible Reynolds number. In the second paper of this series, we will give an entirely different proof based on Godunov-Ryabenki analysis which not only gives the optimal convergence results with minimum assumptions, but also exhibits clearly the effect of noncommutativity of the various operators involved.

We add here a few general remarks before ending this introduction. There appears to be considerable amount of confusion in the subject of physical and numerical boundary conditions for incompressible flows. Part of this is due to the lack of a well-established model problem which can be used as a benchmark test problem for the various proposals. Very often tests are done for a few selected physical problems and conclusions are drawn from results of these particular runs with a fixed set of numerical and physical parameters (grid size, CFL number, Reynolds number, etc), instead of the asymptotic limit as these parameters vary. It is not surprising that these conclusions often contradict each other.

Also responsible to the state of the confusion is some of the crude and sometimes irrelevant analysis presented in the literature. Extrapolated conclusions based on crude analysis may very well contribute more to the confusion rather than understanding. The numerical phenomena involved in the projection method is sufficiently complex that soft arguments can hardly touch the heart of the matter, neither does a simple convergence theorem or crude error estimates.

For convenience, we list here the content of the rest of the paper:

Section 2. Review of the projection methods

Section 3. Summary of results and outline of proofs

Section 4. First order schemes without spatial discretization

Section 5. Effect of numerical boundary conditions

Section 6. Second order schemes without spatial discretization

Section 7. Generalizations

Appendix 1. First order schemes with spatial discretization

Appendix 2. Second order schemes with spatial discretization

Appendix 3. Post-processing for the pressure.

§2. Review of the Projection Methods

In primitive variables, NSE takes the following form

$$\left\{\begin{aligned} &\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \Delta \boldsymbol{u}\,, \\ &\nabla\cdot\boldsymbol{u} = 0\,. \end{aligned}\right. \tag{2.1}$$

Here $\boldsymbol{u} = (u, v)$ is the velocity, and p is the pressure. For simplicity, we will only consider the case when the no-slip boundary condition is supplemented to (2.1):

$$\boldsymbol{u} = 0 \qquad \text{on } \partial\Omega \tag{2.2}$$

where Ω is an open domain in $\boldsymbol{R}^2$ with smooth or piecewise smooth boundary.

§2.1. Time discretization

As a first step toward the construction of an efficient numerical scheme for (2.1)-(2.2), we discretize (2.1) in time using backward Euler:

$$\left\{\begin{aligned} &\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\triangle t} + (\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n + \nabla p^{n+1} = \Delta \boldsymbol{u}^{n+1}\,, \\ &\nabla\cdot\boldsymbol{u}^{n+1} = 0\,. \end{aligned}\right. \tag{2.3}$$

We do not hesitate to use implicit schemes since the NSE is intrinsically implicit anyway. Alternately we can discretize (2.1) using the trapezoidal rule, resulting in the Crank-Nicholson scheme:

$$\left\{\begin{aligned} &\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\triangle t} + (\boldsymbol{u}^{n+1/2}\cdot\nabla)\boldsymbol{u}^{n+1/2} + \nabla p^{n+1} = \Delta \frac{\boldsymbol{u}^{n+1} + \boldsymbol{u}^n}{2}\,, \\ &\nabla\cdot\boldsymbol{u}^{n+1} = 0\,. \end{aligned}\right. \tag{2.4}$$

It is not important at this point to specify the discretization for the convection terms. (2.3) and (2.4) are solved together with the boundary condition :

$$\boldsymbol{u}^{n+1} = 0 \qquad \text{on } \partial\Omega\,. \tag{2.5}$$

However, both schemes are highly inefficient since they require, at each time step, the solution of (2.3) or (2.4) which are coupled systems of Stokes-like equations for $(\boldsymbol{u}^{n+1}, p^{n+1})$. This is precisely the reason for proposing the projection method, as a numerical device to decouple the computation of $\boldsymbol{u}^{n+1}$ and p^{n+1} [4, 21]. Instead of simultaneously satisfying the momentum equation and the incompressibility constraint, projection method proceeds by first ignoring the incompressibility constraint, compute an intermediate velocity field $\boldsymbol{u}^*$ using the momentum equation and then project $\boldsymbol{u}^*$ back to the space of incompressible vector fields to obtain $\boldsymbol{u}^{n+1}$ and p^{n+1}. The actual realization of this procedure for the first order scheme can be summarized as :

First order scheme:

Step 1:

$$\begin{cases} \dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\triangle t} + (\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n = \Delta \boldsymbol{u}^*\,, \\ \\ \boldsymbol{u}^* = 0\,, \qquad \text{on } \partial\Omega\,. \end{cases} \tag{2.6}$$

Step 2:

$$\begin{cases} \boldsymbol{u}^* = \boldsymbol{u}^{n+1} + \triangle t \nabla p^{n+1}\,, \\ \nabla \cdot \boldsymbol{u}^{n+1} = 0\,. \end{cases} \tag{2.7}$$

The boundary condition for $\boldsymbol{u}^*$ in (2.6) is rather natural, at least for the first order scheme. The agonizing decision to be made is the boundary condition for (2.7). If we take the inner product of (2.1) with the unit normal and tangent vectors at $\partial\Omega$, $\boldsymbol{n}$ and $\boldsymbol{t}$ respectively, we arrive at

$$\frac{\partial p}{\partial \boldsymbol{n}} = \boldsymbol{n} \cdot \Delta \boldsymbol{u}\,, \qquad \frac{\partial p}{\partial \boldsymbol{t}} = \boldsymbol{t} \cdot \Delta \boldsymbol{u}\,, \qquad \text{on } \partial\Omega\,. \tag{2.8}$$

So both the Neumann and Dirichlet boundary conditions seem plausible for the pressure in (2.7). The prevailing point of view for resolving this ambiguity is the following [6]. The boundary condition in (2.7) is part of the specification of the projection operator. If one requires that the space of divergence-free vector fields be orthogonal (with respect to the usual L^2 inner project) to the space of irrotational vector fields, then the divergence-free fields has to satisfy the boundary condition :

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0\,, \qquad \text{on } \partial\Omega\,. \tag{2.9}$$

Therefore for (2.7) one has

$$\boldsymbol{u}^{n+1}\cdot\boldsymbol{n} = 0\,, \quad \text{or} \quad \frac{\partial p^{n+1}}{\partial \boldsymbol{n}} = 0\,, \qquad \text{on } \partial\Omega\,. \tag{2.10}$$

In this case (2.7) is none other than the standard Helmholtz decomposition. This boundary condition is strongly favored in the literature. The question to be addressed then is whether orthogonality is really important.[3]

The bottomline is that in most situations, large errors will be introduced at the boundary, either on velocity or on pressure, because of the inconsistency of the boundary conditions. The hope is that these large errors will be restricted to a boundary layer and not affect the accuracy in the interior. Whether this actually happens is precisely the question to be addressed here.

To give an indication that the numerical solution contains boundary layers, let us consider the linear case. Without the nonlinear term, (2.6), (2.7) and (2.10) combine to give

$$\begin{cases} (I - \Delta t \Delta)\Delta p^{n+1} = 0\,, \\ \dfrac{\partial p^{n+1}}{\partial \boldsymbol{n}} = 0\,, \qquad \text{on } \partial\Omega\,. \end{cases} \tag{2.11}$$

In contrast, the linear Stokes equations implies

$$\Delta p = 0$$

without boundary condition on p. Therefore if $p^{n+1}(\boldsymbol{x})$ has any chance of being close to $p(\boldsymbol{x}, (n+1)\Delta t)$, there must be numerical boundary layers in p^{n+1} with thickness $O(\Delta t^{1/2})$. This is indeed the case as will be seen in §3 and §4.

Second order schemes:

[3] Here and in the following, the term "projection" should be understood in a more general sense than the Helmholtz decomposition since more general boundary conditions are allowed.

There are at least three different ways to decouple the system (2.4) to get a formally second order scheme. These are respectively projection methods bases on: (1) accurate boundary conditions for the intermediate velocity field [13]; (2) accurate pressure boundary conditions [16]; (3) pressure increment formulation [2, 24]. Below is a summary of these methods.

(1) Projection method based on accurate boundary conditions for the intermediate velocity field (Kim and Moin's method [13]):

$$
\begin{cases}
\dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\triangle t} + (\boldsymbol{u}^{n+1/2}\cdot\nabla)\boldsymbol{u}^{n+1/2} = \Delta\dfrac{\boldsymbol{u}^* + \boldsymbol{u}^n}{2}\,, \\
\boldsymbol{u}^* + \boldsymbol{u}^n = \triangle t \nabla p^{n-1/2}\,, \qquad \text{on } \partial\Omega\,, \\
\boldsymbol{u}^* = \boldsymbol{u}^{n+1} + \triangle t \nabla p^{n+1/2}\,, \\
\nabla\cdot\boldsymbol{u}^{n+1} = 0\,, \\
\dfrac{\partial p^{n+1/2}}{\partial \boldsymbol{n}} = 0\,, \qquad \text{on } \partial\Omega\,.
\end{cases}
\tag{2.12}
$$

In this formulation, homogeneous Neumann boundary condition for pressure is retained. An inhomogeneous boundary condition for $\boldsymbol{u}^*$ is introduced so that the slip velocity of $\boldsymbol{u}^{n+1}$ at the boundary is of order $\triangle t^2$.

Remark. The nonlinear convection term $(\boldsymbol{u}^{n+1/2}\cdot\nabla)\boldsymbol{u}^{n+1/2}$ can be treated in many ways. In Theorems 2 and 4, we use an explicit Adams-Bashforth formula, $\frac{3}{2}(\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n - \frac{1}{2}(\boldsymbol{u}^{n-1}\cdot\nabla)\boldsymbol{u}^{n-1}$, which is the one used by Kim and Moin.

It is readily seen that the projection step enforces

$$
\frac{\partial p^{n+1}}{\partial \boldsymbol{n}} = \frac{\partial p^{n}}{\partial \boldsymbol{n}} = \cdots = \frac{\partial p^{0}}{\partial \boldsymbol{n}} = 0\,, \qquad \text{on } \partial\Omega
\tag{2.13}
$$

for the numerical solution. In general this is not satisfied by the exact solution of (2.1). Therefore we expect that $\frac{\partial p^n}{\partial \boldsymbol{n}}$ has $O(1)$ error at the boundary. As will be seen in §4, this causes $\boldsymbol{u}^*$ and p^n to have numerical boundary layers.

(2) Projection method based on accurate pressure boundary condition [16]:

$$
(2.14)\qquad \begin{cases} \dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\triangle t} + (\boldsymbol{u}^{n+1/2}\cdot\nabla)\boldsymbol{u}^{n+1/2} = \Delta\dfrac{\boldsymbol{u}^* + \boldsymbol{u}^n}{2}\,, \\ \boldsymbol{u}^* = 0\,, \qquad \text{on } \partial\Omega\,, \\ \boldsymbol{u}^* = \boldsymbol{u}^{n+1} + \triangle t \nabla p^{n+1/2}\,, \\ \nabla\cdot\boldsymbol{u}^{n+1} = 0\,, \\ \dfrac{\partial p^{n+1/2}}{\partial \boldsymbol{n}} = -\boldsymbol{n}\cdot[\nabla\times(\nabla\times\boldsymbol{u}^*)]\,, \qquad \text{on } \partial\Omega\,. \end{cases}
$$

In this formulation, the homogeneous Dirichlet boundary condition for the intermediate state $\boldsymbol{u}^*$ is retained. An inhomogeneous Neumann boundary condition for pressure is introduced so that the slip velocity of $\boldsymbol{u}^{n+1}$ at the boundary is of order $O(\triangle t^2)$.

The boundary condition for pressure in (2.14) is motivated by the first relation in (2.8). Notice that imposing (2.8) directing may not be consistent with the Poisson equation for pressure

$$
(2.15)\qquad \Delta p^{n+1} = \frac{1}{\triangle t}\nabla\cdot\boldsymbol{u}^*
$$

which implies

$$
(2.16)\qquad \int_{\partial\Omega} \frac{\partial p^{n+1}}{\partial \boldsymbol{n}}\, ds = 0\,.
$$

However, the revised form of the pressure boundary condition is guaranteed to be consistent with the above relation. For more discussion see the end of §5.

(3) Projection method based on the pressure increment formulation: [2, 3, 24]

$$
(2.17)\qquad \begin{cases} \dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\triangle t} + (\boldsymbol{u}^{n+1/2}\cdot\nabla)\boldsymbol{u}^{n+1/2} + \nabla p^{n-1/2} = \Delta\dfrac{\boldsymbol{u}^* + \boldsymbol{u}^n}{2}\,, \\ \boldsymbol{u}^* = 0\,, \qquad \text{on } \partial\Omega\,, \\ \boldsymbol{u}^* = \boldsymbol{u}^{n+1} + \triangle t(\nabla p^{n+1/2} - \nabla p^{n-1/2})\,, \\ \nabla\cdot\boldsymbol{u}^{n+1} = 0\,, \\ \dfrac{\partial p^{n+1/2}}{\partial \boldsymbol{n}} = 0\,, \qquad \text{on } \partial\Omega\,. \end{cases}
$$

Again the spurious slip velocity of $\boldsymbol{u}^{n+1}$ at the boundary is of order $\triangle t^2$, and the numerical solutions satisfy (2.13). If we let $\hat{\boldsymbol{u}} = \boldsymbol{u}^* - \triangle t \nabla p^{n-1/2}$ in (2.12), then we have

$$
\begin{cases}
\dfrac{\hat{\boldsymbol{u}} - \boldsymbol{u}^n}{\triangle t} + (\boldsymbol{u}^{n+1/2}\cdot\nabla)\boldsymbol{u}^{n+1/2} + \nabla p^{n-1/2} = \Delta \dfrac{\hat{\boldsymbol{u}} + \boldsymbol{u}^n}{2} + \dfrac{\Delta t}{2}\Delta\nabla p^{n-1/2}\,, \\
\hat{\boldsymbol{u}} + u^n = 0\,, \qquad \text{on } \partial\Omega\,, \\
\boldsymbol{u}^* = \boldsymbol{u}^{n+1} + \triangle t(\nabla p^{n+1/2} - \nabla p^{n-1/2})\,, \\
\nabla\cdot\boldsymbol{u}^{n+1} = 0\,, \\
\dfrac{\partial p^{n+1/2}}{\partial \boldsymbol{n}} = 0\,, \qquad \text{on } \partial\Omega\,.
\end{cases}
\tag{2.18}
$$

Except the last term is the first equation, this is basically the same as (2.17). This suggests that (2.17) should behavior similarly to (2.12). Surprisingly enough, (2.17) exhibits some peculiarities not shared by either (2.12) or (2.14). This will be the subject of a subsequent paper [8].

§2.2. Spatial discretization

The remaining task is to solve the Poisson type equations in (2.6)-(2.7) etc, instead of the coupled system of Stokes-like equation in (2.3) and (2.4). Any of the popular methods, such as finite difference, finite element, spectral, or spectral element, can be used for this purpose. In many cases, fast Poisson solvers or domain decomposition methods can be used to drastically speed up the calculation. When the Reynolds number is large, the NSE are effectively convection-dominated. One can then borrow the techniques developed in the numerical solutions of hyperbolic equations or compressible flows. Such examples can be found in [2, 3, 17].

As an example of how the fully discrete schemes can be analyzed in the same fashion as the spatially continuous schemes, we will consider in the Appendices the well-known spatial discretization scheme: centered difference on a staggered grid (also known as the MAC mesh), coupled with the time-splitting schemes.

An illustration of the MAC mesh near the boundary is given in Figure 1, following the presentation of [1]. Here pressure is evaluated at the square points (i, j), the u velocity at the triangle points $(i \pm 1/2, j)$, and the v velocity at the circle points $(i, j \pm 1/2)$. The

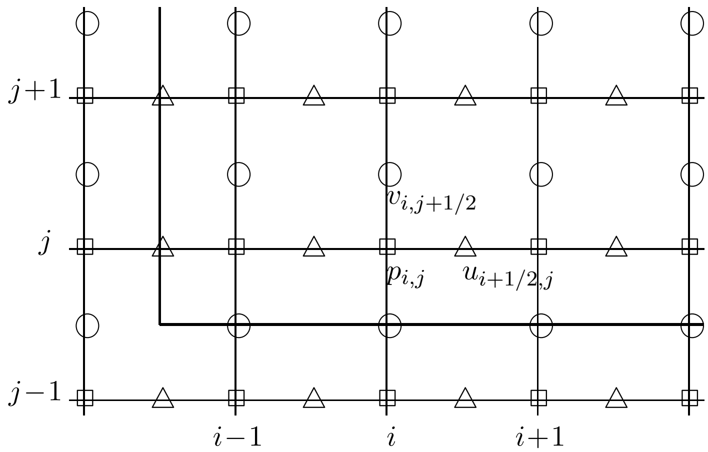

Figure 1: The MAC mesh

discrete divergence is computed at the square points:

$$(\nabla\cdot\boldsymbol{u})_{i,j} = \frac{u_{i+1/2,j} - u_{i-1/2,j}}{\triangle x} + \frac{v_{i,j+1/2} - v_{i,j-1/2}}{\triangle y}\,. \tag{2.19}$$

Other differential operators are discretized as:

(2.20)

$$(\Delta u)_{i+1/2,j} = \frac{u_{i+3/2,j} - 2u_{i+1/2,j} + u_{i-1/2,j}}{\triangle x^2} + \frac{u_{i+1/2,j+1} - 2u_{i+1/2,j} + u_{i+1/2,j-1}}{\triangle y^2}\,,$$

$$(\Delta v)_{i,j+1/2} = \frac{v_{i+1,j+1/2} - 2u_{i,j+1/2} + u_{i-1,j+1/2}}{\triangle x^2} + \frac{v_{i,j+3/2} - 2v_{i,j+1/2} + v_{i,j-1/2}}{\triangle y^2}\,,$$

$$\begin{aligned} (p_x)_{i+1/2,j} &= \frac{p_{i+1,j} - p_{i,j}}{\triangle x}\,, \\ (p_y)_{i,j+1/2} &= \frac{p_{i,j+1} - p_{i,j}}{\triangle y}\,, \end{aligned} \tag{2.21}$$

$$\begin{aligned} \bar{u}_{i,j+1/2} &= \tfrac{1}{4}(u_{i+1/2,j} + u_{i-1/2,j} + u_{i+1/2,j+1} + u_{i-1/2,j+1})\,, \\ \bar{v}_{i+1/2,j} &= \tfrac{1}{4}(v_{i+1,j+1/2} + v_{i+1,j-1/2} + v_{i,j+1/2} + v_{i,j-1/2})\,. \end{aligned} \tag{2.22}$$

$$
(2.23) \qquad \begin{aligned}
(\boldsymbol{u}\cdot\nabla a)_{i+1/2,j} &= u_{i+1/2,j}\frac{a_{i+3/2,j}-a_{i-1/2,j}}{2\triangle x} + \bar{v}_{i+1/2,j}\frac{a_{i+1/2,j+1}-a_{i+1/2,j-1}}{2\triangle y} \\
(\boldsymbol{u}\cdot\nabla b)_{i,j+1/2} &= \bar{u}_{i,j+1/2}\frac{b_{i+1,j+1/2}-b_{i-1,j+1/2}}{2\triangle x} + v_{i,j+1/2}\frac{b_{i,j+3/2}-b_{i,j-1/2}}{2\triangle y}
\end{aligned}
$$

$$
(2.24) \qquad \mathcal{N}_h(\boldsymbol{u},\boldsymbol{a}) = ((\boldsymbol{u}\cdot\nabla a)_{i+1/2,j}, (\boldsymbol{u}\cdot\nabla b)_{i,j+1/2})
$$

Clearly the truncation errors of these approximations are of second order.

The boundary condition $u = 0$ is imposed at the vertical physical boundary, whereas $v = 0$ is imposed at the "ghost" circle points which are $\triangle x/2$ to the left or right of the physical boundary. Similarly the boundary condition $v = 0$ is imposed at the horizontal physical boundary, but $u = 0$ is imposed at the "ghost" triangle points with a distance of $\triangle y/2$ away from the physical boundary.

Notations: We will use C to denote generic constants which may depend on the norms of the exact solutions. Norms will be taken over the entire domain Ω.

§3. Summary of Results and Outline of Proofs

For simplicity of presentation, we will concentrate on the situation when $\Omega = [-1,1] \times [0,2\pi]$ with periodic boundary condition in the y direction and no-slip boundary condition in the x-direction: $\boldsymbol{u}(x,0,t) = \boldsymbol{u}(x,2\pi,t), \boldsymbol{u}(-1,y,t) = 0, \boldsymbol{u}(1,y,t) = 0$. We will use $\partial'\Omega$ to denote the part of the boundary at $x = \pm 1$. We will always assume that $\triangle x \sim \triangle y$ and $h = \min(\triangle x, \triangle y)$. Extensions to general domains will be discussed in §7. We will concentrate our discussions on the spatially continuous schemes since the main issue is in the time-discretization, as we have illustrated above.

The main results of this paper are the following (the constants are independent of $\triangle t$ and h):

Theorem 1. *Let* $(\boldsymbol{u}, p)$ *be a smooth solution of the Navier-Stokes equation* (2.1) *with smooth initial data* $\mathbf{u}^0(\boldsymbol{x})$ *and let* $(\boldsymbol{u}_{\triangle t}, p_{\triangle t})$ *be the numerical solution for the semi-discrete projection method* (2.6), (2.7) *and* (2.10). *Then we have*

$$
(3.1) \qquad \|\boldsymbol{u}-\boldsymbol{u}_{\triangle t}\|_{L^\infty(0,T;L^2)} + \triangle t^{1/2}\,\|p-p_{\triangle t}\|_{L^2(0,T;L^2)} \le C\triangle t\,,
$$

Furthermore, if $\boldsymbol{u}^0(\boldsymbol{x})$ *satisfies the compatibility condition*

$$\boldsymbol{u}^0(\boldsymbol{x}) = 0\,, \quad \partial_y p(\boldsymbol{x},0) = \partial^2_{xy} p(\boldsymbol{x},0) = 0\,, \quad \text{on } \partial^{'}\Omega, \tag{3.2}$$

then we have

$$\|\boldsymbol{u} - \boldsymbol{u}_{\triangle t}\|_{L^\infty} + \triangle t^{1/2} \|p - p_{\triangle t}\|_{L^\infty} \le C \triangle t\,, \tag{3.3}$$

$$\|p - p_{\triangle t} - p_c\|_{L^\infty} \le C \triangle t \tag{3.4}$$

where

$$\begin{aligned} p_c(\boldsymbol{x},t) \; &\equiv \triangle t^{1/2} \frac{e}{e-1} e^{-|x-1|/\triangle t^{1/2}}\, \partial_x p_{\triangle t}(x - \triangle t^{1/2}, y, t) \\ &+ \triangle t^{1/2} \frac{e}{e-1} e^{-|x+1|/\triangle t^{1/2}}\, \partial_x p_{\triangle t}(x + \triangle t^{1/2}, y, t)\,. \end{aligned}$$

Remark. It is rather common to require compatibility conditions on the initial data for the convergence of numerical schemes, although here we require more than necessary. We refer to the work of Heywood and Rannacher [11], and Okamoto [15] on discussions of minimum compatibility assumptions.

Theorem 2. *Let* $(\boldsymbol{u}, p)$ *be a smooth solution of the Navier-Stokes equation* (2.1) *with smooth initial data* $\boldsymbol{u}^0(\boldsymbol{x})$ *and let* $(\boldsymbol{u}_{\triangle t}, p_{\triangle t})$ *be the numerical solution for the semi-discrete projection method* (2.12)*. Then we have*

$$\|\boldsymbol{u} - \boldsymbol{u}_{\triangle t}\|_{L^\infty(0,T;L^2)} + \triangle t \|p - p_{\triangle t}\|_{L^\infty(0,T;L^2)} \le C \triangle t^2\,, \tag{3.5}$$

Furthermore, if $\boldsymbol{u}^0(\boldsymbol{x})$ *satisfies the compatibility condition*

$$\partial_x^{\alpha_1} \partial_y^{\alpha_2} \boldsymbol{u}^0(\boldsymbol{x}) = 0\,, \quad \text{on } \partial^{'}\Omega\,, \text{for } \alpha_1 + \alpha_2 \le 6\,, \tag{3.6}$$

then we have

$$\|\boldsymbol{u} - \boldsymbol{u}_{\triangle t}\|_{L^\infty} + \triangle t^{3/2} \|p - p_{\triangle t}\|_{L^\infty} \le C \triangle t^2\,, \tag{3.7}$$

$$\max_{dist(\boldsymbol{x} - \partial^{'}\Omega) \ge \triangle t^{1/2}} |p - p_{\triangle t}| \le C \triangle t^2\,, \tag{3.8}$$

$$\|p - p_{\triangle t} - p_c\|_{L^\infty} \le C\triangle t\,, \tag{3.9}$$

where

$$\begin{aligned} p_c(\boldsymbol{x},t) \;&\equiv \sqrt{\frac{\triangle t}{2}}\,\frac{e^{\sqrt{2}}}{e^{\sqrt{2}}-1}e^{-\sqrt{2}|x-1|/\triangle t^{1/2}}\,\partial_x p_{\triangle t}(x-\triangle t^{1/2},y,t) \\ &+\sqrt{\frac{\triangle t}{2}}\,\frac{e^{\sqrt{2}}}{e^{\sqrt{2}}-1}e^{-\sqrt{2}|x+1|/\triangle t^{1/2}}\,\partial_x p_{\triangle t}(x+\triangle t^{1/2},y,t). \end{aligned}$$

Remark. Appendix 3 contains a discussion on how to remove the next order boundary layer errors in the numerical approximations of pressure to get uniform $O(\triangle t^2)$ convergence rate.

