Numerical Methods for Viscous Incompressible Flows: Some Recent Advances

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Dedicated to Professor Hong-ci Huang on occasion of his retirement

1. Introduction. The purpose of this article is to review the recent work of J. G. Liu and author on numerical methods for viscous incompressible flows. I have neglected many important contributions to this field from other people, most notably are the work done on Stokes solvers, including fast Stokes solver [12, 8], Jie Shen's work on the pseudo-compressibility method, and lattice Boltzmann methods [1].

Consider the Navier-Stokes equation

$$(1.1) \qquad \begin{cases} \boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \frac{1}{Re}\Delta \boldsymbol{u} + \boldsymbol{f} \\ \nabla \cdot \boldsymbol{u} = 0 \end{cases}$$

together with the simplest physical boundary condition

$$(1.2) \qquad \boldsymbol{u}|_\Gamma = 0$$

The problem of interest here is to find numerical solutions of (1.1)-(1.2). This is a central issue in the subject of computational fluid dynamics, dating back to at least the 60's, when the MAC scheme [11] and the projection method [2, 3] were invented. MAC scheme and staggered grid enjoyed immediate success and similar ideas were also applied to electromagnetics (the Yee scheme). On the other hand, it wasn't until the late 80's and early 90's, did we begin to understand the mysteries surrounding the projection method and realize its potential. By now projection method is by far the most popular method in this field.

The main difficulty in numerically solving (1.1)-(1.2) is the lack of proper evolutionary equation for pressure. After all for incompressible flows, pressure does not carry its usual thermodynamic meaning. It is there mainly as a Lagrange multiplier for the constraint of incompressibility. This translates to a lack of proper boundary condition, when equations for pressure are derived from (1.1)-(1.2). For example if we take divergence of the momentum equation, we get

$$(1.3) \qquad -\Delta p = \text{Tr}(\nabla \boldsymbol{u})^2$$

with no obvious choice of boundary condition for p. This problem is also present in the other popular reformulation of (1.1)-(1.2), namely the vorticity formulation:

$$(1.4) \qquad \boldsymbol{\omega}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} = \frac{1}{Re}\Delta\boldsymbol{\omega} + \nabla \times \boldsymbol{f}$$

where

$$(1.5) \qquad \boldsymbol{\omega} = \nabla \times \boldsymbol{u}$$

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To evolve (1.4), it would be much more convenient if we had at our disposal boundary conditions for $\boldsymbol{\omega}$. But this is not the case. Instead, the boundary condition is given in (1.2) in terms of $\boldsymbol{u}$.

This problem of boundary condition, either for p or for $\boldsymbol{\omega}$, is the central issue in this area. It is also the main issue that we will address in this article. To begin our discussion, let us first remark that there are three different regimes as far as numerics for (1.1)-(1.2) is concerned.

(1). $Re >> 1$. Usually in 2D if $Re \geq 100$, the flow is in this regime. The boundary in 3D is less clear since there is less work done in 3D so far. This includes moderate to high Reynolds number flows and therefore turbulent flows. Here we must distinguish the issue of physical complexity of the flows and the numerical issue of designing schemes that have the right stability properties. This regime is often regarded in the literature as the most difficult one. We will argue that as far as numerics is concerned, this is the simplest regime. This is the regime where the new vorticity schemes developed by E and Liu [4, 5, 7, 13] work the best. This is also the regime where explicit methods work the best.

(2). $Re \doteq O(1)$. In this regime, the viscous forces and the inertial forces are comparable. The viscous term needs to be treated implicitly to have good stability properties for the numerical method. This is the regime where project method is at its best.

(3). $Re << 1$. This is the creeping flow regime. In this regime, the governing equation is almost linear - the Stokes equation. This means that the non-locality is also the strongest. Many biological flows, non-Newtonian flows belong to this regime. This is the regime where fast Stokes solvers are desired. Currently there are two main directions for developing fast Stokes solvers. The first is to find ways to adapt traditional methods such as multi-grid and preconditioned conjugate gradient methods to solve the algebraic equations from finite difference or finite element discretizations. The second is to use integral equations [12].

For lack of space, we will concentrate on the first regime in the present paper. The plan is to first discuss the vorticity formulation and in particular discuss the local vorticity boundary conditions and the issue of cell Reynolds number, even though the second issue is not special to the vorticity formulation but is common to all central schemes for equations of advection-diffusion type. We then comment on the relation between vorticity and velocity formulations and how these ideas can be used for the velocity-pressure formulation.

