Statistical Theory for the Stochastic Burgers Equation in the Inviscid Limit

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Abstract

A statistical theory is developed for the stochastic Burgers equation in the inviscid limit. Master equations for the probability density functions of velocity, velocity difference, and velocity gradient are derived. No closure assumptions are made. Instead, closure is achieved through a dimension reduction process; namely, the unclosed terms are expressed in terms of statistical quantities for the singular structures of the velocity field, here the shocks. Master equations for the environment of the shocks are further expressed in terms of the statistics of singular structures on the shocks, namely, the points of shock generation and collisions. The scaling laws of the structure functions are derived through the analysis of the master equations. Rigorous bounds on the decay of the tail probabilities for the velocity gradient are obtained using realizability constraints. We also establish that the probability density function $Q(\xi)$ of the velocity gradient decays as $|\xi|^{-7/2}$ as $\xi \to -\infty$. © 2000 John Wiley & Sons, Inc.

Contents

1 Introduction

Consider the randomly forced Burgers equation

$$u_t + uu_x = \nu u_{xx} + f\,, \tag{1.1}$$

Should there be a comma in $\langle f(x,t)f(y,s)\rangle$?

where $f(x,t)$ is a zero-mean, Gaussian, statistically homogeneous, and white-in-

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time random process with covariance

$$\langle f(x,t)f(y,s)\rangle = 2B(x-y)\delta(t-s)\,,$$

and $B(x)$ is a smooth function. In the language of stochastic differential equations, (1.1) must be interpreted as

$$du = (-uu_x + \nu u_{xx})dt + dW(x,t)\,,$$

where $dW(x,t)$ is a white-noise process satisfying

$$dW(x,t)dW(y,t) = 2B(x-y)dt\,.$$

We will be interested in the statistical behavior of the stationary states (invariant measures) of (1.1) if they exist, or the transient states with possibly random initial data. There are two main reasons for considering this problem. The first is that (1.1) and its multidimensional version are among the simplest nonlinear models in nonequilibrium statistical mechanics. As such they serve as qualitative models for a wide variety of problems including charge density waves [19], vortex lines in high-temperature superconductors [5], dislocations in disordered solids and kinetic roughening of interfaces in epitaxial growth [29], formation of large-scale structures in the universe [35, 39], etc. The connection between these problems and (1.1) can be understood as follows: Consider an elastic string in a random potential $V(x,s)$. The string is assumed to be *directed* in the sense that there is a timelike direction, assumed to be s, such that the configuration of the string is a graph over the s-axis. Let $Z(x,t)$ be the partition function for the configurations of the string in the interval $0 \le s \le t$, pinned at position x at time t,

$$Z(x,t) = \left\langle \mathrm{e}^{-\beta\int_0^t V(w(s),s)ds} \,\middle|\, w(t) = x \right\rangle,$$

where $\langle \cdot \mid w(t) = x\rangle$ denotes the expectation over all Brownian paths $w(\cdot)$ such that $w(t) = x$, $\beta = 1/kT$, k is the Boltzmann constant, and T is the temperature. The free energy $\varphi(x,t) = \ln Z(x,t)$ then satisfies

$$\varphi_t + \frac{1}{2}|\nabla\varphi|^2 = \nu\Delta\varphi + V\,,$$

where $\nu = kT$. This is the well-known Kardar-Parisi-Zhang equation [27]. In one dimension, if we let $u = \varphi_x$ and $f = V_x$, we obtain (1.1).

The second reason for studying (1.1) is in some sense a technical one. For a long time, (1.1) has served as the benchmark for field-theoretic techniques such as the direct interaction approximation or the renormalization group methods developed for solving the problem of hydrodynamic turbulence. This role of (1.1) is made more evident by the recent flourish of activities introducing fairly sophisticated techniques in field theory to hydrodynamics [8, 25, 32]. In this context, (1.1) is often referred to as Burgers turbulence. Since the phenomenology of the so-called Burgers turbulence is far simpler than that of real turbulence, one hopes that exact results can be obtained which can then be used to benchmark the methods. However, so far our experience has proved otherwise: The problem of Burgers turbulence is complicated enough that a wide variety of predictions have been made as a consequence of the wide variety of techniques used [1, 2, 3, 4, 6, 7, 9, 10, 11, 12, 13, 17, 22, 23, 24, 25, 26, 28, 32, 34, 36, 37, 39].

The main purpose of the present paper is to clarify this situation and to obtain exact results that are expected for Burgers turbulence. Along the way we will also develop some technical aspects that we believe will be useful for other problems. The main issues that interest us are the scaling of the structure functions and the asymptotic behavior of the probability density functions (PDF) in the inviscid limit. The former is well-understood heuristically, but we will derive the results from self-consistent asymptotics on the master equation for the PDF of the velocity difference. The latter is at the moment very controversial, and we hope to settle the controversy by deriving exact results on the asymptotic behavior for the PDFs.

From a technical point of view we insist on working with master equations for the PDFs and making no closure assumptions. The unclosed terms are expressed in terms of the statistics of singular dissipative structures of the field, here the shocks. We then derive a master equation for the statistics of the environment of the shocks by relating them to the singular structures on the shocks, namely, the points of shock creation and collisions. These are then amenable to local analysis. In this way we achieve closure through dimension reduction. We then extract information on the asymptotic behavior of PDFs using realizability constraints and self-consistent asymptotics. We certainly hope that this philosophy will be useful for other problems.

One main issue that will be addressed in this paper is the behavior of the PDF of the velocity gradient. Assuming statistical homogeneity, let $Q^\nu(\xi,t)$ be the PDF of $\xi = u_x$. Q^ν satisfies

$$Q^\nu_t = \xi Q^\nu + \left(\xi^2 Q^\nu\right)_\xi + B_1 Q^\nu_{\xi\xi} - \nu\left(\langle \xi_{xx} \mid \xi\rangle Q^\nu\right)_\xi,$$

where $B_1 = -B_{xx}(0)$ and $\langle \xi_{xx} \mid \xi\rangle$ is the average of ξ_{xx} conditional on ξ. This equation is unclosed since the explicit form of the last term, representing the effect of the dissipation, is unknown. We are interested in Q^ν at the inviscid limit:

$$\underline{Q}(\xi,t) = \lim_{\nu\to 0} Q^\nu(\xi,t).$$

In order to derive an equation for $\underline{Q}$, one needs to evaluate

$$F(\xi,t) = -\lim_{\nu\to 0} \nu\left(\langle \xi_{xx} \mid \xi\rangle Q^\nu\right)_\xi.$$

This is where the difficulty arises.

Remembering that $u_x(x,t) = \lim_{\Delta x\to 0}(u(x+\Delta x,t) - u(x,t))/\Delta x$, the procedure outlined above pertains to the process of first taking the limit as $\Delta x \to 0$, then the limit as $\nu \to 0$. It is natural to consider also the other situation, when the limit $\nu \to 0$ is taken first. In this case the limiting form of (1.1) (notice that we now write the nonlinear term in a conservative form),

$$u_t + \frac{1}{2}\left(u^2\right)_x = f\,, \tag{1.2}$$

has to be interpreted in a weak sense by requiring

$$\iint \left\{ u\varphi_t + \frac{1}{2}u^2\varphi_x + f\varphi \right\} dx\,dt = 0 \tag{1.3}$$

for all compactly supported smooth functions φ. The solutions u satisfying (1.3) are called weak solutions. In order to ensure a well-defined dynamics for (1.2), i.e., existence and uniqueness of solutions with given initial data, an additional entropy condition has to be imposed on weak solutions. This amounts to requiring

$$u(x+,t) \le u(x-,t) \tag{1.4}$$

for all (x,t). The entropy condition (1.4) is the effect of the viscous term in the inviscid limit. There is a huge mathematical literature on (1.2) for the deterministic case. Standard references are [30, 31, 38]. The random case was studied recently in the paper by E, Khanin, Mazel, and Sinai [14].

The first step in the present paper is to derive master equations for single and multipoint PDFs of u satisfying (1.2). In particular, we derive an equation for the PDF of $\eta(x,y,t) = (u(x+y) - u(x,t))/y$, $Q^\delta(\eta,x,t)$. We are interested in

$$Q(\xi,t) = \lim_{x\to 0} Q^\delta(\xi,x,t)\,.$$

One natural question is whether

$$Q = \underline{Q}\,. \tag{1.5}$$

We will present a very strong argument that (1.5) holds for generic initial data and for the type of forces described after (1.1). Some of our results also apply to Q for the more general case when it is possibly different from $\underline{Q}$.

The issue now reduces to the evaluation or approximation of $F(\xi,t)$. Several different proposals have been made; each leads at statistical steady state ($Q = 0$) to an asymptotic expression of the form

$$Q(\xi) \sim \begin{cases} C_-|\xi|^{-\alpha} & \text{as } \xi \to -\infty \\ C_+\xi^\beta \mathrm{e}^{-\xi^3/(3B_1)} & \text{as } \xi \to +\infty, \end{cases}$$

but with a variety of values for the exponents α and β (here the $C_\pm$'s are constants). By invoking operator product expansion, Polyakov [32] suggested that $F = aQ + b\xi Q$, with $a = 0$ and $b = -\frac{1}{2}$. This leads to $\alpha = \frac{5}{2}$ and $\beta = \frac{1}{2}$. Boldyrev [6, 7] considered the same closure with $-1 \le b \le 0$, which gives $2 \le \alpha \le 3$ and $\beta = 1 + b$. Based on heuristic arguments, Bouchaud and Mézard [9] introduced a Langevin equation for velocity gradient, which gives $2 \le \alpha \le 3$, $\beta = 0$. The instant-on analysis [4, 18, 25] predicts the right tail of Q without giving a precise value for β, and it does not give any specific prediction for the left tail. E et al. [13] made a geometrical evaluation of the effect of F based on the observation that large negative gradients are generated near shock creation. Their analysis gives a rigorous upper bound for α: $\alpha \le \frac{7}{2}$. In [13], it was claimed that this bound is actually reached, i.e., $\alpha = \frac{7}{2}$. Finally, Gotoh and Kraichnan [24] argued that the viscous term is negligible to leading order for large $|\xi|$, i.e., $F \approx 0$ for $|\xi| \gg B_1^{1/3}$, giving rise to $\alpha = 3$ and $\beta = 1$.

In this paper, we will first give a more detailed proof of the bound

$$\alpha > 3\,,$$

announced in [17]. We will then show that

$$\alpha = \frac{7}{2}$$

by studying the master equation for the environment of the shocks.

This relies on knowing the local behavior near the most singular structures (here the points of shock creation). Points of shock creation were isolated in [14] as a mechanism for obtaining large negative values of u_x and resulted in the prediction that $\alpha = \frac{7}{2}$. From this point of view, the present paper provides the missing step establishing the fact that points of shock creation provide the leading-order contribution to the left tail of Q.

Concerning the style of presentation, for the most part our working assumptions will be the following:

1. Solutions of (1.2) are piecewise smooth.
2. Shock paths are smooth except at the points of collision.
3. Shocks are created at zero amplitude and shock strengths add up at collision.

The theorems and lemmas are proved under these assumptions. Fully rigorous proofs of these statements are not yet available, even though these are considered part of the folklore in this subject. On the other hand, our primary purpose in this paper is to develop an approach according to which various asymptotic limiting behaviors can be calculated. We will therefore leave the full proof of these statements to future work.

Before ending this introduction, we make some remarks about notations and nomenclature. In analogy with fluid mechanics, u will be referred to as the velocity field. We will denote the multipoint PDFs of $u(\cdot,t)$ as $Z^\nu(u_1,x_1,\dots,u_n,x_n,t)$, i.e.,

$$\text{Prob}(a_1 < u(x_1,t) \le b_1,\dots,a_n < u(x_n,t) \le b_n) = \int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} Z^\nu(u_1,x_1,\dots,u_n,x_n,t)du_1\cdots du_n\,.$$

The superscript ν refers to the viscous case, $\nu > 0$. In the inviscid limit we will denote the multipoint PDFs of $u(\cdot,t)$ by $Z(u_1,x_1\dots,u_n,x_n,t)$. Statistical stationary values will be denoted by the subscript ∞, e.g., Z^ν_∞ or Z_∞. We reserve the special notations $R^\nu(u,x,t) = Z^\nu(u,x,t)$ and $R(u,x,t) = Z(u,x,t)$ for the one-point PDF of $u(x,t)$. $Q^\nu(\xi,t)$ denotes the PDF of $\xi(x,t)$, and its inviscid limit is $Q(\xi,t)$.

Statistical symmetries, or equality in law, will be denoted by $\stackrel{\text{(d)}}{=}$. Two such symmetries will be used repeatedly. The first is statistical homogeneity,

$$u(x,t) \stackrel{\text{(d)}}{=} u(x+y,t) \quad \text{for all } y. \tag{1.6}$$

(1.6) holds if for all measurable, real-valued functionals $\Phi(u(\cdot,t))$, we have

$$\langle \Phi(u(\cdot,t)) \rangle = \langle \Phi(u(\cdot+y,t)) \rangle \quad \text{for all } y.$$

Note that from (1.1) and the assumptions on f, it follows that (1.6) holds for all times if $u_0(x) \stackrel{\text{(d)}}{=} u_0(x+y)$, where $u_0(\cdot) = u(\cdot,0)$ is the initial velocity field. Also, if a statistical steady state exists and is unique, then it satisfies (1.6). The second symmetry will be referred to as statistical parity invariance and is related to the invariance of (1.1) under the transformation $x \to -x$, $u \to -u$. This implies that

$$u(x,t) \stackrel{\text{(d)}}{=} -u(-x,t) \tag{1.7}$$

or

$$\langle \Phi(u(\cdot,t)) \rangle = \langle \Phi(-u(-(\cdot),t)) \rangle.$$

(1.7) holds for all times if $u_0(x) \stackrel{\text{(d)}}{=} -u_0(-x)$. (1.7) is also satisfied at statistical steady state.

We will use (Ω, H, P) to denote the probability space for the forcing f, and (Ω_0, H_0, P_0) to denote the probability space for the initial data u_0, where Ω_0 is the Skorohod space $D(I)$ on an interval I and H_0 is the Borel σ-algebra. P_0 is assumed to be independent of P. When I is a finite interval, we assume periodic boundary conditions. Note that the existence of stationary states is proved only in this case [14].

2 Velocity and Velocity Differences

Let $u^\nu(x,t,f,u_0)$ be the solution of (1.1) with forcing f and initial data u_0. It follows from standard results (for the deterministic case, see [30, 31]) that for fixed t

$$u^\nu(\cdot,t,f,u_0) \to u(\cdot,t,f,u_0) \quad \text{as } \nu \to 0,$$

$P \times P_0$-almost surely, in $L^2_{\text{loc}}(I)$. Hence

$$Z^\nu \to Z$$

weakly. For the statistically stationary states, it was established in [14] that we have

$$Z^\nu_\infty \to Z_\infty$$

weakly. Therefore to study the inviscid limit of statistical quantities of u, such as the PDFs of u and $\delta u(x,y,t) = u(x+y,t) - u(y,t)$, it is enough to consider (1.2).

It is useful, however, to write this equation in a modified form more convenient for calculations. To this end, let

$$u_{\pm}(x,t) = u(x\pm,t)\,.$$

For any function $g(u) = g(u(x,t))$, we define two average functions

$$[g(u)]_{\mathrm{A}} = \frac{1}{2}(g(u_+) + g(u_-))\,, \qquad [g(u)]_{\mathrm{B}} = \int_0^1 g(u_- + \beta(u_+ - u_-))d\beta\,. \tag{2.1}$$

It is shown in [40] that (1.2) is equivalent to

$$u_t + [u]_{\mathrm{A}} u_x = f\,, \tag{2.2}$$

or, in the language of stochastic differential equations,

$$du = -[u]_{\mathrm{A}} u_x dt + dW(x,t)\,. \tag{2.3}$$

Equation (2.1) assigns an unambiguous meaning to the quantity $[u]_{\mathrm{A}} u_x$ when the solutions of (2.2) develop shocks. Moreover, the following chain and product rules hold:

$$g_x(u) = [g_u(u)]_{\mathrm{B}} u_x\,, \tag{2.4}$$

$$(g(u)h(u))_x 7 = [g(u)]_{\mathrm{A}} h_x(u) + [h(u)]_{\mathrm{A}} g_x(u)\,. \tag{2.5}$$

Similar rules apply for derivatives in t. As an example of these rules, we have that the integral of the term $[u]_{\mathrm{A}} u_x$ across a shock located at $x = y$ is given by (using $[u]_{\mathrm{A}} = [u]_{\mathrm{B}}$)

$$\int_{y-}^{y+} [u]_{\mathrm{A}} u_x\, dx = \int_{y-}^{y+} \frac{1}{2}(u^2)_x\, dx = \frac{1}{2}\left(u_+^2 - u_-^2\right).$$

It is also convenient to define

$$\bar{u}(x,t) = \frac{1}{2}(u_+(x,t) + u_-(x,t))\,, \qquad s(x,t) = u_+(x,t) - u_-(x,t)\,.$$

In terms of $(\bar{u}, s)$, the two averages in (2.1) are

$$[g(u)]_{\mathrm{A}} = \frac{1}{2}\left(g\left(\bar{u} + \frac{s}{2}\right) + g\left(\bar{u} - \frac{s}{2}\right)\right), \qquad [g(u)]_{\mathrm{B}} = \int_{-1/2}^{1/2} g(\bar{u} + \beta s)d\beta\,.$$

We will denote by $\{y_j\}$ the set of shock positions at time t. Hence the set

$$\{(y_j, \bar{u}(y_j,t), s(y_j,t))\}$$

quantifies the shocks at time t.

2.1 Master Equation for the Inviscid Burgers Equation

We now turn to $R(u,x,t)$, the one-point PDF of the solution of (2.2). We have

THEOREM 2.1 *R satisfies*

$$R_t = -uR_x - \int_{\mathbb{R}} K(u-u')R_x(u',x,t)du' + B_0 R_{uu} + G\,, \tag{2.6}$$

where $B_0 = B(0)$, $K(u) = (H(u) - H(-u))/2$, $H(\cdot)$ is the Heaviside function, and $G(u,x,t)$ is given by

$$G(u,x,t) = \left(\int_{\mathbb{R}^-} \int_{u+s/2}^{u-s/2} (u-\bar{u})\varrho(\bar{u},s,x,t)d\bar{u}\,ds \right)_u . \tag{2.7}$$

Here $\varrho(\bar{u},s,x,t)$ is defined such that $\varrho(\bar{u},s,x,t)d\bar{u}\,ds\,dx$ gives the average number of shocks in $[x,x+dx)$ with $\bar{u}(y,t) \in [\bar{u},\bar{u}+d\bar{u})$ and $s(y,t) \in [s,s+ds)$, where $y \in [x,x+dx)$ is the shock location.

