PDFs for Velocity and Velocity Gradients in Burgers Turbulence

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Abstract

We characterize the tails of the probability distribution functions for the solution of Burgers' equation with Gaussian initial data and its derivatives $\frac{\partial^k v(x,\, t)}{\partial x^k}$, $k \;=\; 0, 1, 2....$ The tails are "stretched exponentials" of the form $\mathrm{P}(\theta) \;\propto\; \exp\,(-(Re)^{-p}\, t^q\, \theta^r)$, where Re is the Reynolds number. The exponents p , q and r depend on the initial spectrum as well as on the order of differentiation, k. These exact results are compared with those obtained using the mapping closure technique.

1. Introduction

Burgers' equation with a random initial data or a with a random stirring force is often used to test various proposals of Turbulence Theory, such as Kraichnan's Direct Interaction Approximation [12], the Renormalization Group [5] and the Mapping Closure [6], [7], [8]. Although the physics of "Burgers Turbulence" and of fluid turbulence are rather different, compressible turbulence is somewhat analogous to Burgers' equation at small scales, since both systems exhibit shock-wave precursors which dissipate energy. Burgers' equation with random initial data can be viewed as an extremely simplified model for acoustic turbulence [11], [9].

In this paper, we study the one-point velocity statistics of solutions to the Burgers' equation with random initial data. Specifically, we characterize the tails of the probability distribution functions (PDFs) for the velocity and its higher-order derivatives. The systematic study of PDFs in turbulence was initiated by Kraichnan [6] and Pope [7] to quantify the deviation from Gaussianity arisng from nonlinear momentum transport. Their work and various nunerical results [4] show a noticeable difference between the PDF tails of the velocity and those of the velocity gradient, attributed to the intermittency of the underlying vorticity field.

In the special case of Burgers' equation, the PDFs of the velocity and its gradient were studied using the mapping closure and numerical simulations by [6] and [8]. Their results indicate that the PDF for the velocity derivative has approximately exponential tails, at least for small Reynolds numbers (Re $\approx$ 10).

We analyze here the PDFs of the velocity and its higher-order derivatives from first principles, using its explicit quadrature via the Hopf-Cole transformation [3] We show that the PDFs of $\frac{\partial^k v(x,\ t)}{\partial x^k}$ have tails that can be fitted to "stretched exponentials" of the form $\mathrm{P}(\theta)\ \propto\ \exp\,(-(Re)^{-p}\, t^q\, \theta^r)$, with fractional values of $p,\ q$ and r depending on the initial velocity spectrum and on the order of differentiation. We also give an estimate on the "range of validity" of these stretched exponential tails in terms of the Reynolds number. Since these results are exact, they can be compared with other predictions on the PDF tails. The first conclusion that emerges is that the shape of PDF tails is highly non-universal: it depends sensitively on the initial velocity spectrum. The second is that the Burgers velocity field will typically have shorter-than-Gaussian tails due to strong energy dissipation through shocks. The PDF tails of derivatives are generally fatter, and more so as the order of differentiation increases. However, the tails for the PDFs of velocity derivatives are shorter than those predicted by Gotoh and Kraichnan using the mapping closure. A possible explanation for this is the *a priori* assumption of Gaussian velocity statistics made in the mapping closure theory. Although this assumption appears justifiable for incompressible turbulence, the Burgers picture suggests that compressible turbulence has shorter tails due to the stronger energy dissipation through shocks.

We consider solutions of the non-dimensional viscous Burgers equations

$$\frac{\partial v(x,\ t)}{\partial t} \ +\ v(x,\ t)\ \frac{\partial v(x,\ t)}{\partial x} \ =\ \frac{1}{Re}\frac{\partial^2 v(x,\ t)}{\partial x^2} \tag{1}$$

with

$$v(x,\ 0)\ =\ v_0(x) \tag{2}$$

being a a random stationary Gaussian field with mean zero and self-similar spectrum

$$E(q)\ =\ E_0\,|q/q_0|^{\epsilon}\ ,\qquad -1\ <\ \epsilon\ <\ +1\ , \tag{3}$$

or with the "ultraviolet-regularized" spectrum

$$E(q)\ =\ E_0\,|q/q_0|^{\epsilon}\ e^{-|q/q_1|^2}\ ,\qquad -1\ <\ \epsilon\ <\ \infty\ . \tag{4}$$

