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BOUNDARY LAYER THEORY AND THE ZERO-VISCOSITY LIMIT OF THE NAVIER-STOKES EQUATION

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Abstract: A central problem in the mathematical analysis of fluid dynamics is the asymptotic limit of the fluid flow as viscosity goes to zero. This is particularly important when boundaries are present since vorticity is typically generated at the boundary as a result of boundary layer separation. The boundary layer theory, developed by Prandtl about a hundred years ago, has become a standard tool in addressing these questions. Yet at the mathematical level, there is still a lack of fundamental understanding of these questions and the validity of the boundary layer theory. In this article, we review recent progresses on the analysis of Prandtl's equation and the related issue of the zero-viscosity limit for the solutions of the Navier-Stokes equation. We also discuss some directions where progress is expected in the near future.

1 Introduction

Consider

$$\begin{cases} \boldsymbol{u}_t^\varepsilon + (\boldsymbol{u}^\varepsilon \cdot \nabla)\boldsymbol{u}^\varepsilon + \nabla p^\varepsilon = \varepsilon \Delta \boldsymbol{u}^\varepsilon \\ \nabla \cdot \boldsymbol{u}^\varepsilon = 0 \text{ on } \Omega \end{cases} \tag{1}$$

$$\boldsymbol{u}^\varepsilon \cdot \boldsymbol{n}|_{\partial\Omega} = 0, \quad \boldsymbol{u}^\varepsilon \cdot \tau|_{\partial\Omega} = 0, \tag{2}$$

with

$$\boldsymbol{u}^\varepsilon(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}) \tag{3}$$

Here $\boldsymbol{n}$ and τ are the unit normal and tangent vectors at the boundary $\partial\Omega$. We will restrict our attention to two space dimensions.

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The question of interest is the limit of $\{\boldsymbol{u}^\varepsilon\}$ as $\varepsilon \to 0$.

At a first sight one might guess that as $\varepsilon \to 0$, the term $\varepsilon \Delta \boldsymbol{u}^\varepsilon$ is negligible and $\boldsymbol{u}^\varepsilon$ converges to $\boldsymbol{u}^0$, the solution of Euler's equation

$$\begin{cases} \boldsymbol{u}_t^0 + (\boldsymbol{u}^0 \cdot \nabla)\boldsymbol{u}^0 + \nabla p^0 = 0 \\ \nabla \cdot \boldsymbol{u}^0 = 0 \end{cases} \tag{4}$$

$$\boldsymbol{u}^0 \cdot \boldsymbol{n}|_{\partial\Omega} = 0 \tag{5}$$

and

$$\boldsymbol{u}^0(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}) \tag{6}$$

An immediate problem with this idea is that the NS and Euler's equations admit different kinds of boundary conditions. Besides the condition of no normal flow at the boundary, NS also imposes the condition of no-slip: $\boldsymbol{u}^\varepsilon \cdot \tau = 0$. Prandtl, in 1904 [15], proposed that a thin boundary layer exists in a neighborhood of $\partial\Omega$ where $\boldsymbol{u}^\varepsilon$ goes through a sharp transition from a solution to the Euler's equation to the no-slip boundary condition necessary for NS. In other words,

$$\boldsymbol{u}^\varepsilon \doteq \boldsymbol{u}^0 + \boldsymbol{u}_{BL} \tag{7}$$

where $\boldsymbol{u}_{BL}$ is small except at a small neighborhood of $\partial\Omega$. He went on further to derive the simplified equations in the boundary layer. Prandtl's work formed the basis of the boundary layer theory. Since its inception, boundary layer theory has become a major tool and a major success in hydrodynamics and many other subjects. It has become one of the most powerful techniques in asymptotic analysis. Many theoretical results have also been established to justify the validity of boundary layer theory. Yet for the case of incompressible flow where the original boundary layer theory was proposed, its validity is still very much of an issue. The purpose of the present paper is to review the current status of these problems for incompressible flow. After a brief derivation of Prandtl's equation, we will review the results obtained on the following questions:

1. Well-posedness of Prandtl's equation.

2. Finite time singularity formation.

3. Convergence of solutions of NS to solutions of Euler and Prandtl's equations.

One important aspect of the boundary layer theory is to understand boundary layer separation which has often been associated with a finite time singularity formation in Prandtl's equation. We will discuss this in Section 3.