Theorem 3. *Let* $(\boldsymbol{u},p)$ *be a solution of the Navier-Stokes equation* (2.1) *with smooth initial data* $\boldsymbol{u}^0(\boldsymbol{x})$ *satisfying the compatibility condition*

$$\boldsymbol{u}^0(\boldsymbol{x}) = 0\,, \quad \partial_y p(\boldsymbol{x},0) = \partial^2_{xy} p(\boldsymbol{x},0) = 0\,, \quad \text{on } \partial^{'}\Omega, \tag{3.10}$$

Let $(\boldsymbol{u}_h,p_h)$ *be the numerical solution of the projection method* (2.6), (2.7) *and* (2.10) *coupled with the MAC spatial discretization. Assume that* $\triangle t << h$. *Then we have*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^\infty} + \triangle t^{1/2}\|p-p_h\|_{L^\infty} \le C(\triangle t + h^2)\,, \tag{3.11}$$

$$\|p - p_h - p_c\|_{L^\infty} \le C(\triangle t + h^2)\,, \tag{3.12}$$

where

$$\begin{aligned} p_c(\boldsymbol{x},t) \;&\equiv \triangle t^{1/2}\,\beta\frac{e^{\alpha}}{e^{\alpha}-1}\,e^{-\alpha|x-1|/\triangle t^{1/2}}\;D^x_+ p_h(x-\triangle t^{1/2},y,t) \\ &+\triangle t^{1/2}\,\beta\frac{e^{\alpha}}{e^{\alpha}-1}\,e^{-\alpha|x+1|/\triangle t^{1/2}}\;D^x_+ p_h(x+\triangle t^{1/2},y,t), \end{aligned} \tag{3.13}$$

$$\alpha = \frac{\triangle t^{1/2}}{\triangle x}\text{arccosh}\,\left(1+\frac{\triangle x^2}{2\triangle t}\right), \qquad \beta = \frac{\triangle x}{\triangle t^{1/2}}\left(1-e^{-\alpha\triangle x/\triangle t^{1/2}}\right)^{-1}.$$

Theorem 4. *Let* $(\boldsymbol{u},p)$ *be a smooth solution of the Navier-Stokes equation* (2.1) *with smooth initial data* $\boldsymbol{u}^0(\boldsymbol{x})$ *satisfying the compatibility condition*

$$\partial_x^{\alpha_1}\partial_y^{\alpha_2}\boldsymbol{u}^0(\boldsymbol{x}) = 0\,, \quad \text{on } \partial\Omega\,, \text{for } \alpha_1+\alpha_2 \le 6\,, \tag{3.14}$$

Let $(\boldsymbol{u}_h, p_h)$ be the numerical solution of the projection method (2.12) coupled with the MAC spatial discretization. Assume that $\triangle t^2 << h$. Then we have

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^\infty} + \triangle t^{3/2}\|p-p_h\|_{L^\infty} + \triangle t\|p-p_h\|_{L^\infty(0,T,L^2)} \le C(\triangle t^2 + h^2)\,. \tag{3.15}$$

$$\|p-p_h-p_c\|_{L^\infty} \le C(\triangle t + h^2)\,, \tag{3.16}$$

where

$$\begin{aligned} p_c \ &\equiv \triangle t^{1/2}\,\beta\frac{e^\alpha}{e^\alpha-1}\,e^{-\alpha|x-1|/\triangle t^{1/2}}\ D_+^x p_h(x-\triangle t^{1/2},y,t) \\ &+\triangle t^{1/2}\,\beta\frac{e^\alpha}{e^\alpha-1}\,e^{-\alpha|x+1|/\triangle t^{1/2}}\ D_+^x p_h(x+\triangle t^{1/2},y,t), \end{aligned} \tag{3.17}$$

$$\alpha = \frac{\triangle t^{1/2}}{\triangle x}\text{arccosh}\,(1+\frac{\triangle x^2}{\triangle t})\,, \qquad \beta = \frac{\triangle x}{\triangle t^{1/2}}(1-e^{-\alpha\triangle x/\triangle t^{1/2}})^{-1}\,.$$

Remark. We refer to §7 for extensions to general domains.

There are three major steps in the proofs of these results. Here we illustrate these steps for the first order scheme (2.6), (2.7) and (2.10).

Steps 1: Using boundary layer analysis, we construct approximate solutions of the form ($t^n = n\triangle t$):

$$\begin{aligned} U^*(\boldsymbol{x},t^n) &= \boldsymbol{u}_0^*(\boldsymbol{x},t^n) + \triangle t^{1/2}\ \boldsymbol{u}_1^*(\boldsymbol{x},(x\pm 1)/\triangle t^{1/2},t^n) + \cdots\,, \\ U^n(\boldsymbol{x},t^n) &= \boldsymbol{u}_0(\boldsymbol{x},t^n) + \triangle t^{1/2}\ \boldsymbol{u}_1(\boldsymbol{x},(x\pm 1)/\triangle t^{1/2},t^n) + \cdots\,, \\ P^n(\boldsymbol{x},t^n) &= p_0(\boldsymbol{x},t^n) + \triangle t^{1/2}\ p_1(\boldsymbol{x},(x\pm 1)/\triangle t^{1/2},t^n) + \cdots \end{aligned} \tag{3.18}$$

satisfying the numerical scheme to high order accuracy:

$$\begin{cases} \dfrac{U^*-U^n}{\triangle t} + (U^n\cdot\nabla)U^n = \Delta U^* + \triangle t^\alpha \boldsymbol{f}^n\,, \\ U^* = 0\,, \qquad \text{on } \partial\Omega\,, \\ \dfrac{U^{n+1}-U^*}{\triangle t} + \nabla P^n = \triangle t^\alpha \boldsymbol{g}^n\,, \\ \nabla\cdot U^{n+1} = 0\,, \\ U^{n+1}\cdot\boldsymbol{n} = 0\,, \qquad \text{on } \partial\Omega\,, \\ U^0 = \boldsymbol{u}^0 + \triangle t^\alpha \boldsymbol{w}^0 \end{cases} \tag{3.19}$$

where α is a pre-determined number.

Step 2: The L^2-stability of these numerical schemes can be proved using energy estimates. Together with (2.6), (2.7), (2.10) and (3.19), we get

$$\begin{aligned} \|\boldsymbol{u}^n - U^n\|_{L^2} &\le C^* \triangle t^{\alpha} , \\ \|\boldsymbol{u}^* - U^*\|_{L^2} &\le C^* \triangle t^{\alpha} , \\ \|p^n - P^n\|_{L^2} &\le C^* \triangle t^{\alpha-1} \end{aligned} \tag{3.20}$$

where the constant C^* depends on either

$$\|\boldsymbol{u}^n\|_{L^\infty} = \sup_{0\le t\le T} \|\boldsymbol{u}^n(\cdot,t)\|_{L^\infty} ,$$

or

$$\|\boldsymbol{u}^n\|_{W^{1,\infty}} = \sup_{0\le t\le T} \|\boldsymbol{u}^n(\cdot,t)\|_{W^{1,\infty}} .$$

Step 3: To complete the proof, we need to:

(1) Establish a priori estimates on $\|\boldsymbol{u}^n\|_{L^\infty}$ or $\|\boldsymbol{u}^n\|_{W^{1,\infty}}$;

(2) Convert the L^2 estimates in (3.20) to L^∞ estimates.

The standard way of achieving (1) and (2) in fully-discrete methods is to use the inverse inequality:

$$\|\boldsymbol{u}_h\|_{L^\infty} \le h^{-d/2} \|\boldsymbol{u}_h\|_{L^2}$$

where h is the spatial mesh size, d is the dimension. This is also the major component of Strang and Michelson's analysis. This standard trick is used to prove Theorems 3 and 4 for the fully-discrete schemes. However, this trick cannot be used to prove Theorems 1 and 2 which deal with the spatially continuous schemes. In this case, we get (1) and (2) directly by using careful a priori estimates and the regularity theory for elliptic equations.

The actual proofs are quite complicated. In the next section, we provide the detailed proof of Theorem 1. The proof of Theorem 2 is analogous although some details in estimates are different. This is done in §6. The fully-discrete schemes and the proofs of Theorems 3 and 4 are left to the Appendices.

§4. First Order Schemes Without Spatial Discretization

We will concentrate on the following version of the first order projection method:

$$
\begin{cases}
\dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\triangle t} + (\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n = \Delta \boldsymbol{u}^* \,, \\[2ex]
\boldsymbol{u}^* = 0 \,, \qquad \text{on } \partial\Omega \,, \\
\boldsymbol{u}^* = \boldsymbol{u}^{n+1} + \triangle t \nabla p^n \,, \\
\nabla \cdot \boldsymbol{u}^{n+1} = 0 \,, \\
\dfrac{\partial p^n}{\partial \boldsymbol{n}} = 0 \,, \qquad \text{on } \partial\Omega \,.
\end{cases}
\tag{4.1}
$$

The corresponding fully discrete scheme with the standard MAC spatial discretization will be studied in Appendix 1. Many variants of (4.1) are possible. Some of them are discussed in the next section.

§4.1 Asymptotic Analysis of the Numerical Solutions

Denote the solutions of (4.1) as $(\boldsymbol{u}_{\triangle t}, \boldsymbol{u}^*_{\triangle t}, p_{\triangle t})$. Motivated by the discussions in §2, we make the following ansatz, valid at $t^n = n\triangle t$, $n = 1, 2, \cdots$

$$
\begin{cases}
\boldsymbol{u}^*_{\triangle t}(\boldsymbol{x}, t) = \boldsymbol{u}^*_0(\boldsymbol{x}, t) + \displaystyle\sum_{j=1} \varepsilon^j [\boldsymbol{u}^*_j(\boldsymbol{x}, t) + \boldsymbol{a}^*_j(\xi, y, t)] \,, \\[2ex]
\boldsymbol{u}_{\triangle t}(\boldsymbol{x}, t) = \boldsymbol{u}_0(\boldsymbol{x}, t) + \displaystyle\sum_{j=1} \varepsilon^j \boldsymbol{u}_j(\boldsymbol{x}, t) \,, \\[2ex]
p_{\triangle t}(\boldsymbol{x}, t) = p_0(\boldsymbol{x}, t) + \varphi_0(\xi, y, t) + \displaystyle\sum_{j=1} \varepsilon^j [p_j(\boldsymbol{x}, t) + \varphi_j(\xi, y, t)] \,.
\end{cases}
\tag{4.2}
$$

Here $\varepsilon = \triangle t^{1/2}$, $\xi = (x+1)/\varepsilon, \boldsymbol{u}^*_j = (u^*_j, v^*_j), \boldsymbol{a}^*_j = (a^*_j, b^*_j), \boldsymbol{u}_j = (u_j, v_j)$. We assume that the ξ-dependent functions decay super-algebraically as $\xi \to +\infty$. In doing so, we have committed our attention to the left boundary at $x = -1$. Clearly a similar analysis can be done at the right boundary $\{x = 1\}$. Our purpose is to find the coefficients in this expansion such that the truncated series satisfies (4.1) to high order accuracy. Using the notation $\nabla_\xi = (\partial_\xi, 0)$, $\nabla_y = (0, \partial_y)$, we have

$$
\Delta \boldsymbol{u}^*_{\triangle t} = \Delta_{\boldsymbol{x}} \boldsymbol{u}^*_0 + \sum_{j=1} \varepsilon^j (\Delta_{\boldsymbol{x}} \boldsymbol{u}^*_j + \varepsilon^{-2} \partial^2_\xi \boldsymbol{a}^*_j + \partial^2_y \boldsymbol{a}^*_j) \,,
\tag{4.3}
$$

$$\nabla\cdot\boldsymbol{u}_{\triangle t} = \nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}_0 + \sum_{j=1}\varepsilon^j\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u}_j\,, \tag{4.4}$$

$$\nabla p_{\triangle t} = \nabla_{\boldsymbol{x}} p_0 + \varepsilon^{-1}\nabla_\xi\varphi_0 + \nabla_y\varphi_0 + \sum_{j=1}\varepsilon^j(\nabla_{\boldsymbol{x}} p_j + \varepsilon^{-1}\nabla_\xi\varphi_j + \nabla_y\varphi_j)\,, \tag{4.5}$$

$$\begin{aligned}\boldsymbol{u}^{n+1}(\boldsymbol{x}) &= \boldsymbol{u}_0(\boldsymbol{x},t^{n+1}) + \sum_{j=1}\varepsilon^j\boldsymbol{u}_j(\boldsymbol{x},t^{n+1})\\ &= \sum_{k=0}\frac{1}{k!}\varepsilon^{2k}\boldsymbol{u}_0^{(k)}(\boldsymbol{x},t^n) + \sum_{j=1}\varepsilon^j\sum_{k=0}\frac{1}{k!}\varepsilon^{2k}\boldsymbol{u}_j^{(k)}(\boldsymbol{x},t^n)\,.\end{aligned} \tag{4.6}$$

In the following we will omit the subscript $\boldsymbol{x}$ for differential operators with respect to $\boldsymbol{x}$.

Next we substitute these relations into (4.1) in order to determine the coefficients of ε^j in (4.2). We get hierarchies of equations by collecting equal powers of ε.

The first equation in (4.1) gives:

$$\boldsymbol{u}_0^* = \boldsymbol{u}_0\,, \tag{4.7}$$

$$\boldsymbol{u}_1^* + \boldsymbol{a}_1^* - \boldsymbol{u}_1 = \partial_\xi^2\boldsymbol{a}_1^*\,, \tag{4.8}$$

$$\boldsymbol{u}_2^* + \boldsymbol{a}_2^* - \boldsymbol{u}_2 + (\boldsymbol{u}_0\cdot\nabla)\boldsymbol{u}_0 = \Delta^2\boldsymbol{u}_0^* + \Delta_\xi^2\boldsymbol{a}_2^*\,. \tag{4.9}$$

For $j \geq 1$,

$$\boldsymbol{u}_{j+2}^* + \boldsymbol{a}_{j+2}^* - \boldsymbol{u}_{j+2} + \sum_{k=0}^{j}(\boldsymbol{u}_k\cdot\nabla)\boldsymbol{u}_{j-k} = \Delta\boldsymbol{u}_j^* + \partial_\xi^2\boldsymbol{a}_{j+2}^* + \partial_y^2\boldsymbol{a}_j^*\,. \tag{4.10}$$

The second equation in (4.1) implies

$$\boldsymbol{u}_0^* = \boldsymbol{u}_0\,, \tag{4.11}$$

$$\boldsymbol{u}_1^* + \boldsymbol{a}_1^* = \boldsymbol{u}_1 + \nabla_\xi\varphi_0\,, \tag{4.12}$$

$$\boldsymbol{u}_2^* + \boldsymbol{a}_2^* = \boldsymbol{u}_2 + \partial_t\boldsymbol{u}_0 + \nabla p_0 + \nabla_\xi\varphi_1 + \nabla_y\varphi_0\,. \tag{4.13}$$

For $j = 2\ell - 1$, $\ell \geq 1$,

$$\boldsymbol{u}_{j+2}^* + \boldsymbol{a}_{j+2}^* = \boldsymbol{u}_{j+2} + \partial_t\boldsymbol{u}_j + \nabla p_j + \nabla_\xi\varphi_{j+1} + \nabla_y\varphi_j + \sum_{k=2}^{\ell}\frac{1}{k!}\boldsymbol{u}_{j-2k+2}^{(k)}\,. \tag{4.14}$$

For $j = 2\ell$, $\ell \geq 1$,

$$\boldsymbol{u}^*_{j+2}+\boldsymbol{a}^*_{j+2} = \boldsymbol{u}_{j+2}+\partial_t \boldsymbol{u}_j+\nabla p_j+\nabla_\xi \varphi_{j+1}+\nabla_y \varphi_j+\frac{1}{(\ell+1)!}\boldsymbol{u}_0^{(\ell+1)}+\sum_{k=2}^{\ell}\frac{1}{k!}\boldsymbol{u}^{(k)}_{j-2k+2}\,. \tag{4.15}$$

From the third equation in (4.1), we obtain

$$\nabla\cdot \boldsymbol{u}_j = 0\,, \qquad j = 0, 1, \cdots\,. \tag{4.16}$$

The boundary conditions become

$$\boldsymbol{u}^*_0 = 0, \quad \partial_\xi \varphi_0 = 0, \quad \text{at} \quad x = -1, \quad \xi = 0\,, \tag{4.17}$$

$$\boldsymbol{u}^*_j + \boldsymbol{a}^*_j = 0\,, \quad \partial_x p_{j-1} + \partial_\xi \varphi_j = 0\,, \quad \text{at} \quad x = -1, \quad \xi = 0\,, \tag{4.18}$$

for $j > 0$.

Our next task is to analyze these equations to see whether they are solvable. We begin by noticing that (4.8) and (4.12) imply

$$\boldsymbol{u}^*_1 = \boldsymbol{u}_1\,, \tag{4.19}$$

$$\boldsymbol{a}^*_1 = \partial_\xi^2 \boldsymbol{a}^*_1 = \nabla_\xi \varphi_0 \tag{4.20}$$

since $\boldsymbol{u}^*_1$ and $\boldsymbol{u}_1$ do not depend on ξ. From (4.17), we get

$$\boldsymbol{a}^*_1 = 0\,, \quad \varphi_0 = 0\,. \tag{4.21}$$

Next we collect the ξ-independent part of (4.9), (4.13) and (4.16) to obtain

$$\begin{cases} \partial_t \boldsymbol{u}_0 + \nabla p_0 + (\boldsymbol{u}_0\cdot\nabla)\boldsymbol{u}_0 = \Delta \boldsymbol{u}_0\,, \\ \nabla\cdot\boldsymbol{u}_0 = 0 \end{cases} \tag{4.22}$$

The remaining part of the these equations give

$$\boldsymbol{a}^*_2 = \partial_\xi^2 \boldsymbol{a}^*_2\,, \tag{4.23}$$

$$\boldsymbol{a}^*_2 = \nabla_\xi \varphi_1 + \nabla_y \varphi_0\,. \tag{4.24}$$

Not surprisingly, the leading order terms in (4.2) satisfy the original NSE (4.22), with the boundary condition: $\boldsymbol{u}_0 = 0$ on $\partial^{'}\Omega$. It is natural to associate (4.22) with the initial condition: $\boldsymbol{u}_0(\boldsymbol{x}, 0) = \boldsymbol{u}^0(\boldsymbol{x})$. It is easy to see that (4.23)-(4.24) are satisfied if we choose

$$b_2^* = 0\,, \quad a_2^* = \partial_\xi \varphi_1\,, \tag{4.25}$$

$$\varphi_1 = \partial_\xi^2 \varphi_1\,. \tag{4.26}$$

The boundary condition for (4.26) can be obtained from (4.18) with $j = 1$:

$$\partial_\xi \varphi_1 + \partial_x p_0 = 0\,, \quad \text{at} \quad \xi = 0, \quad x = -1\,. \tag{4.27}$$

(4.26) and (4.27) imply

$$\varphi_1(\xi, y, t) = \partial_x p_0(-1, y, t)\, e^{-\xi}\,. \tag{4.28}$$

So far we have obtained solutions for $\boldsymbol{u}_0^*, \boldsymbol{u}_0, p_0, \boldsymbol{a}_1^*, \varphi_0, \boldsymbol{a}_2^*, \varphi_1$. Let $j = 1$ in (4.10), $j = 3$ in (4.14) and $j = 1$ in (4.16), we get:

$$\begin{cases} \partial_t \boldsymbol{u}_1 + \nabla p_1 + (\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{u}_1 + (\boldsymbol{u}_1 \cdot \nabla)\boldsymbol{u}_0 = \Delta \boldsymbol{u}_1\,, \\ \nabla \cdot \boldsymbol{u}_1 = 0\,, \end{cases} \tag{4.29}$$

$$\begin{cases} \boldsymbol{a}_3^* = \partial_\xi^2 \boldsymbol{a}_3^*\,, \\ \boldsymbol{a}_3^* = \nabla_\xi \varphi_2 + \nabla_y \varphi_1\,. \end{cases} \tag{4.30}$$

The boundary condition for (4.29) is $\boldsymbol{u}_1|_{\partial'\Omega} = 0$. The initial data for (4.29) is $\boldsymbol{u}_1\,|_{t=0} = 0$. Therefore we have

$$\boldsymbol{u}_1 = 0\,, \quad p_1 = 0\,. \tag{4.31}$$

From (4.18) and (4.30), we have

$$\partial_\xi \varphi_2 = 0\,, \quad \text{at} \quad x = -1, \quad \xi = 0\,. \tag{4.32}$$

The solutions of (4.30) and (4.32) are given by:

$$\varphi_2 = 0\,, \quad b_3^* = \partial_y \varphi_1\,, \quad a_3^* = 0\,. \tag{4.33}$$

These give us $(\boldsymbol{u}_1^*, \boldsymbol{u}_1, p_1, \boldsymbol{a}_3^*, \varphi_2)$. Similarly, we have the next set of equations:

$$
\left\{
\begin{aligned}
& \partial_t \boldsymbol{u}_2 + \nabla p_2 + (\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{u}_2 + (\boldsymbol{u}_2 \cdot \nabla)\boldsymbol{u}_0 = \Delta \boldsymbol{u}_2 + \Delta(\partial_t \boldsymbol{u}_0 + \nabla p_0) - \tfrac{1}{2}\partial_t^2 \boldsymbol{u}_0 \,, \\
& \nabla \cdot \boldsymbol{u}_2 = 0 \,,
\end{aligned}
\right.
\tag{4.34}
$$

$$
\left\{
\begin{aligned}
& \boldsymbol{a}_4^* = \partial_\xi^2 \boldsymbol{a}_4^* + \partial_y^2 \boldsymbol{a}_2^* \,, \\
& \boldsymbol{a}_4^* = \nabla_\xi \varphi_3 + \nabla_y \varphi_2 \,.
\end{aligned}
\right.
\tag{4.35}
$$

The boundary conditions can be obtained from (4.18) and (4.13):

$$
\boldsymbol{u}_2 + \nabla p_0 + \nabla_\xi \varphi_1 = 0 \,, \quad \text{at} \quad \xi = 0, \quad x = -1,
\tag{4.36}
$$

$$
\partial_\xi \varphi_3 + \partial_x p_2 = 0 \,, \quad \text{at} \quad \xi = 0, \quad x = -1 \,.
\tag{4.37}
$$

However, choosing the right initial data for (4.34) is a rather subtle issue. We will defer the discussion to the end of this subsection.

Notice that (4.36) and (4.27) imply that $\boldsymbol{u}_2 \cdot \boldsymbol{n} = 0$ on $\partial' \Omega$. In general, (4.14), (4.15) and (4.18) imply that this is true for all $\boldsymbol{u}_j$.

Solutions of (4.35), (4.37) are given by

$$
b_4^* = 0 \,, \quad a_4^* = \partial_\xi \varphi_3 \,.
\tag{4.38}
$$

$$
\varphi_3(\xi, y, t) = (\partial_x p_2 + \frac{1}{2}\partial_x \partial_y^2 p_0) \mid_{x=-1} e^{-\xi} + \frac{1}{2}\partial_x \partial_y^2 p_0 \mid_{x=-1} \xi e^{-\xi} \,.
\tag{4.39}
$$

This set of equations determines $(\boldsymbol{u}_2^*, \boldsymbol{u}_2, p_2, \boldsymbol{a}_4^*, \varphi_3)$. If we now look at the next equation in each of the groups (4.10), (4.14) and (4.16), we obtain

$$
\left\{
\begin{aligned}
& \partial_t \boldsymbol{u}_3 + \nabla p_3 + (\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{u}_3 + (\boldsymbol{u}_3 \cdot \nabla)\boldsymbol{u}_0 = \Delta \boldsymbol{u}_3 + \Delta \nabla p_2 \,, \\
& \nabla \cdot \boldsymbol{u}_3 = 0 \,,
\end{aligned}
\right.
\tag{4.40}
$$

$$
\boldsymbol{a}_5^* = \partial_\xi^2 \boldsymbol{a}_5^* + \partial_y^2 \boldsymbol{a}_3^* \,,
\tag{4.41}
$$

$$
a_5^* = \partial_\xi \varphi_4, \quad b_5^* = \partial_y \varphi_3 \,.
\tag{4.42}
$$

Usually one would expect that $(\boldsymbol{u}_3, p_3) = 0$, since they are the coefficients of $\triangle t^{3/2}$ terms. But the boundary layer in p gives rise to a nonzero boundary condition for (4.40). This is seen from (4.14) which for $j = 3$ reads:

$$\boldsymbol{u}_3^* + \boldsymbol{a}_3^* = \boldsymbol{u}_3 + \partial_t \boldsymbol{u}_1 + \nabla p_1 + \nabla_\xi \varphi_2 + \nabla_y \varphi_1 = \boldsymbol{u}_3 + \nabla_y \varphi_1 \,. \tag{4.43}$$

Therefore (4.18) implies that

$$\boldsymbol{u}_3 + \nabla_y \varphi_1 = 0 \,, \quad \text{at} \quad x = -1, \quad \xi = 0 \,. \tag{4.44}$$

(4.40) and (4.44), together with a suitable initial condition that matches (4.44) at $t = 0$, determine $(\boldsymbol{u}_3, p_3)$. This in turn determines the boundary condition for φ_4

$$\partial_\xi \varphi_4 = -\partial_x p_3 \quad \text{at} \quad \xi = 0, \quad x = -1 \,. \tag{4.45}$$

(4.41) and (4.45) can be solved for φ_4, etc. This set of equations determines $(\boldsymbol{u}_3^*, \boldsymbol{u}_3,\ p_3,$ $\boldsymbol{a}_5^*, \varphi_4)$. Obviously this procedure can be continued and we obtain

$$a_j^* = \partial_\xi \varphi_{j-1} \,, \quad b_j^* = \partial_y \varphi_{j-2} \,, \tag{4.46}$$

$$\varphi_j = \partial_\xi^2 \varphi_j + \partial_y^2 \varphi_{j-2} \,. \tag{4.47}$$

$$\varphi_j = \sum_{k=0}^{[j/2]} F_{j,k}(y) \xi^k e^{-\xi} \,, \tag{4.48}$$

Now if we let

$$\begin{cases} U^* = \boldsymbol{u}_0^* + \displaystyle\sum_{j=1}^{2N} \varepsilon^j (\boldsymbol{u}_j^* + \boldsymbol{a}_j^*) \,, \\ U^n = \boldsymbol{u}_0 + \displaystyle\sum_{j=1}^{2N} \varepsilon^j \boldsymbol{u}_j \,, \\ P^n = p_0 + \displaystyle\sum_{j=1}^{2N} \varepsilon^j (p_j + \varphi_j) + \varepsilon^{2N+1} \varphi_{2N+1} \,, \end{cases} \tag{4.49}$$

then we have

$$
\begin{cases}
\dfrac{U^* - U^n}{\triangle t} + (U^n \cdot \nabla) U^n = \Delta U^* + \triangle t^{N-1/2} \boldsymbol{f}_N \,, \\[2ex]
U^* = 0 \,, \qquad \text{on } \partial' \Omega \,, \\
U^* = U^{n+1} + \triangle t \nabla P^n + \triangle t^{N+1/2} \boldsymbol{g}_N \,, \\
\nabla \cdot U^{n+1} = 0 \,, \\
\dfrac{\partial P^n}{\partial \boldsymbol{n}} = \boldsymbol{n} \cdot U^{n+1} = 0 \,, \qquad \text{on } \partial' \Omega
\end{cases}
\tag{4.50}
$$

where the coefficients $\boldsymbol{f}_N$ and $\boldsymbol{g}_N$ are functionals of $(\boldsymbol{u}_0, p_0)$. They are bounded and smooth if $(\boldsymbol{u}_0, p_0)$ are sufficiently smooth.

