PDFs for the Random Forced Burgers Equation

Weinan E, [1] Konstantin Khanin, [2] Alexandre Mazel [3] and Yakov Sinai [4]

Abstract: We propose a new approach for the analysis of stationary correlation functions of 1D Burgers equation driven by a random force. We use this to study the asymptotic behavior of the probability distribution of velocity gradients and velocity increments.

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Statistical properties of solutions of random forced Burgers equation have been a subject of intensive studies recently (see [1, 2, 3, 4, 5, 6]). Of particular interest are the asymptotic properties of probability distribution functions associated with velocity gradients and velocity increments. Aside from the fact that such issues are of direct interest to a large number of problems such as the growth of random surfaces [1], it is also hoped that the field-theoretic techniques developed for the Burgers equation will eventually be useful for understanding more complex phenomena such as turbulence.

In this paper, we propose a new and direct approach to analyze the scaling properties of the various distribution functions for the random forced Burgers equation. We will consider the problem

$$u_t + \frac{1}{2}(u^2)_x = \nu u_{xx} - V_x(x,t) \tag{1}$$

Most of our discussion will be limited to the inviscid case when $\nu = 0$. But we will summarize at the end the necessary changes for the case when $0 < \nu << 1$. The potential V of the force

[1]Courant Institute of Mathematical Sciences, New York University.

[2]Heriot-Watt University, Edinburgh and Landau Institute, Moscow.

[3]Princeton University, Princeton.

[4]Princeton University, Princeton and Landau Institute, Moscow.

is a random function

$$V(x,t) = \sum_{|k|\le k_0} \{C_k^{(1)} \cos\frac{2\pi kx}{L} B_k^{(1)}(t) + C_k^{(2)} \sin\frac{2\pi kx}{L} B_k^{(2)}(t)\} \tag{2}$$

Here $B_k^{(1)}$ and $B_k^{(2)}$ are identically distributed independent white noises, L is the size of the system. We will only consider periodic solutions of (1) with period L. It is clear that the statistical behavior of the solutions depend on the decay properties of the coefficients $\{C_k^{(1)}\}$ and $\{C_k^{(2)}\}$. In this paper we deal with the simplest case when the forcing is limited to a finite number of modes.

Our study of stationary correlation functions is based on an idea appeared earlier in [7]. We will construct a statistically stationary functional of the forces $u(x,t) = \Psi(\{B_k^{(1)}(\tau), B_k^{(2)}(\tau), \tau \le t\})$ such that u is a solution of (1). The stationary distribution of (1) is then given by the distribution of $\Psi_0 = \Psi(\{B_k^{(1)}(\tau), B_k^{(2)}(\tau), \tau \le 0\})$.

Our construction of Ψ_0 (and hence Ψ by time-translation) is based on the following idea. Given an initial data, (1) as a hyperbolic differential equation can be solved using the method of characteristics. The chacteristics satify Newton's equation

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = -V_x(x,t) \tag{3}$$

The solution u at (x,t) is given by the velocity of of the characteristic which reaches x at time t. Of course the solution as well as the field of characteristics depends heavily on the initial data. To get the stationary distribution of solutions, we need to select special initial data which amounts to selecting special field of characteristics. This special class of characteristics is given by what we call minimizers [8]. A curve $\{(x(t),t), t \le 0\}$ is a minimizer if it minimizes the action

(4)
$$A(\{x(t)\}) = \int_{-\infty}^{0} \left(\frac{1}{2}(\frac{dx}{d\tau})^2 + \sum_{|k|\le k_0} \{C_k^{(1)} \cos\frac{2\pi kx}{L} B_k^{(1)}(\tau) + C_k^{(2)} \sin\frac{2\pi kx}{L} B_k^{(2)}(\tau)\} \right) d\tau$$

with respect to arbitrary perturbations on finite time intervals. It is easy to see that minimizers satisfy Newton's equation for characteristics.