The range of ϵ in (3) corresponds to self-similar velocity potentials $\phi_0(x)\ =\ \int_0^x v_0(x')\,dx'$ satisfying the coarseness condition

$$<\ (\phi_0(x)\ -\ \phi_0(y))^2\ >\propto\ (x\ -\ y)^{2H}\ , \tag{5}$$

with Hurst exponent $H\ =\ (1\ -\ \epsilon)\ /\ 2$ in the range $0\ <\ H\ <\ 1$. The spectra in (4) correspond to regularized velocities with potentials satisfying (5) for $|x-y|\ \gg\ 1$, if $-1\ <\ \epsilon\ <\ +1$, or with stationary potentials with

$$<\ (\phi_0(x))^2\ >=\ \text{constant} \tag{6}$$

when $\epsilon\ >\ 1$. The critical value $\epsilon\ =\ 1$ corresponds to logarithmic growth of the potential mean-square differences [1].

Our results on the PDF tails are as follows: if $-1\ <\ \epsilon\ <\ +1$ $(0\ <\ H\ <\ 1)$ with initial spectra (3) or (4), then

$$\text{Prob.}\{\ \ |\partial_x^k\ v(x,\ t)|\ >\ \theta\ \}\ \approx\ \exp\left[\,-C\,(Re)^{-p}\ t^q\ \theta^r\ \right]\ , \tag{7}$$

for $\theta\ \gg\ \text{Max}\,[(Re)^{-1}\,,\ (Re)^k\,]$, with

$$p\ =\ \frac{(3\ +\ \epsilon)\ k}{k\ +\ 1}\qquad ,\quad q\ =\ 1\ +\ \epsilon\ , \tag{8}$$

and

$$r\ =\ \frac{3\ +\ \epsilon}{k+1}\ . \tag{9}$$

For $\epsilon\ >\ 1$ and spectra (4), the PDFs have "stretched exponential" tails of

[1]The class of initial velocities with spectra (3) and (4) span the full range of stationary Gaussian functions with power-type infrared scaling.

the form (7) with

$$p \;=\; \frac{4\;k}{k\;+\;1} \qquad , \;\; q \;=\; 2\;, \tag{10}$$

and

$$r \;=\; \frac{4}{k+1}\;. \tag{11}$$

In the marginal case $\epsilon \;=\; 1$, we obtain a logarithmic correction to (7):

$$\text{Prob.}\{\;\;|\partial_x^k\; v(x,\;t)|\;>\;\theta\;\}\;\approx\;\exp\Big[\;\frac{-C\;(Re)^{-p}\;t^q\;\theta^r}{\ln[\theta^{1/(k+1)}\,(Re)^{-k/(k+1)}\,t\;]}\;\Big]\;, \tag{12}$$

where $p,\; q$ and r are given in (10) and (11). [2]

For all spectra considered here, the extreme tails for the velocity PDF are shorter than Gaussian. Notice that Gaussian tails are obtained formally in the limit $\epsilon\;\rightarrow\;-1$, which is the threshold where the initial velocity ceases to be statistically homogeneous. The exponent for the velocity derivative, obtained by setting $k = 1$ in (9) and (11), is $r\;\;=\;\;\text{Max}[\;(3+\epsilon)/2\;,\;2\;]$. Exponential tails (with exponent 1) are not achieved for stationary processes; they correspond again to the limit $\epsilon\;\;\rightarrow\;\;-1$. Gaussian-type tails for the derivative ($r\;=\;2$) are obtained when the velocity potential is stationary ($\epsilon\;>\;1\;;\;H\;=\;0$).

In the case of delta-correlated or short-range correlated initial velocities ($\epsilon\;=\;0$ or $H\;=\;1/2$), we obtain from (7)

$$\text{Prob.}\{\;\;|\partial_x^k\; v(x,\;t)|\;>\;\theta\;\}\;\approx\;\exp[\;\;-C\;(Re)^{-3k/(k+1)}\;t\;\theta^{3/(k+1)}\;]\;, \tag{13}$$

for $\theta\;\;\gg\;\;\text{Max}\,[(Re)^{-1}\,,\,(Re)^k\,]$. Thus, for white-noise initial data, the velocity tails are the stretched exponentials $\exp\,[-C\,t\;\theta^3\,]$ and the velocity derivative tails are given by $\exp\,[\,-C\,(Re)^{-1.5}\,t\,\theta^{1.5}\,]$.

