Asymptotic Theory for the Probability Density Functions in Burgers Turbulence

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A rigorous study is carried out for the randomly forced Burgers equation in the inviscid limit. No closure approximations are made. Instead the probability density functions of velocity and velocity gradient are related to the statistics of quantities defined along the shocks. This method allows one to compute the anomalies, as well as asymptotics for the structure functions and the probability density functions. It is shown that the left tail for the probability density function of the velocity gradient has to decay faster than $|\xi|^{-3}$. A further argument confirms the prediction of E *et al.* [Phys. Rev. Lett. **78**, 1904 (1997)] that it should decay as $|\xi|^{-7/2}$.

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In this Letter, we focus on statistical properties of solutions of the randomly forced Burgers equation

$$u_t + uu_x = \nu u_{xx} + f, \tag{1}$$

where f is a zero-mean, statistically homogeneous, white-in-time Gaussian process with covariance

$$\langle f(x,t)f(y,s)\rangle = 2B(x-y)\delta(t-s), \tag{2}$$

where $B(x)$ is smooth. We are particularly interested in the probability density function (pdf) of the velocity gradient $\xi(x,t) = u_x(x,t)$, since it depends heavily on the intermittent events created by the shocks. Assuming statistical homogeneity, and letting $Q(\xi;t)$ be the pdf of $\xi(x,t)$, it can be shown that Q satisfies

$$Q_t = \xi Q + \big(\xi^2 Q\big)_\xi + B_1 Q_{\xi\xi} - \nu\big(\langle \xi_{xx}|\xi\rangle Q\big)_\xi, \tag{3}$$

where $B_1 = -B_{xx}(0)$. $\langle \xi_{xx}|\xi\rangle$ is the ensemble-average of ξ_{xx} conditional on ξ. The explicit form of this term is unknown, leaving (3) unclosed. There have been several proposals on how to approximately evaluate the quantity

$$F(\xi;t) = -\lim_{\nu\to 0}\nu\big(\langle \xi_{xx}|\xi\rangle Q\big)_\xi. \tag{4}$$

At steady state, they all lead to an asymptotic expression of the form

$$Q \sim \begin{cases} C_-|\xi|^{-\alpha} & \text{as}\quad \xi\to-\infty,\\ C_+\xi^\beta \mathrm{e}^{-\xi^3/(3B_1)} & \text{as}\quad \xi\to+\infty,\end{cases} \tag{5}$$

for Q, but with a variety of values for the exponents α and β (here the $C_\pm$'s are numerical constants). By invoking the operator product expansion, Polyakov [1] suggested that $F = aQ + b\xi Q$, with $a = 0$ and $b = -1/2$. This leads to $\alpha = 5/2$ and $\beta = 1/2$. Boldyrev [2] considered the same closure with $-1 \le b \le 0$, which gives $2 \le \alpha \le 3$ and $\beta = 1 + b$. The instanton analysis [3,4] predicts the right tail of Q without giving a precise value for β, but has not given any specific prediction for the left tail. E *et al.* [5] made a geometrical evaluation of the effect of F, based on the observation that large negative gradients are generated near shock creation. Their analysis gives a rigorous upper-bound for α: $\alpha \le 7/2$. In [5], it was claimed that this bound is actually reached, i.e., $\alpha = 7/2$. Finally Gotoh and Kraichnan [6] argued that the viscous term is negligible to leading order for large $|\xi|$, i.e. $F\approx 0$ for $|\xi| \gg B_1^{1/3}$. This approximation leads to $\alpha = 3$ and $\beta = 1$. For other approaches, see e.g. [7,8]. In this letter we proceed at an exact evaluation of (4) and we prove that α has to be strictly larger than 3 (a result which does not require that steady state be reached). At steady state, we prove that $\beta = 1$ and we give an argument which supports strongly the prediction of [5], namely, $\alpha = 7/2$.