Remarks. 1. G can be referred to as a *dissipative anomaly*. It can be written more explicitly as

$$\begin{aligned} G(u,x,t) &= \frac{1}{2}\int_{\mathbb{R}^-} s\left(\varrho\left(u-\frac{s}{2},s,x,t\right) + \varrho\left(u+\frac{s}{2},s,x,t\right)\right)ds \\ &\quad - \int_{-1/2}^{1/2}\int_{\mathbb{R}^-} s\varrho(u+\beta s,s,x,t)ds\,d\beta\,. \end{aligned} \tag{2.8}$$

2. $\varrho(\bar{u},s,x,t)$ can be equivalently defined as

$$\varrho(\bar{u},s,x,t) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}\times\mathbb{R}} e^{i\lambda\bar{u}+i\mu s}\hat{\varrho}(\lambda,\mu,x,t)d\lambda\,d\mu\,,$$

where

$$\hat{\varrho}(\lambda,\mu,x,t) = \left\langle \sum_j e^{-i\lambda\bar{u}(y_j,t)-i\mu s(y_j,t)}\delta(x-y_j) \right\rangle .$$

Equations (2.6) and (2.7) do not require statistical homogeneity, nor that the number density of shocks $\rho = \rho(t)$ be finite. For homogeneous situations such that ρ is finite, these equations simplify. Since $R_x = 0$, (2.6) reduces to

$$R_t = B_0 R_{uu} + G \tag{2.9}$$

where $G(u,x,t) = G(u,t)$. Since the shock characteristics are independent of its location, we have

$$\varrho(\bar{u},s,x,t) = \rho S(\bar{u},s,t)$$

where $S(\bar{u},s,t)$ is the PDF of $(\bar{u}(y_0,t),s(y_0,t))$, conditional on the property that y_0 is a shock position (because of the statistical homogeneity, y_0 is a dummy variable). Thus (2.7) reduces to

$$G(u,t) = \rho \left(\int_{\mathbb{R}^-} \int_{u+s/2}^{u-s/2} (u-\bar{u}) S(\bar{u},s,t) d\bar{u} ds \right)_u . \tag{2.10}$$

At the statistical stationary state, $R_t = 0$ in (2.9), and one readily verifies from this equation that

$$R_\infty(u) = \frac{\rho}{2B_0} \int_{\mathbb{R}^-} \int_{u+s/2}^{u-s/2} \left(\frac{s^2}{4} - (u-\bar{u})^2 \right) S_\infty(\bar{u},s) d\bar{u} ds . \tag{2.11}$$

Clearly, $R_\infty \geq 0$ since $S_\infty(\bar{u},s) \geq 0$. In addition, by direct computation we obtain

$$\int_{\mathbb{R}} R_\infty \, du = -\frac{\rho}{12B_0} \langle s^3 \rangle .$$

Thus, from the requirement that R_∞ be normalized to unity, we get

$$B_0 = -\frac{\rho}{12} \langle s^3 \rangle . \tag{2.12}$$

PROOF OF THEOREM 2.1: Let $\theta(\lambda,x,t) = \mathrm{e}^{-i\lambda u(x,t)}$. $\langle\theta\rangle$ is the characteristic function of $u(x,t)$, and R is given by

$$R(u,x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{i\lambda u} \langle \theta(\lambda,x,t) \rangle d\lambda .$$

Using

$$dW(x,t)dW(y,t) = 2B(x-y)dt ,$$

it follows from (2.3), (2.4), and (2.5) that

$$d\theta = \big(-i\lambda [u]_{\mathrm{A}} u_x [\theta]_{\mathrm{B}} - \lambda^2 B_0 \theta \big) dt - i\lambda\theta dW(x,t) ;$$

thus

$$\langle\theta\rangle_t = i\lambda \langle [u]_{\mathrm{A}} u_x [\theta]_{\mathrm{B}} \rangle - \lambda^2 B_0 \langle\theta\rangle .$$

Note that the terms involving the force contain θ, not $[\theta]_{\mathrm{B}}$. This is because $\theta = [\theta]_{\mathrm{B}}$ except when there is a shock, and this is a set of zero probability. Of course, this argument does not apply for the convection term, since u_x is infinite at shocks. To average the convective term $i\lambda[u]_{\mathrm{A}} u_x [\theta]_{\mathrm{B}}$, we use

$$\theta_x = -i\lambda u_x [\theta]_{\mathrm{B}}$$

to get

$$i\lambda \big\langle [u]_{\mathrm{A}} u_x [\theta]_{\mathrm{B}} \big\rangle = -\big\langle [u]_{\mathrm{A}} \theta_x \big\rangle = -\big\langle u\theta \big\rangle_x + \big\langle u_x [\theta]_{\mathrm{A}} \big\rangle .$$

For convenience, we write this equation as

$$i\lambda\big\langle [u]_{\mathrm{A}} u_x [\theta]_{\mathrm{B}}\big\rangle = -i\langle\theta\rangle_{x\lambda} + i\lambda^{-1}\langle\theta\rangle_x + \big\langle u_x\big([\theta]_{\mathrm{A}} - [\theta]_{\mathrm{B}}\big)\big\rangle\,.$$

Combining these expressions gives

$$\langle\theta\rangle_t = -i\langle\theta\rangle_{x\lambda} + i\lambda^{-1}\langle\theta\rangle_x - \lambda^2 B_0\langle\theta\rangle + \hat{G}, \tag{2.13}$$

where $\hat{G}(\lambda,x,t)$ is given by

$$\hat{G}(\lambda,x,t) = \big\langle u_x\big([\theta]_{\mathrm{A}} - [\theta]_{\mathrm{B}}\big)\big\rangle\,.$$

To proceed with the evaluation of $\hat{G}$, note that the only contributions to this term are from the shocks, since $[\theta]_{\mathrm{A}} = [\theta]_{\mathrm{B}}$ except at shocks. Let $\{y_j\}$ denote the positions of the shocks at time t. Since $u_x(y_j,t) = s(y_j,t)\delta(x-y_j)$ at the shocks, $\hat{G}$ can be understood as

$$\hat{G}(\lambda,x,t) = \Big\langle \sum_j s_j\delta(x-y_j)\Big([\theta(\lambda,y_j,t)]_{\mathrm{A}} - [\theta(\lambda,y_j,t)]_{\mathrm{B}}\Big)\Big\rangle, \tag{2.14}$$

where $s_j = s(y_j,t)$. Using (2.1),

$$\hat{G}(\lambda,x,t) = \Big\langle \sum_j s_j\delta(x-y_j)\mathrm{e}^{-i\lambda\bar{u}_j}\Big(\mathrm{e}^{i\lambda s_j/2} + \mathrm{e}^{-i\lambda s_j/2} - 2\int_{-1/2}^{1/2}\mathrm{e}^{-i\lambda\beta s_j}\,d\beta\Big)\Big\rangle,$$

where $\bar{u}_j = \bar{u}(y_j,t)$. The use of $\varrho(\bar{u},s,x,t)$ shows that this average is

$$\hat{G}(\lambda,x,t) = \int_{\mathbb{R}\times\mathbb{R}^-} \frac{s}{2}\mathrm{e}^{-i\lambda\bar{u}}\Big(\mathrm{e}^{i\lambda s/2} + \mathrm{e}^{-i\lambda s/2} - 2\int_{-1/2}^{1/2}\mathrm{e}^{-i\lambda\beta s}\,d\beta\Big)\varrho(\bar{u},s,x,t)\,d\bar{u}\,ds\,.$$

Going back to the variable u, we get (2.8), hence (2.6). □

Remark. Under the assumption of ergodicity with respect to spatial translations, an alternative derivation of (2.10) is to go back to (2.14) and use the equivalence between ensemble average and spatial average. Then

$$\begin{aligned}\hat{G}(\lambda,t) &= \lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L}\sum_j s_j\delta(x-y_j)\big([\theta(\lambda,y_j,t)]_{\mathrm{A}} - [\theta(\lambda,y_j,t)]_{\mathrm{B}}\big)dx\\ &= \lim_{L\to\infty}\frac{N}{2L}\frac{1}{N}\sum_{j=1}^{N} s_j\big([\theta(\lambda,y_j,t)]_{\mathrm{A}} - [\theta(\lambda,y_j,t)]_{\mathrm{B}}\big)\,,\end{aligned}$$

where N is the number of shocks in the interval $[-L,L]$. By using the ergodicity again, it follows that the sum is equal to

$$\hat{G}(\lambda,t) = \rho\int_{\mathbb{R}\times\mathbb{R}^-}\frac{s}{2}\mathrm{e}^{-i\lambda\bar{u}}\Big(\mathrm{e}^{i\lambda s/2} + \mathrm{e}^{-i\lambda s/2} - 2\int_{-1/2}^{1/2}\mathrm{e}^{-i\lambda\beta s}\,d\beta\Big)S(\bar{u},s,t)\,d\bar{u}\,ds,$$

where we used $\lim_{L\to\infty} N/(2L) = \rho$. Going back to the variable u, we get (2.10).

2.2 An Alternative Derivation

The derivation of (2.6) (or (2.9) for statistically homogeneous situations) given above is rigorous but rather unintuitive. In fact, the effect of the viscous term is buried in the definition of the two averages in (2.1), and it is not at all clear at this stage whether (2.6) arises in the limit of the equation for R^ν as $\nu \to 0$. Recall that R^ν satisfies (see (A.4) in the appendix)

$$R_t^\nu = B_0 R_{uu}^\nu - \nu\big(\langle u_{xx} \mid u\rangle R^\nu\big)_u,$$

assuming statistical homogeneity. Here we give another, less rigorous but more intuitive, derivation of (2.9) and (2.10) by working with the equation for R^ν, and calculating directly the limit of the viscous term as $\nu \to 0$. In particular, we will compute explicitly that the dissipative anomaly (2.10) is given by

$$G(u,t) = -\lim_{\nu\to 0} \nu\big(\langle u_{xx} \mid u\rangle R\big)_u\,. \tag{2.15}$$

This will give strong support to the claim that $R = \lim_{\nu\to 0} R^\nu$ satisfies (2.9).

Assuming spatial ergodicity, the average of the dissipative term can be expressed as

$$\begin{aligned}\nu\langle u_{xx} \mid u\rangle R^\nu &= \nu\langle u_{xx}(x,t)\delta(u-u(x,t))\rangle \\ &= \nu \lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{L} u_{xx}(x,t)\delta(u-u(x,t))dx\,.\end{aligned} \tag{2.16}$$

Clearly, in the limit as $\nu \to 0$, only small intervals around the shocks will contribute to the integral. In these intervals, boundary layer analysis can be used to obtain an accurate approximation of $u(x,t)$.

The basic idea is to split u into the sum of an inner solution near the shock and an outer solution away from the shock, and using systematic matched asymptotics to construct uniform approximation of u (for details, see, e.g., [21]). For the outer solution, we look for an approximation in the form of a series in ν:

$$u = u^{\text{out}} = u_0 + \nu u_1 + O(\nu^2)\,.$$

Then u_0 satisfies

$$u_{0t} + u_0 u_{0x} = f\,,$$

i.e., the Burgers equation without the dissipation term. In order to deal with the inner solution around the shock, let $y = y(t)$ be the position of a shock, define the stretched variable $z = (x-y)/\nu$, and let

$$u^{\text{in}}(x,t) = v\left(\frac{x-y}{\nu} + \delta, t\right),$$

where δ is a perturbation of the shock position to be determined later. Then v satisfies

$$\nu v_t + (v - \bar{u} + \nu\gamma)v_z = v_{zz} + \nu f\,, \tag{2.17}$$

where $\bar{u} = dy/dt$, $\gamma = d\delta/dt$, and, to $O(\nu^2)$, νf can be evaluated at $x = y$ (and can thus be considered as a function of t only).

We study (2.17) by regular perturbation analysis. We look for a solution in the form

$$v = v_0 + \nu v_1 + O\big(\nu^2\big)\,.$$

To leading order, from (2.17) we get for v_0 the equation

$$(v_0 - \bar{u})v_{0z} = v_{0zz}\,.$$

The boundary condition for this equation arises from the matching condition with $u^{\text{out}} = u_0 + \nu u_1 + O(\nu^2)$:

$$\lim_{z\to\pm\infty} v_0 = \lim_{x\to y} u_0 \equiv \bar{u} \pm \frac{s}{2}\,,$$

where $s = s(t)$ is the shock strength. It is understood that for small ν, matching takes place for small values of $|x-y|$ and large values of $|z| = |x-y|/\nu$. This gives

$$v_0 = \bar{u} - \frac{s}{2}\tanh\left(\frac{sz}{4}\right).$$

These results show that, to $O(\nu)$, (2.16) can be estimated as

$$\nu\langle u_{xx} \mid u\rangle R^\nu = \nu \lim_{L\to\infty} \frac{N}{2L}\frac{1}{N}\sum_j \int_{\ell_j} u^{\text{in}}_{xx}\delta(u - u^{\text{in}}(x,t))dx\,,$$

where ℓ_j is an interval centered at y_j with width $\gg O(\nu)$. Going to the stretched variable $z = (x-y)/\nu$ and taking the limit as $L\to\infty$, we get

$$\nu\langle u_{xx} \mid u\rangle R^\nu = \rho \int_{\mathbb{R}\times\mathbb{R}^-} S(\bar{u},s,t)\int_{\mathbb{R}} v_{0zz}(z,t)\delta(u - v_0(z,t))dz\,d\bar{u}\,ds\,,$$

where $S(\bar{u},s,t)$ is the PDF of $(\bar{u}(y_0,t), s(y_0,t))$ conditional on y_0 being a shock location. The z-integral can be evaluated exactly using

$$v_{0zz}\,dz = \frac{v_{0zz}}{v_{0z}}\,dv_0 = (v_0 - \bar{u})dv_0\,,$$

where we used the equation $(v_0 - \bar{u})v_{0z} = v_{0zz}$. This leads to

$$\nu\langle u_{xx} \mid u\rangle R^\nu = -\rho \int_{\mathbb{R}\times\mathbb{R}^-} S(\bar{u},s,t)\int_{\bar{u}+s/2}^{\bar{u}-s/2} (v_0 - \bar{u})\delta(u - v_0)dv_0\,d\bar{u}\,ds\,.$$

Hence,

$$\lim_{\nu\to 0}\nu\langle u_{xx} \mid u\rangle R^\nu = -\rho\int_{\mathbb{R}^-}\int_{u+s/2}^{u-s/2}(u-\bar{u})S(\bar{u},s,t)\,d\bar{u}\,ds\,.$$

Using this expression in (2.15), we get (2.7).

2.3 Computing the Anomalies

Here we derive equations for the moments of R. For simplicity, consider a statistically homogeneous case and assume that $u(x,t) \stackrel{(d)}{=} -u(-x,t)$. Then $S(\bar{u},s,t) = S(-\bar{u},s,t)$, $G(u,t) = G(-u,t)$, and $R(u,t) = R(-u,t)$. This means that all moments of odd orders of R are zero. For moments of even orders, we get from (2.9) the following equations ($n \in \mathbb{N}_0$):

$$\frac{d}{dt}\left\langle u^{2n}\right\rangle = 2n(2n-1)B_0\left\langle u^{2n-2}\right\rangle + h_{2n},$$

where h_{2n} is the anomaly term

$$h_{2n} = \int_{\mathbb{R}} u^{2n} G\,du = -\frac{\rho}{2^{2n}(2n+1)}\left(\left\langle (2\bar{u}-s)^{2n}(\bar{u}+ns)\right\rangle - \left\langle (2\bar{u}+s)^{2n}(\bar{u}-ns)\right\rangle\right)$$

($h_{2n+1} = 0$ by parity). An alternative definition of h_{2n} is

$$h_{2n} = 2n \lim_{\nu\to 0} \nu\left\langle u^{2n-1}u_{xx}\right\rangle = -2n(2n-1)\lim_{\nu\to 0}\nu\left\langle u^{2n-2}u_x^2\right\rangle.$$

This gives, for instance,

$$h_2 = -2\lim_{\nu\to 0}\nu\langle u_x^2\rangle = \frac{\rho}{6}\left\langle s^3\right\rangle.$$

At statistical steady state, it gives

$$\left\langle u^{2n}\right\rangle = \frac{C_{2n}\rho}{B_0}\left(\left\langle (2\bar{u}-s)^{2n+2}(\bar{u}+(n+1)s)\right\rangle - \left\langle (2\bar{u}+s)^{2n+2}(\bar{u}-(n+1)s)\right\rangle\right),$$

where $C_n = n!/2^{n+2}(n+3)!$. These expressions can also be obtained from (2.11). In particular, for $n = 2$, we again obtain (2.12).

2.4 Multipoint PDF

We now turn to $Z(u_1,x_1,\ldots,u_n,x_n,t)$, the multipoint PDF of u. We have the following:

THEOREM 2.2 *Z satisfies*

$$\begin{aligned} Z_t = &-\sum_{p=1}^{n} u_p Z_{x_p} + \sum_{p,q=1}^{n} B(x_p-x_q) Z_{u_p u_q} \\ &-\sum_{p=1}^{n}\int_{\mathbb{R}} K(u_p-u')Z_{x_p}(u_1,x_1,\ldots,u',x_p,\ldots,u_n,x_n,t)du' \\ &+\sum_{p=1}^{n} G(u_p,x_p,u_2,x_2\ldots,u_{p-1},x_{p-1},u_{p+1},x_{p+1},\ldots,u_n,x_n,t), \end{aligned} \tag{2.18}$$

where $K(u) = (H(u) - H(-u))/2$, $H(\cdot)$ is the Heaviside function, and G is given by

$$G(u_1,x_1,\dots,u_n,x_n,t) = \left(\int_{\mathbb{R}^-} \int_{u_1+s/2}^{u_1-s/2} (u_1-\bar{u})\varrho(\bar{u},s,x_1,u_2,x_2,\dots,u_n,x_n,t)d\bar{u}ds \right)_{u_1} . \tag{2.19}$$

Here $\varrho(\bar{u},s,x_1,u_2,x_2\dots,u_n,x_n,t)$ is defined such that

$$\varrho(\bar{u},s,x_1,u_2,x_2,\dots,u_n,x_nt)d\bar{u}ds\,du_2\cdots du_n\,dx$$

gives the average number of shocks in $[x_1,x_1+dx)$ with $\bar{u}(y,t)\in[\bar{u},\bar{u}+d\bar{u})$, $s(y,t)\in[s,s+ds)$ where $y\in[x_1,x_1+dx)$ is a shock location, and $u(x_2,t)\in[u_2,u_2+du_2)$, $\dots$, $u(x_n,t)\in[u_n,u_n+du_n)$.