2. Hopf-Cole analysis

[2]The asymptotic relations (7) and (12) are logarithmically accurate, in the sense that algebraic prefactors depending on Re , t and θ are not included. Also, C represents a constant depending on $E_0\,, q_0$ and sometimes q_1, which we do not calculate explicitly.

The shapes of the PDF tails can be derived from simple considerations involving the Hopf-Cole formula [3]

$$v(x\ ,t)\ =\ x/t\ -\ \frac{\int_{-\infty}^{+\infty} \frac{y}{t}\ e^{-\frac{Re}{2}\ [\ \frac{(x-y)^2}{2t}\ +\ \phi_0(y)\ ]}\ dy}{\int_{-\infty}^{+\infty} e^{-\frac{Re}{2}\ [\ \frac{(x-y)^2}{2t}\ +\ \phi_0(y)\ ]}\ dy}\ . \tag{14}$$

If $Re \gg 1$, it is well-known that the Hopf-Cole formula can be analyzed by the method of steepest descent. The potential

$$\Phi(x,\ y\ ,t)\ =\ \frac{(x-y)^2}{2t}\ +\ \phi_0(y) \tag{15}$$

controls the shape of $v(x\ ,t)$. For each x, the main contribution to the right-hand side of (14) comes from the points y at which $\Phi(x,\ y\ ,t)$ achieves its overall minimum value. Steep ramps, corresponding to shock precursors, are formed at spatial locations x where this minimum is achieved at two or more points. Large gradients develop only in the immediate neighborhood of shock precursors. The strength of the shock is given by $\delta y\,/t$, where δy is the distance separating the two minimizers of (15). Moreover, the width of the transition layer around such a shock is proportional to

$$w\ =\ \frac{t}{Re\,\delta y}. \tag{16}$$

Returning to the case of *finite* Reynolds numbers, we observe that a similar steepest descent argument can be made in a vicinity of a steep ramp provided that the shock strength $\delta y/t$ much larger than $(Re)^{-1}$. This is done by rewriting Burgers' equation in scaled variables so that the velocity is of order one in the vicinity of such a steep ramp. The resulting Hopf-Cole formula exhibits then a large "effective" Reynolds number of order $Re\,\delta y$. If we assume without loss of generality that $x\ =\ 0$, we obtain the asymptotic expression

$$v(x,\ t)\ \approx\ \frac{\zeta}{t}\ +\ \frac{x}{t}\ -\ \frac{\delta y}{2t}\ \tanh(\frac{Re\ \delta y\ x}{4t}\ ) \tag{17}$$

valid for $\delta y\ \gg\ (Re)^{-1}\,t$ and for $|x|\ \leq\ \frac{t}{Re\,\delta y}$, where ζ is the center of the interval determined by the minima.

This formula shows that the solution of Burgers equation satisfes, to leading order, the scaling laws

$$\frac{\partial^k v(x,\ t)}{\partial x^k} \ \approx\ (\ \frac{\delta y}{t}\ )^{k+1}\ (Re)^k \quad ,\ k = 0, 1, 2, ..., \text{etc.} \tag{18}$$

Recalling that (17) holds whenever $\delta y/t \ \gg\ (Re)^{-1}$, we conclude the range of values of the velocity and its derivatives for which the statistics will be determined by such "shock precursors" is

$$|\frac{\partial^k v(x,\ t)}{\partial x^k}|\ \gg\ (Re)^{-1} \tag{19}$$

The analysis of the tails of the distributions of the velocity and its derivatives in the range (19) reduces to estimating the tails of the distribution of δy. This problem was studied rigorously by the last two authors in [2] and [1] (1994), for the case of white-noise initial data ($\epsilon\ =\ 0$). In his forthcoming thesis, Ryan [10] obtained rigorous estimates for the tails of the PDF of δy over the entire range of ϵ, as well as a mathematical proof for the scaling argument presented here. The next paragraph contains the essence of this argument.