To begin with, let us remark that it is established in the mathematics literature that the inviscid limit

$$u^0(x,t) = \lim_{\nu\to 0} u(x,t), \tag{6}$$

exists for almost all (x,t). Since u^0 will in general develop shocks, say, at $x = y$, we may have $u_x^0 \propto \delta(x-y)$, and one cannot simply drop the viscous term in the Burgers equation without giving some meaning to $u^0u_x^0$ at shocks. This can be done using *BV-calculus* [9], which allows one to write an equation for u^0 and gives rules for manipulating the terms entering this equation and computing the effect of the viscous term in the inviscid limit. An alternative, more intuitive, way of accessing the effect of the viscous shock on the velocity profile outside the shock is to carry out an asymptotic analysis near and inside the shock. Here we will take the second approach and refer the interested reader to [10] for the first approach with BV-calculus. It is important to remark that the two approaches lead to the same results.

Before considering velocity gradient, it is helpful to study the statistics of velocity itself. Let $R(u;t)$ be the pdf of $u(x,t)$. Assuming statistical homogeneity, R satisfies

$$R_t = B_0 R_{uu} - \nu\big(\langle u_{xx}|u\rangle R\big)_u, \tag{7}$$

where $B_0 = B(0)$. To compute $-\nu\big(\langle u_{xx}|u\rangle R\big)_u$, let us note that for $\nu \ll 1$, the solutions of (1) consist of smooth pieces where the viscous effect is negligible, separated by thin shock layers inside which the viscous effect is important. Let $u_{\rm out}(x,t)$ be the solution of the Burgers equation outside the viscous shock layer; $u_{\rm out}$ can be obtained as a series expansion in ν. To leading order in ν, $u_{\rm out}$ satisfies Riemann's equation, $u_t+uu_x=f$. In order to deal with the shock layer, say at $x=y$, define

$$u_{\rm in}(x,t)=v\left(\frac{x-y}{\nu},t\right), \tag{8}$$

and write $v=v_0+\nu v_1+O(\nu^2)$. To leading order, $v_0(z,t)$ satisfies $(v_0-\bar u)v_{0z}=v_{0zz}$, yielding $v_0(z,t)=\bar u-(s/2)\tanh(sz/4)$ where $\bar u=dy/dt$ and $s\le 0$ is the jump across the shock. Consequently we have the following generic velocity profile inside the shock layer:

$$u_{\rm in}(x,t)=\bar u-\frac{s}{2}\tanh\left(\frac{s(x-y)}{4\nu}\right)+O(\nu). \tag{9}$$

The actual values of $\bar u$ and s are obtained from the matching conditions between $u_{\rm in}$ and $u_{\rm out}$. In terms of v and the stretched variable z, they are

$$\lim_{z\to\pm\infty}v_0=\lim_{x-y\to 0^\pm}u_{\rm out}=\bar u\pm\frac{s}{2}. \tag{10}$$

We will use (9) to evaluate the viscous term in (7). By definition [11],

$$\nu\langle u_{xx}|u\rangle R=\nu\lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L}dx\ u_{xx}\,\delta[u-u(x,t)]. \tag{11}$$

In the limit $\nu\to 0$ only small intervals around the shocks will contribute to the integral. So, we can split the integral into small pieces involving only the shock layers and use the generic form of $u_{\rm in}$ in the layers to evaluate these integrals. To $O(\nu)$, this gives

$$\begin{aligned}&\nu\langle u_{xx}|u\rangle R\\&=\nu\lim_{L\to\infty}\frac{N}{2L}\frac{1}{N}\sum_j\int_{\rm j-th\ layer}dx\ u_{{\rm in}\,xx}\,\delta[u-u_{\rm in}(x,t)]\\&=\rho\int dsd\bar u\ T(\bar u,s;t)\int_{-\infty}^{+\infty}dz\ v_{0zz}\,\delta[u-v_0(z,t)],\end{aligned} \tag{12}$$

where in the second integral we picked any particular shock layer and we went to the stretched variable $z=(x-y)/\nu$. Here N denotes the number of shocks in $[-L,L]$, $\rho=\rho(t)=\lim_{L\to\infty}N/2L$ is the shock density, and $T(\bar u,s;t)$ is the probability density of $\bar u(y,t)$ and $s(y,t)$ conditional on the property that there is a shock at position y (T is independent of y because of statistical homogeneity). The last integral in (12) can of course be evaluated using the explicit form of v_0. Another, more elegant, way to proceed is to use the equation for v_0, $(v_0-\bar u)v_{0z}=v_{0zz}$, and change the integration variable from z to v_0 using $dzv_{0zz}=dv_0v_{0zz}/v_{0z}=dv_0(v_0-\bar u)$. The result is