2. Cell Reynolds number. What makes this regime relatively easy numerically is the fact that we can afford to treat the viscous term explicitly. In fact this should be taken as the definition of this regime. This point of view is rather different from traditional engineering approach to (1.1)-(1.2) which seems to always favor implicit treatment of the viscous term. Let us consider the two time step constraints from the convective and diffusive terms respectively.

$$(2.1) \qquad \Delta t_1 = \frac{\Delta x}{U}, \qquad \Delta t_2 = \frac{\Delta x^2}{\nu}, \qquad \nu = \frac{1}{Re}$$

U is the velocity scale. We have neglected numerical factors such as $\frac{1}{2d}$ where d is the spatial dimension. The ratio of these two time scales:

$$(2.2) \qquad \frac{\Delta t_2}{\Delta t_1} = \frac{U\Delta x}{\nu} = Rc$$

is the so-called cell Reynolds number. In this regime, which includes high Reynolds number flows, the main challenge is to design numerical methods that work for large values of Rc, in which case Δt_2 is larger than Δt_1. Therefore the diffusive time step constraint is much less severe than the convective time step constraint. Hence we can afford to treat the viscous term explicitly without sacrificing the stability property of the overall scheme. We remark that in practice we always honor the convective time step constraint for accuracy purpose, in order to faithfully follow the dynamics.

Why do we want large Rc? First of all if our numerical method works only when $Rc = O(1)$, it means that on a 1024^2 grid, we can only compute flows at Reynolds number $O(10^3)$. This would be about the best we can do in 2D. For the square driven cavity problem, which is the simplest realistic example, the flow is fairly boring at Reynolds number 10^3. It does not develop interesting dynamics until Re reaches about 10^4.

Secondly from accuracy considerations, the smallest active scale in a 2D incompressible flow is of $O(Re^{-1/2}) = O(\nu^{1/2})$. Therefore to resolve the flow we need $\Delta x = O(\nu^{1/2})$. This means $Rc = O(\nu^{-1/2}) = O(Re^{1/2})$, and can be very large for large Re. In other words, accuracy alone does not present a constraint on the size of Rc. This is not the same for compressible flows. In that case, the smallest active scale, the width of the viscous shock layer, is of order $O(\nu)$ and hence $Rc = O(1)$, from accuracy alone. Indeed if we want to perform absolutely resolved calculations for viscous compressible flows, our abilities are very limited. Therefore the art of computing compressible flows has always been to find ways to under-resolve the flow.

Why should there be a problem with respect to cell Reynolds number? Consider the simplest convection equation

$$(2.3) \qquad u_t + au_x = 0$$

If we solve this using forward Euler in time, and centered difference in space

$$(2.4) \qquad \frac{u_j^{n+1} - u_j^n}{\Delta t} + a\,\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0$$

the amplification factor in the von Neumann analysis is

$$(2.5) \qquad g(\xi) = 1 - ia\,(\sin\xi)\frac{\Delta t}{\Delta x}$$

at wavenumber ξ. For stability, we need

$$(2.6) \qquad |g(\xi)| \le 1 + \overline{C}\Delta t$$

for all ξ. This translates to

$$(2.7) \qquad a^2\,\frac{\Delta t}{\Delta x^2} \le 2\overline{C}$$

This is a unnecessarily severe constraint on the size of Δt for (2.3). Furthermore, the error would grow as $e^{\overline{C}t}$.

This phenomenon has repercussions when diffusion term is added. Consider now

$$(2.8) \qquad u_t + au_x = \nu u_{xx}$$

The analog of Reynolds number is $Re = \frac{aL}{\nu}$. Similar to (2.4), consider

$$(2.9) \qquad \frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = \nu\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

The amplification factor becomes

$$(2.10) \qquad g(\xi) = 1 + \Delta t \left\{ -a \frac{\sin \xi}{\Delta x} + \nu \frac{2(\cos \xi - 1)}{\Delta x^2} \right\}$$

(2.6) is satisfied with $\overline{C} = \frac{a^2}{\nu}$ as long as the standard diffusive stability constraint $\frac{\Delta t}{\nu \Delta x^2} < \frac{1}{2}$ is satisfied. If we had treated the diffusion term implicitly,

$$(2.11) \qquad \frac{u_j^{n+1} - u_j^n}{\Delta t} + a \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = \nu \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2}$$

(2.6) would be satisfied with $\overline{C} = \frac{2a^2}{\nu}$ but no other conditions on Δt other than $\Delta t \ll 1$. In any case, these schemes are useless at high Reynolds number even though they are stable at fixed Reynolds number since the error grows as $e^{\overline{C}t} = e^{\frac{a^2}{\nu}t}$. This means that to design schemes that work for high Reynolds number, the standard stability concept is not enough as a designing principle. What we need is uniform stability with respect to Reynolds number, i.e. $\overline{C}$ in (2.6) should be independent of Re. It is easy to see that in order to satisfy this requirement, we must have, for (2.9),

$$(2.12) \qquad Rc \leq 2$$

This is the cell Reynolds number constraint.