We now come to the choice of initial conditions. If we do not require extra compatibility conditions for the initial data $\boldsymbol{u}^0(\boldsymbol{x})$, then to have solutions $(\boldsymbol{u}_2, p_2)$ that are smooth at $t = 0$, we need to choose an initial data for $\boldsymbol{u}_2$ that matches (4.36). While there is no difficulty in doing this, it restricts the approximation of the initial data to

$$
U^0(\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}, 0) + \triangle t \boldsymbol{w}^0(\boldsymbol{x}) \tag{4.51}
$$

where $\boldsymbol{w}^0$ is a bounded function. This is enough for proving the L^2 estimate, but not enough for proving the L^∞ estimate.

However, if we assume that the initial data $\boldsymbol{u}^0(\boldsymbol{x})$ for NSE (2.1) satisfies the following compatibility condition

$$
\boldsymbol{u}^0(\boldsymbol{x}) = 0 \,, \quad \partial_y p_0 = 0 \,, \quad \text{on } \partial' \Omega \,, \tag{4.52}
$$

then we can choose

$$
\boldsymbol{u}_2(\boldsymbol{x}, 0) = 0 \tag{4.53}
$$

since

$$
\boldsymbol{u}_2 \mid_{x=-1, t=0} = -\nabla p_0 \mid_{x=-1, t=0} - \nabla_\xi \varphi_1 \mid_{\xi=0, t=0} = (0, -\partial_y p_0 \mid_{x=-1, t=0}) = 0 \,. \tag{4.54}
$$

Likewise, if we assume

$$
\partial_x \partial_y p_0 = 0, \qquad \text{on } \partial' \Omega \tag{4.55}
$$

then we have

$$\boldsymbol{u}_3(\boldsymbol{x},0) = 0\,. \tag{4.56}$$

Hence we have

$$U^0(\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x},0) + \triangle t^2 \boldsymbol{w}^0(\boldsymbol{x}) \tag{4.57}$$

where $\boldsymbol{w}^0$ is a bounded function.

§4.2. Proof of Theorem 1

Proposition 1. *Let $\boldsymbol{u}^n$, $\boldsymbol{u}^*$ and p^n be the solution of* (4.1)*. Let U^n, U^*, and P^n be the constructed approximate solution satisfying*

$$\begin{cases} \dfrac{U^* - U^n}{\triangle t} + (U^n\cdot\nabla)\, U^n = \Delta U^* + \triangle t^{\alpha}\ \boldsymbol{f}^n\,, \\ U^* = 0\,, \qquad \text{on } \partial\Omega\,, \\ \dfrac{U^{n+1} - U^*}{\triangle t} + \nabla P^n = \triangle t^{\alpha} \boldsymbol{g}^n\,, \\ \nabla\cdot U^{n+1} = 0\,, \\ \dfrac{\partial P^n}{\partial \boldsymbol{n}} = U^{n+1}\cdot\boldsymbol{n} = 0\,, \qquad \text{on } \partial\Omega\,, \\ U^0 = \boldsymbol{u}^0 + \triangle t^{\alpha}\ \boldsymbol{w}^0 \end{cases} \tag{4.58}$$

and

$$\max_{0\le n\le\left[\frac{T}{\triangle t}\right]+1} \|U^n(\cdot)\|_{W^{1,\infty}} \le C^*\,, \qquad \alpha > 1/2\,. \tag{4.59}$$

Then for $0\le t\le T$ we have

$$\|\boldsymbol{u}^n - U^n\|_{L^2} + \triangle t^{1/2} \left(\sum_n \|p^n - P^n\|^2_{H^1}\triangle t\right)^{1/2} \le C_1\ \triangle t^{\alpha} \tag{4.60}$$

and

$$\|\boldsymbol{u}^n - U^n\|_{L^\infty} + \triangle t\ \|p^n - P^n\|_{W^{1,\infty}} \le C_1\ \triangle t^{\alpha - 1/2} \tag{4.61}$$

where

$$C_1 = C\ \|\boldsymbol{w}^0\|_{L^2} + C(C^*) \left(\sum_n \triangle t(\|\boldsymbol{f}^n\|^2_{L^2} + \|\boldsymbol{g}^n\|^2_{L^2} + \triangle t\|\boldsymbol{g}^n\|^2_{H^2})\right)^{1/2}. \tag{4.62}$$

Here C and C^* are constants and $n \leq \left[\frac{T}{\triangle t}\right] + 1$.

Proof: Assume a priori that

$$\max_{0 \leq t^n \leq T} \|\boldsymbol{u}^n\|_{L^\infty} \leq \tilde{C}\,. \tag{4.63}$$

In the following estimates, the constant will sometimes depend on C^* and $\tilde{C}$. Later on we will estimate $\tilde{C}$.

Step 1. *Basic Energy Estimates.* Let

$$\boldsymbol{e}^n = U^n - \boldsymbol{u}^n\,, \qquad \boldsymbol{e}^* = U^* - \boldsymbol{u}^*\,, \qquad q^n = P^n - p^n\,. \tag{4.64}$$

Subtracting (4.58) from (4.1) we get the following error equation

$$\begin{cases} \dfrac{\boldsymbol{e}^* - \boldsymbol{e}^n}{\triangle t} + (\boldsymbol{e}^n \cdot \nabla)\, U^n + (\boldsymbol{u}^n \cdot \nabla)\, \boldsymbol{e}^n = \Delta \boldsymbol{e}^* + \triangle t^\alpha \boldsymbol{f}^n\,, \\ \boldsymbol{e}^* = 0\,, \qquad \text{on } \partial\Omega\,, \\ \dfrac{\boldsymbol{e}^{n+1} - \boldsymbol{e}^*}{\triangle t} + \nabla q^n = \triangle t^\alpha \boldsymbol{g}^n\,, \\ \nabla \cdot \boldsymbol{e}^{n+1} = 0\,, \\ \boldsymbol{e}^{n+1} \cdot \boldsymbol{n} = 0\,, \qquad \text{on } \partial\Omega\,, \\ \boldsymbol{e}^0 = \triangle t^\alpha\ \boldsymbol{w}^0\,. \end{cases} \tag{4.65}$$

Taking the scalar product of the first equation of (4.65) with $2\boldsymbol{e}^*$ and integrating by parts, we obtain

(4.66)

$$\begin{aligned} &\|\boldsymbol{e}^*\|^2 - \|\boldsymbol{e}^n\|^2 + \|\boldsymbol{e}^* - \boldsymbol{e}^n\|^2 + 2\triangle t\, \|\nabla \boldsymbol{e}^*\|^2 \\ &\quad \leq \triangle t^{2\alpha+1} \|\boldsymbol{f}^n\|^2 + \triangle t\ \|\boldsymbol{e}^*\|^2 - 2\triangle t\ \int_\Omega \boldsymbol{e}^* \cdot (\boldsymbol{e}^n \cdot \nabla)\ U^n\, d\boldsymbol{x} - 2\triangle t\ \int_\Omega \boldsymbol{e}^* \cdot (\boldsymbol{u}^n \cdot \nabla)\ \boldsymbol{e}^n\, d\boldsymbol{x} \\ &\quad \leq \triangle t^{2\alpha+1} \|\boldsymbol{f}^n\|^2 + \triangle t\ \|\boldsymbol{e}^*\|^2 + C\, \triangle t\ \|\boldsymbol{e}^*\|\ \|\boldsymbol{e}^n\| + 2\triangle t\ \int_\Omega \boldsymbol{e}^n \cdot (\boldsymbol{u}^n \cdot \nabla)\ \boldsymbol{e}^*\, d\boldsymbol{x} \\ &\quad \leq \triangle t^{2\alpha+1} \|\boldsymbol{f}^n\|^2 + C^*\, \triangle t\ \|\boldsymbol{e}^*\|^2 + (C^* + \tilde{C}^2)\, \triangle t\ \|\boldsymbol{e}^n\|^2 + \triangle t\ \|\nabla \boldsymbol{e}^*\|^2\,. \end{aligned}$$

Taking the scalar product of the second equation of (4.65) with $2\boldsymbol{e}^{n+1}$ yields

$$\|\boldsymbol{e}^{n+1}\|^2 - \|\boldsymbol{e}^*\|^2 + \|\boldsymbol{e}^{n+1} - \boldsymbol{e}^*\|^2 \leq \triangle t\ \|\boldsymbol{e}^{n+1}\|^2 + \triangle t^{2\alpha+1} \|\boldsymbol{g}^n\|^2\,. \tag{4.67}$$

Combining (4.66) and (4.67), we get

$$\|\boldsymbol{e}^{n+1}\|^2 - \|\boldsymbol{e}^n\|^2 + \|\boldsymbol{e}^* - \boldsymbol{e}^n\|^2 + \|\boldsymbol{e}^{n+1} - \boldsymbol{e}^*\|^2 + \triangle t\ \|\nabla \boldsymbol{e}^*\|^2$$
$$\le C\,\triangle t\ (\|\boldsymbol{e}^{n+1}\|^2 + \|\boldsymbol{e}^n\|^2) + \triangle t^{2\alpha+1}(\|\boldsymbol{f}^n\|^2 + \|\boldsymbol{g}^n\|^2)\,. \tag{4.68}$$

Applying the discrete Gronwall lemma to the last inequality, we arrive at

$$\|\boldsymbol{e}^n\| + \triangle t^{1/2}\ \|\nabla \boldsymbol{e}^*\| + \left(\sum_n (\|\boldsymbol{e}^* - \boldsymbol{e}^n\|^2 + \|\boldsymbol{e}^{n+1} - \boldsymbol{e}^*\|^2)\right)^{1/2} \le C_1 \triangle t^\alpha\,. \tag{4.69}$$

Hence, from the second equation of (4.65), we have

$$\|\boldsymbol{e}^n\| + \triangle t^{1/2}\left(\sum_n \|q^n\|^2_{H^1}\ \triangle t\right)^{1/2} \le C_1 \triangle t^\alpha\,.$$

We have proved (4.60), assuming that $\tilde{C}$ in (4.63) is bounded independent of $\triangle t$.

Step 2. L^∞*-norm Estimates.* Taking the divergence of the third equation of (4.65) we obtain

$$\begin{cases} \Delta q^n = \dfrac{\nabla\cdot \boldsymbol{e}^*}{\triangle t} - \triangle t^\alpha \nabla\cdot \boldsymbol{g}^n\,, \\[2ex] \dfrac{\partial q^n}{\partial \boldsymbol{n}} = 0\,, \qquad \text{on } \partial\Omega\,. \end{cases} \tag{4.70}$$

Without lose of generality, we can normalize the pressure, such that $\int_\Omega q^n\, d\boldsymbol{x} = 0$. Applying standard regularity theorems to the above Neumann problem, we arrive at

$$\|q^n\|_{H^2} \le C\,\triangle t^{-1}\|\nabla \boldsymbol{e}^*\| + C\,\triangle t^\alpha \|\boldsymbol{g}^n\|_{H^1} \le C_1 \triangle t^{\alpha - 3/2}\,. \tag{4.71}$$

From the second equation in (4.65) we also have

$$\|\nabla \boldsymbol{e}^{n+1}\| \le \|\nabla \boldsymbol{e}^*\| + \triangle t\ \|q^n\|_{H^2} + \triangle t^{\alpha+1}\|\boldsymbol{g}^n\|_{H^1} \le C_1 \triangle t^{\alpha - 1/2}\,. \tag{4.72}$$

From the first equation of (4.65) and (4.71), (4.72), we obtain

$$\|\boldsymbol{\Delta} \boldsymbol{e}^*\| \le \triangle t^{-1}\|\boldsymbol{e}^* - \boldsymbol{e}^n\| + C\,(\|\boldsymbol{e}^n\| + \|\nabla \boldsymbol{e}^n\| + \triangle t^\alpha \|\boldsymbol{f}^n\|) \le C_1 \triangle t^{\alpha-1}\,. \tag{4.73}$$

This implies

$$\|\boldsymbol{e}^*\|_{H^2} \le C_1 \triangle t^{\alpha-1}\,. \tag{4.74}$$

Using Sobolev inequality, we get

$$\|\boldsymbol{e}^*\|_{L^\infty} \leq \|\boldsymbol{e}^*\|^{1/2} \ \|\boldsymbol{e}^*\|_{H^2}^{1/2} \leq C_1 \triangle t^{\alpha-1/2}\,. \tag{4.75}$$

From (4.70), we have

$$\|\Delta q^n\|_{H^1} \leq \triangle t^{-1} \|\boldsymbol{e}^*\|_{H^2} + \triangle t^{\alpha} \|\boldsymbol{g}^n\|_{H^2} \leq C_1 \triangle t^{\alpha-2}\,. \tag{4.76}$$

This implies

$$\|q^n\|_{H^3} \leq C_1 \triangle t^{\alpha-2}\,. \tag{4.77}$$

Notice that the second equation of (4.65) gives

$$\|\nabla q^n\|_{L^2} \leq \triangle t^{-1} \|\boldsymbol{e}^{n+1} - \boldsymbol{e}^*\| + \triangle t^{\alpha} \|\boldsymbol{g}^n\|_{L^2} \leq C_1 \triangle t^{\alpha-1}\,. \tag{4.78}$$

Therefore with Sobolev inequality, Poincare inequality and (4.77) we have

$$\|\nabla q^n\|_{L^\infty} \leq \|\nabla q^n\|^{1/2} \ \|\nabla q^n\|_{H^2}^{1/2} \leq C_1 \triangle t^{\alpha-3/2}\,. \tag{4.79}$$

Using the second equation of (4.65) one more time, we get

$$\|\boldsymbol{e}^{n+1}\|_{L^\infty} \leq \|\boldsymbol{e}^*\|_{L^\infty} + \triangle t \ \|\nabla q^n\|_{L^\infty} + \triangle t^{\alpha+1} \|\boldsymbol{g}^n\|_{L^\infty} \leq C_1 \triangle t^{\alpha-1/2}\,. \tag{4.80}$$

Since $\alpha > \frac{1}{2}$, if we choose $\triangle t$ small enough, we will always have

$$\|\boldsymbol{e}^{n+1}\|_{L^\infty} \leq 1\,. \tag{4.81}$$

Therefore in (4.63) we can choose

$$\tilde{C} = 1 + \max_{n \leq 1 + \left[\frac{T}{\triangle t}\right]} \|U^n(\cdot)\|_{L^\infty} \tag{4.82}$$

which depends only on the exact solution $(\boldsymbol{u}, p)$. This proves (4.61) and (4.62).

Proof of Theorem 1: Now, we simple use the above proposition and chose $N = 3$ in the expansion (4.49) we have

$$\|\boldsymbol{u}^n - U^n\|_{L^\infty(0,T;L^2)} + \triangle t^{1/2} \|p^n - P^n\|_{L^2(0,T;L^2)} \leq C \triangle t \tag{4.83}$$

But the boundary layer terms in P^n can be estimated as

(4.84)

$$\begin{aligned}\|p(\cdot,t)-P^n\|_{L^2(0,T;L^2)} &\leq \left(\sum_n \|\triangle t^{1/2}\varphi_1^n\|^2\triangle t\right)^{1/2} \\ &= \triangle t\left(\sum_n \|\varphi_1^n\|^2\right)^{1/2} = \triangle t^{3/4}\left(\sum_n \|\partial_x p_0^n\|^2\triangle t\right)^{1/2} \leq C\triangle t^{3/4}\,.\end{aligned}$$

Combining (4.83) and (4.84) we obtain (3.1). Clearly (3.3) is a directly consequence of Proposition 1.

Recall the expansion

$$p_{\triangle t}(\boldsymbol{x},t) = p_0(\boldsymbol{x},t) + \triangle t^{1/2}\partial_x p_0(-1,y,t)e^{-\xi} + O(\triangle t) \tag{4.85}$$

To get a uniform approximation for the pressure we need to subtract from $p_{\triangle t}$ the second term in the right hand side. Note that this term involves p_0 which is not known. We need to approximate it by the numerical solution $p_{\triangle t}$. This can be done using (4.85) evaluated at $x=-1+\triangle t^{1/2}$:

$$|\partial_x p_{\triangle t}(-1+\triangle t^{1/2},y,t) - (1-e^{-1})\partial_x p_0(-1,y,t)| \leq \triangle t^{1/2} \tag{4.86}$$

Hence we get

$$p_{\triangle t}(\boldsymbol{x},t) = p_0(\boldsymbol{x},t) + \triangle t^{1/2}\frac{e}{e-1}\partial_x p_{\triangle t}(-1+\triangle t^{1/2},y,t)e^{-\xi} + O(\triangle t) \tag{4.87}$$

This proves (3.4).

§5. Effects of Numerical Boundary Conditions

In this section we focus on the issue which is that main source of confusion in the subject of projection methods: the boundary condition for pressure at the projection step. We will examine the effect of different boundary conditions on the accuracy of the numerical approximations using the explicit asymptotic analysis presented in the last section. As we have seen earlier, the Neumann boundary condition for pressure leads to numerical solutions with the following asymptotic form ($t = n\triangle t$):

$$
\begin{aligned}
&\boldsymbol{u}^n(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x},t) + \triangle t\, \boldsymbol{u}_2(\boldsymbol{x},t) + \triangle t^2 \boldsymbol{u}_4 + \cdots\,,\\
(5.1)\qquad &\boldsymbol{u}^*(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x},t) + \triangle t\,[\boldsymbol{u}_2^*(\boldsymbol{x},t,\triangle t) + \boldsymbol{a}_2^*((x\pm 1)/\triangle t^{1/2}, y, t)] + \cdots\,,\\
&p^n(\boldsymbol{x}) = p(\boldsymbol{x},t) + \triangle t^{1/2}\,\varphi_1((x\pm 1)/\triangle t^{1/2}, y, t) + \triangle t\, p_2(\boldsymbol{x},t) + \cdots\,.
\end{aligned}
$$

We see that boundary layer terms of the order $\triangle t^{1/2}$ and $\triangle t$ appear respectively in the pressure approximation and the intermediate velocity field, whereas the projected velocity field does not have numerical boundary layers.

Let us now replace the Neumann boundary condition (2.10) by a Dirichlet boundary condition:

$$
\frac{\partial p^n}{\partial \boldsymbol{t}} = 0\,, \quad \text{on } \partial\Omega\,, \tag{5.2}
$$

or more specifically

$$
p^n = 0\,, \quad \text{on } \partial\Omega\,. \tag{5.3}
$$

To analyze the boundary layer structure of the resulted scheme, we proceed as in §4.1 and make the same ansatz as (4.2). Equations (4.7)-(4.16) remain valid. However, the boundary conditions are changed to

$$
\boldsymbol{u}_0^* = 0\,, \quad p_0 + \varphi_0 = 0\,, \quad \text{at} \quad x = -1, \quad \xi = 0\,, \tag{5.4}
$$

$$
\boldsymbol{u}_j^* + \boldsymbol{a}_j^* = 0\,, \quad p_j + \varphi_j = 0\,, \quad \text{at} \quad x = -1, \quad \xi = 0\,, \tag{5.5}
$$

for $j \geq 1$.

We still have (4.22), which together with the boundary condition $\boldsymbol{u}_0 \mid_{\partial\Omega} = 0$ and initial condition determines $\boldsymbol{u}_0$ and p_0. This in turn gives the boundary condition for φ_0:

$$
\varphi_0 \mid_{\xi=0} = -p_0 \mid_{x=-1}\,. \tag{5.6}
$$

Going back to (4.20), we obtain

$$a_1^* = \partial_\xi \varphi_0\,, \quad b_1^* = 0\,, \tag{5.7}$$

$$\varphi_0(\xi, y, t) = -p_0(-1, y, t)\, e^{-\xi}\,, \tag{5.8}$$

Although $(\boldsymbol{u}_1, p_1)$ still satisfies the same equations (4.29), $\boldsymbol{u}_1$ no longer vanishes at the boundary. Instead, we have

$$(u_1, v_1)\,|_{x=-1} = (-\partial_\xi \varphi_0\,|_{\xi=0}, 0) = (-p_0(-1, y, t), 0)\,. \tag{5.9}$$

This implies that in general, we will have $(\boldsymbol{u}_1, p_1) \neq 0$. Therefore the numerical solution with the boundary condition (5.3) will have the following form ($t = n\triangle t$):

$$\begin{cases} \boldsymbol{u}^n(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}, t) + \triangle t^{1/2} \boldsymbol{u}_1(\boldsymbol{x}, t) + \cdots\,, \\ \boldsymbol{u}^*(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}, t) + \triangle t^{1/2} (\boldsymbol{u}_1^*(\boldsymbol{x}, t) + \boldsymbol{a}_1^*(\xi, y, t)) + \cdots\,, \\ p^n(\boldsymbol{x}) = p(\boldsymbol{x}, t) + \varphi_0(\xi, y, t) + \triangle t^{1/2} (p_1(\boldsymbol{x}, t) + \varphi_0(\xi, y, t)) + \cdots\,. \end{cases} \tag{5.10}$$

As a result of using the Dirichlet boundary condition (5.3), not only the accuracy of the pressure approximation deteriorates to order zero because of the appearance of $O(1)$ numerical boundary layer, the overall accuracy of the velocity approximation is also reduced to $O(\triangle t^{1/2})$. Note also that the leading order error term in the velocity is not of boundary layer type. Clearly the boundary condition (5.3) is a bad choice.

A potentially better choice is suggested by (2.8):

$$\frac{\partial p^n}{\partial \boldsymbol{t}} = \boldsymbol{t} \cdot \Delta \boldsymbol{u}^n\,, \qquad \text{on } \partial\Omega\,. \tag{5.11}$$

This may not be consistent since $\oint_{\partial\Omega} \frac{\partial p^n}{\partial \boldsymbol{t}} ds = 0$, whereas the line integral of $\boldsymbol{t} \cdot \Delta \boldsymbol{u}^n$ over $\partial\Omega$ may not be zero. Therefore we replace (5.11) by

$$\frac{\partial p^n}{\partial \boldsymbol{t}} = \boldsymbol{t} \cdot \Delta \boldsymbol{u}^n - \frac{1}{|\partial\Omega|} \int_{\partial\Omega} (\boldsymbol{t} \cdot \Delta \boldsymbol{u}^n)\, ds\,, \qquad \text{on } \partial\Omega \tag{5.12}$$

whereas $|\partial\Omega|$ denote the total length of $\partial\Omega$. For the geometry we are considering, this becomes

$$\varphi^n(\pm 1, y, t) = \int_0^y \Delta v^n(\pm 1, z, t)\, dz - \frac{y}{2\pi} \int_0^{2\pi} \Delta v^n(\pm 1, z, t)\, dz\,. \tag{5.13}$$

To see the effect of this boundary condition, we follow the same procedure as described above. Again, (4.7)-(4.16) remain valid whereas (4.17)-(4.18) are changed to

$$\boldsymbol{u}_0^* = 0\,, \quad \boldsymbol{u}_j^* + \boldsymbol{a}_j^* = 0\,, \quad \text{at} \quad x = -1, \quad \xi = 0\,, \tag{5.14}$$

$$(p_j + \varphi_j)(-1, y, t) = \int_0^y \Delta v_j(-1, z, t)\, dz - \frac{y}{2\pi} \int_0^{2\pi} \Delta v_j(-1, z, t)\, dz\,, \tag{5.15}$$

for $j \geq 0$. The leading order $(\boldsymbol{u}_0, p_0)$ still satisfies (4.22) which in turn determines the boundary condition for φ_0. Notice that at $x = -1$, (4.22) implies

$$\partial_y p_0 = \Delta v_0 \tag{5.16}$$

Consequently we have (from the periodicity in y)

$$\int_0^{2\pi} \Delta v_0(-1, y, t)\, dy = 0\,, \quad p_0(-1, y, t) = \int_0^y \Delta v_0(-1, z, t)\, dz\,. \tag{5.17}$$

Hence we obtain

$$\varphi_0 = 0 \quad \text{at} \quad \xi = 0\,. \tag{5.18}$$

Going back to (4.20), we get

$$\boldsymbol{a}_1^* = 0\,, \quad \varphi_0 = 0\,. \tag{5.19}$$

We now turn to the next order terms. Obviously we still have

$$\boldsymbol{u}_1 = 0\,, \quad p_1 = 0\,. \tag{5.20}$$

Hence we get from (4.23), (4.24) and (5.15)

$$\varphi_1 = 0\,, \quad \boldsymbol{a}_2^* = 0\,. \tag{5.21}$$

In general we will have

$$\varphi_2 \neq 0\,, \quad \boldsymbol{a}_3^* \neq 0\,, \tag{5.22}$$

so are the higher order terms.

We conclude that with the boundary condition (5.12) or (5.13), the numerical solutions take the following form ($t = n\triangle t$):

$$
\begin{cases}
\boldsymbol{u}^*(\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x},t) + \triangle t\ \boldsymbol{u}_2(\boldsymbol{x},t) + O(\triangle t^{3/2})\,, \\
\boldsymbol{u}^n(\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x},t) + \triangle t\ \boldsymbol{u}_2(\boldsymbol{x},t) + O(\triangle t^{2})\,, \\
p^n(\boldsymbol{x}) = p_0(\boldsymbol{x},t) + \triangle t\,[p_2(\boldsymbol{x},t) + \varphi_2(\xi,y,t)) + O(\triangle t^{3/2}]\,.
\end{cases}
\tag{5.23}
$$

We see that the effect of (5.13) is to suppress the leading order boundary layer terms in (5.1).