3. Method of steepest descent and the PDF of δy

We claim that, in general, the PDF of δy satisfies the scaling relations

$$\text{Prob}\{\ \delta y\ >\ L\ \}\ \approx\ \begin{cases} \exp[\ -C\, L^{3+\epsilon}/t^2\ ] & \text{if } -1\ <\ \epsilon\ <\ 1 \\ \\ \exp[\ -C\, L^4/t^2\ ] & \text{if } \epsilon\ >\ 1 \end{cases} \tag{20}$$

for t fixed and $L\ \gg\ t$. It is easy to check that the characterization of the PDFs for the velocity and its derivatives, (7), follows from these asymptotic formulas and the scaling relations, (18), derived in the previous section. However, it is important to notice that the asymptotics for δy presented here apply only for $\delta y\ \gg\ t$, which translates, using (18), into

$$|\frac{\partial^k v(x,t)}{\partial x^k}|\ \gg\ (Re)^k\ . \tag{21}$$

Together with (19), this condition gives the range of validity of the stretched exponentials in (7), $\theta \gg \text{Max}\,[\,(Re)^{-1}\,, (Re)^{k}\,]$.

Let us make explicit the scaling relations in (20). The probability that given realization of the velocity potential $\phi_0 = \phi$ occurs is proportional to

$$\exp[\, - \, S(\phi)\,]\ , \qquad S(\phi) \;=\; \frac{1}{2}\int |\widehat{\dot\phi(q)}|^2\, \frac{1}{E(q)}\, dq\ . \tag{22}$$

Here, $S(\phi)$ is the classical action associated with the the Gaussian field with energy spectrum $E(q)$ given in (3) or (4) and $\widehat{\dot\phi(q)}$ denotes the Fourier transform of the derivative of the path ϕ .[3]

The probability that $\delta y \;>\; L$ for $L \;\gg\; 1$ can be estimated by the method of steepest descent, i.e. by finding the minimum action among the set of all paths satisfying this condition. For this purpose, observe that if $\delta y \;>\; L$, and if (without loss of generality) the interval between the two minimizers contains the Lagrangian point $y = 0$, then the potential

$$\Phi(0,\ y,\ t) \;=\; \frac{y^2}{2t} \;+\; \phi_0(y) \tag{23}$$

must be negative for some $|y| \;\geq\; L/2$, whence

$$\phi_0(y) \;=\; -y^2/2t \qquad\qquad \text{for some } |y| \;\geq\; L/2\ . \tag{24}$$

A straightforward calculation shows that the minimum action over the set of all paths paths satisfying $\phi \;=\; 0$ and $\phi(y) \;=\; -y^2/2t$ is

$$S_{min} \;=\; \frac{y^4}{8t^2}\,\frac{1}{\int\ \frac{\sin^2(qy/2)}{q^2}\,E(q)\,dq}\ . \tag{25}$$

Notice that the denominator in this last expression depends on y and behaves differently according to whether $\epsilon \,<\, 1$ or $\epsilon \,>\, 1$. Specifically, for $y \;\gg\; 1$,

$$S_{min} \;\approx\; \frac{y^{3+\epsilon}}{8t^2}\,\frac{1}{\int\ \frac{\sin^2(q/2)}{q^2}\,E_0\,|q/q_0|^{\epsilon}\,dq} \tag{26}$$

[3]Recall that ϕ is the integral of v_0. Hence the derivative of ϕ appears in the integral in (22).

if $-1 \ < \ \epsilon \ < \ 1$, whereas

$$S_{min} \ \approx \ \frac{y^4}{4t^2} \frac{1}{\int \ \frac{1}{q^2} \, E_0 \, |q/q_0|^{\epsilon} \, e^{-|q/q_1|^2} dq} \tag{27}$$

if $\epsilon \ > \ 1$. Optimizing over $y \ > \ L/2$, we deduce the asymptotic upper bounds

$$\text{Prob}\,\{ \ \delta y \ > \ L \ \} \ \leq \begin{cases} \exp\,[\ -C\, L^{3+\epsilon}/t^2\ ] & \text{if } -1 \ < \ \epsilon \ < \ 1 \\ \\ \exp\,[\ -C\, L^{4}/t^2\ ] & \text{if } \epsilon \ > \ 1. \end{cases} \tag{28}$$

for $L \ \gg \ 1$.