What causes this problem is that the stability region of forward Euler does not cover any part of the imaginary axis where the spectrum of the centered difference operator lies. This can be easily fixed by switching to a time stepping scheme whose stability region covers a neighborhood of the imaginary axis near the origin. Examples of such schemes include the 3rd and 4th order Runge-Kutta, 3rd and 4th order Adams-Bashforth formulas.

Same remarks apply to (2.11) when the diffusion term is treated implicitly. Remember that the origin of the problem is in the convective term.

There are other ways to get around the cell Reynolds number issue. For example, one can discretize the convective term using upwind schemes. The problem with such an approach is that it adds complexity and reduces accuracy. Furthermore, it is not necessary as we already discussed.

3. The local vorticity boundary conditions. The advantage of the vorticity methods is most easily seen in 2D. In this case we can introduce the streamfunction ψ and represent velocity as $\boldsymbol{u} = (u, v) = (-\psi_y, \psi_x)$. The boundary condition $(u, v) = (0, 0)$ becomes $\psi = \frac{\partial \psi}{\partial n} = 0$. The vorticity-streamfunction formulation reads

$$(3.1) \qquad \begin{cases} \partial_t \omega + (\boldsymbol{u} \cdot \nabla)\omega = \frac{1}{Re}\Delta\omega + \nabla \times \boldsymbol{f} \\ \Delta\psi = \omega \end{cases}$$

The basic idea is to use the Dirichlet boundary condition for the Poisson equation $\Delta\psi = \omega$ to solve for ψ, and convert the remaining boundary $\frac{\partial \psi}{\partial n} = 0$ as a boundary condition for ω. To illustrate this, we look at the simplest finite difference setting on a regular grid, as shown in Figure 1. The PDEs are discretized using standard centered difference formulas:

$$(3.2) \qquad \begin{cases} \partial_t \omega + (\boldsymbol{u} \cdot \nabla_h)\omega = \frac{1}{Re}\Delta_h\omega + g \\ \Delta_h\psi = \omega \end{cases}$$
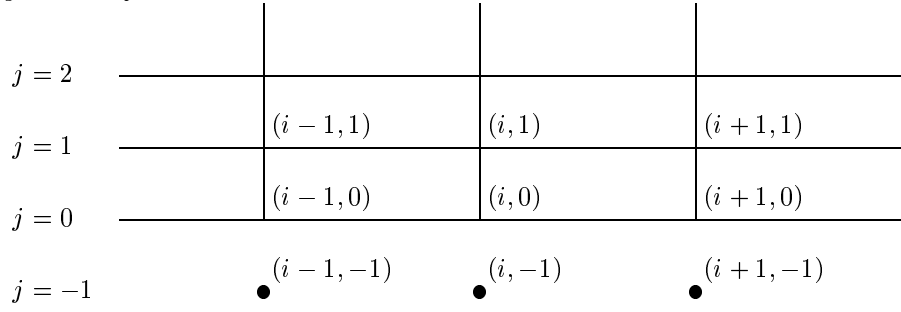
where $g = \nabla_h \times \boldsymbol{f}$.

Figure 1

In order to derive numerical boundary conditions for ω, introduce the "ghost points" represented by the $j = -1$ grid line. We will approximate $\frac{\partial \psi}{\partial n}|_\Gamma = 0$ by

$$(3.3) \qquad \frac{\psi_{i,1} - \psi_{i,-1}}{2h} = 0$$

i.e.

$$(3.4) \qquad \psi_{i,-1} = \psi_{i,1}$$

We also have

$$(3.5) \qquad \psi_{i,0} = 0$$

Then use the second equation in (3.3)

$$(3.6) \qquad \omega_{i,0} = \frac{1}{h^2} \left(\psi_{i+1,0} + \psi_{i-1,0} + \psi_{i,-1} + \psi_{i,1} - 4\psi_{i,0} \right)$$

Using (3.3) and (3.4), this gives

$$(3.7) \qquad \omega_{i,0} = \frac{1}{h^2} \left(\psi_{i,1} + \psi_{i,-1} \right) = \frac{2\psi_{i,1}}{h^2}$$

This is the classical Thom's formula [17].