To obtain an improved Neumann boundary condition based on the first relation in (2.8), let us observe that p^n satisfies the Poisson equation

$$\Delta p^n = \frac{1}{\triangle t}\nabla\cdot\boldsymbol{u}^* \tag{5.24}$$

which implies

$$\int_{\partial\Omega}\frac{\partial p^n}{\partial\boldsymbol{n}}\,ds = -\frac{1}{\triangle t}\int_{\partial\Omega}\boldsymbol{u}^*\cdot\boldsymbol{n}\,ds = 0\,. \tag{5.25}$$

Direct imposition of

$$\frac{\partial p^n}{\partial\boldsymbol{n}} = \boldsymbol{n}\cdot\Delta\boldsymbol{u}^*\,, \quad \text{or} \quad \boldsymbol{n}\cdot\Delta\boldsymbol{u}^n\,, \qquad \text{on } \partial\Omega \tag{5.26}$$

may not be consistent with (5.25). However, since

$$\Delta\boldsymbol{u} = \nabla(\nabla\cdot\boldsymbol{u}) - \nabla\times(\nabla\times\boldsymbol{u}) \tag{5.27}$$

and $\nabla\cdot\boldsymbol{u}^n = 0$, $\nabla\cdot\boldsymbol{u}^* \sim 0$, we can use instead

$$\frac{\partial p^n}{\partial\boldsymbol{n}} = -\boldsymbol{n}\cdot[\nabla\times(\nabla\times\boldsymbol{u}^*)]\,, \qquad \text{on } \partial\Omega\,, \tag{5.28}$$

or

$$\frac{\partial p^n}{\partial\boldsymbol{n}} = -\boldsymbol{n}\cdot[\nabla\times(\nabla\times\boldsymbol{u}^n)]\,, \qquad \text{on } \partial\Omega\,. \tag{5.29}$$

It is easy to check that both (5.28) and (5.29) are consistent with (5.25) and lead to (5.23). However, to rigorously justify these asymptotic analysis is still an open question.

§6. Second Order Schemes without Spatial Discretization

In this section we carry out the same program as in §4 for Kim and Moin's method, (2.12). Again, we will concentrate on the time-discretized version and leave the fully discrete scheme to Appendix 2. The second order projection method with pressure increment formulation will be dealt with in a subsequent paper [8]. Analysis for the improved pressure boundary conditions still remains open.

§6.1. Asymptotic Analysis of Kim and Moin's Method

Here we will leave out the nonlinear term since it does not affect the major steps but complicates substantially the presentation. The reader can readily fill in the missing terms when any standard second order approximation of the nonlinear term is added in.

We begin with the following ansatz:

$$\left\{\begin{aligned} &\boldsymbol{u}^*(\boldsymbol{x}) = \boldsymbol{u}_0^*(\boldsymbol{x}, t^n) + \sum_{j=1} \varepsilon^j [\boldsymbol{u}_j^*(\boldsymbol{x}, t^n) + \boldsymbol{a}_j^*(\xi, y, t^n)]\,, \\ &\boldsymbol{u}^n(\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}, t^n) + \sum_{j=1} \varepsilon^j \boldsymbol{u}_j(\boldsymbol{x}, t^n)\,, \\ &p^{n-1/2}(\boldsymbol{x}) = p_0(\boldsymbol{x}, t^{n-1/2}) + \sum_{j=1} \varepsilon^j [p_j(\boldsymbol{x}, t^{n-1/2}) + \varphi_j(\xi, y, t^{n-1/2})]\,. \end{aligned}\right. \tag{6.1}$$

Here again we set $\varepsilon = \triangle t^{1/2}$, $\xi = (x+1)/\varepsilon$, $t^n = n\triangle t$, $t^{n-1/2} = (n-1/2)\triangle t$, $n = 1, 2, \cdots$. Substituting (6.1) into (2.12) and collecting equal powers of ε, we get the following equations:

From the first equation in (2.12), we get

$$\boldsymbol{u}_0^* = \boldsymbol{u}_0\,, \tag{6.2}$$

$$\boldsymbol{u}_1^* + \boldsymbol{a}_1^* - \boldsymbol{u}_1 = \frac{1}{2}\partial_\xi^2 \boldsymbol{a}_1^*\,, \tag{6.3}$$

$$\boldsymbol{u}_2^* + \boldsymbol{a}_2^* - \boldsymbol{u}_2 = \frac{1}{2}(\Delta \boldsymbol{u}_0^* + \partial_\xi^2 \boldsymbol{a}_2^* + \Delta \boldsymbol{u}_0)\,. \tag{6.4}$$

For $j \geq 1$,

$$\boldsymbol{u}_{j+2}^* + \boldsymbol{a}_{j+2}^* - \boldsymbol{u}_{j+2} = \frac{1}{2}(\Delta \boldsymbol{u}_j^* + \partial_\xi^2 \boldsymbol{a}_{j+2}^* + \partial_y^2 \boldsymbol{a}_j^* + \Delta \boldsymbol{u}_j)\,. \tag{6.5}$$

From the third equation in (2.12), we get

$$\boldsymbol{u}_1^* + \boldsymbol{a}_1^* = \boldsymbol{u}_1\,, \tag{6.6}$$

$$\boldsymbol{u}_2^* + \boldsymbol{a}_2^* = \boldsymbol{u}_2 + \partial_t \boldsymbol{u}_0 + \nabla p_0 + \nabla_\xi \varphi_1\,, \tag{6.7}$$

$$\boldsymbol{u}_3^* + \boldsymbol{a}_3^* = \boldsymbol{u}_3 + \partial_t \boldsymbol{u}_1 + \nabla p_1 + \nabla_\xi \varphi_2 + \nabla_y \varphi_1\,. \tag{6.8}$$

For $j = 2\ell, \ell \geq 1$,

$$\begin{aligned}
\boldsymbol{u}_{j+2}^* + \boldsymbol{a}_{j+2}^* \quad &= \boldsymbol{u}_{j+2} + \partial_t \boldsymbol{u}_j + \nabla p_j + \nabla_\xi \varphi_{j+1} + \nabla_y \varphi_j \\
&\quad + \frac{1}{(\ell+1)!}\boldsymbol{u}_0^{(\ell+1)} + \sum_{k=2}^{\ell} \frac{1}{k!}\boldsymbol{u}_{j-2k+2}^{(k)} + \frac{1}{2^\ell \ell!}\nabla p_0^{(\ell)} + \sum_{k=1}^{\ell-1} \frac{1}{2^k k!}\nabla p_{j-2k}^{(k)} \\
&\quad + \sum_{k=1}^{\ell} \frac{1}{2^k k!}(\nabla_\xi \varphi_{j-2k+1}^{(k)} + \nabla_y \varphi_{j-2k}^{(k)})\,.
\end{aligned} \tag{6.9}$$

For $j = 2\ell+1, \ell \geq 1$,

$$\begin{aligned}
\boldsymbol{u}_{j+2}^* + \boldsymbol{a}_{j+2}^* \quad &= \boldsymbol{u}_{j+2} + \partial_t \boldsymbol{u}_j + \nabla p_j + \nabla_\xi \varphi_{j+1} + \nabla_y \varphi_j \\
&\quad + \sum_{k=2}^{\ell+1} \frac{1}{k!}\boldsymbol{u}_{j-2k+2}^{(k)} + \sum_{k=1}^{\ell} \frac{1}{2^k k!}(\nabla p_{j-2k}^{(k)} + \nabla_\xi \varphi_{j-2k+1}^{(k)} + \nabla_y \varphi_{j-2k}^{(k)})\,.
\end{aligned} \tag{6.10}$$

From the incompressibility condition, we get

$$\nabla \cdot \boldsymbol{u}^j = 0\,, \quad \text{for} \quad j \geq 0\,. \tag{6.11}$$

The boundary conditions imply that for $x = -1$, $\xi = 0$,

$$\boldsymbol{u}_0 + \boldsymbol{u}_0^* = 0\,, \tag{6.12}$$

$$\boldsymbol{u}_1 + \boldsymbol{u}_1^* + \boldsymbol{a}_1^* = 0\,, \tag{6.13}$$

$$\boldsymbol{u}_2 + \boldsymbol{u}_2^* + \boldsymbol{a}_2^* = \nabla p_0 + \nabla_\xi \varphi_1\,. \tag{6.14}$$

$$\boldsymbol{u}_3 + \boldsymbol{u}_3^* + \boldsymbol{a}_3^* = \nabla p_1 + \nabla_\xi \varphi_2 + \nabla_y \varphi_1\,; \tag{6.15}$$

for $j = 2\ell$, $\ell \geq 2$

$$\begin{aligned} \boldsymbol{u}_j + \boldsymbol{u}_j^* + \boldsymbol{a}_j^* \quad &= \nabla p_{j-2} + \nabla_\xi \varphi_{j-1} + \nabla_y \varphi_{j-2} + \frac{(-1)^{\ell-1}}{2^{\ell-1}} \frac{1}{(\ell-1)!} \nabla p_0^{(\ell-1)} \\ &\quad + \sum_{k=1}^{\ell-2} \frac{(-1)^k}{2^k k!} \nabla p_{j-2k-2}^{(k)} + \sum_{k=1}^{\ell-1} \frac{(-1)^k}{2^k k!} (\nabla_\xi \varphi_{j-2k-1}^{(k)} + \nabla_y \varphi_{j-2k-2}^{(k)}); \end{aligned} \tag{6.16}$$

for $j = 2\ell + 1$, $\ell \geq 2$

$$\begin{aligned} \boldsymbol{u}_j + \boldsymbol{u}_j^* + \boldsymbol{a}_j^* \quad &= \nabla p_{j-2} + \nabla_\xi \varphi_{j-1} + \nabla_y \varphi_{j-2} \\ &\quad + \sum_{k=1}^{\ell-1} \frac{(-1)^k}{2^k k!} (\nabla p_{j-2k-2}^{(k)} + \nabla_\xi \varphi_{j-2k-1}^{(k)} + \nabla_y \varphi_{j-2k-2}^{(k)}); \end{aligned} \tag{6.17}$$

and for $j \geq 0$

$$\partial_x p_j + \partial_\xi \varphi_{j+1} = 0\,. \tag{6.18}$$

Next we go through all these equations, order by order, to see if they are solvable. Since this is very similar to what we did in §4.1, we will only give a summary of results.

The coefficients in the expansions (6.1) can be obtained successively in the following order:

$$\boldsymbol{u}_0^*(\boldsymbol{x}, t) = \boldsymbol{u}_0(\boldsymbol{x}, t)\,, \tag{6.19}$$

$$\boldsymbol{u}_1^*(\boldsymbol{x}, t) = \boldsymbol{u}_1(\boldsymbol{x}, t)\,, \tag{6.20}$$

$$\boldsymbol{a}_1^* \equiv 0\,, \tag{6.21}$$

$$\begin{cases} \partial_t \boldsymbol{u}_0 + \nabla p_0 = \Delta \boldsymbol{u}_0\,, \\ \nabla\cdot \boldsymbol{u}_0 = 0\,, \\ \boldsymbol{u}_0 = 0\,, \qquad \text{on } \partial^{'}\Omega\,, \end{cases} \tag{6.22}$$

$$\boldsymbol{u}_2^* = \boldsymbol{u}_2 + \partial_t \boldsymbol{u}_0 + \nabla p_0 \,, \tag{6.23}$$

$$\begin{cases} \varphi_1 = \frac{1}{2}\partial_\xi^2 \varphi_1 \,, \\ \partial_\xi \varphi_1 \mid_{\xi=0} \, = -\partial_x p_0 \mid_{x=-1} \,, \end{cases} \tag{6.24}$$

$$\varphi_1 = \frac{1}{\sqrt{2}} \partial_x p_0 \mid_{x=-1} e^{-\sqrt{2}\xi} \,, \tag{6.25}$$

$$a_2^* = \partial_\xi \varphi_1, \quad b_2^* = 0 \,, \tag{6.26}$$

$$\begin{cases} \partial_t \boldsymbol{u}_1 + \nabla p_1 = \Delta \boldsymbol{u}_1 \,, \\ \nabla \cdot \boldsymbol{u}_1 = 0 \,, \\ \boldsymbol{u}_1 \mid_{\partial' \Omega} = 0 \,, \quad \boldsymbol{u}_1(\boldsymbol{x}, 0) = 0 \,. \end{cases} \tag{6.27}$$

This implies

$$\boldsymbol{u}_1 = 0 \,, \quad p_1 = 0 \,. \tag{6.28}$$

We next have:

$$\boldsymbol{u}_3^* = \boldsymbol{u}_3 \,, \tag{6.29}$$

$$\varphi_2 = 0 \,, \quad a_3^* = 0 \,, \quad b_3^* = \partial_y \varphi_1 \,, \tag{6.30}$$

$$\begin{cases} \partial_t \boldsymbol{u}_2 + \nabla p_2 = \Delta \boldsymbol{u}_2 + \frac{1}{2}\Delta(\partial_t \boldsymbol{u}_0 + \nabla p_0) - \frac{1}{2}\partial_t^2 \boldsymbol{u}_0 - \frac{1}{2}\partial_t \nabla p_0 \,, \\ \nabla \cdot \boldsymbol{u}_2 = 0 \,, \\ \boldsymbol{u}_2 \mid_{\partial' \Omega} = 0 \,, \quad \boldsymbol{u}_2(\boldsymbol{x}, 0) = 0 \,. \end{cases} \tag{6.31}$$

This also implies

$$\boldsymbol{u}_2 = 0 \,, \quad p_2 = 0 \,, \tag{6.32}$$

$$\begin{cases} \varphi_3 = \frac{1}{2}(\partial_\xi^2 \varphi_3 + \partial_y^2 \varphi_1) \,, \\ \partial_\xi \varphi_3 \mid_{\xi=0} \, = 0 \,, \end{cases} \tag{6.33}$$

$$\varphi_3(\xi, y, t) = \frac{1}{2}\partial_{xyy} p_0 \mid_{x=-1} (\frac{1}{\sqrt{2}} + \xi)\ e^{-\sqrt{2}\xi}\,, \tag{6.34}$$

$$a_4^* = \partial_\xi \varphi_3 + \frac{1}{2}\partial_\xi \varphi_1^{(1)}\,, \quad b_4^* = 0\,, \tag{6.35}$$

$$\boldsymbol{u}_4^* = \boldsymbol{u}_4 + \frac{1}{2}\boldsymbol{u}_0^{(2)} + \frac{1}{2}\nabla p_0^{(1)}\,, \tag{6.36}$$

$$\boldsymbol{u}_3 = 0\,, \quad p_3 = 0\,, \tag{6.37}$$

$$\varphi_4 = 0\,, \quad a_5^* = 0\,, \quad b_5^* = \partial_y \varphi_3 + \frac{1}{2}\partial_y \varphi_1^{(1)}\,, \tag{6.38}$$

$$\boldsymbol{u}_5^* = \boldsymbol{u}_5\,, \tag{6.39}$$

$$\begin{cases} \partial_t \boldsymbol{u}_4 + \nabla p_4 = \Delta \boldsymbol{u}_4 + \frac{1}{4}\Delta(\boldsymbol{u}_0^{(2)} + \nabla p_0^{(1)}) - \frac{1}{6}\boldsymbol{u}_0^{(3)} - \frac{1}{8}\nabla p_0^{(2)}\,, \\ \nabla \cdot \boldsymbol{u}_4 = 0\,, \\ \boldsymbol{u}_4 \mid_{x=-1} = -\frac{1}{2}\partial_t(\nabla p_0 + \frac{1}{2}\nabla_\xi \varphi_1) \mid_{x=-1,\xi=0}\ . \end{cases} \tag{6.40}$$

In the last equation, there is a similar boundary condition at $x = 1$. With a suitable initial data, (6.40) has a smooth solution. Again we will defer the discussions on choosing the initial data until the end of this subsection.

Continue in this fashion, we obtain

$$\begin{cases} \varphi_5 = \frac{1}{2}(\partial_\xi^2 \varphi_5 + \partial_y^2 \varphi_3)\,, \\ \partial_\xi \varphi_5 \mid_{\xi=0} = -\partial_x p_4 \mid_{x=-1}\,, \end{cases} \tag{6.41}$$

$$a_6^* = \partial_\xi \varphi_5 + \frac{1}{2}\partial_\xi \partial_t \varphi_3\,, \quad b_6^* = 0\,, \tag{6.42}$$

$$\boldsymbol{u}_6^* = \boldsymbol{u}_6 + \boldsymbol{u}_4^{(1)} + \frac{1}{6}\boldsymbol{u}_0^{(3)} + \nabla p_4 + \frac{1}{8}\nabla p_0^{(2)}\,, \tag{6.43}$$

$$\begin{cases} \partial_t \boldsymbol{u}_5 + \nabla p_5 = \Delta \boldsymbol{u}_5\,, \\ \nabla \cdot \boldsymbol{u}_5 = 0\,, \\ \boldsymbol{u}_5 \mid_{x=-1} = -\frac{1}{2}\partial_t \nabla_y \varphi_1 \mid_{x=-1,\xi=0}\ . \end{cases} \tag{6.44}$$

Notice that as in the case of the first order scheme, we generally have $(\boldsymbol{u}_5, p_5) \neq 0$, because of the contributions from the boundary.

$$(6.45)\qquad \begin{cases} \varphi_6 = \frac{1}{2}(\partial_\xi^2 \varphi_6 + \partial_y^2 \varphi_4)\,, \\ \partial_\xi \varphi_6 \mid_{\xi=0} = -\partial_x p_5 \mid_{x=-1}\,, \end{cases}$$

$$(6.46)\qquad a_7^* = \partial_\xi \varphi_6\,, \quad b_7^* = \partial_y \varphi_5 + \frac{1}{2}\partial_y \varphi_3^{(1)}\,,$$

$$(6.47)\qquad \boldsymbol{u}_7^* = \boldsymbol{u}_7 + \partial_t \boldsymbol{u}_5 + \nabla p_5\,,$$

$$(6.48)\qquad \begin{cases} \partial_t \boldsymbol{u}_6 + \nabla p_6 = \Delta \boldsymbol{u}_6 + \frac{1}{2}\Delta(\partial_t \boldsymbol{u}_4 + \nabla p_4 + \frac{1}{6}\boldsymbol{u}_0^{(3)} + \frac{1}{8}\nabla p_0^{(2)}) \\ \qquad -\dfrac{1}{4!}\boldsymbol{u}_0^{(4)} - \dfrac{1}{2}\boldsymbol{u}_4^{(2)} - \dfrac{1}{2^3 3!}\nabla p_0^{(3)} - \nabla p_4^{(1)} \\ \nabla\cdot \boldsymbol{u}_6 = 0\,, \\ \boldsymbol{u}_6 \mid_{x=-1} = \frac{1}{4}\nabla p_0^{(2)} \mid_{x=-1} - \frac{1}{8}\nabla_\xi \varphi_1^{(2)} \mid_{\xi=0}\ . \end{cases}$$

$$(6.49)\qquad \begin{cases} \varphi_7 = \frac{1}{2}(\partial_\xi^2 \varphi_7 + \partial_y^2 \varphi_5)\,, \\ \partial_\xi \varphi_7 \mid_{\xi=0} = -\partial_x p_6 \mid_{x=-1}\,, \end{cases}$$

$$(6.50)\qquad a_8^* = \sum_{k=0}^{3} \frac{1}{2^k k!}\partial_\xi \varphi_{j-2k}^{(k)}\,, \quad b_8^* = \partial_y \varphi_6\,,$$

$$(6.51)\qquad \boldsymbol{u}_8^* = \boldsymbol{u}_8 + \boldsymbol{u}_6^{(1)} + \frac{1}{2}\boldsymbol{u}_4^{(2)} + \frac{1}{4!}\boldsymbol{u}_0^{(4)} + \nabla p_6 + \frac{1}{2}\nabla p_4^{(1)} + \frac{1}{48}\nabla p_0^{(3)}\,,$$

$$(6.52)\qquad \begin{cases} \partial_t \boldsymbol{u}_7 + \nabla p_7 = \Delta \boldsymbol{u}_7 + \frac{1}{2}\Delta(\partial_t \boldsymbol{u}_5 + \nabla p_5) - \frac{1}{2}\boldsymbol{u}_5^{(2)} - \frac{1}{2}\nabla p_5^{(1)}\,, \\ \nabla\cdot \boldsymbol{u}_7 = 0\,, \\ \boldsymbol{u}_7 \mid_{x=-1} = -\frac{1}{2}\nabla_y(\varphi_5 - \varphi_3^{(1)} + \frac{5}{8}\varphi_1^{(2)}) \mid_{x=-1,\xi=0}\ . \end{cases}$$

In general, if we let

$$(6.53)\qquad \psi_j = \sum_{k=0}^{[j/2]} \frac{1}{2^k k!}\varphi_{j-2k}^{(k)} = \varphi_j + \frac{1}{2}\partial_t \varphi_{j-2} + \frac{1}{8}\partial_t^2 \varphi_{j-4} + \cdots$$

such that $\{\psi_j\}_{j\geq 4}$ satisfies

$$\begin{cases} \varphi_j = \frac{1}{2}(\partial_\xi^2\varphi_j + \partial_y^2\varphi_{j-2})\,, \\ \partial_\xi\varphi_j\,|_{\xi=0} = -\partial_x p_{j-1}\,|_{x=-1} \end{cases} \tag{6.54}$$

and φ_j decays exponentially as $\xi \to +\infty$. Then we have

$$a_j^* = \partial_\xi\psi_{j-1}\,, \quad b_j^* = \partial_y\psi_{j-2}\,. \tag{6.55}$$

Clearly $\boldsymbol{a}_j^*$ also decays exponentially as $\xi \to +\infty$. On the hand $(\boldsymbol{u}_j, p_j)$ solves a system of linear Stokes equations with source terms.

Now if we let

$$\begin{cases} U^* = \boldsymbol{u}_0 + \displaystyle\sum_{j=1}^{2N} \varepsilon^j(\boldsymbol{u}_j^* + \boldsymbol{a}_j^*)\,, \\ U^n = \boldsymbol{u}_0 + \displaystyle\sum_{j=1}^{2N} \varepsilon^j \boldsymbol{u}_j\,, \\ P^{n-1/2} = p_0 + \displaystyle\sum_{j=1}^{2N} \varepsilon^j(p_j + \varphi_j) + \varepsilon^{2N+1}\varphi_{2N+1}\,, \end{cases} \tag{6.56}$$

then we have

$$\begin{cases} \dfrac{U^* - U^n}{\triangle t} = \Delta\dfrac{U^* + U^n}{2} + \triangle t^{N-1/2}\boldsymbol{f}_N\,, \\ U^* + U^n = \triangle t \nabla P^{n-1/2}\,, \qquad \text{on } \partial\Omega\,, \\ U^* = U^{n+1} + \triangle t \nabla P^{n+1/2} + \triangle t^{N+1/2}\boldsymbol{g}_N\,, \\ \nabla\cdot U^{n+1} = 0\,, \\ \dfrac{\partial P^{n+1/2}}{\partial \boldsymbol{n}} = \boldsymbol{n}\cdot U^{n+1} = 0\,, \qquad \text{on } \partial\Omega \end{cases} \tag{6.57}$$

where $\boldsymbol{f}_N$, $\boldsymbol{g}_N$ are bounded and smooth if $(\boldsymbol{u}_0, p_0)$ is sufficiently smooth.

As in §4.1, if we do not assume any compatibility condition for u^0, then the smoothness of $(\boldsymbol{u}_4, p_4)$ at $t = 0$ requires us to choose initial data for (6.40) such that it matches the boundary condition in (6.40). This restricts the approximation at $t = 0$ to

$$U^0(\boldsymbol{x}) = \boldsymbol{u}^0(\boldsymbol{x}) + \triangle t^2 \boldsymbol{w}^0(\boldsymbol{x}) \tag{6.58}$$

where $\boldsymbol{w}^0$ is a bounded function.

However, it is straightforward to check that under the compatibility conditions stated in Theorem 2, we can choose

$$\boldsymbol{u}_4(\boldsymbol{x},0) = \boldsymbol{u}_5(\boldsymbol{x},0) = \boldsymbol{u}_6(\boldsymbol{x},0) = \boldsymbol{u}_7(\boldsymbol{x},0) = 0. \tag{6.59}$$

Consequently, we have

$$U^0(\boldsymbol{x}) = \boldsymbol{u}^0(\boldsymbol{x}) + \triangle t^4 \boldsymbol{w}^0(\boldsymbol{x}) \tag{6.60}$$

where $\boldsymbol{w}^0$ is a bounded function.

§6.2. Proof of Theorem 2

As in the proof of Theorem 1, Theorem 2 is a direct consequence of the following result, together with (6.56) with $N = 5$.

Proposition 2. *Let $\boldsymbol{u}^n$, $\boldsymbol{u}^*$ and p^n be the solution of (2.12) with initial data $\boldsymbol{u}^0$. Let U^n, U^*, and P^n be the constructed approximate solution satisfying*

$$\begin{cases} \dfrac{U^* - U^n}{\triangle t} = \Delta \dfrac{U^* + U^n}{2} - \dfrac{3}{2}(U^n \cdot \nabla)\, U^n + \dfrac{1}{2}(U^{n-1} \cdot \nabla)\, U^{n-1} + \triangle t^\alpha \boldsymbol{f}^n\,, \\ U^* + U^n = \triangle t\ \nabla P^{n-1/2}\,, \qquad \text{on } \partial\Omega\,, \\ \dfrac{U^{n+1} - U^*}{\triangle t} + \nabla P^{n+1/2} = \triangle t^\alpha \boldsymbol{g}^n\,, \\ \nabla \cdot U^{n+1} = 0\,, \\ U^{n+1} \cdot \boldsymbol{n} = 0\,, \qquad \text{on } \partial\Omega\,, \\ U^0 = \boldsymbol{u}^0 + \triangle t^\alpha \boldsymbol{w}^0 \end{cases} \tag{6.61}$$

and

$$\max_{0 \le n \le \left[\frac{T}{\triangle t}\right]+1} \|U^n(\cdot)\|_{W^{1,\infty}} \le C^*\,, \qquad \alpha > 7/4\,. \tag{6.62}$$

Then we have

$$\|\boldsymbol{u}^n - U^n\|_{L^2} + \triangle t\, \|p^n - P^n\|_{H^1} \le C_1 \triangle t^\alpha \tag{6.63}$$

and

$$\|\boldsymbol{u}^n - U^n\|_{L^\infty} + \triangle t\,\|p^n - P^n\|_{W^{1,\infty}} \le C_1 \triangle t^{\alpha-7/4} \tag{6.64}$$

where C_1 is same in Proposition 1.