Using a limiting procedure for action-minimizing characteristics on increasingly large time intervals, we can show that with probability 1, minimizers exist, i.e. for each x, there exists $v = v(x)$ such that the solution of (3) with initial value $(x, v(x))$ gives rise to a minimizer. Furthermore, two different minimizers never intersect in the past,i.e. if $x_1(\tau)$ and $x_2(\tau)$ are two different minimizers, then $x_1(\tau) \neq x_2(\tau)$ for all $\tau < 0$. This remarkable property is a consequence of the randomness and endows minimizers with an intrinsic meaning: the minimizers are unique except for a set E of countably many points. This is because the minimizers can only intersect at $t = 0$. If x_0 is a point where two minimizers $x_1(\tau)$ and $x_2(\tau)$ intersect, then x_1 and x_2 enclose an interval at $t = -1$. Intervals correspond to different x_0's do not intersect. Hence there can only be countably many points of intersection. Now the functional Ψ_0 can be defined by the initial velocity of the minimizers: $u(x, 0) = \Psi_0(\{B_k^{(1)}(\tau), B_k^{(2)}(\tau), \tau \leq 0\}) = v(x)$. This is well-defined everywhere except on E which correspond to the locations of shocks where v is discontinuous. At each location of shocks, there are at least two minimizers corresponding to limits of velocities from the left and right.

Another basic property of the minimizers is the hyperbolicity well-known in the theory of dynamical systems. Assume that $u(x, 0)$ given by Ψ_0 is continuous on an interval $[x_1, x_2]$. Hyperbolicity means that the minimizers emanating from x_1 and x_2 converge exponentially fast to each other in the past, i.e.

$$|x_1(\tau) - x_2(\tau)| \leq C|x_1 - x_2|e^{-\mu|\tau|}$$

where C is a random constant and μ is the Lyapunov exponent of the field of minimizers. Using the terminology of the dynamical systems theory, one can say that each continuous component of the solution $u(x, 0)$ is an unstable manifold of any point on the graph $(x, u(x, 0)), x_1 \leq x \leq x_2$. This implies that there are only finitely many shocks at each time.

We now show how this construction can be used to study stationary probability distribution of velocity gradients and velocity increments. The velocity gradient $\frac{\partial u}{\partial x}$ can be represented as a sum of a continuous component $\frac{\partial u_1}{\partial x}$ and a sum of delta functions representing contribution from the shocks: $\frac{\partial u_2}{\partial x} = \sum_i w_i \delta(x - x_i(t))$ where $(x_i(t), t)$ is in D – the set of shocks in the (x,t) plane. Since shocks can only be created and merge but never disappear, the shock set D should basically have a spine structure with a skeleton shock running from $t=-\infty$ to $t=+\infty$ and newly created shocks forming finite ribs that eventually merge with each other and with the skeleton shock.

Since we are concerned with stationary probabilities, we can restrict ourselves to $t=0$. First consider the probability $P\{\frac{\partial u_1}{\partial x} > z\}$ for large z. These are associated with steep ramps and are due to large fluctuations of the force. To estimate this probability, let $x_1 < x_2$ be close. For $\tau < 0$, the minimizers passing through $(x_1, 0)$ and $(x_2, 0)$ satisfy

$$v_1(\tau) = v_1(0) - \int_\tau^0 V_x(x_1(s), s)ds$$

$$v_2(\tau) = v_2(0) - \int_\tau^0 V_x(x_2(s), s)ds$$

with $v_1(0) < v_2(0)$, and

$$x_1(\tau) = x_1 + v_1(0)\tau - \int_\tau^0 dt \int_t^0 V_x(x_1(s), s)ds$$

$$x_2(\tau) = x_2 + v_2(0)\tau - \int_\tau^0 dt \int_t^0 V_x(x_2(s), s)ds$$

If $\frac{v_2(0)-v_1(0)}{x_2-x_1}$ is large, in the absence of forces these characteristics would intersect very quickly in the past at time $-\frac{x_1-x_2}{v_1(0)-v_2(0)}$. However, since they are minimizers, they cannot intersect. Therefore the action of the forces results in the inequality

$$0 < x_2(\tau) - x_1(\tau) = x_2 - x_1 + (v_2(0) - v_1(0))\tau - \int_\tau^0 dt \int_t^0 (V_x(x_2(s), s) - V_x(x_1(s), s)ds$$

i.e.

$$1 + \frac{v_2(0) - v_1(0)}{x_2 - x_1}\tau > \int_\tau^0 dt \int_t^0 \frac{V_x(x_2(s), s) - V_x(x_1(s), s)}{x_2 - x_1} ds$$

$$= \int_\tau^0 dt \int_t^0 V_{xx}(x^*(s), s)\frac{x_2(s) - x_1(s)}{x_2 - x_1} ds \tag{5}$$