How is (3.3) used in practice? The two most popular approaches are the following:

1. The Backward Euler approach is to discretize in time as

$$(3.8) \qquad \begin{cases} \dfrac{\omega_{i,j}^{n+1} - \omega_{i,j}^n}{\Delta t} + \left(\boldsymbol{u}_{i,j}^n \cdot \nabla_h \right) \omega_{i,j}^n = \dfrac{1}{Re} \Delta_h \omega_{i,j}^{n+1} + g \\ \begin{cases} \Delta_h \psi_{i,j}^{n+1} = \omega_{i,j}^{n+1} \\ \psi_{i,0}^{n+1} = 0 \end{cases} \\ \omega_{i,0}^{n+1} = \dfrac{2\psi_{i,1}^{n+1}}{h^2} \end{cases}$$

This is a fully coupled system. Indeed this is the equivalent of the Uzawa scheme in vorticity formulation. Solving this coupled system of equations at each time step for $(\omega^{n+1}, \psi^{n+1})$ turned out to be a nightmare. The iterative methods failed miserably in the 60's and 70's. This dramatic failure was discussed by Orszag and Israeli [14]. In the 80's, direct methods began to replace the iterative methods. This eliminated the problem of divergence, but at the cost of introducing considerable complexity which

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rendered the methods essentially useless in 3D. Most influential among these direct methods is the influence matrix method. One can write down equivalent forms of (3.3), in the form

$$(3.9) \qquad\qquad A\omega_b = \beta$$

where ω_b is the numerical values of ω at the boundary, A is the influence matrix: $A = (a_{i,l})$ and $a_{i,l}$ is the slip velocity at the grid point $(i, 0)$ generated by a discrete vorticity distribution which is equal to 1 at the grid point $(l, 0)$, and zero at all other grid points. Obviously A is a full matrix. This kind of equations are called "global vorticity boundary conditions" for obvious reasons.

The influence matrix method is very popular in the engineering community. We refer to [15] for an extensive discussion of these ideas. However, we believe that this is not the correct solution to the original problem. The coupling between interior values of vorticity, boundary values of vorticity and the streamfunction is caused by the implicit treatment of the viscous term. As we have argued earlier, this is unnecessary if the cell Reynolds number is not too small, as is the case for the regime we are considering. This coupling can be avoided if the viscous term is treated explicitly. If the cell Reynolds number has to be small, as in the other two regimes, the issue of efficiently using the vorticity formulation in numerical computation is still pretty much open.

2. Forward Euler approach. Given $\{\omega^n\}$ up to the boundary, $\{\omega^{n+1}\}$ is computed via the following steps

Step 1. Update ω^{n+1} in the interior using

$$(3.10) \qquad\qquad \frac{\omega_{i,j}^{n+1} - \omega_{i,j}^n}{\Delta t} + (\boldsymbol{u}_{i,j}^n \cdot \nabla_h)\omega_{i,j}^n = \frac{1}{Re}\Delta_h \omega_{i,j}^n + g$$

Step 2. Solving

$$(3.11) \qquad\qquad \begin{cases} \Delta_h \psi_{i,j}^{n+1} = \omega_{i,j}^{n+1} \\ \psi_{i,0}^{n+1} = 0 \end{cases}$$

Notice that since this is a Dirichlet problem, only interior values of ω^{n+1} are needed. Boundary values of ω^{n+1} are not needed.

Step 3. Update ω^{n+1} at the boundary using Thom's formula.

The simplicity of this approach is obvious. There is no coupling between the interior and boundary values of vorticity. Only one standard Poisson equation has to be solved at each time step.

However, as simple as this method is, it does not work for high Reynolds number flows. The basic reason was discussed earlier. This method is based on a scheme that has severe cell Reynolds number constraints. The remedy to this, as we discussed earlier, is simply to replace forward Euler by a time stepping procedure that are convectively stable. Examples includes 3rd and 4th order Runge-Kutta schemes and Adams-Bashforth.

These discoveries, made in [4], are extremely simple and elementary, but also very fundamental. Indeed as was demonstrated in [4], Thom's formula coupled with a convectively stable time-stepping procedure such as the classical Runge-Kutta, is already a remarkably efficient method. What is more important, it is extremely simple. They revived the interest on using vorticity formulation for high Reynolds number flows, and paved the way for designing more accurate, flexible vorticity methods.