Proof: As in the proof of Proposition 1, we assume a priori that

$$\|\boldsymbol{u}^n\|_{L^\infty} \le \tilde{C} \tag{6.65}$$

for $n \le \left[\frac{T}{\triangle t}\right] + 1$.

Step 1. *Equation for Error Functions.* We first reformulated the Kim and Moin's scheme (2.12) by introducing the following new intermediate variables

$$\begin{aligned} &\boldsymbol{u}^* + \boldsymbol{u}^n - \triangle t\ \nabla p^{n-1/2} \to 2\hat{\boldsymbol{u}}^*\,, \\ &U^* + U^n - \triangle t\ \nabla P^{n-1/2} \to 2\hat{U}^*\,. \end{aligned} \tag{6.66}$$

(2.12) becomes

$$\begin{cases} \dfrac{2(\hat{\boldsymbol{u}}^* - \boldsymbol{u}^n)}{\triangle t} + \nabla\left(p^{n-1/2} - \dfrac{1}{2}\triangle t\ \Delta p^{n-1/2}\right) \\ \qquad = \Delta \hat{\boldsymbol{u}}^* - \dfrac{3}{2}(\boldsymbol{u}^n\cdot\nabla)\ \boldsymbol{u}^n + \dfrac{1}{2}(\boldsymbol{u}^{n-1}\cdot\nabla)\ \boldsymbol{u}^{n-1}\,, \\ \hat{\boldsymbol{u}}^* = 0\,, \quad \text{on } \partial\Omega\,, \\ \dfrac{\boldsymbol{u}^{n+1} + \boldsymbol{u}^n - 2\hat{\boldsymbol{u}}^*}{\triangle t} + \nabla(p^{n+1/2} - p^{n-1/2}) = 0\,, \\ \nabla\cdot\boldsymbol{u}^{n+1} = 0\,, \\ \boldsymbol{u}^{n+1}\cdot\boldsymbol{n} = 0\,, \quad \text{on } \partial\Omega\,. \end{cases} \tag{6.67}$$

The approximation solution (6.57) changes similarly. Let functions

$$\boldsymbol{e}^n = U^n - \boldsymbol{u}^n\,, \qquad \boldsymbol{e}^* = \hat{U}^* - \hat{\boldsymbol{u}}^*\,, \qquad q^n = P^{n-1/2} - p^{n-1/2}\,. \tag{6.68}$$

Subtracting the reformulated form of (6.57) from (6.67), we get an equation for the error functions:

$$
(6.69)\qquad \begin{cases} \dfrac{2(\boldsymbol{e}^* - \boldsymbol{e}^n)}{\triangle t} + \nabla\left(q^n - \dfrac{1}{2}\triangle t\ \Delta q^n\right) = \Delta \boldsymbol{e}^* + \dfrac{1}{2}(\boldsymbol{e}^{n-1}\cdot\nabla)\ U^{n-1} \\ \qquad\qquad + \dfrac{1}{2}(\boldsymbol{u}^{n-1}\cdot\nabla)\ \boldsymbol{e}^{n-1} - \dfrac{3}{2}(\boldsymbol{e}^{n}\cdot\nabla)\ U^{n} - \dfrac{3}{2}(\boldsymbol{u}^{n}\cdot\nabla)\ \boldsymbol{e}^{n} + \triangle t^\alpha \boldsymbol{f}^n\,, \\ \boldsymbol{e}^* = 0\,, \quad \text{on } \partial\Omega\,, \\ \dfrac{\boldsymbol{e}^{n+1} + \boldsymbol{e}^n - 2\boldsymbol{e}^*}{\triangle t} + \nabla(q^{n+1} - q^n) = \triangle t^\alpha \boldsymbol{g}^n\,, \\ \nabla\cdot\boldsymbol{e}^{n+1} = 0\,, \\ \boldsymbol{e}^{n+1}\cdot\boldsymbol{n} = 0\,, \quad \text{on } \partial\Omega\,, \\ \boldsymbol{e}^0 = \triangle t^\alpha \boldsymbol{w}^0\,. \end{cases}
$$

Step 2. *Basic Energy Estimate.* Taking the scalar product of the first equation of (6.69) with $\boldsymbol{e}^*$ and integrating by parts, we get

$$
(6.70)\qquad \begin{aligned} &\|\boldsymbol{e}^*\|^2 - \|\boldsymbol{e}^n\|^2 + \|\boldsymbol{e}^* - \boldsymbol{e}^n\|^2 + \triangle t\ \|\nabla \boldsymbol{e}^*\|^2 \\ &\quad \le -\triangle t \int_\Omega \boldsymbol{e}^*\cdot\nabla(q^n - \frac{1}{2}\triangle t \Delta q^n)\,d\boldsymbol{x} + C\,\triangle t^{2\alpha+1}\|\boldsymbol{f}^n\|^2 \\ &\qquad + C\,\triangle t\ (\|\boldsymbol{e}^n\|^2 + \|\boldsymbol{e}^{n-1}\|^2 + \|\boldsymbol{e}^*\|^2) + \tfrac{1}{2}\triangle t\ \|\nabla \boldsymbol{e}^*\|^2\,. \end{aligned}
$$

Taking the scalar product of the second equation of (6.69) with $\boldsymbol{e}^{n+1}$, we obtain

$$
(6.71)\qquad \begin{aligned} &\|\boldsymbol{e}^{n+1}\|^2 - \|\boldsymbol{e}^*\|^2 + \|\boldsymbol{e}^{n+1} - \boldsymbol{e}^*\|^2 - \tfrac{1}{2}(\|\boldsymbol{e}^{n+1}\|^2 - \|\boldsymbol{e}^n\|^2) - \tfrac{1}{2}\|\boldsymbol{e}^{n+1} - \boldsymbol{e}^n\|^2 \\ &\quad \le C\,\triangle t^{2\alpha+1}\|\boldsymbol{g}^n\|^2 + C\,\triangle t\ \|\boldsymbol{e}^{n+1}\|^2\,. \end{aligned}
$$

Combining the these two estimates we obtain

$$
(6.72)\qquad \begin{aligned} &\frac{1}{2}(\|\boldsymbol{e}^{n+1}\|^2 - \|\boldsymbol{e}^n\|^2) + \|\boldsymbol{e}^* - \boldsymbol{e}^n\|^2 + \|\boldsymbol{e}^{n+1} - \boldsymbol{e}^*\|^2 \\ &\quad -\tfrac{1}{2}\|\boldsymbol{e}^{n+1} - \boldsymbol{e}^n\|^2 + \tfrac{1}{2}\triangle t\ \|\nabla \boldsymbol{e}^*\|^2 \\ &\quad \le -\triangle t\ \int_\Omega \boldsymbol{e}^*\cdot\nabla(q^n - \frac{1}{2}\triangle t \boldsymbol{\Delta} q^n)\,d\boldsymbol{x} + C\,\triangle t^{2\alpha+1}(\|\boldsymbol{f}^n\|^2 + \|\boldsymbol{g}^n\|^2) \\ &\quad + C\,\triangle t\ (\|\boldsymbol{e}^n\|^2 + \|\boldsymbol{e}^{n-1}\|^2 + \|\boldsymbol{e}^*\|^2 + \|\boldsymbol{e}^{n+1}\|^2)\,. \end{aligned}
$$

Since

$$
(6.73)\qquad 2\|\boldsymbol{e}^* - \boldsymbol{e}^n\|^2 + 2\|\boldsymbol{e}^{n+1} - \boldsymbol{e}^*\|^2 = \|\boldsymbol{e}^{n+1} - \boldsymbol{e}^n\|^2 + \|\boldsymbol{e}^{n+1} + \boldsymbol{e}^n - 2\boldsymbol{e}^*\|^2\,,
$$

we get

$$
\begin{aligned}
&\|\boldsymbol{e}^{n+1}\|^2 - \|\boldsymbol{e}^n\|^2 + \|\boldsymbol{e}^{n+1} + \boldsymbol{e}^n - 2\boldsymbol{e}^*\|^2 + \triangle t\ \|\nabla \boldsymbol{e}^*\|^2 \\
&\le -2\triangle t\ \int_\Omega \boldsymbol{e}^* \cdot \nabla(q^n - \frac{1}{2}\triangle t \Delta q^n)\, d\boldsymbol{x} + C\,\triangle t\ (\|\boldsymbol{e}^n\|^2 + \|\boldsymbol{e}^{n-1}\|^2 + \|\boldsymbol{e}^{n+1}\|^2) \\
&\quad + C\,\triangle t^{2\alpha+1}(\|\boldsymbol{f}^n\|^2 + \|\boldsymbol{g}^n\|^2)\,.
\end{aligned}
\tag{6.74}
$$

To estimate the first term on the right hand of (6.74), we let

$$
\begin{aligned}
I &\equiv -2\triangle t \int_\Omega \boldsymbol{e}^* \cdot \nabla(q^n - \frac{1}{2}\triangle t\ \Delta q^n)\, d\boldsymbol{x} \\
&= -2\triangle t \int_\Omega \boldsymbol{e}^* \cdot \nabla q^n\, d\boldsymbol{x} - \triangle t^2 \int_\Omega (\nabla \cdot \boldsymbol{e}^*) \Delta q^n\, d\boldsymbol{x} \equiv I_1 + I_2\,.
\end{aligned}
\tag{6.75}
$$

Using the second equation and integrating by parts, we can write the first term as

$$
\begin{aligned}
I_1 &= -2\triangle t \int_\Omega \boldsymbol{e}^* \cdot \nabla q^n\, d\boldsymbol{x} \\
&= -\triangle t^2 \int_\Omega \nabla(q^{n+1} - q^n) \nabla q^n\, d\boldsymbol{x} - \triangle t^{\alpha+2} \int_\Omega \boldsymbol{g}^n \cdot \nabla q^n\, d\boldsymbol{x} \\
&= -\frac{1}{2}\triangle t^2 (\|\nabla q^{n+1}\|^2 - \|\nabla q^n\|^2) \\
&\quad + \frac{1}{2}\triangle t^2 \|\nabla(q^{n+1} - q^n)\|^2 - \triangle t^{\alpha+2} \int_\Omega \boldsymbol{g}^n \cdot \nabla q^n\, d\boldsymbol{x}\,.
\end{aligned}
\tag{6.76}
$$

Since

$$
\begin{aligned}
\frac{1}{2}\triangle t^2 \|\nabla(q^{n+1} - q^n)\|^2 &= \frac{1}{2}\|\boldsymbol{e}^{n+1} + \boldsymbol{e}^n - 2\boldsymbol{e}^*\|^2 \\
&\quad + \frac{1}{2}\triangle t^{2\alpha+2}\|\boldsymbol{g}^n\|^2 + \triangle t^{\alpha+1} \int_\Omega \boldsymbol{g}^n \cdot (\boldsymbol{e}^{n+1} + \boldsymbol{e}^n - 2\boldsymbol{e}^*)\, d\boldsymbol{x}\,.
\end{aligned}
\tag{6.77}
$$

We have

(6.78)

$$
\begin{aligned}
I_1 &= -\tfrac{1}{2}\triangle t^2 (\|\nabla q^{n+1}\|^2 - \|\nabla q^n\|^2) + \tfrac{1}{2}\|\boldsymbol{e}^{n+1} + \boldsymbol{e}^n - 2\boldsymbol{e}^*\|^2 \\
&\quad + \frac{1}{2}\triangle t^{2\alpha+2}\|\boldsymbol{g}^n\|^2 + \triangle t^{\alpha+1} \int_\Omega \boldsymbol{g}^n \cdot (\boldsymbol{e}^{n+1} + \boldsymbol{e}^n - 2\boldsymbol{e}^*)\, d\boldsymbol{x} - \triangle t^{\alpha+2} \int_\Omega \boldsymbol{g}^n \cdot \nabla q^n\, d\boldsymbol{x}\,.
\end{aligned}
$$

Next we rewrite the second term as

$$
\begin{aligned}
I_2 &= -\triangle t^2 \int_\Omega (\nabla\cdot\boldsymbol{e}^*)\Delta q^n \, d\boldsymbol{x} \\
&= -\frac{1}{2}\triangle t^3 \int_\Omega \Delta(q^{n+1}-q^n)\Delta q^n \, d\boldsymbol{x} - \frac{1}{2}\triangle t^{\alpha+3}\int_\Omega (\nabla\cdot\boldsymbol{g}^n)\Delta q^n \, d\boldsymbol{x} \\
&= -\frac{1}{4}\triangle t^3(\|\Delta q^{n+1}\|^2 - \|\Delta q^n\|^2) + \frac{1}{4}\triangle t^3\|\Delta(q^{n+1}-q^n)\|^2 \\
&\qquad -\frac{1}{2}\triangle t^{\alpha+3}\int_\Omega (\nabla\cdot\boldsymbol{g}^n)\Delta q^n \, d\boldsymbol{x} \\
&= -\frac{1}{4}\triangle t^3(\|\Delta q^{n+1}\|^2 - \|\Delta q^n\|^2) + \triangle t \, \|\nabla\cdot\boldsymbol{e}^*\|^2 + \frac{1}{4}\triangle t^{2\alpha+3}\|\nabla\cdot\boldsymbol{g}^n\|^2 \\
&\quad -\triangle t^{\alpha+2}\int_\Omega (\nabla\cdot\boldsymbol{g}^n)(\nabla\cdot\boldsymbol{e}^*)\, d\boldsymbol{x} - \frac{1}{2}\triangle t^{\alpha+3}\int_\Omega (\nabla\cdot\boldsymbol{g}^n)\Delta q^n \, d\boldsymbol{x}\,.
\end{aligned} \tag{6.79}
$$

Combining these two terms we arrive at

$$
\begin{aligned}
I = &-\frac{1}{2}\triangle t^2(\|\nabla q^{n+1}\|^2 - \|\nabla q^n\|^2) - \frac{1}{4}\triangle t^3(\|\Delta q^{n+1}\|^2 - \|\Delta q^n\|^2) \\
&+\frac{1}{2}\|\boldsymbol{e}^{n+1}+\boldsymbol{e}^n - 2\boldsymbol{e}^*\|^2 + \triangle t \, \|\nabla\cdot\boldsymbol{e}^*\|^2 + \triangle t^{\alpha+1}\int_\Omega \boldsymbol{g}^n\cdot(\boldsymbol{e}^{n+1}+\boldsymbol{e}^n-2\boldsymbol{e}^*)\, d\boldsymbol{x} \\
&-\triangle t^{\alpha+2}\int_\Omega \boldsymbol{g}^n\cdot\nabla q^n \, d\boldsymbol{x} - \triangle t^{\alpha+2}\int_\Omega (\nabla\cdot\boldsymbol{g}^n)(\nabla\cdot\boldsymbol{e}^*)\, d\boldsymbol{x} \\
&-\frac{1}{2}\triangle t^{\alpha+3}\int_\Omega (\nabla\cdot\boldsymbol{g}^n)\Delta q^n \, d\boldsymbol{x} + \frac{1}{4}\triangle t^{2\alpha+3}\|\nabla\cdot\boldsymbol{g}^n\|^2 + \frac{1}{2}\|\triangle t^{\alpha+1}\boldsymbol{g}^n\|^2\,.
\end{aligned} \tag{6.80}
$$

This gives

$$
\begin{aligned}
I \le &-\tfrac{1}{2}\triangle t^2(\|\nabla q^{n+1}\|^2 - \|\nabla q^n\|^2) - \tfrac{1}{4}\triangle t^3(\|\Delta q^{n+1}\|^2 - \|\Delta q^n\|^2) \\
&+\tfrac{1}{2}\|\boldsymbol{e}^{n+1}+\boldsymbol{e}^n - 2\boldsymbol{e}^*\|^2 + \triangle t \, \|\nabla\boldsymbol{e}^*\|^2 + \triangle t\|\boldsymbol{e}^{n+1}+\boldsymbol{e}^n-2\boldsymbol{e}^*\|^2 \\
&+2\triangle t^3\|\nabla q^n\|^2 + 2\triangle t^4\|\Delta q^n\|^2 + 2\triangle t^{2\alpha+1}(\|\boldsymbol{g}^n\|^2 + \triangle t\|\boldsymbol{g}^n\|^2_{H^1})\,.
\end{aligned} \tag{6.81}
$$

Going back to (6.74) we obtain

$$
\begin{aligned}
&\|\boldsymbol{e}^{n+1}\|^2 - \|\boldsymbol{e}^n\|^2 + \tfrac{1}{2}\|\boldsymbol{e}^{n+1}+\boldsymbol{e}^n-2\boldsymbol{e}^*\|^2 + \triangle t \, \|\nabla\boldsymbol{e}^*\|^2 \\
&\quad +\tfrac{1}{2}\triangle t^2(\|\nabla q^{n+1}\|^2 - \|\nabla q^n\|^2) + \tfrac{1}{4}\triangle t^3(\|\Delta q^{n+1}\|^2 - \|\Delta q^n\|^2) \\
&\le \triangle t^3\|\nabla q^n\|^2 + \triangle t^4\|\Delta q^n\|^2 + C\,\triangle t \, (\|\boldsymbol{e}^n\|^2 + \|\boldsymbol{e}^{n-1}\|^2 + \|\boldsymbol{e}^{n+1}\|^2) \\
&\quad +C\,\triangle t^{2\alpha+1}(\|\boldsymbol{f}^n\|^2 + \triangle t\|\boldsymbol{g}^n\|^2_{H^1})\,.
\end{aligned} \tag{6.82}
$$

Gronwall lemma gives

$$\|\boldsymbol{e}^n\| + \|\boldsymbol{e}^*\| + \triangle t\ \|\nabla q^n\| + \triangle t^{3/2}\ \|\Delta q^n\| + \triangle t^{1/2}\|\nabla \boldsymbol{e}^*\| \le C_1 \triangle t^{\alpha}\,. \tag{6.83}$$

Step 3. *L^∞-norm Estimate.* Taking the divergence to the second equation of (6.69), we obtain

$$\begin{cases} \Delta(q^{n+1}-q^n) = 2\dfrac{\nabla\cdot\boldsymbol{e}^*}{\triangle t} + \triangle t^{\alpha}\nabla\cdot\boldsymbol{g}^n\,, \\ \dfrac{\partial(q^{n+1}-q^n)}{\partial\boldsymbol{n}} = 0\,, \qquad \text{on } \partial^{'}\Omega\,. \end{cases} \tag{6.84}$$

We can always normalize pressure such that $\int_\Omega (q^{n+1}-q^n)\,d\boldsymbol{x} = 0$. Applying standard regularity theorem to (6.84) and using (6.83), we have

$$\|q^{n+1}-q^n\|_{H^2} \le C\,\|\Delta(q^{n+1}-q^n)\| \le C\triangle t^{\alpha-3/2}\,. \tag{6.85}$$

The second equation of (6.69) implies directly

$$\begin{aligned} \|\nabla(\boldsymbol{e}^{n+1}+\boldsymbol{e}^n)\| &\le 2\|\nabla\boldsymbol{e}^*\| + \triangle t\|q^{n+1}-q^n\|_{H^2} + \triangle t^{\alpha+1}\|\boldsymbol{g}^n\|_{H^1} \\ &\le C\,\triangle t^{\alpha-1/2}(C_1 + \triangle t^{3/2}\|\boldsymbol{g}^n\|_{H^1})\,. \end{aligned} \tag{6.86}$$

Obviously, we have

$$\|\nabla\boldsymbol{e}^n\| \le \sum_{k=0}^{n}\|\nabla(\boldsymbol{e}^k+\boldsymbol{e}^{k-1})\| \le C_1\,\triangle t^{\alpha-3/2}\,. \tag{6.87}$$

From the first equation of (6.69) we obtain can derive

$$\begin{aligned} \|\Delta\boldsymbol{e}^*\| &\le \triangle t^{-1}\|\boldsymbol{e}^*-\boldsymbol{e}^n\| + C\,\|\boldsymbol{e}^n\| + \|\nabla\boldsymbol{e}^n\| + C\,\triangle t^{\alpha}\|\boldsymbol{f}^n\| \\ &\le C_1\triangle t^{\alpha-3/2}\,. \end{aligned} \tag{6.88}$$

Consequently, we have

$$\|\boldsymbol{e}^*\|_{H^2} \le C_1\triangle t^{\alpha-3/2}\,, \qquad \|\boldsymbol{e}^*\|_{L^\infty} \le C_1\triangle t^{\alpha-3/4}\,. \tag{6.89}$$

From the second equation of (6.69), we have

$$\|\nabla(q^{n+1}-q^n)\| \le C_1\triangle t^{\alpha-1} \tag{6.90}$$

and

$$
\begin{aligned}
\|\Delta(q^{n+1}-q^n)\|_{H^1} &\leq \triangle t^{-1}\|\boldsymbol{e}^*\|_{H^2} + \triangle t^{\alpha}\|\boldsymbol{g}^n\|_{H^2} \\
&\leq C\,\triangle t^{\alpha-5/2}(C_1 + \triangle t^2\|\boldsymbol{g}^n\|_{H^2})\,.
\end{aligned}
\tag{6.91}
$$

Hence, we have

$$
\|q^{n+1}-q^n\|_{H^3} \leq C_1\triangle t^{\alpha-5/2} \tag{6.92}
$$

and

$$
\|\nabla(q^{n+1}-q^n)\|_{L^\infty} \leq C_1\triangle t^{\alpha-7/4}\,. \tag{6.93}
$$

Using the second equation of (6.69) once more, we get

$$
\begin{aligned}
\|\boldsymbol{e}^{n+1}+\boldsymbol{e}^n\|_{L^\infty} &\leq \|\boldsymbol{e}^*\|_{L^\infty} + \triangle t\ \|\nabla(q^{n+1}-q^n)\|_{L^\infty} + \triangle t^{\alpha+1}\|\boldsymbol{g}^n\|_{L^\infty} \\
&\leq C_1\triangle t^{\alpha-3/4}\,.
\end{aligned}
\tag{6.94}
$$

Hence we have

$$
\|\boldsymbol{e}^n\|_{L^\infty} \leq C_1\triangle t^{\alpha-7/4}\,. \tag{6.95}
$$

As in §4.2, if we chose $\triangle t$ small enough, we have $\|\boldsymbol{e}^n\|_{L^\infty} \leq 1$. Hence in (6.65) we can choose

$$
\tilde{C} = 1 + \max_{n\leq\left[\frac{T}{\triangle t}\right]+1} \|U^n(\cdot)\|_{L^\infty}
$$

which depends only on the exact solution $(\boldsymbol{u}, p)$. Combining (6.66), (6.68) and (6.95), we get

$$
\|\boldsymbol{u}^n - U^n\|_{L^\infty} + \|\boldsymbol{u}^* - U^*\|_{L^\infty} + \triangle t\|p^n - P^n\|_{W^{1,\infty}} \leq C_1\triangle t^{\alpha-7/4}\,. \tag{6.96}
$$

This completes the proof of the proposition.

§7. Generalizations

Our goal is not to prove the most general theorems possible, but rather to elucidate the numerical phenomena involved. Nevertheless, we will mention here briefly some possibilities of generalizing the main results. The proofs of these statements are more or less straightforward, following the ideas presented above, although the actual details can be very tedious.

(1). There is no difficulty in generalizing Theorems 1-4 to three dimensional problems. Only obvious changes are required for the statement of the results and their proofs. This also marks an advantage of the projection method: In going from two to three dimensions, the formulation basically remains the same.

(2). More general spatial discretizations can be considered, such as the spectral method, finite element method, or more general finite difference method. However, one has to be careful in the projection step since it is in the mixed formulation:

$$\begin{cases} \boldsymbol{u}^{n+1} + \triangle t \nabla p^{n+1} = \boldsymbol{u}^* \\ \nabla \cdot \boldsymbol{u}^{n+1} = 0 \\ \dfrac{\partial p^{n+1}}{\partial \boldsymbol{n}} \,|_{\partial\Omega} = 0 \end{cases} \tag{7.1}$$

The basic stability criteria for mixed problems such as the inf-sup condition has to be satisfied. In other words, the null space of the discrete Laplacian for pressure may contain functions other than the constant functions. These so-called "parasitic modes" have to be subtracted to obtain the pressure approximation (see [1]).

(3). More interesting is the generalization to general domains. Obviously the stability and a priori estimates in §4.2 and §6.2 require no change. The changes required for the asymptotic analysis are described below.