Take $\tau \sim -\frac{1}{z}$ (which is the time that the curves would intersect in the absence of forces), e.g. $\tau = -\frac{3}{z}$ in (5), then $\frac{v_2(0)-v_1(0)}{x_2-x_1}$ is close to $\frac{\partial u}{\partial x}$. Therefore the left hand side is negative with absolute value greater than 1. Thus

$$\left| \int_{-\frac{3}{z}}^0 dt \int_t^0 V_{xx}(x^*(s), s)\frac{x_2(s) - x_1(s)}{x_2 - x_1} ds \right| > 1 \tag{6}$$

The ratio $\frac{x_2(s)-x_1(s)}{x_2-x_1}$ is continuous and bounded, and the distribution of the random variable inside the absolute value sign in (6) is roughly the same as the distribution of the integral of Brownian motion over an interval of length $\tau \sim \frac{1}{z}$ which is Gaussian with variance proportional to z^{-3}. This gives the estimate

$$(7) \qquad P\{\int_{-\frac{3}{z}}^{0} dt \int_{t}^{0} V_{xx}(x^*(s), s) \frac{x_2(s)-x_1(s)}{x_2-x_1} ds| > 1\} \sim \exp\{-\text{Const}\, z^3\}$$

The constant in the last expression is not universal and depends on the details of the distribution of the force. A similar result was obtained in [3, 5] using entirely different argument.

Next we consider the case when $\frac{\partial u}{\partial x} < z << -1$. These are associated with pre-shock events, i.e. places on the (x,t)-plane where shocks are generated. (See also [9] where the concept of pre-shock events are discussed). Take such a point (x_0, t_0). At (x_0, t_0), $\frac{\partial u}{\partial x} = -\infty$. We will describe in more detail the set of (x,t) near (x_0, t_0) where $\frac{\partial u}{\partial x} < z$.

Generically we have $\triangle u = u(x, t_0) - u(x_0, t_0) = C(x-x_0)^{\frac{1}{3}}$ near x_0 where C is a random constant. For small $|t-t_0|$, the action of forces is negligible. Therefore the solution near (x_0, t_0) satisfies

$$(8) \qquad u(x,t) = u_0(y), \quad y = x - u(x,t)(t-t_0)$$

From (8) we have

$$(9) \qquad \frac{\partial u}{\partial x} = \frac{\frac{du_0}{dy}}{1 + \frac{du_0}{dy}(t-t_0)}$$

This shows that near (x_0, t_0) the set $\{(x,t), \frac{\partial u}{\partial x} < z\}$ is a curvilinear region centered at (x_0, t_0) and bounded by the curves $x = x_0 + u_0(x_0)(t-t_0) \pm \left(-\frac{3z}{C(1+z|t-t_0|)}\right)^{-\frac{3}{2}} \mp C\left(-\frac{3z}{C(1+z|t-t_0|)}\right)^{-\frac{1}{2}}$ $(t-t_0)$. The width of this region behaves as $O(|z|^{-\frac{3}{2}})$ and the height is roughly $O(|z|^{-1})$. Therefore its area is approximately $O(|z|^{-\frac{5}{2}})$.

The set of pre-shocks forms a stationary random point field in the (x,t)-plane with density $q dx dt$. Stationarity means that q is independent of t, i.e. $q = q(x)$. Hence we conclude that $P\{\frac{\partial u}{\partial x} < z\}$ is proportional to $Q|z|^{-\frac{5}{2}}$, $Q = \int_0^L q(x)dx$. The density of this distribution decays as $Q|z|^{-\frac{7}{2}}$.