Next, we address the accuracy issue. It is fairly easy to see that the presence of the nonlinear term is not essential (aside from the fact that it adds some minor technicalities) and it does not change much the discussion. So we will neglect it and consider instead the linear problem

$$(3.12) \qquad \begin{cases} \partial \omega_t = \frac{1}{Re}\Delta\omega + \boldsymbol{f} \\ \omega = \Delta\psi \\ \psi|_\Gamma = 0, \quad \frac{\partial\psi}{\partial n}|_\Gamma = 0 \end{cases}$$

At a first sight, the accuracy of Thom's formula is highly suspicious because of the h^2 term in the denominator. It is even more questionable when Fromm's formula

$$(3.13) \qquad \omega_{o,j} = \frac{1}{h^2}\psi_{i,j}$$

was put forward as an alternative to Thom's formula. Fromm's formula is derived if one uses one-sided difference to approximate $\frac{\partial\psi}{\partial n}|_\Gamma = 0$. Moreover, since we do not have to insist on using the 5-point formula when we evaluate $\omega = \Delta\psi$ at the boundary, many variants of Thom's formula were derived. We summarize them in the following table.

Reference	$\omega - \psi$ formulation	MAC scheme
Thom 1933	$\omega_{0,j} = \frac{2}{\triangle x^2}\psi_{1,j}$	$v_{-\frac{1}{2},j} = -v_{\frac{1}{2},j}$
	$\psi_{-1,j} = \psi_{1,j}$	
Woods 1954	$\omega_{0,j} = \frac{3}{\triangle x^2}\psi_{1,j} - \frac{1}{2}\omega_{1,j}$	$v_{-\frac{1}{2},j} = -\frac{5}{2}v_{\frac{1}{2},j} + \frac{1}{2}v_{\frac{3}{2},j}$ $-\frac{1}{2\triangle y}(u_{1,j+\frac{1}{2}} - u_{1,j-\frac{1}{2}})$
Fromm 1963	$\omega_{0,j} = \frac{1}{\triangle x^2}\psi_{1,j}$	$v_{-\frac{1}{2},j} = 0$
	$\psi_{-1,j} = 0$	
Wilkes 1963 Pearson 1965	$\omega_{0,j} = \frac{1}{2\triangle x^2}(8\psi_{1,j} - \psi_{2,j})$	$v_{-\frac{1}{2},j} = -\frac{5}{2}v_{\frac{1}{2},j} + \frac{1}{2}v_{\frac{3}{2},j}$
	$\psi_{-1,j} = 3\psi_{1,j} - \frac{1}{2}\psi_{2,j}$	
Orszag-Israeli 1974	$\omega_{0,j} = \frac{1}{3\triangle x^2}(10\psi_{1,j} - \psi_{2,j})$	$v_{-\frac{1}{2},j} = -2v_{\frac{1}{2},j} + \frac{1}{3}v_{\frac{3}{2},j}$
	$\psi_{-1,j} = \frac{7}{3}\psi_{1,j} - \frac{1}{3}\psi_{2,j}$	
Orszag-Israeli 1974	$\omega_{0,j} = \frac{1}{13\triangle x^2}(35\psi_{1,j} - \psi_{3,j})$	$v_{-\frac{1}{2},j} = -\frac{21}{13}v_{\frac{1}{2},j} + \frac{1}{13}v_{\frac{3}{2},j} + \frac{1}{13}v_{\frac{5}{2},j}$
	$\psi_{-1,j} = \frac{22}{13}\psi_{1,j} - \frac{1}{13}\psi_{3,j}$	

Table 1. Local vorticity boundary condition

It turns out that Thom's formula, coupled with the centered difference scheme described earlier, gives second order accuracy, even for vorticity. The correct way to see this is to write the numerical method (say the semi-discrete version) in the following equivalent form:

$$(3.14) \qquad \begin{cases} \frac{\partial}{\partial t}\Delta_h\psi_h = \frac{1}{Re}\Delta_h^2\psi_h + f \\ \psi_h|_\Gamma = 0 \\ D_h\psi|_\Gamma = 0 \end{cases}$$

where Δ_h is the standard 5-point formula and D_h is the centered difference approximation to $\frac{\partial}{\partial n}$. At the same time we can write the PDE as

$$(3.15) \qquad \begin{cases} \frac{\partial}{\partial t}\Delta_h\psi = \frac{1}{Re}\Delta_h^2\psi + f + O(h^2) \\ \psi|_\Gamma = 0 \\ D_h\psi|_\Gamma = O(h^2) \end{cases}$$