Let $\boldsymbol{x} = \boldsymbol{R}(s) + \varepsilon\rho\boldsymbol{n}$, where $\boldsymbol{R}(s)$ is a point at $\partial\Omega$, s is the arclength of $\partial\Omega$ from a reference point to $\boldsymbol{R}(s)$, $\boldsymbol{n}$ is the inward normal of $\partial\Omega$ at $\boldsymbol{R}(s)$. We will use (s, ρ) as our coordinates for the boundary layer terms, and denote by $\boldsymbol{e}_s$ and $\boldsymbol{e}_\rho$ the unit coordinate vectors. This is a well-defined coordinate system near the boundary. It is an orthogonal system. The scaling factors h_1 and h_2 are given by

$$h_1 = (\frac{\partial \boldsymbol{x}}{\partial s} \cdot \frac{\partial \boldsymbol{x}}{\partial s})^{1/2} = 1 + \varepsilon\rho\kappa(s)\,, \quad h_2 = (\frac{\partial \boldsymbol{x}}{\partial \rho} \cdot \frac{\partial \boldsymbol{x}}{\partial \rho})^{1/2} = \varepsilon \tag{7.2}$$

where κ is the curvature of $\partial\Omega$ at $\boldsymbol{R}(s)$, positive for a convex curve. In this coordinate system, the differential operators take the following form:

$$\triangle u(s,\rho) = \frac{1}{\varepsilon(1+\varepsilon\rho\kappa)}[\frac{\partial}{\partial s}(\frac{\varepsilon}{1+\varepsilon\rho\kappa}\frac{\partial u}{\partial s}) + \frac{\partial}{\partial \rho}(\frac{1+\varepsilon\rho\kappa}{\varepsilon}\frac{\partial u}{\partial \rho})] \tag{7.3}$$

$$\nabla p = \frac{1}{1+\varepsilon\rho\kappa}\frac{\partial p}{\partial s}\boldsymbol{e}_s + \frac{1}{\varepsilon}\frac{\partial p}{\partial \rho}\boldsymbol{e}_\rho \tag{7.4}$$

$$\nabla \cdot (u\boldsymbol{e}_s + v\boldsymbol{e}_\rho) = \frac{1}{\varepsilon(1+\varepsilon\rho\kappa)}\{\varepsilon\frac{\partial u}{\partial s} + \frac{\partial}{\partial \rho}[(1+\varepsilon\rho\kappa)v]\} \tag{7.5}$$

Now we can repeat the analysis in §4.1 and §6.1 using these formulas. Here we will only outline the necessary changes for the first order scheme analyzed in §4. The interested reader can fill in the details for the other cases.

The ansatz remains the same as (4.2), with ξ replaced by ρ, y replaced by s in the boundary layer terms. We should keep in mind that (4.2) is only valid near the boundary and the vectors are decomposed using the basis $\{\boldsymbol{e}_s, \boldsymbol{e}_\rho\}$. In the interior of the domain, the numerical solution admits a regular perturbation expansion.

It is easy to see that $\boldsymbol{u}_j^*$, $\boldsymbol{u}_j$, p_j, $j = 0, 1, 2, \cdots$ still satisfy the same equations as in §4, whereas the equations for the boundary layer terms are changed as follows:

$$\boldsymbol{a}_1^* = 0 \tag{7.6}$$

$$\boldsymbol{a}_2^* = \frac{\partial^2 \boldsymbol{a}_2^*}{\partial \rho^2}\,, \quad \boldsymbol{a}_2^* = \frac{\partial \varphi_1}{\partial \rho}\boldsymbol{e}_\rho \tag{7.7}$$

$$\frac{\partial \varphi_1}{\partial \rho}\mid_{\rho=0} = -\frac{\partial p_0}{\partial \boldsymbol{n}}\mid_{\partial\Omega} \tag{7.8}$$

Therefore we have

$$\varphi_1(\rho, s) = \frac{\partial p_0}{\partial \boldsymbol{n}}\mid_{\partial\Omega} e^{-\rho}. \tag{7.9}$$

For $\boldsymbol{a}_3^*$ and φ_2 we have

$$(7.10) \qquad \boldsymbol{a}_3^* = \frac{\partial \varphi_2}{\partial \rho}\boldsymbol{e}_\rho + \frac{\partial \varphi_1}{\partial s}\boldsymbol{e}_s\,, \quad \boldsymbol{a}_3^* = \frac{\partial \partial^2 \boldsymbol{a}_3^*}{\partial \partial \rho^2} + \kappa \frac{\partial \boldsymbol{a}_2^*}{\partial \rho,}$$

$$(7.11) \qquad \frac{\partial \varphi_2}{\partial \rho}\,|_{\rho=0}= 0.$$

From (7.10) and (7.11), we get

$$(7.12) \qquad \varphi_2(\rho, s) = \frac{1}{2}u(s)\frac{\partial p_0}{\partial \boldsymbol{n}}\,|_{\partial\Omega}\,(1+\rho)e^{-\rho}.$$

We next have

$$(7.13) \qquad \boldsymbol{a}_4^* = \frac{\partial \varphi_3}{\partial \rho}\boldsymbol{e}_\rho + (\frac{\partial \varphi_2}{\partial s} - \rho\kappa\frac{\partial \varphi_1}{\partial s})\boldsymbol{e}_s$$

$$(7.14) \qquad \boldsymbol{a}_4^* = \frac{\partial^2 \boldsymbol{a}_4^*}{\partial \rho^2} + \kappa\frac{\partial \boldsymbol{a}_3^*}{\partial \rho} - \rho\kappa^2\frac{\partial \boldsymbol{a}_2^*}{\partial \rho} + \frac{\partial^2 \boldsymbol{a}_2^*}{\partial s^2}$$

$$(7.15) \qquad \frac{\partial \varphi_3}{\partial \rho}\,|_{\rho=0}= -\frac{\partial p_2}{\partial \boldsymbol{n}}\,|_{\partial\Omega}$$

In a priori, it is not clear whether (7.13) and (7.14) are consistent (which means that we might have to introduce boundary layer terms in $\boldsymbol{u}^n$). But if we write $\boldsymbol{a}_4^* = a_4^*\boldsymbol{e}_\rho + b_4^*\boldsymbol{e}_s$, and use the fact that $\frac{\partial \boldsymbol{e}_s}{\partial s} = \kappa\boldsymbol{e}_\rho$, $\frac{\partial \boldsymbol{e}_\rho}{\partial s} = -\kappa\boldsymbol{e}_s$, we see that (7.14) is equivalent to:

$$(7.16) \qquad a_4^* = \frac{\partial^2 a_4^*}{\partial \rho^2} + \kappa\frac{\partial^2 \varphi_2}{\partial \rho^2} - \rho\kappa^2\frac{\partial^2 \varphi_1}{\partial \rho^2} + \frac{\partial^3 \varphi_1}{\partial \rho \partial s^2} - \kappa^2\frac{\partial \varphi_1}{\partial \rho}$$

$$(7.17) \qquad b_4^* = \frac{\partial^2 b_4^*}{\partial \rho^2} + 2\kappa\frac{\partial^2 \varphi_1}{\partial \rho \partial s} + \frac{\partial}{\partial s}(\kappa\frac{\partial \varphi_1}{\partial \rho})$$

(7.16) serves as the equation for φ_3, together with the boundary condition (7.15). (7.17) is satisfied by $b_4^* = \frac{\partial \varphi_2}{\partial s} - \rho\kappa\frac{\partial \varphi_1}{\partial s}$. This procedure can obviously be continued to as high order as we wish.

In summary, we obtain the following extension of Theorem 1.

Theorem 5. *Let $(\boldsymbol{u}, p)$ be a smooth solution of the Navier-Stokes equation* (2.1) *with smooth initial data $\mathbf{u}^0(\boldsymbol{x})$ and let $(\boldsymbol{u}_{\triangle t}, p_{\triangle t})$ be the numerical solution for the semi-discrete projection method* (2.6)*,* (2.7) *and* (2.10)*. Then we have*

$$(7.18) \qquad \|\boldsymbol{u} - \boldsymbol{u}_{\triangle t}\|_{L^\infty(0,T;L^2)} + \triangle t^{1/2}\,\|p - p_{\triangle t}\|_{L^2(0,T;L^2)} \le C\triangle t\,,$$

Furthermore, if $\boldsymbol{u}^0(\boldsymbol{x})$ satisfies the compatibility condition

$$\boldsymbol{u}^0(\boldsymbol{x}) = 0\,, \quad \frac{\partial p}{\partial \boldsymbol{n}}(\boldsymbol{x}, 0) = 0 \quad \text{on } \partial\Omega. \tag{7.19}$$

then we have

$$\|\boldsymbol{u} - \boldsymbol{u}_{\triangle t}\|_{L^\infty} + \triangle t^{1/2}\, \|p - p_{\triangle t}\|_{L^\infty} + \|p - p_{\triangle t} - p_c\|_{L^\infty} \le C\triangle t\,. \tag{7.20}$$

where

$$p_c(\boldsymbol{x}, t) = \triangle t^{1/2} \frac{e}{e-1} e^{-\rho} \frac{\partial p_{\triangle t}}{\partial \boldsymbol{n}}(s, \rho + \triangle t^{1/2}, t).$$

The required change for the extension of Theorem 2 to general domains is more or less the same.

(4). One can also consider the generalization to other type of boundary conditions, including inhomogeneous ones. These generalizations are more or less standard. Of course if the boundary condition is too exotic, then the analysis becomes nearly impossible.

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Appendix 1. First Order Schemes with Spatial Discretization

We will concentrate on the following version of the first order projection method with the standard MAC spatial discretization:

$$\begin{cases} \dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\triangle t} + \mathcal{N}_h(\boldsymbol{u}^n, \boldsymbol{u}^n) = \Delta_h \boldsymbol{u}^*\,, \\ \boldsymbol{u}^* = 0\,, \qquad \text{on } \partial^{'}\Omega\,, \\ \boldsymbol{u}^* = \boldsymbol{u}^{n+1} + \triangle t \nabla_h p^n\,, \\ \nabla_h \cdot \boldsymbol{u}^{n+1} = 0\,, \\ \boldsymbol{n} \cdot \boldsymbol{u}^{n+1} = 0\,, \qquad \text{on } \partial^{'}\Omega \end{cases} \tag{8.1}$$

where notations $\mathcal{N}_h, \Delta_h, \nabla_h, \nabla_h\cdot$ is defined in (2.19-24) and the means of $\boldsymbol{u}^* \mid_{\partial'\Omega} = \boldsymbol{n}\boldsymbol{u}^{n+1} \mid_{\partial'\Omega}$ $= 0$ is also defined in §2.2.

For $\boldsymbol{a} = (a, b), \boldsymbol{c} = (c, d), \boldsymbol{u} = (u, v)$, we define the following discrete inner products on the grid:

(8.2)

$$((\boldsymbol{a}, \boldsymbol{c})) = \triangle x \triangle y \sum_{i=1}^{N-1} \sum_{j=1}^{N} a_{i+1/2,j} c_{i+1/2,j} + \triangle x \triangle y \sum_{i=1}^{N} \sum_{j=1}^{N} b_{i,j+1/2} d_{i,j+1/2}$$

$$((\boldsymbol{u}, \nabla_h p)) = \triangle y \sum_{i=1}^{N-1} \sum_{j=1}^{N} u_{i+1/2,j}(p_{i+1,j} - p_{i,j}) + \triangle x \sum_{i=1}^{N} \sum_{j=1}^{N} v_{i,j+1/2}(p_{i,j+1} - p_{i,j})$$

$$((\nabla_h \cdot \boldsymbol{u}, p)) = \triangle y \sum_{i=1}^{N-1} \sum_{j=1}^{N} (u_{i+1/2,j} - u_{i-1/2,j}) p_{i,j} + \triangle x \sum_{i=1}^{N} \sum_{j=1}^{N} (v_{i,j+1/2} - v_{i,j-1/2}) p_{i,j}$$

and discrete norms

$$\|\boldsymbol{u}\| = ((\boldsymbol{u}, \boldsymbol{u}))^{1/2}, \qquad \|\boldsymbol{u}\|_\infty = \max_{i,j} |\boldsymbol{u}_{i,j}| \tag{8.3}$$

Denote $h = \min(\triangle x, \triangle y)$.

Lemma 8.1 *We have the following*

(i) *Inverse inequality:*

$$\|f\|_\infty \le \frac{1}{h} \|f\|, \tag{8.4}$$

(ii) *Poincare inequality: suppose* $f \mid_{x=\pm 1} = 0$*, then*

$$\|f\| \le \|\nabla_h f\|, \tag{8.5}$$

(iii) *Suppose* $\boldsymbol{n} \cdot \boldsymbol{u} \mid_{x=\pm 1} = 0$ *,then we have*

$$((\boldsymbol{u}, \nabla_h p)) = ((\nabla_h \cdot \boldsymbol{u}, p)) \tag{8.6}$$

(iv) *Suppose* $\boldsymbol{u} \mid_{x=\pm 1} = 0$ *,then we have*

$$2((\boldsymbol{u}, \Delta_h \boldsymbol{u})) \le -\|\nabla_h \boldsymbol{u}\|^2 - \|\nabla_h \cdot \boldsymbol{u}\|^2 \tag{8.7}$$

(v) *Suppose* $\boldsymbol{a}\mid_{x=\pm1}=0$ *and* $\boldsymbol{c}\cdot\boldsymbol{n}\mid_{\boldsymbol{x}=\pm1}=0$*, then we have*

$$|(\!(\boldsymbol{a},\mathcal{N}_h(\boldsymbol{u},\boldsymbol{c}))\!)| \le \|\boldsymbol{c}\|\|\nabla_h\boldsymbol{a}\|\|\boldsymbol{u}\|_{W^{1,\infty}} \tag{8.8}$$

Proof : The proof of (i–iii) is standard. We first show (iv). Summation by parts gives

(8.9)
$$(\!(\boldsymbol{u},\Delta_h\boldsymbol{u})\!) = -\|\nabla\boldsymbol{u}\|^2+\sum_j[v_{0,j+1/2}(v_{1,j+1/2}-v_{0,j+1/2})-v_{N,j+1/2}(v_{N+1,j+1/2}-v_{N,j+1/2})]$$

Since $v\mid_{x=\pm1}=0$, we have

$$v_{1,j+1/2}=-v_{0,j+1/2}\,,\quad v_{N,j+1/2}=-v_{N+1,j+1/2} \tag{8.10}$$

Hence

$$(\!(\boldsymbol{u},\Delta_h\boldsymbol{u})\!) = -\|\nabla_h\boldsymbol{u}\|^2+2\sum_j(v_{0,j+1/2}^2-v_{N,j+1/2}^2) \tag{8.11}$$

But

$$\begin{aligned}\|\nabla_h\boldsymbol{u}\|^2 &\ge \|\nabla_h\cdot\boldsymbol{u}\|^2+\sum_j[(v_{1,j+1/2}-v_{0,j+1/2})^2+(v_{N+1,j+1/2}-v_{N,j+1/2})^2]\\ &= \|\nabla_h\cdot\boldsymbol{u}\|^2+4\sum_j[(v_{0,j+1/2})^2+(v_{N+1,j+1/2})^2]\end{aligned} \tag{8.12}$$

Combination of (8.11) and (8.12) gives (8.7).

To show (v), denote $I=(\!(\boldsymbol{a},\mathcal{N}_h(\boldsymbol{u},\boldsymbol{c}))\!)$. We have

$$\begin{aligned}I &= \triangle x\triangle y\sum_{i,j}a_{i+1/2,j}(u_{i+1/2,j}D_0^x c_{i+1/2,j}+\bar{v}_{i+1/2,j}D_0^y c_{i+1/2,j})\\ &\quad+\triangle x\triangle y\sum_{i,j}b_{i,j+1/2}(\bar{u}_{i,j+1/2}D_0^x d_{i,j+1/2}+v_{i,j+1/2}D_0^y d_{i+1/2,j})\end{aligned} \tag{8.13}$$

Summation by parts gives

$$
\begin{aligned}
I &= -\triangle x \triangle y \sum_{i,j} c_{i+1/2,j}[D_0^x(u_{i+1/2,j}a_{i+1/2,j}) + D_0^y(\bar{v}_{i+1/2,j}a_{i+1/2,j})] \\
&\quad -\triangle x \triangle y \sum_{i,j} d_{i,j+1/2}[D_0^x(\bar{u}_{i,j+1/2}b_{i,j+1/2}) + D_0^y(v_{i,j+1/2}b_{i,j+1/2})] \\
&\quad +\frac{1}{4}\triangle x \triangle y \sum_j (\bar{u}_{N+1,j+1/2}d_{N,j+1/2} - \bar{u}_{N,j+1/2}d_{N+1,j+1/2})D_-^x b_{N+1,j+1/2} \\
&\quad -\frac{1}{4}\triangle x \triangle y \sum_j (\bar{u}_{1,j+1/2}d_{0,j+1/2} - \bar{u}_{0,j+1/2}d_{1,j+1/2})D_+^x b_{0,j+1/2}
\end{aligned}
\tag{8.14}
$$

Here we have used the fact that

$$
b_{1,j+1/2} = -b_{0,j+1/2}\,, \quad b_{N,j+1/2} = -b_{N+1,j+1/2} \tag{8.15}
$$

Now, (8.8) follows directly. This completes the proof of the lemma.

Again we set $\varepsilon = \triangle t^{1/2}$, $\xi = (x+1)/\varepsilon$, $x_i = -1 + i\triangle x$, $\xi_i = i\triangle\xi$, $\triangle\xi = \triangle x/\varepsilon$, $t^n = n\triangle t$, $t^{n-1/2} = (n-1/2)\triangle t$, $n = 1, 2, \cdots$. Clearly, we have

$$
\begin{aligned}
D_+^\xi \boldsymbol{a}(\xi_i, y_j, t) &= \frac{\boldsymbol{a}(\xi_{i+1}, y_j, t) - \boldsymbol{a}(\xi_i, y_j, t)}{\triangle\xi} \\
&= \varepsilon \frac{\boldsymbol{a}(x_{i+1}/\varepsilon, y_j, t) - \boldsymbol{a}(x_i/\varepsilon, y_j, t)}{\triangle x} = \varepsilon D_+^x \boldsymbol{a}(x_i/\varepsilon, y_j, t)
\end{aligned}
\tag{8.16}
$$

This shows that $D_+^\xi = \varepsilon D_+^x$. We will use the notation

$$
D_\xi^2 = D_-^\xi D_+^\xi\,, \quad D_y^2 = D_-^y D_+^y\,, \tag{8.17}
$$

and

$$
\nabla_\xi = (D_+^\xi, 0)\,, \quad \nabla_y = (0, D_+^y)\,, \tag{8.18}
$$

Denote the solutions of (8.1) as $(\boldsymbol{u}_h, \boldsymbol{u}_h^*, p_h)$. Motivated by the discussions in §4, we make the following ansatz, valid at $t^n = n\triangle t$, $n = 1, 2, \cdots$

$$
\left\{
\begin{aligned}
&\boldsymbol{u}_h^*(\boldsymbol{x}, t) = \boldsymbol{u}_0(\boldsymbol{x}, t) + \sum_{j=2} \varepsilon^j [\boldsymbol{u}_j^*(\boldsymbol{x}, t) + \boldsymbol{a}_j^*(\xi, y, t)]\,,\\
&\boldsymbol{u}_h(\boldsymbol{x}, t) = \boldsymbol{u}_0(\boldsymbol{x}, t) + \sum_{j=2} \varepsilon^j \boldsymbol{u}_j(\boldsymbol{x}, t)\,,\\
&p_h(\boldsymbol{x}, t) = p_0(\boldsymbol{x}, t) + \sum_{j=1} \varepsilon^j [p_j(\boldsymbol{x}, t) + \varphi_j(\xi, y, t)]\,.
\end{aligned}
\right. \tag{8.19}
$$

Note that the functions involved are defined only on the numerical grid. So these formulas and the following ones should be understood as being valid on the grid. We have

$$
\Delta_h \boldsymbol{u}_h^* = \Delta_h \boldsymbol{u}_0 + \sum_{j=2} \varepsilon^j (\Delta_h \boldsymbol{u}_j^* + \varepsilon^{-2} D_\xi^2 \boldsymbol{a}_j^* + D_y^2 \boldsymbol{a}_j^*)\,, \tag{8.20}
$$

$$
\nabla_h \cdot \boldsymbol{u}_h = \nabla_h \cdot \boldsymbol{u}_0 + \sum_{j=2} \varepsilon^j \nabla_h \cdot \boldsymbol{u}_j\,, \tag{8.21}
$$

$$
\nabla_h p_h = \nabla_h p_0 + \varepsilon^{-1} \nabla_\xi \varphi_0 + \nabla_y \varphi_0 + \sum_{j=1} \varepsilon^j (\nabla_h p_j + \varepsilon^{-1} \nabla_\xi \varphi_j + \nabla_y \varphi_j)\,, \tag{8.22}
$$

$$
\begin{aligned}
\boldsymbol{u}_h^{n+1}(\boldsymbol{x}) \quad &= \boldsymbol{u}_0(\boldsymbol{x}, t^{n+1}) + \sum_{j=2} \varepsilon^j \boldsymbol{u}_j(\boldsymbol{x}, t^{n+1})\\
&= \sum_{k=0} \frac{1}{k!} \varepsilon^{2k} \boldsymbol{u}_0^{(k)}(\boldsymbol{x}, t^n) + \sum_{j=2} \varepsilon^j \sum_{k=0} \frac{1}{k!} \varepsilon^{2k} \boldsymbol{u}_j^{(k)}(\boldsymbol{x}, t^n)\,.
\end{aligned} \tag{8.23}
$$

Next we substitute these relations into (8.1) in order to determine the coefficients of ε^j in (8.19). We get hierarchies of equations by collecting equal powers of ε.

The first equation in (8.1) gives:

$$
\boldsymbol{u}_2^* + \boldsymbol{a}_2^* - \boldsymbol{u}_2 + \mathcal{N}_h(\boldsymbol{u}_0, \boldsymbol{u}_0) = \Delta_h^2 \boldsymbol{u}_0^* + D_\xi^2 \boldsymbol{a}_2^*\,. \tag{8.24}
$$

For $j \geq 1$,

$$
\boldsymbol{u}_{j+2}^* + \boldsymbol{a}_{j+2}^* - \boldsymbol{u}_{j+2} + \sum_{k=0}^{j} \mathcal{N}_h(\boldsymbol{u}_k, \boldsymbol{u}_{j-k}) = \Delta_h \boldsymbol{u}_j^* + D_\xi^2 \boldsymbol{a}_{j+2}^* + D_y^2 \boldsymbol{a}_j^*\,. \tag{8.25}
$$

The second equation in (8.1) implies

$$
\boldsymbol{u}_2^* + \boldsymbol{a}_2^* = \boldsymbol{u}_2 + \partial_t \boldsymbol{u}_0 + \nabla_h p_0 + \nabla_\xi \varphi_1 + \nabla_y \varphi_0\,. \tag{8.26}
$$

For $j = 2\ell - 1$, $\ell \geq 1$,

$$\boldsymbol{u}^*_{j+2} + \boldsymbol{a}^*_{j+2} = \boldsymbol{u}_{j+2} + \partial_t \boldsymbol{u}_j + \nabla_h p_j + \nabla_\xi \varphi_{j+1} + \nabla_y \varphi_j + \sum_{k=2}^{\ell} \frac{1}{k!} \boldsymbol{u}^{(k)}_{j-2k+2} \,. \tag{8.27}$$

For $j = 2\ell$, $\ell \geq 1$,

$$\boldsymbol{u}^*_{j+2} + \boldsymbol{a}^*_{j+2} = \boldsymbol{u}_{j+2} + \partial_t \boldsymbol{u}_j + \nabla_h p_j + \nabla_\xi \varphi_{j+1} + \nabla_y \varphi_j + \frac{1}{(\ell+1)!} \boldsymbol{u}_0^{(\ell+1)} + \sum_{k=2}^{\ell} \frac{1}{k!} \boldsymbol{u}^{(k)}_{j-2k+2} \,. \tag{8.28}$$

From the third equation in (8.1), we obtain

$$\nabla_h \cdot \boldsymbol{u}_j = 0 \,, \qquad j = 0, 1, \cdots \,. \tag{8.29}$$

The boundary conditions become

$$\boldsymbol{u}^*_j + \boldsymbol{a}^*_j = 0 \,, \quad D^x_+ p_{j-1} + D^\xi_+ \varphi_j = 0 \,, \quad \text{at} \quad x = -1, \quad \xi = 0 \,, \tag{8.30}$$

for $j > 0$.

Next we go through all these equations, order by order, to see if they are solvable. Since this is very similar to what we did in §4.1, we will only give a summary of results.

The coefficients in the expansions (8.19) can be obtained successively in the following order:

$$\begin{cases} \partial_t \boldsymbol{u}_0 + \nabla_h p_0 + \mathcal{N}_h(\boldsymbol{u}_0, \boldsymbol{u}_0) = \Delta_h \boldsymbol{u}_0 \,, \\ \nabla_h \cdot \boldsymbol{u}_0 = 0 \\ \boldsymbol{u}_0 = 0 \,, \qquad \text{at} \quad x = \pm 1 \,, \\ \boldsymbol{u}_0(\cdot, 0) = \boldsymbol{u}^0(\cdot) \end{cases} \tag{8.31}$$

Using the following lemma, we know that (8.31) has a smooth solution in the sense that the divided difference of various orders are bounded. The lemma itself, as well as Lemma 8.3, belongs to the folklore of classical numerical analysis.

Lemma 8.2. *Let* $(\boldsymbol{u}, p)$ *be a solution of the Navier-Stokes equation* (2.1) *with smooth initial data* $\boldsymbol{u}^0(\boldsymbol{x})$ *satisfying some compatibility conditions. Let* $(\boldsymbol{u}_0, p_0)$ *be a solution of* (8.31). *Then* $(\boldsymbol{u}_0, p_0)$ *is smooth in the sense that its discrete derivatives are bounded. Moreover, we have*

$$\|\boldsymbol{u} - \boldsymbol{u}_0\|_{L^\infty} + \|p - p_0\|_{L^\infty} \leq C h^2 \tag{8.32}$$

We next have

$$\boldsymbol{u}_2^* = \boldsymbol{u}_2 + \partial_t \boldsymbol{u}_0 + \nabla_h p_0 \,, \tag{8.33}$$

$$\begin{cases} \varphi_1 = D_\xi^2 \varphi_1 \,, \\ D_+^\xi \varphi_1 \mid_{\xi=0} = -D_+^x p_0 \mid_{x=-1} \,, \end{cases} \tag{8.34}$$

This gives

$$\varphi_1(\xi, y, t) = \beta D_+^x p_0(-1, y, t)\, e^{-\alpha \xi} \,. \tag{8.35}$$

where

$$\alpha = \frac{1}{\triangle \xi} \text{arccosh } (1 + \triangle \xi^2/2) \,, \qquad \beta = \triangle \xi (1 - e^{-\alpha \triangle \xi})^{-1} \,. \tag{8.36}$$

$$a_2^* = D_+^\xi \varphi_1, \quad b_2^* = 0 \,, \tag{8.37}$$

$$\boldsymbol{u}_3^* = \boldsymbol{u}_3 \,, \tag{8.38}$$

$$\varphi_2 = 0 \,, \quad a_3^* = 0 \,, \quad b_3^* = D_+^y \varphi_1 \,, \tag{8.39}$$

$$\begin{cases} \partial_t \boldsymbol{u}_2 + \nabla_h p_2 + \mathcal{N}_h(\boldsymbol{u}_0, \boldsymbol{u}_2) + \mathcal{N}_h(\boldsymbol{u}_2, \boldsymbol{u}_0) \\ \qquad\qquad = \Delta_h \boldsymbol{u}_2 + \Delta_h(\partial_t \boldsymbol{u}_0 + \nabla_h p_0) - \frac{1}{2} \partial_t^2 \boldsymbol{u}_0 \,, \\ \nabla_h \cdot \boldsymbol{u}_2 = 0 \,, \\ \boldsymbol{u}_2 \mid_{x=-1} = -\nabla_h p_0 \mid_{x=-1} - \nabla_\xi \varphi_1 \mid_{\xi=0} \,, \quad \text{on } \partial\Omega \end{cases} \tag{8.40}$$

With a suitable initial data, we know from the following lemma that (8.40) has a smooth solution.