Lower bounds are obtained by estimating the probability of a particular realization $\phi_0 \ = \ \phi$ such that the potential $\Psi(0 \ , y, \ t)$ has two minima separated by a distance of at least L. For instance, if $-1 \ < \ \epsilon \ < \ 1$, we take

$$\phi_0(y) \ \approx \ \phi(y) \ = \begin{cases} 0 & \text{if } y \ \leq \ L/2 \\ \\ -\frac{L}{t}\,(y \ - \ L/2) & \text{if } L/2 \ \leq \ y \ \leq \ L \\ \\ -L^2/2t & \text{if } y \ \geq \ L \ . \end{cases} \tag{29}$$

Adding the term $y^2/2t$ to this function, we obtain a double-well potential $\Psi(0 \ , y, \ t)$ with minima at $y = 0$ and $y = L$. The action of the path $\phi(y)$ in (29) is easily computed since

$$\dot{\phi}(y) \ = \ - \ \frac{L}{t} \ (H(y) \ - \ H(y - L/2) \ , \tag{30}$$

where H is the Heaviside function. Taking the Fourier transform, we obtain

$$\widehat{\dot{\phi}(q)} \ = \ -\frac{L}{t} \frac{1 \ - \ e^{i \frac{Lq}{2}}}{q} \ . \tag{31}$$

In the case of pure power-law spectra (3) we substitute (31) into (22) and obtain

$$S(\phi) \ = \ \frac{L^2}{2E_0 \, t^2} \int \frac{\sin^2(\frac{Lq}{2})}{q^2} |q|^{-\epsilon} \, dq \ = \ \frac{L^{3+\epsilon}}{2E_0 \, t^2} \int \frac{\sin^2(\frac{q}{2})}{q^2} |q|^{-\epsilon} \, dq \ . \tag{32}$$

Thus, the likelihood that Φ has two minima separated by a distance $L \gg t$ is at least of order $\exp\left[-C\, L^{3+\epsilon}/t^2 \right]$, completing the proof of the asymptotic relation (20) for this case. If the spectrum has a high-wavenumber , we should mollify the function in (29) since otherwise the action diverges. This can be done by convolving $\phi(y)$ with a Gaussian pulse $(2\pi\, A)^{-1/2} \exp\left[-y^2/2A\right]$ with variance A larger than $2/q_1^2$. The Fourier transform of this smoother function is then

$$\widehat{\dot{\phi}(q)} \;=\; -\frac{L}{t}\, \frac{1 \;-\; e^{i\frac{Lq}{2}}}{q}\; e^{-A\,q^2\,/2}\;. \tag{33}$$

It is clear that the use of the mollified version of (29) will not affect the double-well structure of Φ for $L \gg 1$. Furthermore, it is easy to verify that the asymptotic behavior of the action for $L \gg t$ is given by (32) to leading order. This completes the proof of (20) for $-1 < \epsilon < 1$.

For $\epsilon > 1$, a different "trial path" is required due to the non-integrability of $1/E(q)$ for small momenta. For $1 < \epsilon < 3$, we consider realizations such that

$$\phi_0(y) \;\approx\; \phi(y) \;=\; \begin{cases} 0 & \text{if } y \;\le\; L-1 \\[2ex] -\frac{L^2}{2t}\,(y \;-\; L \;+\; 1) & \text{if } L-1 \;\le\; y \;\le\; L \\[2ex] \frac{L^2}{2t}\,(y \;-\; L \;-\; 1) & \text{if } L \;\le\; y \;\le\; L+1 \\[2ex] 0 & \text{if } y \;\ge\; L+1\;. \end{cases} \tag{34}$$

Notice that the associated potential Φ has minima at $y = 0$ and $y = L$, with a deep well at the latter point. However, due to the presence of the ultraviolet energy cutoff, the piecewise linear function in (34) has infinite action. This is remedied as before by convolving the piecewise linear function with the Gaussian pulse with variance $A \;>\; 2/q_1$, at the expense of a minor modification of Φ. An explicit calculation then shows that the Fourier transform of the mollified function is

$$\widehat{\dot{\phi}(q)} \;=\; -\frac{L^2}{2t}\, e^{i(L-1)q}\, \frac{(1 \;-\; e^{iq})^2}{q}\; e^{-A\,q^2\,/2}\;, \tag{35}$$

which vanishes at $q = 0$ together with its first derivative. The action corresponding to this path is

$$\frac{L^4}{2E_0\, t^2} \int \frac{\sin^4(\frac{q}{2})}{q^2} |q|^{-\epsilon}\, e^{-(A-2/q_1^2)\, q^2} dq\ . \tag{36}$$