We will omit the proof of Theorem 2.2 since it is a straightforward generalization of the one given for Theorem 2.1.

For the two-point PDF, $Z(u_1,x_1,u_2,x_2,t)$, (2.18) becomes

$$\begin{aligned} Z_t = &-u_1Z_{x_1} - \int_{\mathbb{R}} K(u_1-u')Z_{x_1}(u',x_1,u_2,x_2,t)du' \\ &-u_2Z_{x_2} - \int_{\mathbb{R}} K(u_2-u')Z_{x_2}(u_1,x_1,u',x_2,t)du' \\ &+B_0Z_{u_1u_1} + B_0Z_{u_2u_2} + 2B(x_1-x_2)Z_{u_1u_2} \\ &+G(u_1,x_1,u_2,x_2,t) + G(u_2,x_2,u_1,x_1,t)\,. \end{aligned} \tag{2.20}$$

Assuming statistical homogeneity, (2.20) can be simplified. First,

$$Z(u_1,y,u_2,x+y,t) = Z(u_1,0,u_2,x,t) \equiv Z(u_1,u_2,x,t)\,.$$

Second, assuming that the number density of shocks ρ is finite,

$$\varrho(\bar{u},s,y,u,x+y,t) = \rho T(\bar{u},s,u,x,t)\,,$$

where $T(\bar{u},s,u,x,t)$ is the PDF of

$$(\bar{u}(y_0,t),s(y_0,t),u(y_0+x,t))\,,$$

conditional on y_0 being a shock position. Thus for statistically homogeneous situations, (2.20) reduces to the following equation for $Z(u_1,u_2,x,t)$:

$$
\begin{aligned}
Z_t = &-(u_2-u_1)Z_x - \int_{\mathbb{R}} K(u_2-u')Z_x(u_1,u',x,t)du' \\
&+ \int_{\mathbb{R}} K(u_1-u')Z_x(u',u_2,x,t)du' + B_0 Z_{u_1u_1} + B_0 Z_{u_2u_2} + 2B(x)Z_{u_1u_2} \\
&+ G(u_1,u_2,x,t) + G(u_2,u_1,-x,t)\,,
\end{aligned} \tag{2.21}
$$

where

$$
G(u_1,u_2,x,t) = \rho\left(\int_{\mathbb{R}^-}\int_{u_1+s/2}^{u_1-s/2}(u_1-\bar{u})T(\bar{u},s,u_2,x,t)d\bar{u}ds\right)_{u_1}. \tag{2.22}
$$

2.5 Velocity Difference and Structure Functions

Assuming statistical homogeneity and letting $Z^\delta(\delta u,x,t)$ be the PDF of the velocity difference $\delta u(x,y,t) = u(x+y,t)-u(y,t)$, $Z^\delta(w,x,t)$ is related to $Z(u_1,u_2,x,t)$ by

$$
Z^\delta(w,x,t) = \int_{\mathbb{R}} Z\left(u-\frac{w}{2},\, u+\frac{w}{2},\, x,\, t\right)du\,.
$$

The following corollary then follows immediately from (2.21) and (2.22):

COROLLARY 2.3 *Z^δ satisfies*

$$
\begin{aligned}
Z_t^\delta = &-wZ_x^\delta - 2\int_{\mathbb{R}} H(w'-w)Z_x^\delta(w',x,t)dw \\
&+ 2(B_0-B(x))Z_{ww}^\delta + G^\delta(w,x,t)\,,
\end{aligned} \tag{2.23}
$$

where $K(w) = (H(w)-H(-w))/2$, $H(\cdot)$ is the Heaviside function, and

$$
G^\delta(w,x,t) = \int_{\mathbb{R}} G\left(u-\frac{w}{2},u+\frac{w}{2},x,t\right)du + \int_{\mathbb{R}} G\left(u+\frac{w}{2},u-\frac{w}{2},-x,t\right)du\,. \tag{2.24}
$$

Remark. G^δ can be put into a form that is more convenient for the calculations. Let

$$
\begin{aligned}
\delta u_+(x,y_0,t) &= u(y_0+|x|,t) - u_+(y_0,t)\,,\\
\delta u_-(x,y_0,t) &= u_-(y_0,t) - u(y_0-|x|,t)\,,
\end{aligned}
$$

and let $U_\pm(s,\delta u_\pm,x,t)$ be the PDFs of $(s(y_0,t),\delta u_\pm(x,y_0,t))$ conditional on y being a shock position. Then, assuming the number density of shocks ρ is finite, G^δ can be expressed as

$$
G^\delta(w,x,t) = G_+^\delta(w,x,t) + G_-^\delta(w,x,t)\,,
$$

where

$$G^{\delta}_{\pm}(w,x,t) = \frac{\rho}{2}\int_{\mathbb{R}^-} s(U_\pm(s,\operatorname{sgn}(x)w - s,x,t) + U_\pm(s,\operatorname{sgn}(x)w,x,t))ds$$
$$- \rho\int_{\mathbb{R}^-}\int_0^1 sU_\pm(s,\operatorname{sgn}(x)w-\beta s,x,t)d\beta\, ds\,.$$

We now consider some consequences of (2.23). We have the following two theorems:

THEOREM 2.4 *In the limit as $x \to 0$,*

$$\begin{aligned} Z^{\delta}(w,x,t) &= \delta(w) - |x|(\rho\delta(w) + \langle\xi\rangle\delta^1(w) - \rho S(w,t)) + o(x)\,,\\ &= (1-\rho|x|)\frac{1}{x}Q\left(\frac{w}{x},t\right) + |x|\rho S(w,t) + o(x)\,, \end{aligned} \tag{2.25}$$

where $\delta^1(w) = d\delta(w)/dw$ and $o(x)$ must be interpreted in the sense of weak convergence. $Q(\xi,t)$ is the PDF of $\xi(x,t)$, the regular part of the velocity gradient, i.e.,

$$u_x(x,t) = \xi(x,t) + \sum_j s(y_j,t)\delta(y-y_j)\,,$$

and $S(s,t)$ is the PDF of $s(y_0,t)$ conditional on y_0 being a shock location.

THEOREM 2.5 (Structure Function Scaling) *Let*

$$\langle|\delta u|^a\rangle = \int_{\mathbb{R}} |w|^a Z^{\delta}(w,x,t)dw\,. \tag{2.26}$$

In the limit as $x \to 0$,

$$\langle|\delta u|^a\rangle = \begin{cases} |x|^a\langle|\xi|^a\rangle + o(x^a) & \text{if } 0 \le a < 1\\ |x|\big(\langle|\xi|\rangle + \rho\langle|s|\rangle\big) + o(x) & \text{if } a = 1\\ |x|\rho\langle|s|^a\rangle + o(x) & \text{if } 1 < a\,. \end{cases} \tag{2.27}$$

In terms of the moments, (2.27) is ($n \in \mathbb{N}_0$)

$$\left\langle \delta u^{2n}\right\rangle = |x|\rho\left\langle s^{2n}\right\rangle + o(x)\,, \quad \left\langle \delta u^{2n+1}\right\rangle = x\rho\left\langle s^{2n+1}\right\rangle + o(x)\,. \tag{2.28}$$

Remark. The first moment satisfies $\langle\delta u\rangle = 0$ for all (x,t). This is a consequence of statistical homogeneity, and it is readily verified since multiplying (2.23) by w and integrating gives in $\langle\delta u\rangle_t = 0$. On the other hand, from (2.23), the second moment satisfies

$$\langle\delta u^2\rangle_t = -\frac{1}{3}\langle\delta u^3\rangle_x + \frac{\rho}{3}\langle s^3\rangle + 4(B_0 - B(x))\,. \tag{2.29}$$

At statistical steady state, this equation reduces to the following ordinary differential equation for $\langle\delta u^3\rangle$:

$$\frac{d}{dx}\langle\delta u^3\rangle = \rho\langle s^3\rangle - 12(B_0 - B(x)) = -12(2B_0 - B(x))\,, \tag{2.30}$$

where we used (2.12). The solution of this equation for the boundary condition $\langle \delta u^3(x=0)\rangle = 0$ is

$$\langle \delta u^3 \rangle = -12\int_0^x (2B_0 - B(y))dy\,, \tag{2.31}$$

implying, in particular, that $\langle \delta u^3\rangle$ is analytic in x at statistical steady state.

An equation for $Q(\xi,t)$ will be derived in Section 3. To interpret (2.25), note that Z^δ can be decomposed into

$$Z^\delta(w,x,t) = p_{ns}(x,t)Z^\delta(w,x,t \mid \text{no shock}) + (1-p_{ns}(x,t))Z^\delta(w,x,t \mid \text{shock})\,,$$

where $p_{ns}(x,t)$ is the probability that there is no shock in $[y,y+x)$, $Z^\delta(w,x,t \mid \text{no shock})$ is the PDF of $\delta u(x,y,t)$ conditional on the property that there is no shock in $[y,y+x)$, and $Z^\delta(w,x,t \mid \text{shock})$ is the PDF of $\delta u(x,y,t)$ conditional on the property that there is at least one shock in $[y,y+x)$. Since by definition of ρ we have

$$p_{ns} = 1-\rho|x| + o(x)\,,$$

(2.25) states that

$$Z^\delta(w,x,t \mid \text{no shock}) = (1-\rho|x|)\frac{1}{x}Q\left(\frac{w}{x},t\right) + o(x)\,,$$
$$Z^\delta(w,x,t \mid \text{shock}) = S(w,t) + o(1)\,.$$

This is consistent with the picture that $\delta u(x,y,t) = x\xi(y,t) + o(x)$ if there is no shock in $[y,y+x)$, and $w(x,y,t) = s(y_0,t) + o(1)$ if $y_0 \in [y,x+y)$ is a shock position.

Proof of Theorem 2.4: We will consider the case $x>0$. The case $x<0$ can be treated similarly. Note first that, in the limit as $x\to 0$, we have

$$xZ^\delta(x\xi,x,t) \to Q(\xi,t)$$

weakly. We postpone the proof of this fact until Section 3. It implies that

$$Z^\delta(w,x,t) = \delta(w) + o(1)$$

weakly. Define

$$A(w,t) = \lim_{x\to 0} x^{-1}\big(Z^\delta(w,x,t) - \delta(w)\big) = \lim_{x\to 0} Z^\delta_x(w,x,t)\,.$$

Taking the limit as $x\to 0$ in the equation for Z^δ, it follows that A satisfies

$$0 = -wA - 2\int_{\mathbb{R}} H(w-w')A(w',x,t)dw' + B(w,t)\,, \tag{2.32}$$

where we used $\lim_{x\to 0}(B_0 - B(x)) = 0$ and we defined

$$B(w,t) = \lim_{x\to 0} G^\delta(w,x,t)\,.$$

To evaluate B, note that as $x\to 0$

$$\delta u_\pm(x,y_0,t) \to 0$$

almost surely. This implies that, as $x \to 0$,

$$U_\pm(s,w,x,t) \to S(s,t)\delta(w),$$

where $S(s,t)$ is the PDF of $s(y_0,t)$ conditional on y_0 being a shock location. Hence, from the expression for G^δ,

$$B(w,t) = \rho w S(w,t) + \rho\langle s\rangle\delta(w) + 2\rho\int_{-\infty}^{w} S(w',t)dw' - 2\rho H(w),$$

where $H(\cdot)$ is the Heaviside function and we used $S(s,t) = 0$ for $s > 0$. Inserting this expression in (2.32), the solution of this equation is

$$A(w,t) = -\delta(w) + \rho\langle s\rangle\delta^1(w) + \rho S(w,t).$$

Here we used the identity $w\delta^1(w) = -\delta(w)$. Using (3.4), $\rho\langle s\rangle = -\langle\xi\rangle$, this can be restated as

$$A(w,t) = -\delta(w) - \langle\xi\rangle\delta^1(w) + \rho S(w,t).$$

Combining the above results, we have

$$Z^\delta(w,x,t) = \delta(w) - x(\delta(w) + \langle\xi\rangle\delta^1(w) - \rho S(w,t)) + o(x)$$

weakly. This establishes the first equation in (2.25) for $x > 0$. Reorganizing this expression as

$$Z^\delta(w,x,t) = (1-\rho x)(\delta(w) - x\langle\xi\rangle\delta^1(w)) + x\rho S(w,t) + o(x) \tag{2.33}$$

and using the identity

$$\delta(w) - x\langle\xi\rangle\delta^1(w) = \frac{1}{x}Q\left(\frac{w}{x},t\right) + o(x),$$

we obtain the second equation in (2.25) for $x > 0$. □

PROOF OF THEOREM 2.5: We will prove (2.27) directly for moments of integer order higher than 1. For other values of a, (2.27) follows from (2.25) and the fact that the tails are controlled by higher-order moments. Note first that for $a > 0$

$$\lim_{x\to 0}\langle|\delta u|^a\rangle = 0,$$

since $\delta u(x,y,t) \to 0$ almost surely as $x \to 0$. Now, multiply (2.23) by w^n ($n \in \mathbb{N}$, $n \geq 2$), integrate, and take the limit as $x \to 0\pm$. The result is

$$0 = -a_{n+1}^\pm + \frac{2}{n+1}a_{n+1}^\pm + b_n^\pm,$$

where

$$a_n^\pm = \lim_{x\to 0\pm} x^{-1}\langle\delta u^n\rangle = \lim_{x\to 0\pm}\langle\delta u^n\rangle_x, \qquad b_n^\pm = \lim_{x\to 0\pm}\int_{\mathbb{R}} w^n G^\delta(w,x,t)dw.$$

Note that

$$(\operatorname{sgn}(x))^n\int_{\mathbb{R}} w^n G^\delta(w,x,t)dw$$

$$= \frac{\rho}{2}\langle s(\delta u_+ + s)^n\rangle + \frac{\rho}{2}\langle s\delta u_+^n\rangle - \rho\int_0^1 \langle s(\delta u_+ + \beta s)^n\rangle d\beta + \frac{\rho}{2}\langle s(\delta u_- + s)^n\rangle$$
$$+ \frac{\rho}{2}\langle s\delta u_-^n\rangle - \rho\int_0^1 \langle s(\delta u_- + s)^n\rangle d\beta\,.$$

Since $\delta u_\pm(x,y_0,t) \to 0$ almost surely as $x \to 0$, it follows that

$$b_n^\pm = (\pm 1)^n \rho\langle s^{n+1}\rangle\left(1 - \frac{2}{n+1}\right).$$

Inserting this expression in the equation for $a_n^\pm$ gives

$$a_{n+1}^\pm = \lim_{x\to 0\pm} x^{-1}\langle \delta u^{n+1}\rangle = (\pm 1)^n \rho\langle s^{n+1}\rangle\,.$$

Thus

$$\langle \delta u^{2n}\rangle = |x|\rho\langle s^{2n}\rangle + o(x)\,, \qquad \langle \delta u^{2n+1}\rangle = x\rho\langle s^{2n+1}\rangle + o(x)\,.$$

This proves (2.28).

We now prove (2.27) for $0 \le a \le 1$. The proof for other values of a is similar. Let

$$f^\delta(w,x,t) = (1-\rho|x|)\frac{1}{x}Q\left(\frac{w}{x},t\right) + |x|\rho S(w,t)\,,$$
$$g^\delta(w,x,t) = \delta(w) - |x|(\rho\delta(w) + \langle\xi\rangle\delta^1(w) - \rho S(w,t))\,,$$

and write for $M > 0$

$$\int_{\mathbb{R}} |w|^a(Z^\delta - f^\delta)dw = \int_{|w|\le M} |w|^a(Z^\delta - f^\delta)dw + \int_{|w|>M} |w|^a(Z^\delta - f^\delta)dw\,.$$

The first term at the right-hand side is $o(x)$ because of (2.25). To estimate the second term, note that for M large enough

$$\begin{aligned}\int_{|w|>M} |w|^a Z^\delta\,dw &\le \int_{|w|>M} w^2 Z^\delta\,dw \le \left|\int_{\mathbb{R}} w^2(Z^\delta - g^\delta)dw\right| + \int_{|w|>M} w^2 g^\delta\,dw\\ &= o(x) + |x|\rho\int_{|w|>M} w^2 S(w,t)dw\\ &= o(x) + O(x)\int_{|w|>M} w^2 S(w,t)dw\,.\end{aligned}$$

$$\begin{aligned}\int_{|w|>M} |w|^a f^\delta\,dw &= |x|^a(1-|x|\rho)\int_{|\xi|>M/x} |\xi|^a Q(\xi,t)d\xi + |x|\rho\int_{w>M} |w|^a S(w,t)dw\\ &= o(x^a) + O(x)\int_{w>M} |w|^a S(w,t)dw\,.\end{aligned}$$

Since M can be made arbitrarily large, we get

$$\int_{\mathbb{R}} |w|^a (Z^\delta - f^\delta) dw \le o(x^a) + \delta_M O(x) \,,$$

where $\delta_M \to 0$ as $M \to \infty$. Noting that

$$\int_{\mathbb{R}} |w|^a f^\delta \, dw = \begin{cases} |x|^a \langle |\xi|^a \rangle & \text{if } 0 \le a < 1 \\ |x| (\langle |\xi| \rangle + \rho \langle |s| \rangle) & \text{if } a = 1 \,, \end{cases}$$

we obtain (2.27) for $0 \le a \le 1$. □

3 Velocity Gradient

We now turn to the study of the PDF of the velocity gradient for the solutions of (2.2). Since the solutions typically contain discontinuities, it is already an issue whether the PDF for the velocity gradient is well-defined. Heuristically, it is well-defined since u only fails to be differentiable at no more than countably many points. We will therefore be concerned with the PDF of the regular part of the gradient.

3.1 Master Equation for the PDF of the Velocity Gradient

We focus on the statistically homogeneous case with finite-number density of shocks and derive an equation for $Q(\xi,t)$, the PDF of $\xi(x,t)$, defined as the regular part of the velocity gradient, i.e.,

$$u_x(x,t) = \xi(x,t) + \sum_j s(y_j) \delta(x - y_j) \,,$$

where $\xi(\cdot,t) \in L^1(I)$. We will prove the following:

THEOREM 3.1 *Q satisfies*

$$Q_t = \xi Q + \left(\xi^2 Q\right)_\xi + B_1 Q_{\xi\xi} + F(\xi,t) \,, \tag{3.1}$$

where $B_1 = -B_{xx}(0)$ and

$$F(\xi,t) = \rho \int_{\mathbb{R}^-} s V(s,\xi,t) ds \,. \tag{3.2}$$

Here $V(s,\xi,t) = (V_+(s,\xi,t) + V_-(s,\xi,t))/2$, and $V_\pm(s,\xi_\pm,t)$ are the PDFs of

$$(s(y_0,t), \xi_\pm(y_0,t) = u_x(y_0 \pm, t)) \,,$$

conditional on the property that y_0 is a shock position.