$$\lim_{\nu\to 0}\nu\langle u_{xx}|u\rangle R=-\rho\int ds\int_{u+s/2}^{u-s/2}d\bar u\ (u-\bar u)T(\bar u,s;t). \tag{13}$$

This equation gives an *exact* expression for the viscous contribution in the limit $\nu\to 0$ in terms of certain statistical quantities associated with the shocks. Of course, using (13) in (7) does not lead to a closed equation since T remains to be specified. However, information can already be obtained at this point without resorting to any closure assumption. For instance, using (13) in (7) and taking the second moment of the resulting equation yields $\langle u^2\rangle_t=2B_0-2\epsilon$ with

$$\epsilon=\lim_{\nu\to 0}\nu\langle u_x^2\rangle=\frac{1}{12}\rho\langle|s|^3\rangle. \tag{14}$$

In particular, at steady state $\rho\langle|s|^3\rangle=12B_0$.

Similar calculations can be carried out for multi-point pdf's and, in particular, for $W(w;x,t)$, the pdf of the velocity difference $w(x,z,t)=u(x+z,t)-u(z,t)$. It leads to an equation of the form

$$\begin{aligned}W_t=-wW_x-2\int_{-\infty}^{w}dw'\ W_x(w';x,t)\\+2[B_0-B(x)]W_{ww}+H(w;x,t),\end{aligned} \tag{15}$$

where, to $O(x)$, H is given by

$$\begin{aligned}H=&\rho\big[wS(w;t)+\langle s\rangle\delta(w)\big]\\&+2\rho\int_{-\infty}^{w}dw'\ S(w';t)-2\rho\theta(w)+O(x).\end{aligned} \tag{16}$$

Here $\theta(w)$ is the Heaviside function and $S(s;t)=\int d\bar u\,T(\bar u,s;t)$ is the conditional pdf of $s(y,t)$. By direct substitution it may be shown that the solution of (15) is, to $O(x^2)$, [13]

$$W\sim(1-\rho x)\frac{1}{x}Q\left(\frac{w}{x};t\right)+\rho xS(w;t)+O(x^2). \tag{17}$$

The first term in this expression contains $Q(\xi;t)$, the pdf of the non-singular part of the velocity gradient, to be considered below (see (20)). This term accounts for those realizations of the flow where there is no shock in between z and $x+z$ (an event of probability $1-\rho x+O(x^2)$). This term also leads to the consistency constraint that $\lim_{x\to 0}W=\delta(w)$ (using $\lim_{x\to 0}Q(w/x;t)/x=\delta(w)$). The next term in (17), $\rho xS(w;t)$, accounts for the realizations of the flow where there is a shock in between z and $x+z$ (an event of probability $\rho x+O(x^2)$). Equation (17) can be used to compute the structure functions, $\langle|w|^a\rangle=\int dw\,|w|^aW$. To leading order this gives

$$\langle|w|^a\rangle\sim\begin{cases}x^a\langle|\xi|^a\rangle+O(x) & \text{if }\ 0\le a<1,\\ x\rho\langle|s|^a\rangle+O(x^{1+a}) & \text{if }\ 1<a,\end{cases} \tag{18}$$

where $\langle|\xi|^a\rangle=\int d\xi\,|\xi|^aQ$. Using $\rho\langle|s|^3\rangle=12B_0$, we get Kolmogorov's relation for $a=3$

$$\langle|w|^3\rangle\sim 12xB_0. \tag{19}$$

We now go back to the velocity gradient. Observe first that, in the limit $\nu\to 0$, the velocity gradient can be written as

$$u_x(x,t) = \xi(x,t) + \sum_j s(y_j)\delta(x-y_j), \qquad (20)$$

where the y_j's are the locations of the shocks, ξ is the *non-singular* part of u_x. Assuming homogeneity, a direct consequence of (20) is

$$\langle u_x \rangle = \langle \xi \rangle + \rho \langle s \rangle = 0. \qquad (21)$$

Unlike the viscous case where $\xi = u_x$, hence $\langle \xi \rangle = 0$, we have in the inviscid limit $\langle \xi \rangle = -\rho\langle s \rangle \neq 0$. Note also that the inviscid limit of the solutions of (3) converge to the pdf of ξ only, which is still going to be denoted by Q.