We mention here another review article on this subject [11], summarizing the work done until the early 70's. The present paper also serves as the introduction to the very careful numerical work in [8] on the infinite Reynolds number limit of flow past cylinder.

Comment on notations: We will use Ω to denote the spatial domain which is assumed to have smooth boundary. We will use bold-faced letters to denote vectors: $\boldsymbol{u} = (u, v), \boldsymbol{x} = (x, y)$. We will use the abbreviation NS for the Navier-Stokes equation.

2 Derivation of Prandtl's equation

For simplicity, we will take the domain Ω to be the upper half space: $\Omega = R^2_+ = \{(x, y), y \geq 0\}$. Curvature effects of the boundary do not enter at the leading order except through solutions of the Euler's equation as the far field boundary condition for the boundary layer equation. The basic assumption for the boundary layer theory is that away from $\partial\Omega = \{(x, 0), x \in R^1\}$, $\boldsymbol{u}^\varepsilon$ is accurately approximated by $\boldsymbol{u}^0$, the solution of Euler's equation. Henceforth we will assume that $(\boldsymbol{u}^0, p^0)$ is smooth.

Let

$$U(x, t) = u^0(x, 0, t) \tag{8}$$

In general, $U(x, t) \neq 0$. Hence a thin layer must exist across which $\boldsymbol{u}^\varepsilon$ makes a transition between U and its boundary condition $\boldsymbol{u}^\varepsilon = 0$ at $y = 0$. Denote the thickness of the layer by δ. Inside the boundary layer, introducing the new variables

$$\tilde{t} = t,\ \tilde{x} = x,\ \tilde{y} = \frac{y}{\delta},\ \tilde{u} = u,\ \tilde{v} = \frac{v}{\delta} \tag{9}$$

NS becomes:

$$\tilde{u}_{\tilde{t}} + \tilde{u}\tilde{u}_{\tilde{x}} + \tilde{v}\tilde{u}_{\tilde{y}} + \tilde{p}_{\tilde{x}} = \varepsilon\tilde{u}_{\tilde{x}\tilde{x}} + \frac{\varepsilon}{\delta^2}\tilde{u}_{\tilde{y}\tilde{y}} \tag{10}$$

$$\delta(\tilde{v}_{\tilde{t}} + \tilde{u}\tilde{v}_{\tilde{x}} + \tilde{v}\tilde{v}_{\tilde{y}}) + \frac{1}{\delta}\tilde{p}_{\tilde{y}} = \varepsilon\delta\tilde{v}_{\tilde{x}\tilde{x}} + \frac{\varepsilon}{\delta}\tilde{v}_{\tilde{y}\tilde{y}} \tag{11}$$

$$\tilde{u}_{\tilde{x}} + \tilde{v}_{\tilde{y}} = 0 \tag{12}$$

Inside the boundary layer, the viscous term is important and should be balanced by the convective term. This gives the relation

$$\delta = \sqrt{\varepsilon} \tag{13}$$

Omitting the tildes, the leading order of (10) and (11) gives, respectively

$$u_t + uu_x + vu_y + p_x = u_{yy} \tag{14}$$

$$p_y = 0 \tag{15}$$

(15) says that to leading order the pressure p is constant across the boundary layer. Hence it is given by $p^0(x, 0, t)$, the solution of Euler's equation. We will denote $P(x, t) = p^0(x, 0, t)$.

(15) is the most important and intriguing simplification in the boundary layer theory. (14) together with the incompressibility condition

$$u_x + v_y = 0 \tag{16}$$

constitute the fundamental equation's in boundary layer theory. They are called Prandtl's equation.