The consequences of (3.1) will be studied in Section 3.3. Note that if $u(x,t) \overset{(d)}{=} -u(-x,t)$, then

$$(\xi_+(x,t), s(x,t)) \overset{(d)}{=} (\xi_-(-x,t), s(-x,t))$$

and $V_+(s,\xi,t) = V_-(s,\xi,t) = V(s,\xi,t)$.

PROOF OF THEOREM 3.1: Let $Q^\delta(\eta,x,t)$ be the PDF of $\eta(x,y,t) = (u(x+y,t) - u(y,t))/x$. Q^δ is related to Z and Z^δ by

$$Q^\delta(\eta,x,t) = xZ^\delta(x\eta,x,t) = x\int_{\mathbb{R}} Z\left(u - \frac{x\eta}{2}, u + \frac{x\eta}{2}, x, t\right) du\,.$$

From (2.23) it follows that Q^δ satisfies

$$\begin{aligned} Q^\delta_t &= \eta Q^\delta + \left(\eta^2 Q^\delta\right)_\eta + B_1(x) Q^\delta_{\eta\eta} - x\eta Q^\delta_x \\ &\quad - 2x \int_{\mathbb{R}} H(\eta-\eta') Q^\delta_x(\eta',x,t)\,d\eta' + F^\delta(\eta,x,t)\,, \end{aligned} \tag{3.3}$$

where $B_1(x) = 2(B_0 - B(x))/x^2$ and

$$F^\delta(\eta,x,t) = F^\delta_1(\eta,x,t) + F^\delta_2(\eta,x,t) + F^\delta_3(\eta,x,t)\,,$$

with

$$\begin{aligned} F^\delta_1(\eta,x,t) &= \rho \int_{\mathbb{R}^-} sV^\delta(s,\eta,x,t)\,ds\,, \\ F^\delta_2(\eta,x,t) &= \rho \int_{\mathbb{R}^-} sV^\delta\left(s, \eta - \frac{s}{|x|}, x, t\right) ds\,, \\ F^\delta_3(\eta,x,t) &= -2\rho \int_{\mathbb{R}^-} \int_0^1 sV^\delta\left(s, \eta - \frac{\beta s}{|x|}, x, t\right) d\beta\, ds\,. \end{aligned}$$

Here $V^\delta(s,\eta,x,t) = (V^\delta_+(s,\eta,x,t) + V^\delta_-(s,\eta,x,t))/2$, $V^\delta_\pm(s,\eta,x,t)$ are the PDFs of

$$(s(y_0,t), \eta_\pm(y_0,x,t))\,,$$

with $\eta_\pm(y_0,x,t) = \pm(u(y_0 \pm |x|,t) - u_\pm(y_0,t))/|x|$, conditional on the property that y_0 is a shock position.

Define

$$Q(\xi,t) = \lim_{x\to 0} Q^\delta(\xi,x,t)\,.$$

It is easy to see that

$$\begin{aligned} \int_{\mathbb{R}} F^\delta_1(\eta,x,t)\,d\eta &= \rho\langle s\rangle\,, \\ \int_{\mathbb{R}} F^\delta_2(\eta,x,t)\,d\eta &= \rho \int_{\mathbb{R}\times\mathbb{R}^-} sV^\delta(s,\eta,x,t)\,d\eta\,ds = \rho\langle s\rangle\,, \\ \int_{\mathbb{R}} F^\delta_3(\eta,x,t)\,d\eta &= -2\rho \int_{\mathbb{R}\times\mathbb{R}^-} sV^\delta(s,\eta,x,t)\,d\eta\,ds = -2\rho\langle s\rangle\,, \end{aligned}$$

consistent with the fact that

$$\int_{\mathbb{R}} F^{\delta}(\eta,x,t)d\eta = 0\,.$$

Hence,

$$\frac{d}{dt}\int_{\mathbb{R}} Q^{\delta}(\eta,x,t)d\eta = 0\,, \qquad \int_{\mathbb{R}} \eta Q^{\delta}(\eta,x,t)d\eta = 0\,.$$

In the limit as $x \to 0$, η_+ and η_- converge, respectively, to the gradient of the velocity at the left and right sides of the shock. Moreover, pointwise in η, we have

$$\lim_{x\to 0} F_2^{\delta}(\eta,x,t) = \lim_{x\to 0} F_3^{\delta}(\eta,x,t) = 0\,, \qquad \lim_{x\to 0} F_1^{\delta}(\eta,x,t) = F(\eta,t)\,,$$

with F given by (3.2). Therefore, a standard argument with test functions applied to (3.3) shows that in the limit as $x \to 0$, Q^{δ} converges weakly to Q, solution of (3.1). ☐

Remark. F_2^{δ} and F_3^{δ} are examples of what we will call “ghost terms,” i.e., terms that have finite total moments but in the limit converge pointwise to zero. Due to the ghost terms, it is not clear at the moment whether Q satisfies $\int_{\mathbb{R}} Q(\xi,t)d\xi = 1$. This will be established as a consequence of Lemma 3.2.

3.2 Alternative Limiting Processes

Using BV calculus, one works at $\nu = 0$ and accesses the statistics of the velocity gradient by taking $x \to 0$ in $(u(x+y,t) - u(y,t))/x$. This procedure gives an equation for

$$Q(\xi,t) = \lim_{x\to 0} Q^{\delta}(\xi,x,t)\,,$$

where $Q^{\delta}(\xi,x,t)$ is the PDF of $(u(x+y,t)-u(y,t))/x$. In this section, we revert the order of the limits: We take $x \to 0$ first, working at finite ν, and then let $\nu \to 0$. As in Section 2.2, this is done using boundary layer analysis and matched asymptotics. In this way, we will obtain an equation for

$$\underline{Q}(\xi,t) = \lim_{\nu\to 0} Q^{\nu}(\xi,t)\,,$$

which will turn out to be identical to the one for Q. To the extent that boundary layer analysis can be justified, this strongly suggests that the limits $x \to 0$ and $\nu \to 0$ commute.

For statistically homogeneous situations, recall that $Q^{\nu}(\xi,t)$ satisfies (see (A.5) in the appendix)

$$Q_t^{\nu} = \xi Q^{\nu} + \left(\xi^2 Q^{\nu}\right)_{\xi} + B_1 Q^{\nu}_{\xi\xi} - \nu\left(\langle \xi_{xx} \mid \xi\rangle Q^{\nu}\right)_{\xi}\,.$$

The average of the dissipative term can be expressed as

$$\nu\langle \xi_{xx} \mid \xi\rangle Q^{\nu} = \nu \lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{L} \xi_{xx}(x,t)\delta(\xi - \xi(x,t))dx\,.$$

As in Section 2.2, we will evaluate this integral using the approximation for ξ_{xx} provided by the boundary layer analysis. However, this analysis has to be developed further in order to evaluate the dissipative term for the velocity gradient. Recall that $\xi^{\text{in}} = u^{\text{in}}_x = v_{0z}/\nu + v_{1z} + O(\nu)$. Inside the shock, the $O(1)$ contribution of v_{1z} is clearly negligible compared to the $O(\nu^{-1})$ contribution of v_{0z}/ν. However, the contribution of v_{1z} *is* important at the border of the shock because v_{0z} decays exponentially fast there.

To evaluate v_{1z} at the shock boundaries $z \to \pm\infty$, consider the equation for v_1 that we get from (2.17):

$$v_{0t} + (v_0 - \bar{u})v_{1z} + v_{0z}(v_1 + \gamma) = v_{1zz} + f\,.$$

The general solution of this equation can be expressed as

$$\begin{aligned} v_1 &= C_1 \mathrm{e}^{\int_0^z (v_0(z') - \bar{u})dz'} \\ &\quad + \int_0^z \left(C_2 + (v_0(z') - \bar{u})\gamma + \int_0^{z'} v_{0t}(z'')dz'' - fz' \right) \mathrm{e}^{\int_{z'}^z (v_0(z'') - \bar{u})dz''}\, dz'\,, \end{aligned}$$

where C_1 and C_2 are constants. They, as well as γ, have to be determined by matching with $u^{\text{out}} = u_0 + \nu u_1 + O(\nu^2)$ [21]. We will not dwell on this problem since C_1, C_2, and γ do not enter the expression for v_{1z} as $z \to \pm\infty$. Indeed, using the expression for v_0, direct computation shows that v_1 reduces asymptotically to

$$\begin{aligned} v_1 &= \frac{1}{s}\left(2\frac{d\bar{u}}{dt} - \frac{ds}{dt} - 2f \right) z + \frac{2}{s^2}\left(C_2 s + \frac{ds}{dt} - 2\frac{d\bar{u}}{dt} - \frac{s^2\gamma}{2} + 2f \right) + O\left(\mathrm{e}^{-sz/2} \right) \\ &\quad \text{as } z \to -\infty, \end{aligned}$$

and

$$\begin{aligned} v_1 &= -\frac{1}{s}\left(2\frac{d\bar{u}}{dt} + \frac{ds}{dt} - 2f \right) z - \frac{2}{s^2}\left(C_2 s + \frac{ds}{dt} + 2\frac{d\bar{u}}{dt} + \frac{s^2\gamma}{2} - 2f \right) + O\left(\mathrm{e}^{sz/2} \right) \\ &\quad \text{as } z \to +\infty\,. \end{aligned}$$

Thus

$$\lim_{z\to\pm\infty} v_{1z} = \mp\frac{2}{s}\frac{d\bar{u}}{dt} - \frac{1}{s}\frac{ds}{dt} \pm \frac{2f}{s} \equiv \xi_\pm\,,$$

where the last equality is simply a definition of $\xi_\pm$. Note that these can be reorganized to give

$$\frac{ds}{dt} = -\frac{s}{2}(\xi_+ + \xi_-)\,, \qquad \frac{d\bar{u}}{dt} = -\frac{s}{4}(\xi_+ - \xi_-) + f\,.$$

In the limit as $\nu \to 0$, these are the equations of motion along the shock.

Using these results, to $O(\nu)$, (2.16) can be estimated as

$$\begin{aligned} &\nu\langle \xi_{xx} | \xi \rangle Q^\nu = \\ &\quad \nu\rho \int\limits_{\mathbb{R}\times\mathbb{R}^-\times\mathbb{R}\times\mathbb{R}} S(\bar{u}, s, \xi_+, \xi_-, t) \int\limits_\ell \xi^{\text{in}}_{xx}(x,t)\delta(\xi - \xi^{\text{in}}(x,t))dx\, d\bar{u}\, ds\, d\xi_+\, d\xi_-\,, \end{aligned}$$

where ℓ is a small interval around y with width $\gg O(\nu)$, $S(\bar{u},s,\xi_+,\xi_-,t)$ is the PDF of $(\bar{u}(y_0,t),s(y_0,t),\xi_+(y_0,t),\xi_-(y_0,t))$, conditional on y_0 being a shock location (ξ_+ and ξ_- have to be included for reasons to be made clear later).

We now go to the stretched coordinate $z=(x-y)/\nu$ and use the result of Section 2.2, namely,

$$\xi^{\text{in}} = \nu^{-1} v_{0z} + v_{1z} + O(\nu)\,.$$

To $O(\nu)$, both terms must be included. However, the contribution of v_{1z} is important only at the border of the shock where v_{0z} falls exponentially fast and can be neglected inside the shock. Thus, $\xi^{\text{in}} \approx \bar{\xi} = \nu^{-1}v_{0z}+\xi_\pm$, and we have

$$\nu\langle\xi_{xx}|\xi\rangle Q^\nu = \rho \int\limits_{\mathbb{R}\times\mathbb{R}^-\times\mathbb{R}\times\mathbb{R}} S(\bar{u},s,\xi_+,\xi_-,t)\int\limits_{\mathbb{R}} \bar{\xi}_{zz}(z,t)\delta(\xi-\bar{\xi}(z,t))dz\,d\bar{u}\,ds\,d\xi_+\,d\xi_-\,.$$

To perform the z integral, we change the integration variable to $\xi' = \bar{\xi}-\xi_-$ for $z<0$ and to $\xi'=\bar{\xi}-\xi_+$ for $z>0$. For $z<0$

$$((v_0-\bar{u})(\bar{\xi}-\xi_-))_z = \bar{\xi}_{zz}\,,\quad (v_0-\bar{u}) = \left(\frac{1}{4}s^2+2\nu(\bar{\xi}-\xi_-)\right)^{1/2} = -\frac{s}{2}+O(\nu)\,,$$

and we have

$$\bar{\xi}_{zz}\,dz = \frac{\bar{\xi}_{zz}}{(\xi-\xi_\pm)_z}\,d\xi' = -\frac{(s/2(\bar{\xi}-\xi_\pm))_z}{(\xi-\xi_\pm)_z}\,d\xi' = -\frac{s}{2}\,d\xi'\,.$$

Similarly, for $z>0$

$$((v_0-\bar{u})(\bar{\xi}-\xi_+))_z = \bar{\xi}_{zz}\,,\qquad (v_0-\bar{u}) = -\left(\frac{1}{4}s^2+2\nu(\bar{\xi}-\xi_+)\right)^{1/2} = \frac{s}{2}+O(\nu)\,,$$

and we have

$$\bar{\xi}_{zz}\,dz = \frac{\bar{\xi}_{zz}}{(\xi-\xi_\pm)_z}\,d\xi' = \frac{(s/2(\bar{\xi}-\xi_\pm))_z}{(\xi-\xi_\pm)_z}\,d\xi' = \frac{s}{2}\,d\xi'\,.$$

This leads to

$$\begin{aligned}
&\nu\langle\xi_{xx}\mid\xi\rangle Q^\nu \\
&\quad = \rho\int\limits_{\mathbb{R}\times\mathbb{R}^-\times\mathbb{R}\times\mathbb{R}} S(\bar{u},s,\xi_+,\xi_-,t)\int_{-s^2/(8\nu)}^0 \frac{s}{2}\delta(\xi-\xi'-\xi_-)d\xi'\,d\bar{u}\,ds\,d\xi_+\,d\xi_- \\
&\qquad + \rho\int\limits_{\mathbb{R}\times\mathbb{R}^-\times\mathbb{R}\times\mathbb{R}} S(\bar{u},s,\xi_+,\xi_-,t)\int_{-s^2/(8\nu)}^0 \frac{s}{2}\delta(\xi-\xi'-\xi_+)d\xi'\,d\bar{u}\,ds\,d\xi_+\,d\xi_-\,.
\end{aligned}$$

Letting $\nu\to 0$ and integrating gives

$$\lim_{\nu\to 0}\nu\langle\xi_{xx}\mid\xi\rangle Q^\nu = \rho\int\limits_{\mathbb{R}^-}\int_\xi^\infty sV(s,\xi',t)d\xi'\,ds\,,$$

where we used the consistency constraint

$$\int_{\mathbb{R}\times\mathbb{R}} S(\bar{u}, s, \xi_+, \xi_-, t)\, d\bar{u}\, d\xi_\mp = V_\pm(s, \xi_\pm, t)\,.$$

Thus

$$-\lim_{\nu\to 0} \nu\big(\langle \xi_{xx} \mid \xi\rangle Q^\nu\big)_\xi = \rho \int_{\mathbb{R}^-} sV(s, \xi, t)\, ds = F(\xi, t)\,.$$

3.3 Consequences of the Master Equation

First we observe the following:

LEMMA 3.2

$$\langle \xi \rangle + \rho \langle s \rangle = 0 \quad \textit{or} \quad \int_{\mathbb{R}} \xi Q(\xi, t)\, d\xi = -\rho \langle s \rangle\,. \tag{3.4}$$

PROOF: Denote by u^ν the solution of (1.1) for finite ν. Then $\langle u_x^\nu \rangle = 0$. Let $\varphi(x)$ be a compactly supported smooth function. We have

$$0 = \int \varphi \langle u_x^\nu \rangle dx = \left\langle \int \varphi u_x^\nu dx \right\rangle = -\left\langle \int \varphi_x u^\nu\, dx \right\rangle.$$

In the limit as $\nu \to 0$, $u^\nu \to u$, the solution of (2.2). Thus

$$0 = -\left\langle \int \varphi_x u\, dx \right\rangle.$$

Denote by y_j the location of the shocks. We can write

$$\int \varphi_x u\, dx = -\sum_j \int_{y_j}^{y_{j+1}} \varphi u_x\, dx - \sum_j \varphi(y_j) s(y_j, t)\,.$$

Averaging this result, we get

$$\int \varphi(\langle \xi \rangle + \rho \langle s \rangle) dx = 0$$

for all compactly supported smooth functions φ. Hence (3.4). $\square$

Notice that for finite ν or x, $\int_{\mathbb{R}} \xi Q^\nu(\xi, t) d\xi = \int_{\mathbb{R}} \xi Q^\delta(\xi, x, t) d\xi = 0$. In the language of Kraichnan [28], (3.4) represents a flow of probability of ξ from the smooth part of the velocity field to the shocks. It reflects the fact that no matter how small ν is, the dissipation range has a finite effect on inertial range statistics.

As a consequence of (3.1) and (3.4), we have

$$\frac{d}{dt} \int_{\mathbb{R}} Q(\xi, t)\, d\xi = 0\,;$$

i.e., the normalization of Q is preserved. For the initial data we are interested in, this implies

$$\int_{\mathbb{R}} Q(\xi,t)d\xi = 1\,.$$

LEMMA 3.3

$$\frac{d}{dt}(\rho\langle s\rangle) = -\frac{\rho}{2}(\langle s\xi_+\rangle + \langle s\xi_-\rangle)\,. \tag{3.5}$$

In particular, $\int_{\mathbb{R}} \xi F(\xi,t)d\xi = \rho(\langle s\xi_+\rangle + \langle s\xi_-\rangle)/2$ *exists and is finite.*

PROOF: Since $\rho\langle s\rangle$ and its derivative are finite by assumption, the existence of the integral $\int_{\mathbb{R}} \xi F(\xi,t)d\xi = \rho(\langle s\xi_+\rangle + \langle s\xi_-\rangle)/2$ follows from the argument below and a standard cutoff argument on large values of $\xi_\pm$.