To evaluate F, there are two ways to proceed. One is to rewrite (15) in terms of the pdf of $(u(x+z,t)-u(z,t))/x$ and take the limit as x goes to zero. This is the approach taken in [10]. The other is to evaluate (4) directly. The two approaches amount to different orders of taking the limit $x \to 0, \nu \to 0$, and give the same result. Hence the two limiting processes commute. We will take the second approach and evaluate (4) using the same basic idea as above. Here, however, we have to proceed more carefully with the shock layer analysis. Differentiation of (9) gives

$$\xi_{\rm in}(x,t) = -\frac{s^2}{8\nu}\mathrm{sech}^2\left(\frac{s(x-y)}{4\nu}\right) + O(1). \qquad (22)$$

While the next order term in (9) was negligible in the limit $\nu \to 0$, the $O(1)$ contribution to $\xi_{\rm in}(x,t)$ actually *dominates* the $O(\nu^{-1})$ contribution at the border of the shock layer because the latter falls exponentially fast as the outer region is approached, whereas the former tends to constants, say, $\xi_\pm$. In particular, the matching between $\xi_{\rm out}(x,t)$ and $\xi_{\rm in}(x,t)$ involves the $O(1)$ terms. To see how matching takes place, differentiating the expression for $u_{\rm in}$, we have $\xi_{\rm in} = \nu^{-1}v_{0z} + v_{1z} + O(\nu)$. The matching condition between $\xi_{\rm in}$ and $\xi_{\rm out}$ reads

$$\lim_{z\to\pm\infty} v_{1z} = \lim_{x-y\to 0^\pm} \xi_{\rm out} \equiv \xi_\pm. \qquad (23)$$

The equation for v_1 is

$$v_{0t} + (v_0 - \bar{u})v_{1z} + v_1 v_{0z} = v_{1zz} + f_x, \qquad (24)$$

and, from the above argument, the only information we really need about v_1 is its values at the boundaries $z \to \pm\infty$. Since v_{0z} falls exponentially fast for large $|z|$, (24) reduces to

$$\bar{u}_t \pm \frac{s_t}{2} \pm \frac{s}{2}v_{1z} = v_{1zz} + f_x, \qquad z \to \pm\infty, \qquad (25)$$

where we used the asymptotic values of v_0. Thus, as $z \to \pm\infty$,

$$v_1 \sim \mp\frac{2\bar{u}_t}{s}z - \frac{s_t}{s}z \pm \frac{2f_x}{s}z + c_1^\pm + c_2^\pm e^{\pm sz/2}. \qquad (26)$$

Notice that the exponential terms are irrelevant in these expression since $s \le 0$. Equation (26) implies

$$\lim_{z\to\pm\infty} v_{1z} = \mp\frac{2\bar{u}_t}{s} - \frac{s_t}{s} \pm \frac{2f_x}{s} = \xi_\pm, \qquad (27)$$

where the last equality is just the definition of $\xi_\pm$. Note that (27) can be rewritten as

$$s_t = -\frac{s}{2}\big(\xi_- + \xi_+\big), \qquad \bar{u}_t = \frac{s}{4}\big(\xi_- - \xi_+\big) + f_x. \qquad (28)$$

In the limit $\nu \to 0$ these are the equations of motion along the shock.