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The error $\eta_h = \psi - \psi_h$ satisfies

$$(3.16) \qquad \begin{cases} \frac{\partial}{\partial t} \Delta_h \eta_h = \frac{1}{Re} \Delta_h^2 \eta_h + O(h^2) \\ \eta_h|_\Gamma = 0, \quad D_h \eta_h|_\Gamma = O(h^2) \end{cases}$$

An energy estimate can be proved for this system, giving rise to

$$(3.17) \qquad \|\nabla_h \psi - \nabla_h \psi_h\|_{L^2} \leq Ch^2$$

or

$$(3.18) \qquad \|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2} \leq Ch^2$$

The accuracy for vorticity, even in maximum norm, comes as a consequence of the super-convergence on regular grids. In fact we have

$$(3.19) \qquad \|\partial^\alpha \boldsymbol{u} - \partial_h^\alpha \boldsymbol{u}_h\|_{L^\infty(\Omega_h^\alpha)} \leq Ch^2$$

where ∂_h^α is centered approximation to the operator ∂^α, Ω_h^α is a subset of grid points on which $\partial_h^\alpha \boldsymbol{u}_h$ can be defined, given the values of $\boldsymbol{u}_h$ on all grid points. In particular, for vorticity we have

$$(3.20) \qquad \|\omega - \omega_h\|_{L^\infty(\Omega_h)} \leq Ch^2$$

where Ω_h is the set of all interior grid points.

Applying the same procedure to other formulas in Table 1, we find that Fromm's formula gives first order accuracy. Orszag-Israeli's formulas give second order accuracy. Even though they use higher order accurate formulas at the boundary, the vorticity transport equation is only approximated to second order accuracy. Numerical results also show that they are not more accurate than Thom's formula [4].

4. Comparison with the primitive variable formulation. MAC scheme on the staggered grid is one of earliest and most charming idea in this subject [11]. It works very well for simple geometries. The staggered grid fits naturally with the structure of the Navier-Stokes equation in the primitive variable form. The only catch is in the evaluation of the nonlinear term which requires the values of all components of the velocity field which are not directly available on the staggered grid since different components are defined on different grids. But a simple average will solve the problem.

In 2D there is a nice completion to this story. This is discovered in [4]. One can naturally define vorticity and streamfunction on the original grids, and the MAC scheme is then exactly the same as the centered difference scheme (19). Moreover, the reflection boundary condition for the MAC scheme translates to Thom's formula. This is perhaps the hidden reason why vorticity schemes work so well in 2D. By using a regular grid they accomplish what MAC scheme does on a staggered grid. In Table 1, we have also listed the equivalent forms of other local vorticity boundary conditions for the MAC scheme.

This equivalence between the two formulations also points to another direction of work. The principles we discussed earlier, namely the explicit treatment of the viscous term, convectively stable time-stepping, local numerical boundary conditions, can also be applied to the primitive variable formulation. These "explicit projection methods" are recognized [6] but their performance has not been fully tested. It is likely that they outperform the vorticity schemes in 3D.

5. Extension to three dimensions. The principles discussed above also apply to three dimension. However, there is a fundamental difference between vorticity in 3D and vorticity in 2D. In 3D, vorticity is a vector, whereas in 2D vorticity can be thought of as being a scalar. Consequently in 3D we often have to solve a vector Poisson equation whose components are coupled together. In practice this can be much more complicated than solving scalar Poisson equations. Another problem is the treatment of the nonlinear term. This does not seem to be an issue in 2D [7]. But in 3D it seems significantly affect the performance of the method. Overall, much work needs to be done in 3D in order to access the usefulness of the vorticity formulation, in comparison with the velocity formulation.

Here we will review a few equivalent forms of the Navier-Stokes equation in vorticity formulation. We will then comment on the numerical methods.