Using

$$u_{\pm t} + u_\pm \xi_\pm = f\,,$$

where $\xi_\pm(x,t) = u_x(x\pm,t)$ and $dy_j/dt = \bar{u}(y_j,t)$, we have

$$\begin{aligned}
\frac{d}{dt}u_+(y_j,t) &= \frac{dy_j}{dt}\xi_+(y_j,t) + u_{+t}(y_j,t)\\
&= \bar{u}(y_j,t)\xi_+(y_j,t) - u_+(y_j,t)\xi_+(y_j,t) + f(y_j,t)\\
&= -\frac{1}{2}s(y_j,t)\xi_+(y_j,t) + f(y_j,t)\,.
\end{aligned}$$

Similarly,

$$\frac{d}{dt}u_-(y_j,t) = \frac{1}{2}s(y_j,t)\xi_-(y_j,t) + f(y_j,t)\,.$$

These equations can be reorganized into

$$\frac{d}{dt}\bar{u}(y_j,t) = -\frac{1}{4}s(y_j,t)(\xi_+(y_j,t) - \xi_-(y_j,t)) + f(y_j,t)$$

and

$$\frac{d}{dt}s(y_j,t) = -\frac{1}{2}s(y_j,t)(\xi_+(y_j,t) + \xi_-(y_j,t))\,.$$

Consider now

$$\rho\langle s\rangle = \left\langle \sum_j s(x,t)\delta(x-y_j)\right\rangle = \left\langle \sum_j s(y_j,t)\delta(x-y_j)\right\rangle. \tag{3.6}$$

Using the equation for $s(y_j,t)$ and $dy_j/dt = \bar{u}(y_j,t)$, we have

$$\begin{aligned}
\frac{d}{dt}(\rho\langle s\rangle) = -\frac{\rho}{2}(\langle s\xi_+\rangle + \langle s\xi_-\rangle) - \left\langle \sum_j s(y_j,t)\bar{u}(y_j,t)\delta(x-y_j)\right\rangle_x\\
+ \text{contribution from shock creation and collision}.
\end{aligned}$$

The second term on the right-hand side is zero by homogeneity. Thus to obtain (3.5) it remains to prove that the contribution of shock creation and collision vanish.

To understand how this term arises and why it vanishes, assume a shock is created at position y_1 at time t_1. Then the sum under the average in (3.6) involves a term like

$$T_1 = s(y_1,t)\delta(x-y_1)H(t-t_1)$$

where $H(\cdot)$ is the Heaviside function. Time differentiation gives

$$\begin{aligned}\frac{dT_1}{dt} = &\frac{ds(y_1,t)}{dt}\delta(x-y_1)H(t-t_1) - s(y_1,t)\bar{u}(y_1,t)\delta^1(x-y_1)\\ &+ s(y_1,t)\delta(x-y_1)\delta(t-t_1)\,.\end{aligned}$$

The last term accounts for shock creation. Since the shock amplitude is zero at creation,

$$s(y_1,t)\delta(x-y_1)\delta(t-t_1) = s(y_1,t_1)\delta(x-y_1)\delta(t-t_1) = 0\,.$$

This means that shock creation makes no contribution to the time derivative of (3.6). Consider now the collision events. Assume at time t_1 the shocks located at y_2 and y_3 merge into one shock located at y_1. Obviously $y_1(t_1) = y_2(t_1) = y_3(t_1)$. Such an event contributes to the sum under the average in (3.6) by a term like

$$T_2 = s(y_1,t)\delta(x-y_1)H(t-t_1) + (s(y_2,t)\delta(x-y_2) + s(y_3,t)\delta(x-y_3))H(t_1-t)\,.$$

Time differentiation gives

$$\begin{aligned}\frac{dT_2}{dt} = &\frac{ds(y_1,t)}{dt}\delta(x-y_1)H(t-t_1) + \frac{ds(y_2,t)}{dt}\delta(x-y_2)H(t_1-t)\\ &+ \frac{ds(y_3,t)}{dt}\delta(x-y_3)H(t_1-t) - s(y_1,t)\bar{u}(y_1,t)\delta^1(x-y_1)H(t-t_1)\\ &- \big(s(y_2,t)\bar{u}(y_2,t)\delta^1(x-y_2) + s(y_3,t)\bar{u}(y_3,t)\delta^1(x-y_3)\big)H(t_1-t)\\ &+ \big(s(y_1,t)\delta(x-y_1) - s(y_2,t)\delta(x-y_2) - s(y_3,t)\delta(x-y_3)\big)\delta(t-t_1)\,.\end{aligned}$$

The term involving $\delta(t-t_1)$ arises from shock collision. Since $y_1(t_1) = y_2(t_1) = y_3(t_1)$ and shock amplitudes add up at collision,

$$\begin{aligned}&\lim_{t\to 0+} s(y_1(t_1+t),t_1+t)\delta(x-y_1(t_1+t))\\ &\quad= \lim_{t\to 0+}(s(y_2(t_1-t),t_1-t)\delta(x-y_2(t_1-t))\\ &\qquad\quad+ s(y_3(t_1-t),t_1-t)\delta(x-y_3(t_1-t)))\,,\end{aligned}$$

the term in the equation for T_2 involving $\delta(t-t_1)$ vanishes. This means that shock collision makes no contribution to the time derivative of (3.6). Hence (3.5). □

COROLLARY 3.4 *We have*

$$\lim_{|\xi|\to\infty} |\xi|^3 Q(\xi,t) = 0\,, \tag{3.7}$$

i.e., Q decays faster than $|\xi|^{-3}$ as $\xi \to -\infty$.

PROOF: Taking the first moment of (3.1) leads to

$$\frac{d}{dt}\langle\xi\rangle = \left[\xi^3 Q\right]_{-\infty}^{+\infty} + \int_{\mathbb{R}} \xi F d\xi = \left[\xi^3 Q\right]_{-\infty}^{+\infty} + \frac{\rho}{2}(\langle s\xi_+\rangle + \langle s\xi_-\rangle)\,.$$

Using (3.4) this equation can be written as

$$\frac{d}{dt}\big(\rho\langle s\rangle\big) = -\left[\xi^3 Q\right]_{-\infty}^{+\infty} - \frac{\rho}{2}(\langle s\xi_+\rangle + \langle s\xi_-\rangle)\,.$$

Using (3.5) we obtain that the boundary term must be zero. Since $\xi^3 Q$ has a different sign for large negative and positive values of ξ, one must have separately $\lim_{\xi\to-\infty}|\xi|^3 Q = 0$ and $\lim_{\xi\to+\infty}|\xi|^3 Q = 0$. Hence (3.7). ☐

Remark. Lemma 3.3 uses the facts that shocks are created at zero amplitude and shock strengths add up at collision, which are our working assumptions. It is possible to construct pathological situations, such as the unforced case with piecewise linear initial data [33], in which case shocks are created at finite amplitude. In this case Lemma 3.3 is changed to

LEMMA 3.5

$$\frac{d}{dt}(\rho\langle s\rangle) = -\frac{\rho}{2}(\langle s\xi_+\rangle + \langle s\xi_-\rangle) - D_c\,, \tag{3.8}$$

where

$$D_c = -\sigma_1\langle s_1\rangle - \sigma_2\langle s_2\rangle \geq 0\,.$$

Here σ_1 and σ_2 are, respectively, the space-time number density of shock creation and collision points, $\langle s_1\rangle \leq 0$ is the average shock amplitude at creation, and $\langle s_2\rangle \leq 0$ is the average gain of amplitude at shock collision. In this case

$$Q(\xi,t) \sim D_c|\xi|^{-3} \quad \textit{as } \xi \to -\infty\,. \tag{3.9}$$

Lemma 3.5 follows from a direct adaptation of the proof of Lemma 3.3.

3.4 Realizability and Asymptotics for the Statistical Steady State

We now turn to the study of (3.1) at steady state

$$0 = \xi Q + \left(\xi^2 Q\right)_\xi + B_1 Q_{\xi\xi} + F(\xi)\,, \tag{3.10}$$

where $F(\xi) = \lim_{t\to\infty} F(\xi,t)$. We shall prove the following:

THEOREM 3.6 *The realizability constraint $Q \in L^1(\mathbb{R})$, $Q \geq 0$, implies*

$$\lim_{\xi\to+\infty} \xi^{-2} \mathrm{e}^{\Lambda} F(\xi) = 0\,, \tag{3.11}$$

where $\Lambda = -\xi^3/3B_1$. Assuming (3.11)*, the only positive solution of* (3.10) *can be expressed as*

$$Q_\infty(\xi) = \frac{1}{B_1}\int_{-\infty}^{\xi} \xi' F(\xi')d\xi' - \frac{\xi \mathrm{e}^{-\Lambda}}{B_1}\int_{-\infty}^{\xi} \mathrm{e}^{\Lambda'} G(\xi')d\xi'\,, \tag{3.12}$$

where

$$G(\xi) = F(\xi) + \frac{\xi}{B_1} \int_{-\infty}^{\xi} \xi' F(\xi') d\xi' .$$

Furthermore,

$$Q_\infty(\xi) \sim \begin{cases} |\xi|^{-3} \int_{-\infty}^{\xi} \xi' F(\xi') d\xi' & \text{as } \xi \to -\infty \\ C_+ \xi \mathrm{e}^{-\Lambda} & \text{as } \xi \to +\infty , \end{cases} \tag{3.13}$$

where

$$C_+ = -\frac{1}{B_1} \int_{\mathbb{R}} \mathrm{e}^{\Lambda} G(\xi) d\xi .$$

Under the conditions of Theorem 3.6, we also have the following lemma:

LEMMA 3.7 *Q_∞ can be expressed as*

$$Q_\infty(\xi) = -\frac{\xi \mathrm{e}^{-\Lambda}}{B_1} \int_{-\infty}^{\xi} \xi'^{-2} \mathrm{e}^{\Lambda'} \int_{-\infty}^{\xi'} \xi'' F(\xi'') d\xi'' \, d\xi' \quad \text{for } \xi < 0, \tag{3.14}$$

and

$$Q_\infty(\xi) = C_+ \xi \mathrm{e}^{-\Lambda} - \frac{\xi \mathrm{e}^{-\Lambda}}{B_1} \int_{\xi}^{+\infty} \xi'^{-2} \mathrm{e}^{\Lambda'} \int_{\xi'}^{+\infty} \xi'' F(\xi'') d\xi'' \, d\xi' \quad \text{for } \xi > 0. \tag{3.15}$$

Remarks. 1. $\xi F(\xi)$ is integrable as a consequence of Lemma 3.3. In particular, (3.5) at steady state gives

$$\int_{\mathbb{R}} \xi F d\xi = \frac{\rho}{2} (\langle s \xi_+ \rangle + \langle s \xi_- \rangle) = 0 .$$

2. C_+ is finite if (3.11) holds. To see this, note that because of the factor e^{Λ}, a problem may arise at $\xi = +\infty$ only. (3.11) implies that we can write

$$F(\xi) = \mathrm{e}^{-\Lambda} f(\xi)$$

with $f(\xi) = o(\xi^2)$ as $\xi \to +\infty$. Using $\int_{\mathbb{R}} \xi F d\xi = 0$, we write $G(\xi)$ as

$$G(\xi) = F(\xi) - \frac{\xi}{B_1} \int_{\xi}^{+\infty} \xi' F(\xi') d\xi' = \mathrm{e}^{-\Lambda} f(\xi) - \frac{\xi}{B_1} \int_{\xi}^{+\infty} \xi' \mathrm{e}^{-\Lambda'} f(\xi') d\xi' .$$

Since $\mathrm{e}^{-\Lambda'} = -B_1 \xi'^{-2} (\mathrm{e}^{-\Lambda'})_{\xi'}$, after integration by parts we have

$$G(\xi) = -\xi \int_{\xi}^{+\infty} (\xi'^{-1} f(\xi'))_{\xi'} \mathrm{e}^{-\Lambda'} d\xi' .$$

Moreover, since $f(\xi) = o(\xi^2)$ as $\xi \to +\infty$, $(\xi^{-1} f(\xi))_\xi = o(1)$ and

$$G(\xi) = \int_{\xi}^{+\infty} o(1) \mathrm{e}^{-\Lambda'} d\xi' = o\left(\int_{\xi}^{+\infty} \mathrm{e}^{-\Lambda'} d\xi' \right) = o\left(\xi^{-2} \mathrm{e}^{-\Lambda} \right),$$

where at the last step we used $\mathrm{e}^{-\Lambda} = O((\xi^{-2} \mathrm{e}^{-\Lambda})_\xi)$. Thus

$$\mathrm{e}^{\Lambda} G(\xi) = o(\xi^{-2}) \quad \text{as } \xi \to +\infty ,$$

which implies that C_+ is finite. Similarly, we can show that the last integral in (3.15) is finite if and only if (3.11) holds. Indeed, using $F(\xi) = o(\xi^2 e^{-\Lambda})$ as $\xi \to +\infty$, we have

$$\int_{\xi'}^{+\infty} \xi'' F(\xi'') d\xi'' = \int_{\xi'}^{+\infty} o(\xi''^3 e^{-\Lambda''}) d\xi'' = o\Big(\int_{\xi'}^{+\infty} \xi''^3 e^{-\Lambda''}\, d\xi''\Big) = o(\xi' e^{-\Lambda'}),$$

where we used $\xi^3 e^{-\Lambda} = O((\xi e^{-\Lambda})_\xi)$. Thus

$$\xi'^{-2} e^{\Lambda'} \int_{\xi'}^{+\infty} \xi'' F(\xi'') d\xi'' = o(\xi'^{-1}) \quad \text{as } \xi \to +\infty, \tag{3.16}$$

which is the necessary and sufficient condition for the last integral in (3.15) to be finite.

PROOF OF LEMMA 3.7: To show (3.14), we start from the explicit expression for the integral involving G in (3.12)

$$-\frac{\xi e^{-\Lambda}}{B_1} \int_{-\infty}^{\xi} e^{\Lambda'} G(\xi') d\xi' = \\ -\frac{\xi e^{-\Lambda}}{B_1} \int_{-\infty}^{\xi} e^{\Lambda'} F(\xi') d\xi' - \frac{\xi e^{-\Lambda}}{B_1^2} \int_{-\infty}^{\xi} e^{\Lambda'} \xi' \int_{-\infty}^{\xi'} \xi'' F(\xi'') d\xi'' d\xi'$$

and integrate by parts the second integral using $e^{\Lambda'} = B_1 \xi'^{-2} (e^{\Lambda'})_{\xi'}$. The result is

$$-\frac{\xi e^{-\Lambda}}{B_1} \int_{-\infty}^{\xi} e^{\Lambda'} G(\xi') d\xi' = \\ -\frac{1}{B_1} \int_{-\infty}^{\xi} \xi' F(\xi') d\xi' - \frac{\xi e^{-\Lambda}}{B_1} \int_{-\infty}^{\xi} \xi'^{-2} e^{\Lambda'} \int_{-\infty}^{\xi'} \xi'' F(\xi'') d\xi''\, d\xi' .$$

Inserting this into (3.12) gives (3.14).

To show (3.15), we use $\int_{\mathbb{R}} \xi F d\xi = 0$ and write (3.12) as

$$Q_\infty(\xi) = C_+ \xi e^{-\Lambda} - \frac{1}{B_1} \int_{\xi}^{+\infty} \xi' F(\xi') d\xi' + \frac{\xi e^{-\Lambda}}{B_1} \int_{\xi}^{+\infty} e^{\Lambda'} G(\xi') d\xi' .$$

Using $\int_{\mathbb{R}} \xi F d\xi = 0$, we write explicitly the integral involving G in this expression as

$$\frac{\xi e^{-\Lambda}}{B_1} \int_{\xi}^{+\infty} e^{\Lambda'} G(\xi') d\xi' = \\ \frac{\xi e^{-\Lambda}}{B_1} \int_{\xi}^{+\infty} e^{\Lambda'} F(\xi') d\xi' - \frac{\xi e^{-\Lambda}}{B_1^2} \int_{\xi}^{+\infty} e^{\Lambda'} \xi' \int_{\xi'}^{+\infty} \xi'' F(\xi'') d\xi''\, d\xi'$$

and integrate by parts the second integral using $e^{\Lambda'} = B_1 \xi'^{-2} (e^{\Lambda'})_{\xi'}$:

$$\frac{\xi e^{-\Lambda}}{B_1} \int_{\xi}^{+\infty} e^{\Lambda'} G(\xi') d\xi' =$$

$$\frac{1}{B_1}\int_{\xi}^{+\infty}\xi' F(\xi')d\xi' - \frac{\xi e^{-\Lambda}}{B_1}\int_{\xi}^{+\infty}\xi'^{-2}e^{\Lambda'}\int_{\xi'}^{+\infty}\xi'' F(\xi'')d\xi''\,d\xi'\,.$$

Inserting this in the last expression for Q_∞ gives (3.15) □

PROOF OF THEOREM 3.6: The general solution of (3.10) is $Q = Q_\infty + C_1 Q_1 + C_2 Q_2$, where C_1 and C_2 are constants and Q_1 and Q_2 are two linearly independent solutions of the homogeneous equation associated with (3.10). Two such solutions are

$$Q_1(\xi) = \xi e^{-\Lambda}, \tag{3.17}$$

$$Q_2(\xi) = 1 - \frac{\xi e^{-\Lambda}}{B_1}\int_{-\infty}^{\xi}\xi' e^{\Lambda'}\,d\xi'\,. \tag{3.18}$$

For $\xi < 0$, by using $e^{\Lambda'} = B_1\xi'^{-2}(e^{\Lambda'})_{\xi'}$, after integration by parts Q_2 can be written as

$$Q_2(\xi) = -\xi e^{-\Lambda}\int_{-\infty}^{\xi}\xi'^{-2}e^{\Lambda'}\,d\xi'\,. \tag{3.19}$$

We now show that realizability requires $C_1 = C_2 = 0$. First, one readily checks that $\lim_{\xi\to-\infty} Q_\infty = \lim_{\xi\to-\infty} Q_2 = 0$, while Q_1 grows unbounded as $\xi \to -\infty$. Hence in order that Q be integrable, we must set $C_1 = 0$ and the general solution of (3.10) is

$$Q(\xi) = C_2 Q_2(\xi) + Q_\infty(\xi)\,.$$

We evaluate this solution asymptotically as $\xi \to \pm\infty$. Consider Q_2 for large negative ξ first. Using $e^{\Lambda'} = B_1\xi'^{-2}(e^{\Lambda'})_{\xi'}$, after integration by parts we write (3.19) as

$$Q_2(\xi) = B_1|\xi|^{-3} - 4B_1\xi e^{-\Lambda}\int_{-\infty}^{\xi}\xi'^{-5}e^{\Lambda'}\,d\xi'\,.$$

Since $\xi^{-5}e^{\Lambda} = O((\xi^{-7}e^{\Lambda})_\xi)$ as $\xi \to -\infty$, the integral in this expression is of the order

$$\begin{aligned}\xi e^{-\Lambda}\int_{-\infty}^{\xi}\xi'^{-5}e^{\Lambda'}\,d\xi' &= \xi e^{-\Lambda}\int_{-\infty}^{\xi} O((\xi'^{-7}e^{\Lambda'})_{\xi'})d\xi' \\ &= O\left(\xi e^{-\Lambda}\int_{-\infty}^{\xi}(\xi'^{-7}e^{\Lambda'})_{\xi'}\,d\xi'\right) = O(\xi^{-6})\,.\end{aligned}$$