Lemma 8.3. *Let $(\boldsymbol{u}, p)$ be a solution of the linear ODE*

$$\begin{cases} \partial_t \boldsymbol{u} + \nabla_h p + \mathcal{N}_h(\boldsymbol{u}_0, \boldsymbol{u}) + \mathcal{N}_h(\boldsymbol{u}, \boldsymbol{u}_0) = \Delta_h \boldsymbol{u} + \boldsymbol{f} \,, \\ \nabla_h \cdot \boldsymbol{u} = 0 \,, \\ \boldsymbol{u} = \boldsymbol{g} \,, \qquad \text{at} \quad x = \pm 1 \,, \\ \boldsymbol{u}(\cdot, 0) = \boldsymbol{u}^0(\cdot) \end{cases} \tag{8.41}$$

where $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{u}^0$ smooth and satisfies some compatibility conditions. Then $(\boldsymbol{u}, p)$ is smooth in the sense that its divided differences of various order are bounded.

Continue in this fashion, we get

$$\left\{\begin{array}{l} \varphi_3 = D_\xi^2 \varphi_3 + D_y^2 \varphi_1 \, , \\ D_+^\xi \varphi_3 \mid_{\xi=0} = -D_+^x p_2 \mid_{x=-1} \, , \end{array}\right. \tag{8.42}$$

The solution for (8.42) is

$$\varphi_3(\xi, y, t) = \beta D_+^x p_2 e^{-\alpha\xi} + \beta_1(\xi + \gamma) D_+^x D_y^2 \, p_0 \mid_{x=-1} e^{-\alpha\xi} \, . \tag{8.43}$$

where

$$\beta_1 = \frac{1}{(1 - e^{-\alpha\triangle\xi})(e^{-\alpha\triangle\xi} - e^{\alpha\triangle\xi})} \, , \qquad \gamma = \triangle\xi \frac{e^{-\alpha\triangle\xi}}{1 - e^{-\alpha\triangle\xi}} \tag{8.44}$$

$$b_4^* = 0 \, , \quad a_4^* = D_+^\xi \varphi_3 \, . \tag{8.45}$$

$$\left\{\begin{array}{l} \partial_t \boldsymbol{u}_3 + \nabla_h p_3 + \mathcal{N}_h(\boldsymbol{u}_0, \boldsymbol{u}_3) + \mathcal{N}_h(\boldsymbol{u}_3, \boldsymbol{u}_0) = \Delta_h \boldsymbol{u}_3 + \Delta_h \nabla_h p_2 \, , \\ \nabla_h \cdot \boldsymbol{u}_3 = 0 \, , \\ \boldsymbol{u}_3 \mid_{x=-1} = -\nabla_y \varphi_1 \mid_{\xi=0} \end{array}\right. \tag{8.46}$$

$$\left\{\begin{array}{l} \varphi_4 = D_\xi^2 \varphi_4 \, , \\ D_+^\xi \varphi_4 \mid_{\xi=0} = -D_+^x p_3 \mid_{x=-1} \, , \end{array}\right. \tag{8.47}$$

$$a_5^* = D_+^\xi \varphi_4, \quad b_5^* = D_+^y \varphi_3 \, . \tag{8.48}$$

Obviously this procedure can be continued and we obtain

$$\left\{\begin{array}{l} \varphi_j = D_\xi^2 \varphi_j + D_y^2 \varphi_{j-2} \, , \\ D_+^\xi \varphi_j \mid_{\xi=0} = -D_+^x p_{j-1} \mid_{x=-1} \, , \end{array}\right. \tag{8.49}$$

$$\varphi_j = \sum_{k=0}^{[j/2]} F_{j,k}(y) \xi^k e^{-\alpha\xi} \, , \tag{8.50}$$

$$(8.51) \qquad a_j^* = D_+^{\xi}\varphi_{j-1}\,, \quad b_j^* = D_+^{y}\varphi_{j-2}\,,$$

Now if we let

$$(8.52) \qquad \begin{cases} U^* = \boldsymbol{u}_0^* + \displaystyle\sum_{j=1}^{2N} \varepsilon^{\,j}(\boldsymbol{u}_j^* + \boldsymbol{a}_j^*)\,, \\ U^n = \boldsymbol{u}_0 + \displaystyle\sum_{j=1}^{2N} \varepsilon^{\,j}\boldsymbol{u}_j\,, \\ P^n = p_0 + \displaystyle\sum_{j=1}^{2N} \varepsilon^{\,j}(p_j + \varphi_j) + \varepsilon^{2N+1}\varphi_{2N+1}\,, \end{cases}$$

then we have

$$(8.53) \qquad \begin{cases} \dfrac{U^* - U^n}{\triangle t} + \mathcal{N}_h(U^n, U^n) = \Delta_h U^* + \triangle t^{\alpha}\boldsymbol{f}\,, \\ U^* = 0\,, \qquad \text{at} \quad x = \pm 1\,, \\ U^* = U^{n+1} + \triangle t \nabla_h P^n + \triangle t^{\alpha+1}\boldsymbol{g}\,, \\ \nabla_h \cdot U^{n+1} = 0\,, \\ D_+^x P^n = \boldsymbol{n}\cdot U^{n+1} = 0\,, \qquad \text{at} \quad x = \pm 1\,, \end{cases}$$

where $\alpha = N - 1/2$, f and $\boldsymbol{g}$ are bounded and smooth if $(\boldsymbol{u}_0, p_0)$ is sufficiently smooth. It is easy to see that

$$(8.54) \qquad \max_{0 \le t \le T} \|U^n(\cdot)\|_{W^{1,\infty}} \le C^*\,.$$

For the initial data, we have

$$(8.55) \qquad U^0(\boldsymbol{x}) = \boldsymbol{u}^0(\boldsymbol{x}) + \triangle t \boldsymbol{w}^0(\boldsymbol{x})$$

where $\boldsymbol{w}^0$ is a bounded function. Furthermore under the compatibility condition (3.10), we can construct a better approximate initial data

$$(8.56) \qquad U^0(\boldsymbol{x}) = \boldsymbol{u}^0(\boldsymbol{x}) + \triangle t^{\,2}\boldsymbol{w}^0(\boldsymbol{x}).$$

Proof of Theorem 3: Assume a priori that

$$(8.57) \qquad \max_{0 \le t^n \le T} \|\boldsymbol{u}^n\|_{W^{1,\infty}} \le \tilde{C}\,.$$

In the following estimates, the constant will sometimes depend on C^* and $\tilde{C}$. Later on we will estimate $\tilde{C}$. Let

$$\boldsymbol{e}^n = U^n - \boldsymbol{u}^n\,, \qquad \boldsymbol{e}^* = U^* - \boldsymbol{u}^*\,, \qquad q^n = P^n - p^n\,. \tag{8.58}$$

Subtracting (8.53) from (8.1) we get the following error equation

$$\begin{cases} \dfrac{\boldsymbol{e}^* - \boldsymbol{e}^n}{\triangle t} + \mathcal{N}_h(\boldsymbol{e}^n, U^n) + \mathcal{N}_h(\boldsymbol{u}^n, \boldsymbol{e}^n) = \Delta_h \boldsymbol{e}^* + \triangle t^\alpha \boldsymbol{f}^n\,, \\ \boldsymbol{e}^* = 0\,, \qquad \text{at} \quad x = \pm 1\,, \\ \dfrac{\boldsymbol{e}^{n+1} - \boldsymbol{e}^*}{\triangle t} + \nabla_h q^n = \triangle t^\alpha \boldsymbol{g}^n\,, \\ \nabla_h \cdot \boldsymbol{e}^{n+1} = 0\,, \\ D_+^x q^n = \boldsymbol{e}^{n+1} \cdot \boldsymbol{n} = 0\,, \qquad \text{at} \quad x = \pm 1\,, \\ \boldsymbol{e}^0 = \triangle t^\alpha\ \boldsymbol{w}^0\,. \end{cases} \tag{8.59}$$

Taking the scalar product of the first equation of (8.59) with $2\boldsymbol{e}^*$ and integrating by parts, we obtain

$$\begin{aligned} &\|\boldsymbol{e}^*\|^2 - \|\boldsymbol{e}^n\|^2 + \|\boldsymbol{e}^* - \boldsymbol{e}^n\|^2 + \triangle t\,\|\nabla_h \boldsymbol{e}^*\|^2 \\ &\quad \le \triangle t^{2\alpha+1}\|\boldsymbol{f}^n\|^2 + \triangle t\ \|\boldsymbol{e}^*\|^2 - 2\triangle t\ (\!(\boldsymbol{e}^*, \mathcal{N}_h(\boldsymbol{e}^n, U^n))\!) \\ &\qquad -2\triangle t\ (\!(\boldsymbol{e}^*, \mathcal{N}_h(\boldsymbol{u}^n, \boldsymbol{e}^n))\!) \\ &\quad \le \triangle t^{2\alpha+1}\|\boldsymbol{f}^n\|^2 + C\triangle t(\|\boldsymbol{e}^*\|^2 + \|\boldsymbol{e}^n\|^2) + \tfrac{1}{2}\triangle t\ \|\nabla_h \boldsymbol{e}^*\|^2\,. \end{aligned} \tag{8.60}$$

Here we have used Lemma 8.1. Taking the scalar product of the second equation of (8.59) with $2\boldsymbol{e}^{n+1}$ yields

$$\|\boldsymbol{e}^{n+1}\|^2 - \|\boldsymbol{e}^*\|^2 + \|\boldsymbol{e}^{n+1} - \boldsymbol{e}^*\|^2 \le \triangle t\ \|\boldsymbol{e}^{n+1}\|^2 + \triangle t^{2\alpha+1}\|\boldsymbol{g}^n\|^2\,. \tag{8.61}$$

Combining (8.60) and (8.61), we get

$$\begin{aligned} &\|\boldsymbol{e}^{n+1}\|^2 - \|\boldsymbol{e}^n\|^2 + \|\boldsymbol{e}^* - \boldsymbol{e}^n\|^2 + \|\boldsymbol{e}^{n+1} - \boldsymbol{e}^*\|^2 + \triangle t\ \|\nabla_h \boldsymbol{e}^*\|^2 \\ &\quad \le C\,\triangle t\ (\|\boldsymbol{e}^{n+1}\|^2 + \|\boldsymbol{e}^n\|^2) + \triangle t^{2\alpha+1}(\|\boldsymbol{f}^n\|^2 + \|\boldsymbol{g}^n\|^2)\,. \end{aligned} \tag{8.62}$$

Applying the discrete Gronwall lemma to the last inequality, we arrive at

$$\|\boldsymbol{e}^n\| + \|\boldsymbol{e}^* - \boldsymbol{e}^n\| + \|\boldsymbol{e}^{n+1} - \boldsymbol{e}^*\| + \triangle t^{1/2}\ \|\nabla_h \boldsymbol{e}^*\| \le C_1 \triangle t^{\alpha}\,. \tag{8.63}$$

Using the second equation of (8.59) we have

$$\|e^n\| + \triangle t \|\nabla_h q^n\| \le C_1 \triangle t^{\alpha} \tag{8.64}$$

Now by inverse inequality (8.4) we have

$$\|\boldsymbol{e}^n\|_{L^\infty} + \triangle t\|\nabla_h q^n\|_{L^\infty} + h\|\boldsymbol{e}^n\|_{W^{1,\infty}} \le C_1 \frac{\triangle t^{\alpha}}{h}\,. \tag{8.65}$$

Chose $N = 3$ and $\triangle t^{\alpha} << h^2$, if we choose $\triangle t$ small enough, we will always have

$$\|\boldsymbol{e}^{n+1}\|_{W^{1,\infty}} \le 1\,. \tag{8.66}$$

Therefore in (8.57) we can choose

$$\tilde{C} = 1 + \max_{n \le \left[\frac{T}{\triangle t}\right]+1} \|U^n(\cdot)\|_{W^{1,\infty}} \tag{8.67}$$

which depends only on the exact solution $(\boldsymbol{u}, p)$. This proves

$$\|\boldsymbol{u}_0 - \boldsymbol{u}_h\|_{L^\infty} + \|p_0 - p_h\|_{L^2} + \triangle t^{1/2}\|p_0 - p_h\|_{L^\infty} + \|p_0 - p_h - p_c\|_{L^\infty} \le C\triangle t \tag{8.68}$$

But we also have from Lemma 8.2

$$\|\boldsymbol{u} - \boldsymbol{u}_0\|_{L^\infty} + \|p - p_0\|_{L^\infty} \le Ch^2 \tag{8.69}$$

Thus

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^\infty} + \|p - p_h\|_{L^2} + \triangle t^{1/2}\|p - p_h\|_{L^\infty} + \|p - p_h - p_c\|_{L^\infty} \le C(\triangle t + h^2) \tag{8.70}$$

This completes the proof of Theorem 3.

Appendix 2. Second Order Schemes with Spatial Discretization

In this section we carry out the same program as in §4 for Kim and Moin's method, (2.12) with the standard MAC spatial discretization:

$$\tag{9.1} \begin{cases} \dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\triangle t} = \Delta_h \dfrac{\boldsymbol{u}^* + \boldsymbol{u}^n}{2}\,, \\ \boldsymbol{u}^* + \boldsymbol{u}^n = \triangle t \nabla_h p^{n-1/2}\,, \qquad \text{at} \quad x = \pm 1\,, \\ \boldsymbol{u}^* = \boldsymbol{u}^{n+1} + \triangle t \nabla_h p^{n+1/2}\,, \\ \nabla_h \cdot \boldsymbol{u}^{n+1} = 0\,, \\ \boldsymbol{n} \cdot \boldsymbol{u}^{n+1} = 0\,, \qquad \text{at} \quad x = \pm 1\,. \end{cases}$$

Here we leave out the nonlinear term since it does not affect the major steps but complicates substantially the presentation.

We begin with the following ansatz:

$$\tag{9.2} \begin{cases} \boldsymbol{u}^*(\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}, t^n) + \sum\limits_{j=2} \varepsilon^j [\boldsymbol{u}_j^*(\boldsymbol{x}, t^n) + \boldsymbol{a}_j^*(\xi, y, t^n)]\,, \\ \boldsymbol{u}^n(\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}, t^n) + \sum\limits_{j=4} \varepsilon^j \boldsymbol{u}_j(\boldsymbol{x}, t^n)\,, \\ p^{n-1/2}(\boldsymbol{x}) = p_0(\boldsymbol{x}, t^{n-1/2}) + \varepsilon \varphi_1(\xi, y, t^{n-1/2}) + \varepsilon^3 \varphi_3(\xi, y, t^{n-1/2}) \\ \qquad + \sum\limits_{j=4} \varepsilon^j [p_j(\boldsymbol{x}, t^{n-1/2}) + \varphi_j(\xi, y, t^{n-1/2})]\,. \end{cases}$$

Here again we set $\varepsilon = \triangle t^{1/2}$, $\xi = (x+1)/\varepsilon$, $t^n = n\triangle t$, $t^{n-1/2} = (n-1/2)\triangle t$, $n = 1, 2, \cdots$. The formulas are to be understood as being valid at the grid points. Substituting (9.2) into (9.1) and collecting equal powers of ε, we get the following equations:

From the first equation in (9.1), we get

$$\tag{9.3} \boldsymbol{u}_2^* + \boldsymbol{a}_2^* - \boldsymbol{u}_2 = \frac{1}{2}(\Delta_h \boldsymbol{u}_0^* + D_\xi^2 \boldsymbol{a}_2^* + \Delta_h \boldsymbol{u}_0)\,.$$

For $j \geq 1$,

$$\tag{9.4} \boldsymbol{u}_{j+2}^* + \boldsymbol{a}_{j+2}^* - \boldsymbol{u}_{j+2} = \frac{1}{2}(\Delta_h \boldsymbol{u}_j^* + D_\xi^2 \boldsymbol{a}_{j+2}^* + D_y^2 \boldsymbol{a}_j^* + \Delta_h \boldsymbol{u}_j)\,.$$

From the third equation in (9.1), we get

$$\boldsymbol{u}_2^* + \boldsymbol{a}_2^* = \boldsymbol{u}_2 + \partial_t \boldsymbol{u}_0 + \nabla_h p_0 + \nabla_\xi \varphi_1 \,, \tag{9.5}$$

$$\boldsymbol{u}_3^* + \boldsymbol{a}_3^* = \boldsymbol{u}_3 + \partial_t \boldsymbol{u}_1 + \nabla_h p_1 + \nabla_\xi \varphi_2 + \nabla_y \varphi_1 \,. \tag{9.6}$$

For $j = 2\ell$,

$$\begin{aligned} \boldsymbol{u}_{j+2}^* + \boldsymbol{a}_{j+2}^* &= \boldsymbol{u}_{j+2} + \partial_t \boldsymbol{u}_j + \nabla_h p_j + \nabla_\xi \varphi_{j+1} + \nabla_y \varphi_j \\ &\quad + \frac{1}{(\ell+1)!}\boldsymbol{u}_0^{(\ell+1)} + \sum_{k=2}^{\ell} \frac{1}{k!}\boldsymbol{u}_{j-2k+2}^{(k)} + \frac{1}{2^\ell \ell!}\nabla_h p_0^{(\ell)} + \sum_{k=1}^{\ell-1} \frac{1}{2^k k!}\nabla_h p_{j-2k}^{(k)} \\ &\quad + \sum_{k=1}^{\ell} \frac{1}{2^k k!}(\nabla_\xi \varphi_{j-2k+1}^{(k)} + \nabla_y \varphi_{j-2k}^{(k)}) \,. \end{aligned} \tag{9.7}$$

For $j = 2\ell + 1$,

$$\begin{aligned} \boldsymbol{u}_{j+2}^* + \boldsymbol{a}_{j+2}^* &= \boldsymbol{u}_{j+2} + \partial_t \boldsymbol{u}_j + \nabla_h p_j + \nabla_\xi \varphi_{j+1} + \nabla_y \varphi_j \\ &\quad + \sum_{k=2}^{\ell+1} \frac{1}{k!}\boldsymbol{u}_{j-2k+2}^{(k)} + \sum_{k=1}^{\ell} \frac{1}{2^k k!}(\nabla_h p_{j-2k}^{(k)} + \nabla_\xi \varphi_{j-2k+1}^{(k)} + \nabla_y \varphi_{j-2k}^{(k)}) \,. \end{aligned} \tag{9.8}$$

From the incompressibility condition, we get

$$\nabla_h \cdot \boldsymbol{u}_j = 0 \,, \quad \text{for} \quad j \geq 0 \,. \tag{9.9}$$

The boundary conditions imply that for $x = -1$, $\xi = 0$,

$$\boldsymbol{u}_0 = 0 \,, \tag{9.10}$$

$$\boldsymbol{u}_2 + \boldsymbol{u}_2^* + \boldsymbol{a}_2^* = \nabla_h p_0 + \nabla_\xi \varphi_1 \,, \tag{9.11}$$

$$\boldsymbol{u}_3 + \boldsymbol{u}_3^* + \boldsymbol{a}_3^* = \nabla_h p_1 + \nabla_\xi \varphi_2 + \nabla_y \varphi_1 \,; \tag{9.12}$$

for $j = 2\ell$, $\ell \geq 1$

$$\begin{aligned} \boldsymbol{u}_j + \boldsymbol{u}_j^* + \boldsymbol{a}_j^* \quad &= \nabla_h p_{j-2} + \nabla_\xi \varphi_{j-1} + \nabla_y \varphi_{j-2} + \frac{(-1)^{\ell-1}}{2^{\ell-1}} \frac{1}{(\ell-1)!} \nabla_h p_0^{(\ell-1)} \\ &\quad + \sum_{k=1}^{\ell-2} \frac{(-1)^k}{2^k k!} \nabla_h p_{j-2k-2}^{(k)} + \sum_{k=1}^{\ell-1} \frac{(-1)^k}{2^k k!} (\nabla_\xi \varphi_{j-2k-1}^{(k)} + \nabla_y \varphi_{j-2k-2}^{(k)}) \end{aligned} \tag{9.13}$$

for $j = 2\ell + 1$, $\ell \geq 1$

$$\begin{aligned} \boldsymbol{u}_j + \boldsymbol{u}_j^* + \boldsymbol{a}_j^* \quad &= \nabla_h p_{j-2} + \nabla_\xi \varphi_{j-1} + \nabla_y \varphi_{j-2} \\ &\quad + \sum_{k=1}^{\ell-1} \frac{(-1)^k}{2^k k!} (\nabla_h p_{j-2k-2}^{(k)} + \nabla_\xi \varphi_{j-2k-1}^{(k)} + \nabla_y \varphi_{j-2k-2}^{(k)}); \end{aligned} \tag{9.14}$$

and for $j \geq 0$

$$D_+^x p_j + D_+^\xi \varphi_{j+1} = 0\,. \tag{9.15}$$

Next we go through all these equations, order by order, to see if they are solvable. It can be checked that the coefficients in the expansions (9.2) can be obtained successively in the following order:

$$\begin{cases} \partial_t \boldsymbol{u}_0 + \nabla_h p_0 = \Delta_h \boldsymbol{u}_0\,, \\ \nabla_h \cdot \boldsymbol{u}_0 = 0\,, \\ \boldsymbol{u}_0 = 0\,, \qquad \text{at} \quad x = \pm 1\,. \end{cases} \tag{9.16}$$

$$\boldsymbol{u}_2^* = \boldsymbol{u}_2 + \partial_t \boldsymbol{u}_0 + \nabla_h p_0\,, \tag{9.17}$$

$$\begin{cases} \varphi_1 = \frac{1}{2} D_\xi^2 \varphi_1\,, \\ D_+^\xi \varphi_1 \,|_{\xi=0} = -D_+^x p_0 \,|_{x=-1}\,, \end{cases} \tag{9.18}$$

$$\varphi_1 = \beta D_+^x p_0 \,|_{x=-1}\, e^{-\alpha \xi}\,, \tag{9.19}$$

where

$$\alpha = \frac{1}{\triangle \xi} \text{arccosh}\, (1 + \triangle \xi^2)\,, \qquad \beta = \triangle \xi (1 - e^{-\alpha \triangle \xi})^{-1}\,. \tag{9.20}$$

$$a_2^* = D_+^\xi \varphi_1, \quad b_2^* = 0\,, \tag{9.21}$$

We next have:

$$\boldsymbol{u}_3^* = \boldsymbol{u}_3\,, \tag{9.22}$$

$$\varphi_2 = 0\,, \quad a_3^* = 0\,, \quad b_3^* = D_+^y \varphi_1\,, \tag{9.23}$$

$$\begin{cases} \varphi_3 = \frac{1}{2}(D_\xi^2 \varphi_3 + D_y^2 \varphi_1)\,, \\ D_+^\xi \varphi_3 \mid_{\xi=0} = 0\,, \end{cases} \tag{9.24}$$

The solution for (9.24) is

$$\varphi_3(y,\xi,t) = \beta_1(\xi+\gamma) D_+^x D_y^2\, p_0 \mid_{x=-1} e^{-\alpha\xi}\,. \tag{9.25}$$

where

$$\beta_1 = \frac{1}{2(1-e^{-\alpha\triangle\xi})(e^{-\alpha\triangle\xi}-e^{\alpha\triangle\xi})}\,, \qquad \gamma = \triangle\xi \frac{e^{-\alpha\triangle\xi}}{1-e^{-\alpha\triangle\xi}} \tag{9.26}$$

$$a_4^* = \frac{1}{2} D_+^\xi \partial_t \varphi_1 + D_+^\xi \varphi_3\,, \quad b_4^* = 0\,, \tag{9.27}$$

$$\boldsymbol{u}_4^* = \boldsymbol{u}_4 + \frac{1}{2}\partial_t^2 \boldsymbol{u}_0 + \frac{1}{2}\partial_t \nabla_h p_0\,, \tag{9.28}$$

$$\varphi_4 = 0\,, \quad a_5^* = 0\,, \quad b_5^* = \frac{1}{2} D_+^y \partial_t \varphi_1 + D_+^y \varphi_3\,, \tag{9.29}$$

$$\boldsymbol{u}_5^* = \boldsymbol{u}_5\,, \tag{9.30}$$

$$\begin{cases} \partial_t \boldsymbol{u}_4 + \nabla_h p_4 = \Delta_h \boldsymbol{u}_4 + \frac{1}{4}\Delta_h(\partial_t^2 \boldsymbol{u}_0 + \partial_t \nabla_h p_0) - \frac{1}{6}\partial_t^3 \boldsymbol{u}_0 - \frac{1}{8}\partial_t^2 \nabla_h p_0\,, \\ \nabla_h \cdot \boldsymbol{u}_4 = 0\,, \\ \boldsymbol{u}_4 \mid_{x=-1} = -\frac{1}{2}(\partial_t \nabla_h p_0 + \frac{1}{2}\partial_t \nabla_\xi \varphi_1) \mid_{x=-1,\xi=0}\,. \end{cases} \tag{9.31}$$

Now if we let

$$\begin{cases} U^* = \boldsymbol{u}_0^* + \sum_{j=1}^{2N} \varepsilon^j (\boldsymbol{u}_j^* + \boldsymbol{a}_j^*)\,, \\ U^n = \boldsymbol{u}_0 + \sum_{j=1}^{2N} \varepsilon^j \boldsymbol{u}_j\,, \\ P^{n-1/2} = p_0 + \sum_{j=1}^{2N} \varepsilon^j (p_j + \varphi_j) + \varepsilon^{2N+1}\varphi_{2N+1}\,, \end{cases} \tag{9.32}$$

then we have

$$
(9.33) \qquad \begin{cases} \dfrac{U^* - U^n}{\triangle t} = \Delta_h \dfrac{U^* + U^n}{2} + \triangle t^{\alpha} \boldsymbol{f} \,, \\ U^* + U^n = \triangle t \nabla_h P^{n-1/2} \,, \qquad \text{at} \quad x = \pm 1 \,, \\ U^* = U^{n+1} + \triangle t \nabla_h P^{n+1/2} + \triangle t^{\alpha+1} \boldsymbol{g} \,, \\ \nabla_h \cdot U^{n+1} = 0 \,, \\ D_+^x P^{n+1/2} = \boldsymbol{n} \cdot U^{n+1} = 0 \,, \qquad \text{at} \quad x = \pm 1 \,, \end{cases}
$$

where $\alpha = N - 1/2$, $\boldsymbol{f}$ and $\boldsymbol{g}$ are bounded and smooth if $(\boldsymbol{u}_0, p_0)$ is sufficiently smooth. It is easy to see that

$$
(9.34) \qquad \max_{0 \le t \le T} \| U^n(\cdot) \|_{W^{1,\infty}} \le C^* \,,
$$

For the initial approximation, we have

$$
(9.35) \qquad U^0(\boldsymbol{x}) = \boldsymbol{u}^0(\boldsymbol{x}) + \triangle t^{\,2} \boldsymbol{w}^0(\boldsymbol{x})
$$

without the extra compatibility condition, and

$$
(9.36) \qquad U^0(\boldsymbol{x}) = \boldsymbol{u}^0(\boldsymbol{x}) + \triangle t^{\,4} \boldsymbol{w}^0(\boldsymbol{x})
$$

with the compatibility condition (3.14).