Notice that the integral converges at low momenta for $1 < \epsilon < 3$. This shows that for ϵ between 1 and 3, the probability of interest is at least $\exp\,[-CL^4/t^2]$ for $L \gg t^{1/2}$, and completes the proof of (20) for this range of ϵ.

The treatment of higher values of ϵ is analogous. Realizations with Fourier transform vanishing to any order at $q = 0$, and which correspond to a a double-well potential Φ with minima separated by a distance L, can be constructed as piecewise linear functions of the form

$$\phi(y) \;=\; \sum_{j=1}^{N} \alpha_j\, (y \,-\, \beta_j)^+\ , \tag{37}$$

where $X^+ = 0$ if $X < 0$ and $X^+ = X$ if $X \geq 0$. The function in (34) has this form with $N = 3$, $\alpha_1 = -L^2/2t$, $\alpha_2 = L^2/t$, $\alpha_3 = -L^2/2t$, $\beta_1 = L - 1$, $\beta_2 = L$and $\beta_3 = L + 1$. In general, the function in (37) should have coefficients

$$\alpha_j \;=\; (-1)^{j+1}\, \frac{N!}{j!(N-j)!}\, \frac{L^2}{2t} \qquad\qquad \beta_j \;=\; L - 2 + j\ , \tag{38}$$

for $1 \leq j \leq N$ with N large enough. This concludes the proof of the scaling relations in (20) for all $\epsilon \neq 1$. The critical case $\epsilon = 1$, which corresponds to logarithmic corrections (12), is left to the interested reader.

4. Discussion

The method used to characterize the PDF tails hinged on two observations: first, for finite Reynolds numbers the extreme values of the velocity and its gradients can be obtained from the steep ramps corresponding to shock precursors. Mathematically, this follows from a steepest descent argument applied to the Hopf-Cole formula. The second observation is that the statistics for the occurrence of extreme events for Gaussian processes can be calculated by minimizing the appropriate action, *i.e.* by the method of steepest descent in function space.

To conclude, it is interesting to discuss the overall shape of the PDFs and the range of validity of these "stretched exponential" tails. This can be done by considering first the cases of very small or very large Reynolds numbers. If the Reynolds number is very small, we have $Re^k \gg (Re)^{-1}$ and hence the tails of the distributions of v , $\frac{\partial v}{\partial x}$ etc, will be stretched exponentials for values of θ such that $\theta \gg (Re)^{-1}$. In the intermediate range, $1 \ll \theta \leq (Re)^{-1}$, we expect Gaussian-like profiles since viscous damping is the dominant effect.

On the other hand, in the case of large Reynolds numbers, we have $(Re)^{-1} \ll (Re)^k$. Thus, the stretched exponentials will be observable only for $\theta \gg (Re)^k$. The "intermediate range", $(Re)^{-1} \ll \theta \ll (Re)^k$, makes sense for the PDF of the *derivatives* of v (when $(k \neq 0)$. This range is interesting because even though condition (19) is satisfied, and thus shock precursors dominate viscous damping, θ is not sufficiently large to guarantee that the stretched exponential law for δy in (20) holds. For such values of θ, we do not expect the PDFs to correspond either to a stretched exponential or a Gaussian. It seems plausible that that in that range the PDFs could be fitted to a power-law, associated with the probability of occurrence of *small* shocks in the inviscid Burgers equation. Such analysis, however, is beyond the scope of this paper. This analysis shows that the stretched exponential tails for the valocity should be numerically observable over a wide range of θs if the Reynolds number is not too small. On the other hand, stretched exponential tails for the PDF of the derivatives should be observable numerically only if the Reynolds number is neither large nor small.

We note finally that the method of this paper can also be applied to study the PDF tails for 2D and 3D Burgers equations with random, irrotational initial data.

Acknowledgment. This research was supported by the National Science Foundation and the U.S. Army.

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