The standard vorticity formulation is

$$(5.1) \qquad \begin{cases} \frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) = \frac{1}{Re} \Delta \boldsymbol{\omega} \\ \boldsymbol{\omega} = \nabla \times \boldsymbol{u}, \quad \nabla \cdot \boldsymbol{\omega} = 0, \quad \nabla \cdot \boldsymbol{u} = 0 \\ \boldsymbol{u}|_\Gamma = 0 \end{cases}$$

PROPOSITION 5.1 (Vorticity-vector potential formulation). *Assume that Γ is piecewise flat. (38) is equivalent to*

$$(5.2) \qquad \begin{cases} \frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) = \frac{1}{Re} \Delta \boldsymbol{\omega} \\ -\Delta \boldsymbol{\psi} = \boldsymbol{\omega} \\ \boldsymbol{n} \times \boldsymbol{\psi}|_\Gamma = 0, \quad \frac{\partial}{\partial n}(\boldsymbol{\psi} \cdot \boldsymbol{n})|_\Gamma = 0 \\ \boldsymbol{n} \cdot (\boldsymbol{\omega} - \nabla \times \boldsymbol{u})|_\Gamma = 0, \quad \boldsymbol{n} \times (\nabla \times \boldsymbol{\psi})|_\Gamma = 0 \end{cases}$$

The boundary condition says that the tangential component of $\boldsymbol{\psi}$ is zero and the normal derivative of the normal component is zero.

PROPOSITION 5.2 (Vorticity-vector potential formulation). (1.1) *is equivalent to* (1.2) *with the boundary conditions replaced by*

$$(5.3) \qquad \begin{aligned} \boldsymbol{n} \times \boldsymbol{\psi}|_\Gamma = 0, \quad \nabla \cdot \boldsymbol{\psi}|_\Gamma = 0, \quad \nabla \cdot \boldsymbol{\omega}|_\Gamma = 0 \\ \boldsymbol{n} \times \nabla \times \boldsymbol{\psi}|_\Gamma = 0 \end{aligned}$$

It is now straightforward to derive the analog of Thom's formula in 3D. As an example we will discuss (1.2). As in 2D, we will couple the boundary conditions $\boldsymbol{n} \times \boldsymbol{\psi}|_\Gamma = 0$, $\frac{\partial}{\partial n}(\boldsymbol{\psi} \cdot \boldsymbol{n})|_\Gamma = 0$ to the vector Poisson equation $-\Delta \boldsymbol{\psi} = \boldsymbol{\omega}$, and convert the rest to vorticity boundary conditions. This is only necessary for the tangential components since the normal component of $\boldsymbol{\omega}$ can be readily evaluated using $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$. Assume that the boundary is at the coordinate plane $\{z = z_0\}$, the 3D analog of Thom's formula is

$$(5.4) \qquad \omega_1(x_i, y_j, z_0) = -\frac{2}{\Delta z^2} \psi_1(x_i, y_j, z_1) + \frac{2}{\Delta z} \tilde{D}_x \psi_3(x_i, y_j, z_0)$$

$$(5.5) \qquad \omega_2(x_i, y_j, z_0) = -\frac{2}{\Delta z^2} \psi_2(x_i, y_j, z_1) + \frac{2}{\Delta z} \tilde{D}_y \psi_3(x_i, y_j, z_0)$$

where $\tilde{D}_x$ and $\tilde{D}_y$ are standard centered differences. The derivation and generalization of these formulas are given in [7].

One issue that is often discussed in the literature is the relevance of the accuracy for the conditions

$$\nabla \cdot \boldsymbol{\omega} = \nabla \cdot \boldsymbol{\psi} = \nabla \cdot \boldsymbol{u} = 0 \tag{5.6}$$

We refer to [7] for some results in this direction.

6. Finite element method. So far we have only discussed finite difference schemes. For complex geometries the finite element methods become attractive, and they give rises naturally to "compact schemes" [5]. For simplicity, we will go back to 2D. To formulate finite element methods, we need weak forms of the vorticity formulation. The simplest one is given by: Find $\omega \in H^1(\Omega)$, $\psi \in H^1_0(\Omega)$, such that

$$\tag{6.1}
\begin{aligned}
&(\varphi, \partial_t \omega) - (\nabla \varphi, \omega \boldsymbol{u}) = -\tfrac{1}{Re}(\nabla \varphi, \nabla \omega), \quad \forall \varphi \in H^1_0(\Omega) \\
&(\nabla \varphi, \nabla \psi) = -(\varphi, \omega), \quad \forall \varphi \in H^1(\Omega)
\end{aligned}$$

Here (,) is the standard L^2 inner product for functions.