The boundary condition for (14-16) is as follows. At $y = 0$, we should have

$$u = 0, \quad v = 0 \tag{17}$$

As $y \to +\infty$, u should be matched to U:

$$u(x, y, t) \to U(x, t), \text{ as } y \to +\infty \tag{18}$$

(18) is consistent with (14) because of Bernoulli's law satisfied by $(\boldsymbol{u}^0, p^0)$

$$U_t + UU_x + P_x = 0 \tag{19}$$

For time-dependent problems, an initial condition of the type $u(x, y, 0) = u_0(x, y)$ should be imposed. The steady state version of (14-16) has also been of considerable interest in the engineering literature. It reads

$$\begin{cases} uu_x + vu_y + P_x = u_{yy} \\ u_x + v_y = 0 \end{cases} \tag{20}$$

This is the steady Prandtl's equation. For (20) the initial condition is often replaced by a condition of the type

$$u(0, y) = u_0(y)$$

say for flow past a semi-infinite plate for which $\Omega = \{(x, y), x, y \geq 0\}$

3 Well-posedness and finite time blow-up

3.1 Steady Prandtl's equation

We will assume that on the domain of consideration $\{(x, y), 0 \leq x \leq X, y \geq 0\}$, $U \geq 0, u_0 \geq 0, u > 0$ if $y \neq 0$. It is helpful to think of x as a time-like variable in this situation. We will assume that u_0 is monotonically increasing in y.

From a physical point of view, there are two distinct cases of interest:

1. The case of favorable pressure gradient $P_x \leq 0$
2. The case of adverse pressure gradient $P_x > 0$ which leads eventually to boundary layer separation.

In both cases it is important to introduce the von Mise transformation:

$$(x, y) \to (x, \varphi), \quad w = u^2 \tag{21}$$

where φ is the streamfunction, i.e.

$$\varphi_y = u, \quad \varphi_x = -v \tag{22}$$

$\varphi(x, 0) = 0$. In the (x, φ) coordinates (20) becomes

$$w_x = \sqrt{w} w_{yy} - 2P_x \tag{23}$$

(23) is a degenerate diffusion equation for which the maximum and comparison principles apply. Based on this, Oleinik proved the basic theorem regarding the existence and uniqueness of strong solutions [12].

Theorem 3.1 *Oleinik, 1962*

Assume that u_0, u_{0y}, u_{0yy} are bounded and Hölder continuous. Assume also that u_0 satisfies the compatibility condition

$$u_{0yy}(y) - P_x(0) = O(y^2) \tag{24}$$

as $y \to 0$. Then there exists an X such that (20) has a unique strong solution u in the domain $D_X = \{(x,y), 0 \le x \le X, y \ge 0\}$. Moreover u, u_x, u_y, u_{yy} are continuous and bounded in D_X. If $P_x \le 0$, then $X = +\infty$.

Oleinik's theorem asserts in particular that in the case of favorable pressure gradient, there exists a global strong solution.

What happens when $P_x > 0$? On physical grounds it is expected that global strong solutions to (20) do not exist. Instead there exists an x^* such that the solutions of (20) cannot be extended beyond x^*. Local behavior at x^* has been studied by Goldstein and Stewartson [4, 17], among others, using asymptotic analysis. Their work predicts that

$$u_y = 0 \text{ at } x = x^*, y = 0 \tag{25}$$

Moreover

$$u(x,y) \sim (x^* - x)^{1/2} U_0 \left(\frac{y}{(x^* - x)^{1/4}} \right) \tag{26}$$

x^* is often regarded as the point of boundary layer separation. Physically (25) states that the shear stress vanishes at the point of separation. This has since often been used as the criterion for boundary layer separation. The singularity at x^* is known as the Goldstein singularity.