We can now evaluate the viscous contribution using

$$\nu\langle \xi_{xx}|\xi\rangle Q = \nu \lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L} dx\ \xi_{xx}\,\delta[\xi - \xi(x,t)]. \qquad (29)$$

The calculation is similar to the one for the velocity and eventually leads to

$$F(\xi;t) = \frac{\rho}{2}\int ds\ s\big[V_-(\xi,s;t) + V_+(\xi,s;t)\big], \qquad (30)$$

where $V_\pm(\xi,s;t)$ are the conditional pdf's of $\xi_\pm(y,t)$ and $s(y,t)$. The appearance of $\xi_\pm$ in (30) is of course a direct result of the $O(1)$ term in (22).

We now use (30) in (3) and analyze some consequences of

$$Q_t = \xi Q + \big(\xi^2 Q\big)_\xi + B_1 Q_{\xi\xi} + F(\xi;t). \qquad (31)$$

Taking the first moment of (31) leads to

$$\langle \xi \rangle_t = \big[\xi^3 Q\big]_{-\infty}^{+\infty} + \frac{\rho}{2}\left[\langle s\xi_-\rangle + \langle s\xi_+\rangle\right], \qquad (32)$$

where we used $\int d\xi\ \xi F = \rho[\langle s\xi_-\rangle + \langle s\xi_+\rangle]/2$. On the other hand, averaging the first equation in (28) gives [14]

$$\big(\rho\langle s\rangle\big)_t = -\frac{\rho}{2}\left[\langle s\xi_-\rangle + \langle s\xi_+\rangle\right]. \qquad (33)$$

Since $\langle\xi\rangle_t = -(\rho\langle s\rangle)_t$ from (21), the comparison between (32) and (33) tells us that the boundary term in (32) must be zero. Since $Q \ge 0$, $\xi^3 Q$ has different sign for large positive and large negative values of ξ. Therefore we must have $\lim_{\xi\to+\infty}\xi^3 Q = 0$ *and* $\lim_{\xi\to-\infty}\xi^3 Q = 0$. This proves that Q goes to zero *faster* than $|\xi|^{-3}$ as $\xi \to -\infty$ and $\xi \to +\infty$.

The analysis can be carried out one step further for the stationary case ($Q_t = 0$). In this case, treating (31) as an inhomogeneous second order ordinary differential equation, we can write its general solution as $Q = C_1 Q_1 + C_2 Q_2 + Q_3$, where C_1 and C_2 are constants, Q_1 and Q_2 are two linearly independent solutions of the homogeneous equation associated with (31), and Q_3 is some particular solution of this equation. One such particular solution is

$$Q_3 = \int_{-\infty}^{\xi} d\xi'\ \frac{\xi' F(\xi')}{B_1} - \frac{\xi e^{-\Lambda}}{B_1}\int_{-\infty}^{\xi} d\xi'\ e^{\Lambda'} G(\xi'), \qquad (34)$$

where $\Lambda = \xi^3/(3B_1)$ and

$$G(\xi) = F(\xi) + \xi\int_{-\infty}^{\xi} d\xi'\ \frac{\xi' F(\xi')}{B_1}. \qquad (35)$$

With this particular solution, it can be shown (see [10] for details) that the realizability constraints imply that $C_1 =$

$C_2 = 0$, i.e. the only non-negative, integrable solution is $Q = Q_3$. Furthermore, in order that Q actually be non-negative, F must satisfy

$$0 \geq F \geq C\xi^2 \mathrm{e}^{-\xi^3/(3B_1)} \qquad \text{as} \quad \xi \to +\infty, \tag{36}$$

for some constant $C < 0$. Substituting into (34), we get

$$Q \sim \begin{cases} C_-|\xi|^{-3} \int_{-\infty}^{\xi} d\xi' \; \xi' F(\xi') & \text{as} \quad \xi \to -\infty, \\ C_+ \xi \mathrm{e}^{-\xi^3/(3B_1)} & \text{as} \quad \xi \to +\infty, \end{cases} \tag{37}$$

which confirms the result $Q \sim C_-|\xi|^{-\alpha}$ with $\alpha > 3$ as $\xi \to -\infty$, and gives $\beta = 1$.