Thus

$$Q_2(\xi) = B_1|\xi|^{-3} + O(\xi^{-6}) \quad \text{as } \xi \to -\infty\,.$$

Consider now Q_∞ for large negative ξ. Using $e^{\Lambda'} = B_1\xi'^{-2}(e^{\Lambda'})_{\xi'}$, after integration by parts we write (3.14) as

$$Q_\infty(\xi) = |\xi|^{-3}\int_{-\infty}^{\xi}\xi' F(\xi')d\xi' \xi e^{-\Lambda}\int_{-\infty}^{\xi}\left(\xi'^{-4}\int_{-\infty}^{\xi'}\xi'' F(\xi'')d\xi''\right)_{\xi'} e^{\Lambda'}\,d\xi'\,.$$

As $\xi \to -\infty$,

$$\xi e^{-\Lambda} \int_{-\infty}^{\xi} \Big(\xi'^{-4} \int_{-\infty}^{\xi'} \xi'' F(\xi'') d\xi'' \Big)_{\xi'} e^{\Lambda'} d\xi' = \xi e^{-\Lambda} \int_{-\infty}^{\xi} o(\xi'^{-5}) e^{\Lambda'} d\xi'$$
$$= o\bigg(\xi e^{-\Lambda} \int_{-\infty}^{\xi} \xi'^{-5} e^{\Lambda'} d\xi' \bigg) = o(\xi^{-6}),$$

where we used the estimate above. Thus

$$Q_\infty(\xi) = |\xi|^{-3} \int_{-\infty}^{\xi} \xi' F(\xi') d\xi' + o(\xi^{-6}) \quad \text{as } \xi \to -\infty.$$

Combining these expressions, we have

$$Q(\xi) = C_2 B_1 |\xi|^{-3} + |\xi|^{-3} \int_{-\infty}^{\xi} \xi' F(\xi') d\xi' + O(C_2 \xi^{-6}) + o\big(\xi^{-6}\big) \quad \text{as } \xi \to -\infty. \tag{3.20}$$

Consider now Q_2 for large positive ξ. Write (3.18) as

$$Q_2(\xi) = 1 - \frac{\xi e^{-\Lambda}}{B_1} \int_{-\infty}^{\xi_\star} \xi' e^{\Lambda'} d\xi' - \frac{\xi e^{-\Lambda}}{B_1} \int_{\xi_\star}^{\xi} \xi' e^{\Lambda'} d\xi',$$

where $\xi_\star > 0$ is arbitrary but fixed. Using $e^{\Lambda'} = B_1 \xi'^{-2} (e^{\Lambda'})_{\xi'}$, after integration by parts of the second integral we get

$$Q_2(\xi) = \xi \xi_\star^{-1} e^{-\Lambda + \Lambda_\star} - \frac{\xi e^{-\Lambda}}{B_1} \int_{-\infty}^{\xi_\star} \xi' e^{\Lambda'} d\xi' - \xi e^{-\Lambda} \int_{\xi_\star}^{\xi} \xi'^{-2} e^{\Lambda'} d\xi'$$
$$= -\xi e^{-\Lambda} \int_{\xi_\star}^{\xi} \xi'^{-2} e^{\Lambda'} d\xi' + O(\xi e^{-\Lambda}).$$

Integrating by parts again gives

$$Q_2(\xi) = -B_1 \xi^{-3} - 4 B_1 \xi e^{-\Lambda} \int_{\xi_\star}^{\xi} \xi'^{-5} e^{\Lambda'} d\xi' + O(\xi e^{-\Lambda}).$$

Since $\xi^{-5} e^{\Lambda} = O((\xi^{-7} e^{\Lambda})_\xi)$ as $\xi \to +\infty$, the remaining integral can be estimated as above, yielding

$$Q_2(\xi) = -B_1 \xi^{-3} + O\big(\xi^{-6}\big).$$

Consider now Q_∞ for large positive ξ. We must distinguish two cases. If (3.11) does not hold, then in (3.12) we decompose

$$\frac{\xi e^{-\Lambda}}{B_1} \int_{-\infty}^{\xi} e^{\Lambda'} G(\xi') d\xi' = \frac{\xi e^{-\Lambda}}{B_1} \int_{-\infty}^{\xi_\star} e^{\Lambda'} G(\xi') d\xi' + \frac{\xi e^{-\Lambda}}{B_1} \int_{\xi_\star}^{\xi} e^{\Lambda'} G(\xi') d\xi'$$
$$= \frac{\xi e^{-\Lambda}}{B_1} \int_{\xi_\star}^{\xi} e^{\Lambda'} G(\xi') d\xi' + O(\xi e^{-\Lambda})$$

and write

$$Q_\infty(\xi) = \frac{1}{B_1} \int_{-\infty}^{\xi} \xi' F(\xi') d\xi' - \frac{\xi e^{-\Lambda}}{B_1} \int_{\xi_\star}^{\xi} e^{\Lambda'} G(\xi') d\xi' + O(\xi e^{-\Lambda}).$$

The second integral in this expression is

$$-\frac{\xi e^{-\Lambda}}{B_1}\int_{\xi_\star}^{\xi} e^{\Lambda'} G(\xi')d\xi' = \\ -\frac{\xi e^{-\Lambda}}{B_1}\int_{\xi_\star}^{\xi} e^{\Lambda'} F(\xi')d\xi' - \frac{\xi e^{-\Lambda}}{B_1^2}\int_{\xi_\star}^{\xi} e^{\Lambda'}\xi'\int_{-\infty}^{\xi'}\xi'' F(\xi'')d\xi''\,d\xi'\,.$$

Integrating by parts the second integral using $e^{\Lambda'} = B_1\xi'^{-2}(e^{-\Lambda'})_{\xi'}$ gives

$$-\frac{\xi e^{-\Lambda}}{B_1}\int_{\xi_\star}^{\xi} e^{\Lambda'} G(\xi')d\xi' = \\ -\frac{1}{B_1}\int_{-\infty}^{\xi}\xi' F(\xi')d\xi' - \frac{\xi e^{-\Lambda}}{B_1}\int_{\xi_\star}^{\xi}\xi'^{-2} e^{\Lambda'}\int_{-\infty}^{\xi'}\xi'' F(\xi'')d\xi''d\xi' + O(\xi e^{-\Lambda})\,.$$

Thus

$$Q_\infty(\xi) = -\frac{\xi e^{-\Lambda}}{B_1}\int_{\xi_\star}^{\xi}\xi'^{-2} e^{\Lambda'}\int_{-\infty}^{\xi'}\xi'' F(\xi'')d\xi''d\xi' + O(\xi e^{-\Lambda})\,.$$

Integrating by parts again using $e^{\Lambda'} = B_1\xi'^{-2}(e^{\Lambda'})_{\xi'}$ gives

$$Q_\infty(\xi) = -\xi^{-3}\int_{-\infty}^{\xi}\xi' F(\xi')d\xi' \\ +\xi e^{-\Lambda}\int_{\xi_\star}^{\xi}\left(\xi'^{-4}\int_{-\infty}^{\xi'}\xi''\,F(\xi'')d\xi''\right)_{\xi'} e^{\Lambda'}\,d\xi' + O(\xi e^{-\Lambda})\,.$$

The last integral can be estimated as for the large negative ξ. Using $\int_{\mathbb{R}}\xi F d\xi = 0$, this leads to

$$Q_\infty(\xi) = \xi^{-3}\int_{\xi}^{+\infty}\xi' F(\xi')d\xi' + o\big(\xi^{-6}\big)\,.$$

Hence,

$$Q(\xi) = -C_2B_1\xi^{-3} + \xi^{-3}\int_{\xi}^{+\infty}\xi' F(\xi')d\xi' + O(C_2\xi^{-6}) + o(\xi^{-6}) \\ \text{as } \xi \to +\infty\,. \tag{3.21}$$

In contrast, if (3.11) holds, then we can use (3.15). From (3.16) it follows that the integral term in this expression is $o(\xi e^{-\Lambda})$ as $\xi \to +\infty$. Thus,

$$Q(\xi) = C_+\xi e^{-\Lambda} + o(\xi e^{-\Lambda})$$

and

$$Q_\infty(\xi) = -C_2B_1\xi^{-3} + C_+\xi e^{-\Lambda} + O(C_2\xi^{-6}) + o(\xi e^{-\Lambda}) \quad \text{as } \xi \to +\infty\,. \tag{3.22}$$

In (3.20), (3.21), and (3.22), if the C_2 term at the right-hand side is nonzero, then it will dominate the second term. Since the C_2 term has the opposite sign when $\xi \to \pm\infty$, for Q_∞ to be nonnegative we must have $C_2 = 0$. This proves that $Q = Q_\infty$. Furthermore, since the F term at the right-hand side of (3.21) is negative (recall that $F \le 0$), this solution must be rejected in order that Q be nonnegative. Thus (3.11) must hold, and we get (3.13). $\square$

Remark. If the assumptions of Lemma 3.3 do not apply, i.e., if shocks are not created at zero amplitude or shock strength does not add up at collision, then from (3.8) at steady state we have

$$\int_{\mathbb{R}} \xi F(\xi) d\xi = -D_c .$$

This implies that (3.21) has to be changed to

$$Q(\xi) = -C_2 B_1 \xi^{-3} + D_c \xi^{-3} + \xi^{-3} \int_{\xi}^{+\infty} \xi' F(\xi') d\xi' + O(C_2 \xi^{-6}) + o(\xi^{-6}) \quad \text{as } \xi \to +\infty .$$

This solution cannot be dismissed as being unrealizable since the second term at the right-hand side can balance the first one. In particular, for $C_2 = D_c/B_1$ we obtain from (3.20) $Q(\xi) = D_c|\xi|^{-3} + o(\xi^{-3})$ as $\xi \to -\infty$, consistent with (3.9).

3.5 Statistics for the Environment of the Shocks

We now turn to the statistics for the environment of the shocks. For simplicity we will focus on statistically homogeneous situations such that $u(x,t) \overset{\text{(d)}}{=} -u(-x,t)$. Define

$$s(x, y_0, t) = u\left(y_0 + \frac{x}{2}, t\right) - u\left(y_0 - \frac{x}{2}, t\right),$$

and let $W(s, \xi_+, \xi_-, x, t)$ be the PDF of

$$\left(s(x, y_0, t), \xi\left(y_0 + \frac{x}{2}, t\right), \xi\left(y_0 - \frac{x}{2}, t\right)\right),$$

conditional on y_0 being a shock location. Since $u(x,t) \overset{\text{(d)}}{=} -u(-x,t)$, it follows that

$$W(s, \xi_+, \xi_-, x, t) = W(s, \xi_-, \xi_+, x, t) .$$

$V(s, \xi, t)$ and $W(s, \xi_+, \xi_-, x, t)$ are related by

$$V(s, \xi, t) = \lim_{x \to 0+} \int_{\mathbb{R}} W(s, \xi', \xi, x, t) d\xi' \tag{3.23}$$

(recall that $V = V_+ = V_-$ if $u(x,t) \overset{\text{(d)}}{=} -u(-x,t)$). Thus,

$$F(\xi, t) = \rho \lim_{x \to 0+} \int_{\mathbb{R}^- \times \mathbb{R}} W(s, \xi', \xi, x, t) ds\, d\xi' . \tag{3.24}$$

We have the following:

THEOREM 3.8 *W satisfies*

$$
\begin{aligned}
(\rho W)_t &= -\rho s W_x + 2\rho(B_0 - B(x))W_{ss} \\
&\quad + \frac{\rho}{2}\xi_+ W + \rho\big(\xi_+^2 W\big)_{\xi_+} + \frac{\rho}{2}\xi_- W + \rho\big(\xi_-^2 W\big)_{\xi_-} \\
&\quad + \rho B_1\big(W_{\xi_+\xi_+} + W_{\xi_-\xi_-}\big) + 2\rho B_1(x) W_{\xi_+\xi_-} \\
&\quad - \rho B_2(x)\big(W_{s\xi_+} + W_{s\xi_-}\big) + \varsigma_1 - \varsigma_2 + J\,,
\end{aligned}
\tag{3.25}
$$

with $B_1(x) = -B_{xx}(x)$ and $B_2(x) = B_x(x)$. $\varsigma_1(s,\xi_+,\xi_-,x,t)$ is defined such that

$$\varsigma_1(s,\xi_+,\xi_-,x,t)\,ds\,d\xi_-\,d\xi_+\,dz\,dt$$

gives the average number of shock creation points in $[z,z+dz)\times[t,t+dt)$ with

$$
\begin{aligned}
s(x,y_1,t_1) &\in [s,s+ds)\,, \\
\xi\left(y_1 + \frac{x}{2}, t_1\right) &\in [\xi_+,\xi_+ + d\xi_+)\,, \\
\xi\left(y_1 - \frac{x}{2}, t_1\right) &\in [\xi_-,\xi_- + d\xi_-)\,,
\end{aligned}
$$

conditional on $(y_1,t_1) \in ([z,z+dz)\times[t,t+dt))$ being a point of shock creation (because of the statistical homogeneity, z is a dummy variable). $\varsigma_2(s,\xi_+,\xi_-,x,t)$ is defined such that

$$\varsigma_2(s,\xi_+,\xi_-,x,t)\,ds\,d\xi_-\,d\xi_+\,dz\,dt$$

gives the average number of shock collision points in $[z,z+dz)\times[t,t+dt)$ with

$$
\begin{aligned}
s(x,y_2,t_2) &\in [s,s+ds)\,, \\
\xi\left(y_2 + \frac{x}{2}, t_2\right) &\in [\xi_+,\xi_+ + d\xi_+)\,, \\
\xi\left(y_2 - \frac{x}{2}, t_2\right) &\in [\xi_-,\xi_- + d\xi_-)\,,
\end{aligned}
$$

conditional on $(y_2,t_2) \in ([z,z+dz)\times[t,t+dt))$ being a point of shock collision. Finally, $J(s,\xi_+,\xi_-,x,t)$ accounts for the possibility of having another shock in between $[y_j - x/2, y_j + x/2]$ and satisfies

$$J(s,\xi_+,\xi_-,x,t) = O(x)\,.$$

At statistical steady state, the definitions for ς_1 and ς_2 simplify. Indeed, in the limit as $t \to +\infty$, we have

$$\varsigma_1(s,\xi_+,\xi_-,x,t) \to \sigma_1 S_1(s,\xi_+,\xi_-,x)\,,$$

where σ_1 is the space-time number density of shock creation points, $S_1(s,\xi_+,\xi_-,x)$ is the PDF of

$$\left(s(x,y_1,t_1), \xi\left(y_1 + \frac{x}{2}, t_1\right), \xi\left(y_1 - \frac{x}{2}, t_1\right)\right),$$

conditional on a shock being created at (y_1,t_1), and

$$\varsigma_2^i(s,\xi_+,\xi_-,x,t) \to \sigma_2 S_2(s,\xi_+,\xi_-,x)\,,$$

where σ_2 is the space-time number density of shock collision points and $S_2(s,\xi_+,\xi_-,x)$ is the PDF of

$$\left(s(x,y_2,t_2),\xi\left(y_2+\frac{x}{2},t_2\right),\xi\left(y_2-\frac{x}{2},t_2\right)\right),$$

conditional on two shocks colliding at (y_2,t_2).

Remark (On the Strategy for the Proof of Theorem 3.8). Ideally, in order to prove Theorem 3.8 we should follow the strategy in Section 3 for the derivation of the equation for Q. Let $X(u_1,x_1,\dots,u_6,x_6,t)$ be the PDF of

$$(u(y_0+x_1,t),\dots,u(y_0+x_6,t))\,,$$

conditional on y_0 being a shock location. Knowing the equation for X, one can easily derive an equation for the conditional PDF of

$$\left(u\left(y_0+\frac{x}{2},t\right),u\left(y_0+\frac{x}{2},t\right),\eta\left(y_0+\frac{x}{2},y,t\right),\eta\left(y_0-\frac{x}{2},y,t\right)\right)$$

where $\eta(x,z,t)=(u(x+z,t)-u(z,t))/z$. Letting $z\to 0$, one derives an equation for W. Clearly, this derivation is rather tedious and, as we now show, unnecessary for our purpose.

Recall that

$$\begin{aligned}
&\rho X(u_1,x_1,\dots,u_6,x_6,t)\\
&\quad=\frac{1}{(2\pi)^6}\int\limits_{\mathbb{R}\times\cdots\times\mathbb{R}} e^{i\lambda_1 u_1+\cdots+i\lambda_6 u_6}\\
&\qquad\qquad\times\left\langle\sum_j e^{-i\lambda_1 u(z+x_1,t)-\cdots-i\lambda_6 u(z+x_6,t)}\delta(z-y_j)\right\rangle d\lambda_1\cdots d\lambda_6\,.
\end{aligned}$$

The average under the integral is the characteristic function associated with X, and an equation for this quantity can be derived using the equation for $u(z+x_p,t)$

$$du=-[u]_{\mathrm{A}}u_{x_p}\,dt+dW(z+x_p,t)\,,\qquad p=1,\dots,6\,.$$

Instead of reproducing these straightforward calculations, we note simply that for homogeneous situations the resulting equation for X will contain terms proportional to

$$\rho_2(x_p,t)=\left\langle\sum_{j,k}\delta(x_p+z-y_k)\delta(z-y_j)\right\rangle.$$

These terms account for the probability of having another shock, say y_1, between y_0 and y_0+x_p; they are the origin of J in (3.25). Note also that technically, the

$\rho_2(x_p,t)$'s arise because of the average $[u]_{\mathrm{A}}$ in the equation for $u(z+x_p,t)$. Now, the key point is to note that

$$\rho(x_p,t) = O(x_p)\,.$$

As a direct result,

$$J(s,\xi_+,\xi_-,x,t) = O(x)\,.$$

We are eventually interested in the limit as $x \to 0$ of W. As will be argued in Section 3.6, in this limit the $O(x)$ terms in (3.25) are negligible. Thus, we will not dwell on obtaining an explicit expression for J. Instead, we will derive (3.25) using ($\bar{W}(x,t) = W_x(x,t)$)

$$\begin{aligned} du &= -uu_{x_\pm}\,dt + dW(z+x_\pm,t)\,, \\ d\xi &= -(u\xi_{x_\pm} + \xi^2)dt + d\bar{W}(z+x_\pm,t)\,, \end{aligned} \tag{3.26}$$

as if no shock were present between z and $z+x_\pm$. The errors we are making are accounted for by the term J.