This heuristic picture is to a large extent rigorously established in the unpublished work of Caffarelli and E. Assuming there exists a constant $\alpha > 0$, such that

$$P_{0x} \ge \alpha > 0 \tag{27}$$

They proved [2]

Theorem 3.2 *Caffarelli and E, 1995, unpublished*

Assume that u_0 satisfies

$$u_0^2(y) - \frac{3}{2} u_{0y}(y) \int_0^y u_0(z)dz \geq 0 \tag{28}$$

for $y \geq 0$. Then

1. *There exists an x^*, such that (20) has a unique strong solution on $D_{x^*} = \{(x, y), 0 \leq x \leq x^*, y > 0\}$, but this strong solution cannot be extended to $x > x^*$.*

2. *The sequences of functions $\{u_\lambda\}$ defined by*

$$u_\lambda(x, y) = \frac{1}{\lambda^{1/2}} u(x^* - x\lambda, \lambda^{1/4} y) \tag{29}$$

   is compact in $C^0(D), D = \{(x, y) : x, y \geq 0\}$.

Obviously the second statement confirms the scaling predicted in (26). Also note that (28) is satisfied by standard u_0 having a linear growth at $y = 0$.

The compactness result is based on a one-sided estimate:

$$u^2 - \frac{3}{2}\varphi u_y - 2x(uu_x + vu_y) \geq 0 \tag{30}$$

together with a sharp local existence result for degenerate data u_0:

Lemma 3.1 *There exist absolute constants A_0 and C_0 such that if*

$$u_0(y) \geq Ay^2 \tag{31}$$

for $0 \leq y \leq y_0$, and some $A > A_0$, then the solution of (20) with data $u(0, y) = u_0(y)$ exists for $0 \leq x \leq x_0 = C_0 A\, y_0^2$, $y > 0$.

In the work of Caffarelli and E, the issue whether u_λ has a unique limit as $\lambda \to 0$ was not settled. This is perhaps related to Stewartson's observation that in the asymptotic expansions at the point of singularity, coefficients of high enough orders are not uniquely determined [17].

3.2 Unsteady Prandtl's equation

For the unsteady Prandtl's equation, in place of the von Mise transformation, we have Crocco's transform. To describe Crocco's transform, we first write down (14-16) in terms of $\omega = u_y$. Physically ω carries the meaning of vorticity in the boundary layer setting.

$$\omega_t + u\omega_x + v\omega_y = \omega_{yy} \tag{32}$$

with the boundary condition

$$\omega_y = P_x, \ \text{at } y = 0 \tag{33}$$

$$\omega_y \to 0, \ \text{as } y \to +\infty \tag{34}$$

Crocco's transform requires u to be monotone in y. For example let us assume that u is increasing in y. Then the new independent variables will be (t, x, u). The new dependent variable will be $w = \omega$. (32) can then be rewritten as

$$w_t + uw_x = w^2 w_{uu} \tag{35}$$

and (33) changes to

$$ww_u = P_x, \ \text{at } y = 0 \tag{36}$$

Again Prandtl's equation is transformed into a degenerate diffusion equation. Using this, Oleinik proved [13]

Theorem 3.3 *Oleinik, 1967*
Assume that u_0 satisfies

$$u_{0y}(x, y) > 0 \ \textit{ for } y \geq 0 \tag{37}$$

Then (14-16) has a unique global strong solution.

What happens when the monotonicity assumption is violated? In particular, can we say something about the finite time blow-up of solutions?

In this direction, the only existing result is due to E and Engquist (1997) [3]. They considered the situation when, after a coordinate transformation,

P_x as well as u_0 vanishes on the line $x = 0$. In this case, the solution to (14-16) can be written in the form

$$u(x, y, t) = xb(x, y, t) \tag{38}$$

and a closed equation can be derived for $a(y, t) = b(0, y, t)$:

$$a_t = a_{yy} - a^2 + a_y \int_0^y a(z, t)dz + Q \tag{39}$$

where $Q(t) = -P_{xx}(0, t)$. Under fairly general conditions it can be shown that solutions of (39) blow up in finite time. In particular, for the case when $Q = 0$, we have

Theorem 3.4 *E and Engquist, 1997*
Assume that

$$E(a_0) = \int_0^\infty (\frac{1}{2}a_{0y}^2 + \frac{1}{4}a_0^3)dy < 0 \tag{40}$$

where $a_0(y) = a(y, 0)$. *Then global strong solutions of (14-16) do not exist.*

Theorem 3.4 says that either local solutions of (14-16) do not exist, or if local solutions exist, they blow up in finite time.