Proof of Theorem 4. Assume a priori that

$$
(9.37) \qquad \max_{0 \le t^n \le T} \| \boldsymbol{u}^n \|_{W^{1,\infty}} \le \tilde{C} \,.
$$

As in the proof of Theorem 2, we let

$$
(9.38) \qquad \boldsymbol{e}^n = U^n - \boldsymbol{u}^n \,, \qquad \boldsymbol{e}^* = \hat{U}^* - \hat{\boldsymbol{u}}^* \,, \qquad q^n = P^{n-1/2} - p^{n-1/2} \,.
$$

where

$$
(9.39) \qquad \begin{aligned} 2\hat{\boldsymbol{u}}^* &= \boldsymbol{u}^* + \boldsymbol{u}^n - \triangle t \ \nabla_h p^{n-1/2} \,, \\ 2\hat{U}^* &= U^* + U^n - \triangle t \ \nabla_h P^{n-1/2} \,. \end{aligned}
$$

From (9.1) and (9.33), we get

$$
(9.40)\qquad
\begin{cases}
\dfrac{2(\boldsymbol{e}^{*}-\boldsymbol{e}^{n})}{\triangle t}+\nabla_h\left(q^n-\dfrac{1}{2}\triangle t\ \Delta_h q^n\right)=\Delta_h\boldsymbol{e}^{*}+\dfrac{1}{2}\mathcal{N}_h(\boldsymbol{e}^{n-1},U^{n-1}) \\
\qquad\qquad +\dfrac{1}{2}\mathcal{N}_h(\boldsymbol{u}^{n-1},\boldsymbol{e}^{n-1})-\dfrac{3}{2}\mathcal{N}_h(\boldsymbol{e}^{n},U^{n})-\dfrac{3}{2}\mathcal{N}_h(\boldsymbol{u}^{n},\boldsymbol{e}^{n})+\triangle t^{\alpha}\boldsymbol{f}^{n}\,, \\
\boldsymbol{e}^{*}=0\,,\qquad \text{at}\quad x=\pm 1\,, \\
\dfrac{\boldsymbol{e}^{n+1}+\boldsymbol{e}^{n}-2\boldsymbol{e}^{*}}{\triangle t}+\nabla_h(q^{n+1}-q^{n})=\triangle t^{\alpha}\boldsymbol{g}^{n}\,, \\
\nabla_h\cdot\boldsymbol{e}^{n+1}=0\,, \\
D_{+}^{x}q^{n+1/2}=\boldsymbol{e}^{n+1}\cdot\boldsymbol{n}=0\,,\qquad \text{at}\quad x=\pm 1\,, \\
\boldsymbol{e}^{0}=\triangle t^{\alpha}\boldsymbol{w}^{0}\,.
\end{cases}
$$

Taking the scalar product of the first equation of (9.40) with $\boldsymbol{e}^*$ and integrating by parts, we get

$$
(9.41)\qquad
\begin{aligned}
&\|\boldsymbol{e}^{*}\|^{2}-\|\boldsymbol{e}^{n}\|^{2}+\|\boldsymbol{e}^{*}-\boldsymbol{e}^{n}\|^{2}+\frac{1}{2}\triangle t\ \|\nabla\boldsymbol{e}^{*}\|^{2}+\frac{1}{2}\triangle t\ \|\nabla\cdot\boldsymbol{e}^{*}\|^{2} \\
&\quad\le -\triangle t\int_{\Omega}\boldsymbol{e}^{*}\cdot\nabla(q^{n}-\frac{1}{2}\triangle t\Delta q^{n})\,d\boldsymbol{x}+C\,\triangle t^{2\alpha+1}\|\boldsymbol{f}^{n}\|^{2} \\
&\quad +C\,\triangle t\ (\|\boldsymbol{e}^{n}\|^{2}+\|\boldsymbol{e}^{n-1}\|^{2}+\|\boldsymbol{e}^{*}\|^{2})+\tfrac{1}{2}\triangle t\ \|\nabla\boldsymbol{e}^{*}\|^{2}\,.
\end{aligned}
$$

Taking the scalar product of the second equation of (9.40) with $\boldsymbol{e}^{n+1}$, we obtain

$$
(9.42)\qquad
\begin{aligned}
&\|\boldsymbol{e}^{n+1}\|^{2}-\|\boldsymbol{e}^{*}\|^{2}+\|\boldsymbol{e}^{n+1}-\boldsymbol{e}^{*}\|^{2}-\tfrac{1}{2}(\|\boldsymbol{e}^{n+1}\|^{2}-\|\boldsymbol{e}^{n}\|^{2})-\tfrac{1}{2}\|\boldsymbol{e}^{n+1}-\boldsymbol{e}^{n}\|^{2} \\
&\quad\le C\,\triangle t^{2\alpha+1}\|\boldsymbol{g}^{n}\|^{2}+C\,\triangle t\ \|\boldsymbol{e}^{n+1}\|^{2}\,.
\end{aligned}
$$

Combining the these two estimates, we get

$$
(9.43)\qquad
\begin{aligned}
&\|\boldsymbol{e}^{n+1}\|^{2}-\|\boldsymbol{e}^{n}\|^{2}+\|\boldsymbol{e}^{n+1}+\boldsymbol{e}^{n}-2\boldsymbol{e}^{*}\|^{2}+\triangle t\ \|\nabla_h\boldsymbol{e}^{*}\|^{2}+\triangle t\ \|\nabla_h\cdot\boldsymbol{e}^{*}\|^{2} \\
&\quad\le -2\triangle t\ (\!(\boldsymbol{e}^{*},\nabla_h(q^{n}-\frac{1}{2}\triangle t\Delta_h q^{n}))\!)+C\,\triangle t\ (\|\boldsymbol{e}^{n}\|^{2}+\|\boldsymbol{e}^{n-1}\|^{2}+\|\boldsymbol{e}^{n+1}\|^{2}) \\
&\quad +C\,\triangle t^{2\alpha+1}(\|\boldsymbol{f}^{n}\|^{2}+\|\boldsymbol{g}^{n}\|^{2})\,.
\end{aligned}
$$

To estimate the first term on the right hand of (9.43), we let

$$
(9.44)\qquad
\begin{aligned}
I\ &\equiv -2\triangle t(\!(\boldsymbol{e}^{*},\nabla_h(q^{n}-\frac{1}{2}\triangle t\ \Delta_h q^{n}))\!) \\
&= -2\triangle t(\!(\boldsymbol{e}^{*},\nabla_h q^{n})\!)-\triangle t^{2}(\!(\nabla_h\cdot\boldsymbol{e}^{*},\Delta_h q^{n})\!)\equiv I_1+I_2\,.
\end{aligned}
$$

Using the second equation and integrating by parts, we can write the first term as

$$
\begin{aligned}
I_1 &= -2\triangle t(\!(\boldsymbol{e}^*, \nabla_h q^n)\!) \\
&= -\triangle t^2(\!(\nabla_h(q^{n+1}-q^n), \nabla_h q^n)\!) - \triangle t^{\alpha+2}(\!(\boldsymbol{g}^n, \nabla_h q^n)\!) \\
&= -\frac{1}{2}\triangle t^2(\|\nabla_h q^{n+1}\|^2 - \|\nabla_h q^n\|^2) \\
&\quad +\frac{1}{2}\triangle t^2\|\nabla_h(q^{n+1}-q^n)\|^2 - \triangle t^{\alpha+2}(\!(\boldsymbol{g}^n, \nabla_h q^n)\!)\,.
\end{aligned} \tag{9.45}
$$

Since

$$
\begin{aligned}
\frac{1}{2}\triangle t^2\|\nabla_h(q^{n+1}-q^n)\|^2 &= \frac{1}{2}\|\boldsymbol{e}^{n+1}+\boldsymbol{e}^n-2\boldsymbol{e}^*\|^2 \\
&\quad +\frac{1}{2}\|\triangle t^{\alpha+1}\boldsymbol{g}^n\|^2 + \triangle t^{\alpha+1}(\!(\boldsymbol{g}^n, \boldsymbol{e}^{n+1}+\boldsymbol{e}^n-2\boldsymbol{e}^*)\!)\,.
\end{aligned} \tag{9.46}
$$

We have

$$
\begin{aligned}
I_1 &= -\tfrac{1}{2}\triangle t^2(\|\nabla_h q^{n+1}\|^2 - \|\nabla_h q^n\|^2) + \tfrac{1}{2}\|\boldsymbol{e}^{n+1}+\boldsymbol{e}^n-2\boldsymbol{e}^*\|^2 \\
&\quad +\frac{1}{2}\|\triangle t^{\alpha+1}\boldsymbol{g}^n\|^2 + \triangle t^{\alpha+1}(\!(\boldsymbol{g}^n, \boldsymbol{e}^{n+1}+\boldsymbol{e}^n-2\boldsymbol{e}^*)\!) - \triangle t^{\alpha+2}(\!(\boldsymbol{g}^n, \nabla_h q^n)\!)\,.
\end{aligned} \tag{9.47}
$$

Next we rewrite the second term as

$$
\begin{aligned}
I_2 &= -\triangle t^2(\!(\nabla_h\cdot\boldsymbol{e}^*, \Delta_h q^n)\!) \\
&= -\frac{1}{2}\triangle t^3(\!(\Delta_h(q^{n+1}-q^n), \Delta_h q^n)\!) - \frac{1}{2}\triangle t^{\alpha+3}(\!(\nabla_h\cdot\boldsymbol{g}^n, \Delta_h q^n)\!) \\
&= -\frac{1}{4}\triangle t^3(\|\Delta_h q^{n+1}\|^2 - \|\Delta_h q^n\|^2) + \frac{1}{4}\triangle t^3\|\Delta_h(q^{n+1}-q^n)\|^2 \\
&\quad -\frac{1}{2}\triangle t^{\alpha+3}(\!(\nabla_h\cdot\boldsymbol{g}^n, \Delta_h q^n)\!) \\
&= -\frac{1}{4}\triangle t^3(\|\Delta_h q^{n+1}\|^2 - \|\Delta_h q^n\|^2) + \triangle t\ \|\nabla_h\cdot\boldsymbol{e}^*\|^2 + \frac{1}{4}\triangle t^{2\alpha+3}\|\nabla_h\cdot\boldsymbol{g}^n\|^2 \\
&\quad -\triangle t^{\alpha+2}(\!(\nabla_h\cdot\boldsymbol{g}^n, \nabla_h\cdot\boldsymbol{e}^*)\!) - \frac{1}{2}\triangle t^{\alpha+3}(\!(\nabla_h\cdot\boldsymbol{g}^n, \Delta_h q^n)\!)\,.
\end{aligned} \tag{9.48}
$$

Combining these two terms we arrive at

$$
\begin{aligned}
I = & -\frac{1}{2}\triangle t^2(\|\nabla_h q^{n+1}\|^2 - \|\nabla_h q^n\|^2) - \frac{1}{4}\triangle t^3(\|\Delta_h q^{n+1}\|^2 - \|\Delta_h q^n\|^2) \\
& +\frac{1}{2}\|\boldsymbol{e}^{n+1} + \boldsymbol{e}^n - 2\boldsymbol{e}^*\|^2 + \triangle t\ \|\nabla_h \cdot \boldsymbol{e}^*\|^2 + \triangle t^{\alpha+1}(\!(\boldsymbol{g}^n, \boldsymbol{e}^{n+1} + \boldsymbol{e}^n - 2\boldsymbol{e}^*)\!) \\
& -\triangle t^{\alpha+2}(\!(\boldsymbol{g}^n, \nabla_h q^n)\!) - \triangle t^{\alpha+2}(\!(\nabla_h \cdot \boldsymbol{g}^n, \nabla_h \cdot \boldsymbol{e}^*)\!) \\
& -\frac{1}{2}\triangle t^{\alpha+3}(\!(\nabla_h \cdot \boldsymbol{g}^n, \Delta_h q^n)\!) + \frac{1}{4}\triangle t^{2\alpha+3}\|\nabla_h \cdot \boldsymbol{g}^n\|^2 + \frac{1}{2}\|\triangle t^{\alpha+1}\boldsymbol{g}^n\|^2 .
\end{aligned} \tag{9.49}
$$

This gives

$$
\begin{aligned}
I \leq & -\tfrac{1}{2}\triangle t^2(\|\nabla_h q^{n+1}\|^2 - \|\nabla_h q^n\|^2) - \tfrac{1}{4}\triangle t^3(\|\Delta_h q^{n+1}\|^2 - \|\Delta_h q^n\|^2) \\
& +\tfrac{1}{2}\|\boldsymbol{e}^{n+1} + \boldsymbol{e}^n - 2\boldsymbol{e}^*\|^2 + \triangle t\ \|\nabla_h \cdot \boldsymbol{e}^*\|^2 + \triangle t\ \|\boldsymbol{e}^{n+1} + \boldsymbol{e}^n - 2\boldsymbol{e}^*\|^2 \\
& +2\triangle t^3\|\nabla_h q^n\|^2 + 2\triangle t^4\|\Delta_h q^n\|^2 + 2\triangle t^{2\alpha+1}(\|\boldsymbol{g}^n\|^2 + \triangle t\|\boldsymbol{g}^n\|^2_{H^1}) .
\end{aligned} \tag{9.50}
$$

Going back to (9.43) we obtain

$$
\begin{aligned}
& \|\boldsymbol{e}^{n+1}\|^2 - \|\boldsymbol{e}^n\|^2 + \tfrac{1}{2}\|\boldsymbol{e}^{n+1} + \boldsymbol{e}^n - 2\boldsymbol{e}^*\|^2 + \triangle t\ \|\nabla_h \boldsymbol{e}^*\|^2 \\
& \quad +\tfrac{1}{2}\triangle t^2(\|\nabla_h q^{n+1}\|^2 - \|\nabla_h q^n\|^2) + \tfrac{1}{4}\triangle t^3(\|\Delta_h q^{n+1}\|^2 - \|\Delta_h q^n\|^2) \\
& \leq \triangle t^3\|\nabla_h q^n\|^2 + \triangle t^4\|\Delta_h q^n\|^2 + C\,\triangle t\ (\|\boldsymbol{e}^n\|^2 + \|\boldsymbol{e}^{n-1}\|^2 + \|\boldsymbol{e}^{n+1}\|^2) \\
& \quad +C\,\triangle t^{2\alpha+1}(\|\boldsymbol{f}^n\|^2 + \triangle t\|\boldsymbol{g}^n\|^2_{H^1}) .
\end{aligned} \tag{9.51}
$$

Gronwall lemma gives

$$
\|\boldsymbol{e}^n\| + \|\boldsymbol{e}^*\| + \triangle t\ \|\nabla_h q^n\| + \triangle t^{3/2}\ \|\Delta_h q^n\| + \triangle t^{1/2}\|\nabla_h \boldsymbol{e}^*\| \leq C_1\triangle t^\alpha . \tag{9.52}
$$

Now by inverse inequality (8.4) we have

$$
\|\boldsymbol{e}^n\|_{L^\infty} + h\|\boldsymbol{e}^n\|_{W^{1,\infty}} + \triangle t\|\nabla_h q^n\|_{L^\infty} \leq C_1\frac{\triangle t^\alpha}{h} . \tag{9.53}
$$

Chose $N = 5$ and $\triangle t^\alpha << h^2$, if we choose $\triangle t$ small enough, we will always have

$$
\|\boldsymbol{e}^{n+1}\|_{L^\infty} \leq 1 . \tag{9.54}
$$

Therefore in (9.37) we can choose

$$\tilde{C} = 1 + \max_{n \le \left[\frac{T}{\triangle t}\right]} \|U^n(\cdot)\|_{W^{1,\infty}} \tag{9.55}$$

which depends only on the exact solution $(\boldsymbol{u}, p)$. This proves

$$\|\boldsymbol{u}_0 - \boldsymbol{u}_h\|_{L^\infty} + \|p_0 - p_h\|_{L^2} + \triangle t^{1/2}\|p_0 - p_h\|_{L^\infty} + \|p_0 - p_h - p_c\|_{L^\infty} \le C\triangle t^2 \tag{9.56}$$

From Lemma 8.2, we have

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^\infty} + \|p - p_h\|_{L^2} + \triangle t^{1/2}\|p - p_h\|_{L^\infty} + \|p - p_h - p_c\|_{L^\infty} \le C(\triangle t^2 + h^2) \tag{9.57}$$

This completes the proof of Theorem 4.

Appendix 3. Post-processing for the Pressure

Theorem 2 tells us how to correct the leading order boundary layer error in the numerical approximations of pressure. Here we will show how the next order boundary layer terms can also be corrected. The asymptotic analysis in §6.2 gives

$$p_{\triangle t} = p_0 + \varepsilon\phi_1 + \varepsilon^3\phi_3 + O(\triangle t^2) \tag{10.1}$$

where

$$\phi_1 = \frac{1}{\sqrt{2}} e^{-\sqrt{2}\xi}\, \partial_x p_0 \,|_{x=-1} \tag{10.2}$$

$$\phi_3 = \frac{1}{2}\left(\frac{1}{\sqrt{2}} + \xi\right) e^{-\sqrt{2}\xi}\, \partial_{xyy} p_0 \,|_{x=-1} \tag{10.3}$$

All these and the following formulas are evaluated at $(n - 1/2)\triangle t$. From (10.1) we have

$$\begin{aligned} \partial_x p_{\triangle t} \,|_{x=-1+\triangle t^{1/2}} &= \partial_x p_0 \,|_{x=-1+\triangle t^{1/2}} - e^{-\sqrt{2}} \partial_x p_0 \,|_{x=-1} \\ &\quad - \frac{\triangle t}{\sqrt{2}} e^{-\sqrt{2}} \partial_{xyy} p_0 \,|_{x=-1} + O(\triangle t^{3/2}) \end{aligned} \tag{10.4}$$

Hence we have

$$\partial_{xyy} p_0 \,|_{x=-1} = \frac{e^{\sqrt{2}}}{e^{\sqrt{2}} - 1} \partial_{xyy} p_{\triangle t} \,|_{x=-1+\triangle t^{1/2}} + O(\triangle t^{1/2}) \tag{10.5}$$

Taylor expansion gives

$$\begin{aligned}\partial_x p_0 \mid_{x=-1+\triangle t^{1/2}} &= \partial_x p_0 \mid_{x=-1} - \varepsilon \partial_x^2 p_0 \mid_{x=-1+\triangle t^{1/2}} \\ &\quad - \frac{\varepsilon^2}{2} \partial_x^3 p_0 \mid_{x=-1+\triangle t^{1/2}} + O(\triangle t^{3/2})\end{aligned} \tag{10.6}$$

Again from (10.1) we have

(10.7)

$$\begin{aligned}\varepsilon \partial_x^2 p_{\triangle t} &= \varepsilon \partial_x^2 p_0 + \partial_\xi^2 \phi_1 + \varepsilon^2 \partial_\xi^2 \phi_3 + O(\triangle t^{3/2}) \\ &= \varepsilon \partial_x^2 p_0 + \sqrt{2} e^{-\sqrt{2}\xi} \, \partial_x p_0 \mid_{x=-1} + \varepsilon^2 (\xi - \frac{1}{\sqrt{2}}) e^{-\sqrt{2}\xi} \, \partial_{xyy} p_0 \mid_{x=-1} + O(\triangle t^{3/2})\end{aligned}$$

(10.8)

$$\begin{aligned}\tfrac{\varepsilon^2}{2} \partial_x^3 p_{\triangle t} &= \frac{\varepsilon^2}{2} \partial_x^3 p_0 + \frac{1}{2} \partial_\xi^3 \phi_1 + \frac{\varepsilon^2}{2} \partial_\xi^3 \phi_3 + O(\triangle t^{3/2}) \\ &= \frac{\varepsilon^2}{2} \partial_x^3 p_0 - e^{-\sqrt{2}\xi} \, \partial_x p_0 \mid_{x=-1} + \varepsilon^2 (1 - \frac{\xi}{\sqrt{2}}) e^{-\sqrt{2}\xi} \, \partial_{xyy} p_0 \mid_{x=-1} + O(\triangle t^{3/2})\end{aligned}$$

Evaluating these expressions at $x = -1 + \triangle t^{1/2}$, we get

$$\begin{aligned}\varepsilon \partial_x^2 p_{\triangle t} \mid_{x=-1+\triangle t^{1/2}} &= \varepsilon \partial_x^2 p_0 \mid_{x=-1+\triangle t^{1/2}} + \sqrt{2} e^{-\sqrt{2}} \partial_x p_0 \mid_{x=-1} \\ &\quad + \varepsilon^2 (1 - \frac{1}{\sqrt{2}}) e^{-\sqrt{2}} \partial_{xyy} p_0 \mid_{x=-1} + O(\triangle t^{3/2})\end{aligned} \tag{10.9}$$

$$\begin{aligned}\tfrac{\varepsilon^2}{2} \partial_x^3 p_{\triangle t} \mid_{x=-1+\triangle t^{1/2}} &= \frac{\varepsilon^2}{2} \partial_x^3 p_0 \mid_{x=-1+\triangle t^{1/2}} - e^{-\sqrt{2}} \, \partial_x p_0 \mid_{x=-1} \\ &\quad + \varepsilon^2 (1 - \frac{1}{\sqrt{2}}) e^{-\sqrt{2}} \, \partial_{xyy} p_0 \mid_{x=-1} + O(\triangle t^{3/2})\end{aligned} \tag{10.10}$$

Combining (10.4), (10.6), (10.9) and (10.10), we obtain

(10.11)

$$\begin{aligned}(\partial_x + \varepsilon \partial_x^2 + \frac{\varepsilon^2}{2} \partial_x^3) p_{\triangle t} \mid_{x=-1+\triangle t^{1/2}} &= [1 - (2 - \sqrt{2}) e^{-\sqrt{2}}] \partial_x p_0 \mid_{x=-1} \\ &\quad + \triangle t \frac{2\sqrt{2} - 3}{\sqrt{2}} e^{-\sqrt{2}} \, \partial_{xyy} p_0 \mid_{x=-1} + O(\triangle t^{3/2})\end{aligned}$$

Or

(10.12)

$$\begin{aligned}\partial_x p_0 \mid_{x=-1} &= \frac{e^{\sqrt{2}}}{e^{\sqrt{2}} - 2 + \sqrt{2}}(\partial_x + \varepsilon\partial_x^2 + \frac{\varepsilon^2}{2}\partial_x^3)p_{\triangle t} \mid_{x=-1+\triangle t^{1/2}} \\ &-\triangle t\frac{2\sqrt{2}-3}{\sqrt{2}e^{\sqrt{2}} + 2 - 3\sqrt{2} + 2(\sqrt{2}-1)e^{-\sqrt{2}}}\,\partial_{xyy}p_{\triangle t} \mid_{x=-1+\triangle t^{1/2}} + O(\triangle t^{3/2})\end{aligned}$$

Finally, using (10.5) and (10.12) in (10.1), (10.2) and (10.3), we get

$$p_{\triangle t} = p_0 - p_c + O(\triangle t^2) \tag{10.13}$$

where

$$\begin{aligned}p_c &= \alpha\triangle t^{1/2}e^{-\sqrt{2}\xi}\,(\partial_x + \triangle t^{1/2}\partial_x^2 + \frac{\triangle t}{2}\partial_x^3)p_{\triangle t} \mid_{x=-1+\triangle t^{1/2}} \\ &+(\beta+\gamma\xi)\triangle t^{3/2}e^{-\sqrt{2}\xi}\,\partial_{xyy}p_{\triangle t} \mid_{x=-1+\triangle t^{1/2}}\end{aligned} \tag{10.14}$$

where

$$\begin{aligned}\alpha &= \frac{e^{\sqrt{2}}}{2 - 2\sqrt{2} + \sqrt{2}e^{\sqrt{2}}} \\ \beta &= \frac{1}{2\sqrt{2}}\frac{e^{\sqrt{2}}}{2e^{\sqrt{2}} - 2} - \frac{2\sqrt{2}-3}{\sqrt{2}e^{\sqrt{2}} + 2 - 3\sqrt{2} + 2(\sqrt{2}-1)e^{-\sqrt{2}}} \\ \gamma &= \frac{e^{\sqrt{2}}}{2e^{\sqrt{2}} - 2}\end{aligned} \tag{10.15}$$

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