We turn now to the probability distribution of $w = \frac{\triangle u}{\triangle x} = \frac{u(x_2,t)-u(x_1,t)}{x_2-x_1}$. For large positive values of w, the analysis remains essentially the same and gives the super-exponential behavior $\exp\{-\text{Const}\, w^3\}$. For negative values the distribution of $\frac{\triangle u}{\triangle x}$ remains close to that of $\frac{\partial u}{\partial x}$ until some threshold value w^* after which it deviates due to the contribution from existing small shocks. We can split this contribution into two parts: one part due to shocks generated recently, and one part due to shocks generated in the distant past. This means that for $\triangle u << 1$, the density of the distribution of the size of the shock inside an interval $(x, x+\triangle x)$ can be written as $f(\triangle u, \triangle x) = f_1(\triangle u, \triangle x) + f_2(\triangle u, \triangle x)$ where $f_1(\triangle u, \triangle x)$ is the probability density that a shock of size $\triangle u$ has appeared at approximately $t = \frac{\triangle x}{\triangle u} \sim -\frac{(\triangle u)^3}{\triangle u} = -(\triangle u)^2$ in $(x, x+\triangle x)$. Since the size of the shock after its first appearance grows as $(\triangle t)^{\frac{1}{2}}$ [10], the probability of having shocks of size $\triangle u$ in an interval of length $\triangle x$ should be $q\triangle x \triangle t = q\triangle x(\triangle u)^2$. Taking derivative with respect to $\triangle u$ we get that the probability density $f_1 = q\triangle x\triangle u$.

f_2 comes from solutions having shocks (of size $\triangle u$) which orginated in the distant past and become weaker due to fluctuations of the forces. For small $\triangle u$, f_2 should have a power-like behavior: $f_2(\triangle u, \triangle x) \sim (\triangle x)(\triangle u)^\beta$. So far our analysis has yielded little specific information about the value of β but there are indications that β should depend on details of the distribution of the force and hence is non-universal. Numerical work is going on to validate this assumption. Assuming this, we have $p(\frac{\triangle u}{\triangle x} = w)dw = C_1(\triangle x)^3 w dw + C_2(\triangle x)^{\beta+2} w^\beta dw$.

Case 1: $\beta > 1$. In this case the first term dominates. The crossover (the value of w^*) can be found from $w^{-\frac{7}{2}} \sim (\triangle x)^3 w$ which gives $w^* \sim (\triangle x)^{-\frac{2}{3}}$.

Case 2: $\beta \le 1$. In this case the second term dominates and $w^* \sim (\triangle x)^{-\frac{2\beta+4}{2\beta+7}}$.

The overall picture for the probability distribution of $\frac{\partial u}{\partial x}$ and $\frac{\triangle u}{\triangle x}$ is summarized in Figure 1.

We end this paper with several remarks.

1. The asymptotic behavior of $p(\frac{\triangle u}{\triangle x})$ found above implies intermittency, i.e.

$$\langle (\triangle u)^n \rangle \sim \begin{cases} (\triangle x)^n, & \text{for} \quad n \le 1 \\ \triangle x, & \text{for} \quad n > 1 \end{cases}$$

i.e. the behavior is bi-fractal.

2. Our calculation of the probability distribution of w is only valid for the regime $|\triangle u| << 1$. When $|\triangle u| >> 1$, the behavior is again super-exponential and can be estimated with the same exponent as we did earlier.

3. In the presence of a finite viscosity we can show that the stationary probability distributions converge to that of the inviscid ones as viscosity goes to zero [8]. Therefore for $\nu << 1$, the picture presented in Figure 1 remains valid for $z >> 1$ and $-\nu^{-1} << z << -1$. For $z = O(\nu^{-1})$, viscous corrections to shock profiles have to be taken into account.

4. Our results imply a seemingly erroneous result that $\langle \left(\frac{\partial u}{\partial x}\right)^2 \rangle < \infty$. This is because we avoid contributions from the neighborhood of shocks. In other words, we used $\langle \left(\frac{\partial u}{\partial x}\right)^2 \rangle = \lim_{R\to+\infty} \lim_{\nu\to 0} \langle \left(\frac{\partial u}{\partial x}\right)^2 \rangle_{\nu,R}$ where $\langle \left(\frac{\partial u}{\partial x}\right)^2 \rangle_{\nu,R}$ is computed for non-zero viscosity ν by averaging $\left(\frac{\partial u}{\partial x}\right)^2$ over realizations such that $|\frac{\partial u}{\partial x}| < R$.

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[10] This follows from pure kinematic calculations. $\triangle u \sim (\triangle x)^{\frac{1}{3}}$, $\triangle x = \triangle u \triangle t$. This implies that $\triangle u \sim (\triangle t)^{\frac{1}{2}}$.

Figure Caption

Figure 1. Probability density function for velocity gradients and velocity increments.