Even though Theorem 3.4 is only proved for the case when the solution has the form (38), the mechanism isolated in the analysis seems to reflect what happens in an unsteady boundary layer separation in general. Unlike the steady case, the blow up in the unsteady case is via "shock formation". The nonlinear terms in (39) resembles the nonlinear terms in the Burgers equation after differentiation in x. The blow-up is in the blow-up of the derivative u_x. As the singularity forms, the singular structure is also convected to $y = +\infty$, making numerical computations difficult. It is not yet clear at this stage whether a_y vanishes at $y = 0$ at the time of singularity formation. This is an important issue if the two scenarios for separation, namely the blow up of u_x and the vanishing of shear stress at the wall, are to be reconciled.

4 The zero-viscosity limit of solutions to the NS equation

4.1 Convergence to Euler plus Prandtl

Most naively, one would expect the following convergence result: Consider the solution of the NS equation $\boldsymbol{u}^\varepsilon$ with initial data

$$\boldsymbol{u}^\varepsilon(x,y,0) = \boldsymbol{u}_0(x,y) + \boldsymbol{U}_0(x, \frac{y}{\sqrt{\varepsilon}}) \tag{41}$$

Let

$$\tilde{\boldsymbol{u}}^\varepsilon(x,y,t) = \boldsymbol{u}^0(x,y,t) + \boldsymbol{U}(x, \frac{y}{\sqrt{\varepsilon}}, t) \tag{42}$$

where $\boldsymbol{u}^0$ is the solution to Euler's equation with initial data $\boldsymbol{u}_0$, $\boldsymbol{U}$ is the solution to Prandtl's equation with initial data $\boldsymbol{U}(x,\tilde{y},0) = \boldsymbol{U}_0(x,\tilde{y})$. Assume that $\boldsymbol{U}$ exists up to time t^*. Then we expect to have

$$\| \tilde{\boldsymbol{u}}^\varepsilon(\cdot,t) - \boldsymbol{u}^\varepsilon(\cdot,t) \|_{L^\infty(\Omega)} \to 0 \tag{43}$$

for $t \in [0, t^*)$, or

$$\| \tilde{\boldsymbol{u}}^\varepsilon(\cdot,t) - \boldsymbol{u}^\varepsilon(\cdot,t) \|_{H^1(\Omega)} \to 0 \tag{44}$$

if some decay at infinity is imposed. In fact if $\boldsymbol{u}^\varepsilon$ can be expressed in the form

$$\boldsymbol{u}^\varepsilon(x,y,t) = \tilde{\boldsymbol{u}}^\varepsilon(x,y,t) + \sqrt{\varepsilon}\boldsymbol{u}_1(x,y,\frac{y}{\sqrt{\varepsilon}},t) + \dots \tag{45}$$

as is commonly assumed in boundary layer theory, then we should have

$$\| \tilde{\boldsymbol{u}}^\varepsilon(\cdot,t) - \boldsymbol{u}^\varepsilon(\cdot,t) \|_{L^\infty(\Omega)} \leq C_1(t)\varepsilon^{1/2} \tag{46}$$

$$\| \tilde{\boldsymbol{u}}^\varepsilon(\cdot,t) - \boldsymbol{u}^\varepsilon(\cdot,t) \|_{H^1(\Omega)} \leq C_2(t)\varepsilon^{1/4} \tag{47}$$

Results of this type have only been proved in the case when the data is analytic [16]. It is far from being clear at this moment whether they hold for the case when the data is only assumed to be smooth. We should remark that the obstacle for obtaining these results is not the lack of general existence theorems for the unsteady Prandtl equation. In fact even for the case when Oleinik's theorem guarantees global existence of strong solutions, the validity of such statements still has not been established. This includes both the steady and unsteady situations.