The actual value of the exponent α depends on the asymptotic behavior of F. The latter can be obtained from further considerations on the dynamics of the shock (28). This is rather involved and will be left to [10]. The result gives $\alpha = 7/2$ which confirms the prediction of [5]. Here we will restrict ourselves to an interpretation of the current approach in terms of the geometric picture. Observe that the largest values of $\xi_\pm$ are achieved just after the shock formation. Assume that a shock is created at time $t = 0$, position $x = 0$, and with velocity $u = 0$. Then, locally

$$x = ut - au^3 + \cdots. \tag{38}$$

It follows that for $t \ll 1$ the solutions of $0 = ut - au^3$, $u_\pm$, behave as

$$u_\pm = \mp\sqrt{\frac{t}{a}} \quad \Rightarrow \quad s = -2\sqrt{\frac{t}{a}}, \tag{39}$$

and $\xi_\pm$, solutions of $1 = \xi t - 3au^2\xi$, behave as

$$\xi_\pm = -\frac{1}{2t}. \tag{40}$$

Assuming that these give the dominant contribution to $F(\xi)$ for large negative values of ξ, the asymptotic form of F is

$$F \sim C \int_0^\infty dt \; s(t) \{\delta[\xi - \xi_-(t)] + \delta[\xi - \xi_+(t)]\}, \tag{41}$$

where C is some constant related to the statistics of the shock life-time and a, and $s(t)$, $\xi_\pm(t)$ are given by (39), (40). The evaluation of (41) gives $F \sim C|\xi|^{-5/2}$, and, hence,

$$Q \sim C_-|\xi|^{-7/2} \qquad \text{as} \quad \xi \to -\infty. \tag{42}$$

Even though this argument gives only a lower bound for F at large negative values of ξ, further arguments presented in [10] indicate that this lower bound is actually sharp.

We conclude that the approximations given in [1,2,6] are too simplistic, and the geometric picture presented in [5] is actually sharp for the forced Burgers turbulence. In view of the elegance of the argument presented in [6], one is naturally interested in the implications of neglecting all together the viscous term in (3). This issue is dealt with in [12], and it is concluded there that the resulting equation describes the evolution of the signed probability density function for multi-valued solutions of Riemann's equation: $u_t + uu_x = f$.

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[1] A. M. Polyakov, Phys. Rev. E **52**, 6183 (1995). Here we followed the presentation of [6]. It is worth noting that Polyakov never discusses velocity gradients. Instead he studies the pdf of velocity increment over the inertial range of scales. However, his predictions imply the results stated here, as a consequence of the discussions before (22).

[2] S. A. Boldyrev, Phys. Rev. E **55**, 6907 (1997).

[3] V. Gurarie and A. Migdal, Phys. Rev. E **54**, 4908 (1996).

[4] E. Balkovsky, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. Lett. **78**, 1452 (1997).

[5] W. E, K. Khanin, A. Mazel, and Ya. G. Sinai, Phys. Rev. Lett. **78**, 1904 (1997).

[6] T. Gotoh and R. H. Kraichnan, Phys. Fluids **10**, 2859 (1998).

[7] J.-P. Bouchaud, M. Mézard, and G. Parisi, Phys. Rev. E **52**, 3656 (1995).

[8] J.-P. Bouchaud and M. Mézard, Phys. Rev. E **54**, 5116 (1996).

[9] A. I. Vol'pert, Math. USSR-Sbornik **2**, 225 (1967).

[10] W. E and E. Vanden Eijnden, in preparation.

[11] Here we use ergodicity with respect to spatial average. This restricts us to working on the entire line in which case the existence of a stationary state remains unclear. However, spatial average is used only for simplicity of the argument and can be avoided as is done in [10]. Then one can work on a finite system for which case the existence of a stationary state is proved in: W. E, K. Khanin, A. Mazel, and Ya. G. Sinai, "Invariant measures for the random-forced Burgers equation," submitted to Ann. Math.

[12] W. E and E. Vanden Eijnden, submitted to Phys. Fluids.

[13] Equation (17) uses relation (21).

[14] The fact that ρ appears inside the derivative at the left hand side of (33) is non-trivial (recall that $\rho_t \neq 0$). It is because we average (28) conditional on the property that s, $\xi_\pm$ are evaluated at a shock position (for details, see [10]).