PROOF OF THEOREM 3.8: Define

$$\theta(\lambda_+,\lambda_-,\mu_+,\mu_-,x_+,x_-,z,t) = \\ \mathrm{e}^{-i\lambda_+u(z+x_+,t)-i\lambda_-u(z+x_-,t)-i\mu_+\xi(z+x_+,t)-i\mu_-\xi(z+x_-,t)}\,.$$

Then

$$\begin{aligned} &\rho W(s,\xi_+,\xi_-,x,t) \\ &\quad= \frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}} \mathrm{e}^{-i\lambda s-i\mu_+\xi_+-i\mu_-\xi_-} \\ &\qquad\qquad \times\left\langle \sum_j \theta(\lambda,-\lambda,\mu_+,\mu_-,\frac{x}{2},-\frac{x}{2},z,t)\delta(z-y_j)\right\rangle d\lambda\, d\mu_+\, d\mu_-\,. \end{aligned}$$

We first derive an equation for $\langle\Theta\rangle = \langle\sum_j \theta\delta(z-y_j)\rangle$, then for W. We use the following rules from Ito calculus:

$$\begin{aligned} dW(x,t)dW(y,t) &= 2B(x-y)dt\,, \\ d\bar{W}(x,t)d\bar{W}(y,t) &= 2B_1(x-y)dt\,, \\ dW(x,t)d\bar{W}(y,t) &= -2B_2(x-y)dt\,, \end{aligned}$$

where $B_1(x) = -B_{xx}(x)$ and $B_2(x) = B_x(x)$. Thus, from (3.26), we obtain (using $B_2(0) = 0$)

$$\begin{aligned} d\theta = {}& i(\lambda_+u_+u_{+x_+} + \lambda_-u_-u_{-x_-})\theta\,dt \\ &- (\lambda_+^2B_0 + \lambda_-^2B_0 + 2\lambda_+\lambda_-B(x_+-x_-))\theta\,dt \\ &+ i(\mu_+u_+\xi_{+x_+} + \mu_+\xi_+^2 + \mu_-u_-\xi_{-x_-} + \mu_-\xi_-^2)\theta\,dt \\ &- (\mu_+^2B_1 + \mu_-^2B_1 + 2\mu_+\mu_-B_1(x_+-x_-))\theta\,dt \end{aligned}$$

$$
\begin{aligned}
&-(\lambda_-\mu_+ - \lambda_+\mu_-)B_2(x_+ - x_-)\theta\, dt\\
&-i\lambda_+\theta\, dW(z+x_+,t) - i\lambda_-\theta\, dW(z+x_-,t)\\
&-i\mu_+\theta d\overline{W}(z+x_+,t) - i\mu_-\theta\, d\overline{W}(z+x_-,t)\,,
\end{aligned}
$$

where $u_\pm = u(z+x_\pm,t)$ and $\xi_\pm = \xi(z+x_\pm,t)$. Similarly, using $dy_j/dt = \bar{u}(y_j,t)$, we get

$$
\begin{aligned}
d\sum_j \delta(z-y_j) = -\sum_j \bar{u}(y_j,t)\delta^1(z-y_j)dt + \sum_k \delta(z-y_k)\delta(t-t_k)dt\\
-\sum_l \delta(z-y_l)\delta(t-t_l)dt\,,
\end{aligned}
$$

where the (y_k,t_k)'s are the points of shock creation and the (y_l,t_l)'s are the points of shock collisions. Using

$$
\theta\delta^1(z-y_j) = (\theta\delta(z-y_j))_z - \theta_{x_+}\delta(z-y_j) - \theta_{x_-}\delta(z-y_j)
$$

and noting that $\langle\cdot\rangle_z = 0$ by statistical homogeneity, we find that

$$
\begin{aligned}
\langle\Theta\rangle_t = {}& i\lambda_+\langle u_+u_{+x_+}\Theta\rangle + i\lambda_-\langle u_-u_{-x_-}\Theta\rangle\\
&-(\lambda_+^2B_0 + \lambda_-^2B_0 + 2\lambda_+\lambda_-B(x_+-x_-))\langle\Theta\rangle\\
&+i\mu_+\langle(u_+\xi_{+x_+} + \xi_+^2)\Theta\rangle + i\mu_-\langle(u_-\xi_{-x_-} + \xi_-^2)\Theta\rangle\\
&-(\mu_+^2B_1 + \mu_-^2B_1 + 2\mu_+\mu_-B_1(x_+-x_-))\langle\Theta\rangle\\
&-(\lambda_-\mu_+ - \lambda_+\mu_-)B_2(x_+-x_-)\langle\Theta\rangle\\
&+\left\langle \sum_j \bar{u}(y_j,t)(\theta_{x_+} + \theta_{x_-})\delta(z-y_j)\right\rangle + \Sigma_1 - \Sigma_2\,,
\end{aligned}
$$

where Σ_1 and Σ_2 account, respectively, for shock creation and collision events. These are given by

$$
\begin{aligned}
&\Sigma_1(\lambda_+,\lambda_-,\mu_+,\mu_-,x_+,x_-,t) =\\
&\quad\left\langle \sum_k \mathrm{e}^{-i\lambda_+u_+ - i\lambda_-u_- - i\mu_+\xi_+ - i\mu_-\xi_-}\delta(z-y_k)\delta(t-t_k)\right\rangle,\\
&\Sigma_2(\lambda_+,\lambda_-,\mu_+,\mu_-,x_+,x_-,t) =\\
&\quad\left\langle \sum_l \mathrm{e}^{-i\lambda_+u_+ - i\lambda_-u_- - i\mu_+\xi_+ - i\mu_-\xi_-}\delta(z-y_l)\delta(t-t_l)\right\rangle.
\end{aligned}
$$

To average the convective terms we use

$$
\begin{aligned}
i\lambda_\pm\langle u_\pm u_{\pm x_\pm}\Theta\rangle + i\mu_\pm\langle(u_\pm\xi_{\pm x_\pm}+\xi_\pm^2)\Theta\rangle &= -\langle u_\pm\Theta_{x_\pm}\rangle - i\mu_\pm\langle\Theta\rangle_{\mu_\pm\mu_\pm}\\
&= -\langle u_\pm\Theta\rangle_{x_\pm} + \langle\xi_\pm\Theta\rangle - i\mu_\pm\langle\Theta\rangle_{\mu_\pm\mu_\pm}\\
&= -i\langle\Theta\rangle_{x_\pm\lambda_\pm} + i\langle\Theta\rangle_{\mu_\pm} - i\mu_\pm\langle\Theta\rangle_{\mu_\pm\mu_\pm}\,.
\end{aligned}
$$

For the term involving $\overline{u}(y_j,t) = (u_+(y_j,t)+u_-(y_j,t))/2$, we note that

$$
\begin{aligned}
u_+(y_j,t)\theta_{x_+} &= u(y_j+x_+,t)\theta_{x_+} - x_+\int_0^1 \xi(y_j+\beta x_+,t)d\beta\,\theta_{x_+}\\
&= (u(y_j+x_+,t)\theta)_{x_+} - \xi(y_j+x_+,t)\theta\\
&\quad - x_+\int_0^1 \xi(y_j+\beta x_+,t)d\beta\theta_{x_+}\\
&= i\theta_{x_+\lambda_+} - i\theta_{\mu_+} - x_+\int_0^1 \xi(y_j+\beta x_+,t)d\beta\,\theta_{x_+}\,.
\end{aligned}
$$

A similar expression holds for $u_-(y_j,t)\theta_{x_-}$. Also,

$$
\begin{aligned}
u_+(y_j,t)\theta_{x_-} &= (u(y_j+x_+,t)\theta)_{x_-} - x_+\int_0^1 \xi(y_j+\beta x_+,t)d\beta\theta_{x_-}\\
&= i\theta_{x_-\lambda_+} - x_+\int_0^1 \xi(y_j+\beta x_+,t)d\beta\theta_{x_-}
\end{aligned}
$$

and a similar expression holds for $u_-(y_j,t)\theta_{x_+}$. Thus

$$
\begin{aligned}
&2\left\langle \sum_j \overline{u}(y_j,t)(\theta_{x_+}+\theta_{x_-})\delta(z-y_j)\right\rangle =\\
&\qquad i\langle\Theta\rangle_{x_+\lambda_+} + i\langle\Theta\rangle_{x_-\lambda_+} + i\langle\Theta\rangle_{x_+\lambda_-} + i\langle\Theta\rangle_{x_-\lambda_-} - i\langle\Theta\rangle_{\mu_+} - i\langle\Theta\rangle_{\mu_-} - R
\end{aligned}
$$

where

$$
\begin{aligned}
&R(\lambda_+,\lambda_-,\mu_+,\mu_-,x_+,x_-)\\
&\quad = x_+\left\langle \sum_j \int_0^1 \xi(y_j+\beta x_+,t)d\beta\ (\theta_{x_+}+\theta_{x_-})\delta(z-y_j)\right\rangle\\
&\qquad + x_-\left\langle \sum_j \int_0^1 \xi(y_j+\beta x_-,t)d\beta\ (\theta_{x_+}+\theta_{x_-})\delta(z-y_j)\right\rangle.
\end{aligned}
$$

Combining the above expressions leads to the following equation for $\langle\Theta\rangle$:

$$\begin{aligned}\langle\Theta\rangle_t = &-\frac{i}{2}(\langle\Theta\rangle_{x_+\lambda_+}+\langle\Theta\rangle_{x_-\lambda_-}-\langle\Theta\rangle_{x_+\lambda_-}-\langle\Theta\rangle_{x_-\lambda_+})\\ &-(\lambda_+^2B_0+\lambda_-^2B_0+2\lambda_+\lambda_-B(x_+-x_-))\langle\Theta\rangle\\ &+\frac{i}{2}(\langle\Theta\rangle_{\mu_+}+\langle\Theta\rangle_{\mu_-})-i\mu_+\langle\Theta\rangle_{\mu_+\mu_+}-i\mu_-\langle\Theta\rangle_{\mu_-\mu_-}\\ &-(\mu_+^2B_1+\mu_-^2B_1+2\mu_+\mu_-B_1(x_+-x_-))\langle\Theta\rangle\\ &-(\lambda_-\mu_+-\lambda_+\mu_-)B_2(x_+-x_-)\langle\Theta\rangle+\Sigma_1-\Sigma_2-R\,.\end{aligned}$$

To obtain an equation for W, we note the following remarkable property of R:

LEMMA 3.9

$$R\left(\lambda_+,\lambda_-,\mu_+,\mu_-,\frac{x}{2},-\frac{x}{2}\right)=0\,.$$

PROOF: To prove this, write

$$\begin{aligned}&R\left(\lambda_+,\lambda_-,\mu_+,\mu_-,\frac{x}{2},-\frac{x}{2}\right)\\ &\quad=\frac{x}{2}\lim_{\bar{x}\to 0}\frac{\partial}{\partial\bar{x}}\Big\langle\sum_j\int_0^1\left(\xi(y_j+\beta\frac{x}{2},t\right)-\xi\left(y_j-\beta\frac{x}{2},t\right)\Big)d\beta\\ &\qquad\qquad\times\theta\left(\lambda_+,\lambda_-,\mu_+,\mu_-,\bar{x}+\frac{x}{2},\bar{x}-\frac{x}{2},z,t\right)\delta(z-y_j)\Big\rangle\,.\end{aligned}$$

We claim that

$$\begin{aligned}A=&\Big\langle\sum_j\int_0^1\left(\xi\left(y_j+\beta\frac{x}{2},t\right)-\xi\left(y_j-\beta\frac{x}{2},t\right)\right)d\beta\\ &\times\theta\left(\lambda_+,\lambda_-,\mu_+,\mu_-,\bar{x}+\frac{x}{2},\bar{x}-\frac{x}{2},z,t\right)\delta(z-y_j)\Big\rangle=0\,.\end{aligned}$$

Indeed, the symmetry $u(x,t)\overset{\text{(d)}}{=}-u(-x,t)$, $\xi(x,t)\overset{\text{(d)}}{=}\xi(-x,t)$ requires that A be invariant under the transformation

$$z\to -z,\quad x\to x,\quad \bar{x}\to-\bar{x},\quad y_j\to -y_j,\quad \lambda_\pm\to-\lambda_\mp,\quad \mu_\pm\to\mu_\mp\,.$$

On the other hand, one checks explicitly that $A\to -A$ under the same transformation. Hence $A=0$ and $R=0$. $\square$

We now continue with the proof of Theorem 3.8. Combining the above expressions, on the subset $\lambda_+ = \lambda$, $\lambda_- = -\lambda$, $x_+ = x/2$, $x_- = -x/2$, $\langle\Theta\rangle$ satisfies

$$\begin{aligned}\langle\Theta\rangle_t &= -i\langle\Theta\rangle_{x\lambda} - 2\lambda^2(B_0 - B(x))\langle\Theta\rangle \\ &\quad + \frac{i}{2}(\langle\Theta\rangle_{\mu_+} + \langle\Theta\rangle_{\mu_-}) - i\mu_+\langle\Theta\rangle_{\mu_+\mu_+} - i\mu_-\langle\Theta\rangle_{\mu_-\mu_-} \\ &\quad - (\mu_+^2 B_1 + \mu_-^2 B_1 + 2\mu_+\mu_- B_1(x))\langle\Theta\rangle \\ &\quad + (\lambda\mu_+ + \lambda\mu_-)B_2(x)\langle\Theta\rangle + \Sigma_1 - \Sigma_2,\end{aligned}$$

where the Σ_1 and Σ_2 are evaluated at $\lambda_+ = \lambda$, $\lambda_- = -\lambda$, $x_+ = x/2, x_- = -x/2$. Going to the variables (s, ξ_+, ξ_-) we obtain (3.25). □

3.6 The Exponent $\frac{7}{2}$

In this section we will derive the following result:

For large negative ξ, F behaves as

$$F(\xi) \sim C|\xi|^{-5/2} \quad \text{as } \xi \to -\infty. \tag{3.27}$$

A direct consequence of (3.27) is

$$Q_\infty(\xi) \sim \begin{cases} C_-|\xi|^{-7/2} & \text{as } \xi \to -\infty \\ C_+\xi e^{-\Lambda} & \text{as } \xi \to +\infty, \end{cases} \tag{3.28}$$

The argument for (3.27) uses the following property of $S_1(s, \xi_+, \overline{\xi}, x)$, the PDF

$$\left(s(y_1, x, t_1), \xi_+\left(y_1 + \frac{x}{2}, t_1\right), \xi_-\left(y_1 - \frac{x}{2}, t_1\right)\right),$$

conditional on a shock being created at (y_1, t_1):

In the limit as $x \to 0$,

$$x^{-1}S_1\left(\overline{s}x^{1/3}, \overline{\xi}_+ x^{-2/3}, \overline{\xi}_- x^{-2/3}, x\right) \to P(\overline{s})\delta\left(\overline{\xi}_+ - \frac{\overline{s}}{3}\right)\delta\left(\overline{\xi}_+ - \overline{\xi}_-\right), \tag{3.29}$$

where $P(\cdot)$ is a PDF supported on $(-\infty, 0]$. Equation (3.29) shows that, in the original variables, S_1 is asymptotically

$$S_1\left(s, \xi_+, \xi_-, x\right) \sim x^{-1/3}P\left(sx^{1/3}\right)\delta\left(\xi_+ - \frac{sx^{-1}}{3}\right)\delta\left(\xi_+ - \xi_-\right). \tag{3.30}$$

We first derive (3.27), then (3.29).

Derivation of (3.27)

Recall that

$$F(\xi) = \rho \lim_{x\to 0+} \int_{\mathbb{R}^-\times\mathbb{R}} W_\infty(s, \xi', \xi, x)\, ds\, d\xi',$$

where W_∞ is the statistical steady state value of W. Thus, evaluating F amounts to evaluating W, which we will do by analyzing (3.25). We note first that this equation describes the process of shock creation, motion, and then collision. Since collision only occurs if shocks are present, it is natural to represent the effect of collision as proportional to the density of shocks in the systems; i.e., we write

$$\varsigma_2 = \rho g(s,\xi_+,\xi_-,x)W$$

for some function $g(s,\xi_-,\xi_+,x)$ that is assumed to be smooth in s, $\xi_\pm$, and x. This amounts to assuming that the characteristics of the shocks around the collision points are not very different from the characteristics of the shocks away from the collision points. For instance, if they were identical, we would have $\varsigma_2 = \sigma_2 W$ and hence $g = \sigma_2/\rho$. Next, since we are interested in W evaluated at $x = 0$, we neglect the terms

$$B_2(x) = O(x)\,, \qquad 2(B_0 - B(x)) = O(x^2)\,, \qquad J = O(x)\,,$$

in (3.25). Finally, since we are interested in the limit as $\xi_\pm \to -\infty$, we neglect the forcing terms in $\xi_\pm$, proportional to B_1 or $B_1(x)$. This amounts to saying that as far as the statistics of the shocks at large negative values of $\xi_\pm$ are concerned, the effect of the forcing is to maintain a statistical steady state.