The subtlety in proving such statements can be appreciated from the following instability result of Grenier [5].

Theorem 4.1 *Grenier, 1998*

For any $N > 0$, there exists a profile $u_* \in C^\infty(R_+)$, such that the solution of the NS equation $\boldsymbol{u}_*^\varepsilon$ with initial data

$$\boldsymbol{u}_*^\varepsilon(x, y, 0) = (u_*(\frac{y}{\sqrt{\varepsilon}}), 0) \tag{48}$$

has the following property: For any $s > 0$, and sufficiently small $\varepsilon > 0$, there exists a solution to the NS equation $\boldsymbol{u}^\varepsilon$, such that

$$\| \boldsymbol{u}^\varepsilon(\cdot, 0) - \boldsymbol{u}_*^\varepsilon(\cdot, 0) \|_{H^s(\Omega)} \leq \varepsilon^N \tag{49}$$

and

$$\| \boldsymbol{u}^\varepsilon(\cdot, T_\varepsilon) - \boldsymbol{u}_*^\varepsilon(\cdot, T_\varepsilon) \|_{H^1} \rightarrow +\infty \tag{50}$$

as $\varepsilon \rightarrow 0$, where

$$T_\varepsilon = O(\sqrt{\varepsilon} \log \frac{1}{\varepsilon}) \tag{51}$$

Grenier's result indicates that boundary layer profiles can be very unstable. The origin of such an instability can be understood as follows. Instead of the usual boundary layer scaling, let us look at the scaling

$$t' = \frac{t}{\sqrt{\varepsilon}}, \; x' = \frac{x}{\sqrt{\varepsilon}}, \; y' = \frac{y}{\sqrt{\varepsilon}} \tag{52}$$

In the new variables, the NS equation becomes (omitting the primes)

$$\begin{aligned} \boldsymbol{u}_t^\varepsilon + (\boldsymbol{u}^\varepsilon \cdot \nabla)\boldsymbol{u}^\varepsilon + \nabla p^\varepsilon &= \sqrt{\varepsilon} \Delta \boldsymbol{u}^\varepsilon \\ \nabla \cdot \boldsymbol{u}^\varepsilon &= 0 \end{aligned} \tag{53}$$

This is the same as the original NS, except that viscosity is increased to $\sqrt{\varepsilon}$. A crucial difference, however, is that boundary layer profiles of the type (48) becomes uniformly smooth in the new coordinates. Therefore we can approximate (53) by the Euler equation.

The well-known Rayleigh's criterion for Euler equation states that a shear flow of the type $(u_*(y), 0)$ maybe unstable if u_* has inflection points. Indeed one can construct explicit unstable shear flow profiles with inflection points. The linearized Euler equation around such a profile has an eigenvalue, λ with positive real part. Typical small perturbations around such a profile grow as

$$\exp(\lambda t') = \exp(\lambda \frac{t}{\sqrt{\varepsilon}}) \tag{54}$$

This gives rise to the instability described in Theorem 4.1.

Grenier's proof consists of controlling the terms generated by the non-linear term, as well as a construction of a viscous sublayer to match the boundary condition.

It should be emphasized that Theorem 4.1 by itself does not invalidate the naive expectation described in the beginning of this section since it relies on initial data of the type

$$\boldsymbol{u}^{\varepsilon}(x, y, 0) = \boldsymbol{u}_0(x, y, \frac{x}{\sqrt{\varepsilon}}, \frac{y}{\sqrt{\varepsilon}}) \tag{55}$$

which is outside the class described by (41).

We should also remark that results of the type (46) and (47) have been proved for the case when the boundary is non-characteristic and when the Euler's equation is replaced by a general system of hyperbolic conservation laws [6, 18, 21]. The viscosity matrix in the NS-like equation is assumed to be non-degenerate. In this case the boundary layer thickness is of order ε. The boundary layer equation becomes an elliptic system. The rate of convergence changes to $O(\varepsilon)$. For the incompressible NS, Temam and Wang proved that if the boundary is non-characteristic, then (46) holds [20]. However, the question is still open for physical systems such as the compressible NS equation, even in one space dimension.