In a finite element discretization, we replace $H^1_0(\Omega)$ by finite dimensional space $X^k_{0,h}$ and $H^1(\Omega)$ by X^k_h. $X^k_{0,h}$ and X^k_h are standard continuous finite element spaces with P_k-elements, i.e. piecewise k-th order polynomials. A finite element discretization of the above equation is: Find $\omega_h \in X^k_h, \psi_h \in X^k_{0,h}$

$$\tag{6.2}
\begin{aligned}
&(\varphi, \partial_t \omega_h) = (\nabla \varphi, \omega_h \boldsymbol{u}_h) = -\tfrac{1}{Re}(\nabla \varphi, \nabla \omega_h), \quad \forall \varphi \in X^k_{0,h} \\
&(\nabla \varphi, \nabla \psi_h) = -(\varphi, \omega_h), \quad \forall \varphi \in X^k_h
\end{aligned}$$

and $\boldsymbol{u}_h = \nabla^\perp \psi_h$.

All these seem pretty standard. The real issue is how to use these to make an efficient time-stepping procedure [13].

Since finite element method amounts to centered schemes, we must use the kind of time discretization procedures discussed in Section 2 in order to avoid severe cell Reynolds number constraint. In practice we use the classical fourth order Runge-Kutta method, which can essentially be written as four forward Euler steps [4]. As before it is enough to illustrate the time-stepping procedure using forward Euler.

Suppose we know the values of ω^n, ψ^n and $\boldsymbol{u}^n$ at t^n. We first compute an auxiliary term $\langle \varphi, \omega^{n+1} \rangle$ for any $\varphi \in X^k_{0,h}$ from

$$\text{(I)} \qquad \langle \varphi, \omega^{n+1} \rangle = \langle \varphi, \omega^n \rangle + \triangle t \langle \nabla \varphi, \omega^n \boldsymbol{u}^n \rangle - \nu \triangle t \langle \nabla \varphi, \nabla \omega^n \rangle \,.$$

Using this auxiliary term, we can solve for streamfunction $\psi^{n+1} \in X^k_{0,h}$ from

$$\text{(II)} \qquad \langle \nabla \varphi, \nabla \psi^{n+1} \rangle = -\langle \varphi, \omega^{n+1} \rangle, \qquad \forall \varphi \in X^k_{0,h} \,.$$

From ψ^{n+1}, we can obtain the vorticity ω^{n+1} by inverting a mass matrix from

$$\text{(III)} \qquad \langle \varphi, \omega^{n+1} \rangle = \langle \nabla \varphi, \nabla \psi^{n+1} \rangle, \qquad \forall \varphi \in X^k_h \,.$$

The right hand side of the above equation does not have to be computed again for each test function $\varphi \in X^k_{0,h}$ since it is equal to the auxiliary term from (II), which has already been computed in (I). Finally we compute the velocity

$$\boldsymbol{u}^{n+1} = \nabla^\perp \psi^{n+1} \,.$$

We should emphasize that in the above time stepping procedure, the momentum equation (I) is completely decoupled from the kinematic equation (II). There is no

iteration required between the vorticity and streamfunction to recover the boundary values for the vorticity.

At a first sight, this procedure seems circular since (II) and (III) seem to use the same equation, but are used to compute different things. So some comments are in order:

1. Step (I) only computes (φ, ω^{n+1}) for $\varphi \in X_{0,h}^k$. To completely determine ω^{n+1} we need to compute (φ, ω^{n+1}) for all $\varphi \in X_h^k$. The computation of (φ, ω^{n+1}) for the degrees of freedom associated with boundary nodes is split into two steps. First in Step (II) we compute the streamfunction ψ^{n+1}. Fortunately, and this is very important for the success of this procedure, knowing (φ, ω^{n+1}) for $\varphi \in X_{0,h}^k$ is enough to compute ψ^{n+1}. This is the same reason why the explicit methods work so well in the finite difference setting. Having ψ^{n+1}, we then compute (φ, ω^{n+1}) for $\varphi \in X_h^k \setminus X_{0,h}^k$ using step (III). However, (III) is also valid for $\varphi \in X_{0,h}^k$ because fo (II).

2. $(\varphi, \omega^{n+1}) = (\nabla\varphi, \nabla\psi^{n+1})$ for $\varphi \in X_h^k \setminus X_{0,h}^k$ can be thought of as the vorticity boundary condition. This is a natural generalization of Thom's formula.

7. Conclusion. In summary, while much progress has been made in the last several years, there are still many important directions that require much more work. Two of the most visible areas are: explicit projection methods and vorticity methods in 3D.

We have only concentrated in this paper on the regime when $Re \gg 1$. A report is in progress on the recent work for the regime $Re = O(1)$.

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