Under these approximations, (3.25) reduces to

$$(\rho W)_t = -\rho s W_x + \frac{\rho}{2}(\xi_+ + \xi_-)W + \rho\big(\xi_+^2 W\big)_{\xi_+} + \rho\big(\xi_-^2 W\big)_{\xi_-} - \rho g W + \varsigma_1\,. \tag{3.31}$$

Assuming no shocks are present at the initial time, the equation must be solved with the initial condition $\rho W(s,\xi_+,\xi_-,x,0) = 0$. The solution is

$$\begin{aligned}\rho W(s,\xi_+,\xi_-,x,t) = \int_0^t &((1-\xi_+\tau)(1-\xi_-\tau))^{-5/2}\\ &\times \exp\left(-\int_0^\tau g\left(s,\frac{\xi_+}{1-\xi_+\tau'},\frac{\xi_-}{1-\xi_-\tau'},x-\tau' s\right)d\tau'\right)\\ &\times \varsigma_1\left(s,\frac{\xi_+}{1-\xi_+\tau},\frac{\xi_-}{1-\xi_-\tau},x-\tau s,t-\tau\right)d\tau\,.\end{aligned}$$

The statistical steady state solution is obtained in the limit as $t \to \infty$ of this expression. Using

$$\lim_{t\to\infty} \varsigma_1(s,\xi_+,\xi_-,x,t) = \sigma_1 S_1(s,\xi_+,\xi_-,x)\,,$$

we obtain

$$\begin{aligned}\rho W_\infty(s,\xi_+,\xi_-,x) = \sigma_1 \int_0^\infty &((1-\xi_+\tau)(1-\xi_-\tau))^{-5/2}\\ &\times \exp\left(-\int_0^\tau g\left(s,\frac{\xi_+}{1-\xi_+\tau'},\frac{\xi_-}{1-\xi_-\tau'},x-\tau' s\right)d\tau'\right)\\ &\times S_1\left(s,\frac{\xi_+}{1-\xi_+\tau},\frac{\xi_-}{1-\xi_-\tau},x-\tau s\right)d\tau\,.\end{aligned}$$

At $x=0$, using (3.30) for S_1, we get

$$\begin{aligned}
&\rho W_\infty(s,\xi_+,\xi_-,0)\\
&\quad=\sigma_1\int_0^\infty((1-\xi_+\tau)(1-\xi_-\tau))^{-5/2}\\
&\qquad\quad\times\exp\left(-\int_0^\tau g\left(s,\frac{\xi_+}{1-\xi_+\tau'},\frac{\xi_-}{1-\xi_-\tau'},-\tau's\right)d\tau'\right)\\
&\qquad\quad\times(|s|\tau)^{-1/3}P\left(-\frac{s^{2/3}}{\tau^{1/3}}\right)\delta\left(\frac{\xi_+}{1-\xi_+\tau}+\frac{1}{3\tau}\right)\delta\left(\frac{\xi_+}{1-\xi_+\tau}-\frac{\xi_-}{1-\xi_-\tau}\right)d\tau\,.
\end{aligned}$$

Since

$$\delta\left(\frac{\xi_+}{1-\xi_+\tau}-\frac{\xi_-}{1-\xi_-\tau}\right)=(1-\xi_+\tau)^2\delta(\xi_+-\xi_-)\,,$$

we have $W_\infty(s,\xi_+,\xi_-,0)\propto\delta(\xi_+-\xi_-)$. Using the relation (3.23) between W and V, this implies that $W_\infty(s,\xi_+,\xi_-,0)=V_\infty(s,\xi_+)\delta(\xi_+-\xi_-)$ and leads to

$$\begin{aligned}
\rho V_\infty(s,\xi)=\sigma_1\int_0^\infty(1-\xi\tau)^{-3}(|s|\tau)^{-1/3}P\left(-\frac{s^{2/3}}{\tau^{1/3}}\right)\delta\left(\frac{\xi}{1-\xi\tau}+\frac{1}{3\tau}\right)\\
\times\exp\left(-\int_0^\tau g\left(s,\frac{\xi}{1-\xi\tau'},\frac{\xi}{1-\xi\tau'},-\tau's\right)d\tau'\right)d\tau\,.
\end{aligned}$$

To perform the integration over τ, we use

$$\delta\left(\frac{\xi}{1-\xi\tau}+\frac{1}{3\tau}\right)=\frac{9}{8\xi^2}\delta\left(\tau+\frac{1}{2\xi}\right).$$

Since we are considering $\xi\ll-1$, the exponential factor evaluated at $\tau=-1/(2\xi)$ is

$$\exp\left(-\int_0^{-1/(2\xi)}g\left(s,\frac{\xi}{1-\xi\tau'},\frac{\xi}{1-\xi\tau'},-\tau's\right)d\tau'\right)=1+O(\xi^{-1})\,.$$

This means that shock collision events make no contribution to leading order, leaving us with

$$\rho V_\infty(s,\xi)=\bar{C}\sigma_1|s|^{-1/3}|\xi|^{-5/3}P\left(-\left(2s^2|\xi|\right)^{1/3}\right),$$

where $\bar{C}=2^{4/3}/3$. Hence,

$$F(\xi)=-\bar{C}\sigma_1\int\limits_{\mathbb{R}^-}s^{2/3}|\xi|^{-5/3}P\left(-\left(2s^2|\xi|\right)^{1/3}\right)ds=-C|\xi|^{-5/2}\,,\tag{3.32}$$

where $C=2^{-1/2}\sigma_1\int_{\mathbb{R}^-}|b|^{3/2}P(b)db$.

Derivation of (3.29)

We use local analysis around the shock creation points [20]. Consider a shock created at y_1 at time t_1 with velocity $u_1 = u(y_1, t_1)$. Assuming x is an analytical function of u, we have locally

$$x = a\left(u\left(y_1 + \frac{x}{2}, t_1\right) - u_1\right)^3 + O\left(u\left(y_1 + \frac{x}{2}, t_1\right) - u_1\right)^4), \tag{3.33}$$

where $a \leq 0$ is a random quantity. Setting $a = (2/b)^3$ gives

Extra) in (3.33). Please delete or add (.

$$u\left(y_1 + \frac{x}{2}, t_1\right) = u_1 + \frac{b}{2} x^{1/3} + O(x^{2/3})\,.$$

Hence

$$s(y_1, x, t_1) = u\left(y_1 + \frac{x}{2}, t_1\right) - u\left(y_1 - \frac{x}{2}, t_1\right) = b x^{1/3} + O(x^{2/3}) \tag{3.34}$$

and

$$\xi\left(y_1 + \frac{x}{2}, t_1\right) = \frac{b}{3} x^{-2/3} + O(x^{-1/3})\,. \tag{3.35}$$

Note that these formulae are only valid if there is no other shock in $[y_1 - x/2, y_1 + x/2]$. Since the probability of having another shock in $[y_1 - x/2, y_1 + x/2]$ is at most $O(x)$, the errors we incur by using (3.34)–(3.35) are of higher order.

Recall that

$$S_1(s, \xi_+, \xi_-, x) = \frac{1}{(2\pi)^3} \int\limits_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}} \mathrm{e}^{i\lambda s + i\mu_+\xi_+ + i\mu_-\xi_-} \Omega(\lambda, \mu_+, \mu_-, x) d\lambda\, d\mu_+\, d\mu_-\,,$$

where

$$\sigma_1 \Omega(\lambda, \mu_+, \mu_-, x) = \left\langle \sum_k \mathrm{e}^{-i\lambda s(y_k, x, t_k) - i\mu_+ \xi(y_k + \frac{x}{2}, t_k) - i\mu_- \xi(y_k - \frac{x}{2}, t_k)} \delta(z - y_k)\delta(t - t_k) \right\rangle.$$

Ω is the characteristic function associated with S_1. Similarly,

$$\Omega(\bar{\lambda} x^{-1/3}, \bar{\mu}_+ x^{2/3}, \bar{\mu}_- x^{2/3}, x)$$

is the characteristic function associated with the rescaled PDF

$$x S_1(\bar{s} x^{1/3}, \bar{\xi}_+ x^{-2/3}, \bar{\xi}_- x^{-2/3}, x)\,.$$

We evaluate $\Omega(\bar{\lambda} x^{-1/3}, \bar{\mu}_+ x^{2/3}, \bar{\mu}_- x^{2/3}, x)$ in the limit as $x \to 0$ using (3.34) and (3.35) for $s(y_1, x, t_1)$, $\xi(y_1 + x/2, t_1)$. This gives

$$\sigma_1 \Omega(\bar{\lambda} x^{-1/3}, \bar{\mu}_+ x^{2/3}, \bar{\mu}_- x^{2/3}, x) = \left\langle \sum_k \mathrm{e}^{-i\bar{\lambda} b - i(\bar{\mu}_+ + \bar{\mu}_-) b/3} \delta(z - y_k)\delta(t - t_k) \right\rangle + O(x^{1/3})\,.$$

In the limit as $x \to 0$, b and (y_k, t_k) are the only random quantities to be averaged over. Furthermore, b is statistically independent of (y_k, t_k) because of statistical homogeneity and stationarity. Let $P(b)$ be the PDF of b. Then

$$\begin{aligned}
&\sigma_1 \lim_{x\to 0} \Omega(\bar{\lambda} x^{-1/3}, \bar{\mu}_+ x^{2/3}, \bar{\mu}_- x^{2/3}, x) \\
&\quad = \left\langle \sum_k \delta(z - y_k)\delta(t - t_k) \right\rangle \int_{\mathbb{R}^-} P(b) \mathrm{e}^{-i\bar{\lambda} b - i(\bar{\mu}_+ + \bar{\mu}_-)b/3}\, db \\
&\quad = \sigma_1 \int_{\mathbb{R}^-} P(b) \mathrm{e}^{-i\bar{\lambda} b - i(\bar{\mu}_+ + \bar{\mu}_-)b/3}\, db\,.
\end{aligned}$$

Direct evaluation of this expression gives (3.29) in the variables $(\bar{s}, \bar{\xi}_+, \bar{\xi}_-)$.

3.7 Connection with the Geometric Picture

Here we compute directly the contribution to F in the neighborhood of shock creation. This is a reformulation of the argument presented in [13] in terms of quantities defined in the present paper. Assume a shock is created at time $t = 0$, position $x = y_1$, and with velocity $u = u_1$. Then locally (compare (3.33))

$$x = y_1 + (u - u_1)t + a(u - u_1)^3 + O((u - u_1)^2 t)\,,$$

where $a \le 0$ is a random quantity. For the purpose of comparison with (3.32), it is useful to set $a = (2/b)^3$. Since for $t \ll 1$ to leading order the shock is located at $x = y_1$, to leading order $u_-(y_1, t)$, $u_+(y_1, t)$ are solutions of $0 = (u - u_1)t + (2(u - u_1)/b)^3$. Thus

$$\begin{aligned}
u_\pm(y_1, t) &= u_1 \mp \left(\frac{|b|^3 t}{8}\right)^{1/2} + O(t)\,, \\
s(y_1, t) &= -\left(\frac{|b|^3 t}{2}\right)^{1/2} + O(t)\,. \qquad (3.36)
\end{aligned}$$

Similarly, to leading order $\xi_-(y_1, t)$ and $\xi_+(y_1, t)$ are solutions of

$$1 = \xi t + 3(2/b)^3 (u_\pm - u_1)^2 \xi\,.$$

Thus

$$\xi_\pm(y_1, t) = -\frac{1}{2t} + O(1)\,. \qquad (3.37)$$

Recall that from (3.24) (using $\xi_+(x,t) \overset{(d)}{=} \xi_-(-x,t)$)

$$\begin{aligned}
F(\xi) &= \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{i\mu\xi} \left\langle \sum_j s(z,t) \mathrm{e}^{-i\mu\xi_+(z,t)} \delta(z - y_j) \right\rangle d\mu \\
&= \left\langle \sum_j s(z,t)\delta(\xi - \xi_+(z,t))\delta(z - y_j) \right\rangle.
\end{aligned}$$

Under the assumption of ergodicity with respect to time translation, $F(\xi)$ can be evaluated from

$$F(\xi) = \lim_{L,T\to+\infty} \frac{1}{2LT} \int_0^T \int_{-L}^{L} \sum_j s(z,t)\delta(\xi - \xi_+(z,t))\delta(z-y_j)\,dz\,dt$$
$$= \lim_{L,T\to+\infty} \frac{1}{2LT} \int_0^T \sum_{j=1}^{N} s(y_j,t)\delta(\xi - \xi_+(y_j,t))\,dt\,,$$

where N is number of shocks in $[-L,L]$. The contribution to F near shock creation points, say F_1, can be evaluated for large negative ξ using (3.36), (3.37) for $s(y_1,t)$, $\xi_+(y_1,t)$. This gives in the limit as $\xi \to -\infty$

$$F_1(\xi) \sim -\sigma_1 \int\limits_{\mathbb{R}^-} P(b) \int\limits_{\mathbb{R}^+} \left(\frac{|b|^3 t}{2}\right)^{1/2} \delta\left(\xi + \frac{1}{2t}\right) dt\,db = -C|\xi|^{-5/2}\,,$$

where $C = 2^{-1/2}\sigma_1 \int_{\mathbb{R}^-} |b|^{3/2} P(b)db$. Comparing with (3.32), we conclude that $F_1(\xi) = F(\xi)$ to leading order.

4 Conclusions

To recapitulate the highlights of this paper, by writing down and working with the master equations in the inviscid limit, we have shown that the scaling of the structure functions is related to the shocks that are the singular structures in the limiting flow. The scaling of the PDFs, on the other hand, is related to the shock creation and collision points, which are singularities on the singular structures.

The present paper provides a framework within which various statistical quantities of the stochastic Burgers equation can be calculated using self-consistent asymptotics without making closure assumptions. The main examples used here are the asymptotic behavior of structure functions and the PDF of the velocity gradient. It seems likely that other statistical quantities, such as the tails of the velocity PDF and the PDF for velocity difference, can also be analyzed in the present framework by exploiting further the source terms in (2.6) and (2.23).

Appendix: Master Equations for the Viscous Case

In this appendix we list results for the PDFs in the viscous case. The master equations below were previously derived, e.g., in [23, 24, 32].

Let $P^\nu(u,\xi,x,t)$ be the PDF of $(u(x,t),\xi(x,t))$ for solutions of (1.1). First we have the following:

LEMMA A.1 *P^ν satisfies*

$$P_t^\nu = -uP_x^\nu + \xi P + \left(\xi^2 P^\nu\right)_\xi + B_0 P_{uu}^\nu + B_1 P_{\xi\xi}^\nu - \nu\left(\langle u_{xx}|u,\xi\rangle P^\nu\right)_u - \nu\left(\langle \xi_{xx}|u,\xi\rangle P^\nu\right)_\xi\,, \tag{A.1}$$

where $B_0 = B(0)$, $B_1 = -B_{xx}(0)$, *and* $\langle \cdot \mid u, \xi \rangle$ *denotes the conditional average on* u *and* ξ.

(A.1) is unclosed since the form of $\langle u_{xx} \mid u,\xi\rangle$ and $\langle \xi_{xx} \mid u,\xi\rangle$ entering the viscous terms is unknown. Most work has resorted to various closure assumptions. Our main goal has been to find ways to extract information from the master equations such as (A.1) without making any closure assumption. Note that from the identity

$$\begin{aligned} P^\nu_{xx} = &-(\langle u_{xx} \mid u,\xi\rangle P^\nu)_u - (\langle \xi_{xx} \mid u,\xi\rangle P^\nu)_\xi \\ &+ (\langle u_x^2 \mid u,\xi\rangle P^\nu)_{uu} + 2(\langle u_x\xi_x \mid u,\xi\rangle P^\nu)_{u\xi} + (\langle \xi_x^2 \mid u,\xi\rangle P^\nu)_{\xi\xi}\,, \end{aligned}$$

the viscous term in (A.1) can also be written as (using $\langle u_x^2 \mid u,\xi\rangle = \xi^2$, $\langle u_x\xi_x \mid u,\xi\rangle = \xi\langle \xi_x \mid u,\xi\rangle$)

$$\begin{aligned} -\nu\big(\langle u_{xx} \mid u,\xi)\rangle P^\nu\big)_u - \nu\big(\langle \xi_{xx} \mid u,\xi)\rangle P^\nu\big)_\xi = \\ \nu P^\nu_{xx} - \nu\xi^2 P^\nu_{uu} - \nu\big(\langle \xi_x^2 \mid u,\xi\rangle P^\nu\big)_{\xi\xi} - 2\nu\big(\xi\langle \xi_x \mid u,\xi\rangle P^\nu\big)_{u\xi}\,. \end{aligned}$$

Thus, viscous effects give rise to *antidiffusion* terms since $\xi^2 \geq 0$ and $\langle \xi_x^2 \mid u,\xi\rangle \geq 0$: This is natural since viscosity tends to shrink the distribution P^ν towards the origin.

Since $\xi = u_x$, we have $\langle a(u)\rangle_x = \langle a_u(u)\xi\rangle$ for all smooth and compactly supported functions $a(\cdot)$. This is expressed as the following:

LEMMA A.2 *The consistency relation*

$$\int_{\mathbb{R}} P^\nu_x d\xi + \int_{\mathbb{R}} \xi P^\nu_u d\xi = 0 \tag{A.2}$$

holds for all time for the solution of (A.1) *if it holds initially.*

Lemma A.2 can be proven upon noting that $A = \int_{\mathbb{R}} P^\nu_x d\xi + \int_{\mathbb{R}} \xi P^\nu_u d\xi$ satisfies $A_t = -uA_x + B_0 A_{uu}$, an equation that can obtained by integration of (A.1). Since $A \equiv 0$ initially, it is zero for all time. In the statistically homogeneous case, (A.2) reduces to $\int_{\mathbb{R}} \xi P^\nu_u d\xi = 0$ (or equivalently $\langle a(u)\rangle_x = \langle a_u(u)\xi\rangle = 0$). (A.2) also ensures that this equation preserves the normalization of P^ν. In fact, we have the following corollary:

COROLLARY A.3 *The solution of* (A.1) *satisfies*

$$\frac{d}{dt}\int_{\mathbb{R}\times\mathbb{R}} P^\nu \, du\, d\xi = 0\,; \tag{A.3}$$

i.e., $\int_{\mathbb{R}\times\mathbb{R}} P^\nu du\, d\xi \ = 1$ *for the initial data we are interested in.*

(A.3) follows immediately from integrating (A.1):

$$\frac{d}{dt}\int_{\mathbb{R}\times\mathbb{R}} P^\nu du d\xi = -\langle u_x\rangle + \langle \xi\rangle + \text{boundary terms} = 0\,.$$

The first two terms on the right-hand side cancel because of (A.3); the boundary terms vanish because for finite ν, P^ν decays faster than algebraically in ξ as $|\xi| \to +\infty$.

Consider the reduced distributions

$$Q^\nu(\xi,x,t) = \int_{\mathbb{R}} P^\nu(u,\xi,x,t)du\,, \qquad R^\nu(u,x,t) = \int_{\mathbb{R}} P^\nu(u,\xi,x,t)d\xi\,.$$

Equation (A.2), written after integration over u as

$$\int_{-\infty}^{u} R_x^\nu(u',x,t)du' + \int_{\mathbb{R}} \xi P^\nu d\xi = 0\,,$$

can be used to derive from (A.1) an equation for R^ν:

$$R_t^\nu = -uR_x^\nu - \int_{-\infty}^{u} R_x^\nu(u',x,t)du' + B_0 R_{uu}^\nu - \nu(\langle u_{xx} \mid u\rangle R^\nu)_u\,. \tag{A.4}$$

For statistically homogeneous situations, $P_x^\nu = 0$ in (A.1). Then, using $\int d\xi\,\xi P^\nu = 0$ and (A.1) leads to the following equations for R^ν and Q^ν:

$$R_t^\nu = B_0 R_{uu}^\nu - \nu\big(\langle u_{xx} \mid u\rangle R^\nu\big)_u\,. \tag{A.5}$$

$$Q_t^\nu = \xi Q^\nu + \big(\xi^2 Q^\nu\big)_\xi + B_1 Q_{\xi\xi}^\nu - \nu\big(\langle \xi_{xx} \mid \xi\rangle Q^\nu\big)_\xi\,. \tag{A.6}$$

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