4.2 Convergence to Euler

Instead of (43) or (44) we may ask for weaker results of the type

$$\| u^\varepsilon(\cdot,t) - u^0(\cdot,t) \|_{L^2(\Omega)} \to 0 \tag{56}$$

The most interesting result in this direction is the following theorem of Kato [9].

Theorem 4.2 *Kato, 1984*
Fix $T > 0$. *Then* $\boldsymbol{u}^\varepsilon(\cdot,t) \to \boldsymbol{u}^0(\cdot,t)$ *in* $L^2(\Omega)$, *uniformly for* $t \in [0,T]$ *if*

$$\varepsilon \int_0^T \int_\Omega |\nabla \boldsymbol{u}^\varepsilon|^2 d^2x dt \to 0$$

or

$$\varepsilon \int_0^T \int_{\Gamma_\varepsilon} |\nabla \boldsymbol{u}^\varepsilon|^2 d^2x dt \to 0 \tag{57}$$

where Γ_ε *is a strip of width* $O(\varepsilon)$ *around* $\partial\Omega$.

The significance of Kato's result is that it points to another length scale of $O(\varepsilon)$ near the boundary where something violent must happen if convergence (56) does not hold. This length scale is much smaller than the boundary layer thickness. This indicates that the issue of convergence to Euler may have nothing to do with Prandtl's equation.

Kato's result was proved by constructing a transition layer, of order ε, where $\boldsymbol{u}^0$ is connected to the physical boundary condition. This result has been improved in a series of papers by Temam and Wang [19]. They proved that Kato's result continues to hold if the full gradient in (57) is replaced by the tangential derivative of either u^ε or v^ε, or if one assumes that the normal derivative of pressure does not grow too fast near the boundary as ε goes to zero.

Masmoudi made the observation that the techniques of Kato can be used to make a clean statement with regard to the validity of (56) if one considers instead NS with anisotropic viscosity:

$$u_t^\varepsilon + u^\varepsilon u_x^\varepsilon + v^\varepsilon u_y^\varepsilon + P_x^\varepsilon = \varepsilon_\| u_{xx}^\varepsilon + \varepsilon_\perp u_{yy}^\varepsilon \tag{58}$$

$$v_t^\varepsilon + u^\varepsilon v_x^\varepsilon + v^\varepsilon v_y^\varepsilon + P_y^\varepsilon = \varepsilon_\| v_{xx}^\varepsilon + \varepsilon_\perp v_{yy}^\varepsilon \tag{59}$$

$$u_x^\varepsilon + v_y^\varepsilon = 0 \tag{60}$$

when $\varepsilon_\perp/\varepsilon_\| \to 0$. He proved [10]

Theorem 4.3 *Masmoudi, 1998*

Fix any $T > 0$. Then

$$\max_{0 \le t \le T} \| \boldsymbol{u}^{\varepsilon}(\cdot, t) - \boldsymbol{u}^{0}(\cdot, t) \|_{L^2(\Omega)} \to 0$$

as $\varepsilon_{\|} \to 0$ if $\varepsilon_{\perp}/\varepsilon_{\|} \to 0$.

Note that in the setting of Theorem 4.3, one may derive the same Prandtl's equation with $\delta = \sqrt{\varepsilon_{\perp}}$ following the logic of Section 2. This result is a reinforcement of the suggestion that as far as (56) is concerned, Prandtl's equation does not play an important role. It may still affect the rate of convergence, however. Similar results are believed not to hold for arbitrary times for the original NS because of the presence of finite size wakes at extremely high Reynolds number. In the wake, Euler's equation should be replaced by some Reynolds averaged equation. This, though, has not been put under tight scrutiny.

5 Conclusions

After this review of the current status of the boundary layer theory and the zero-viscosity limit of the solutions to NS, we arrive at the following suggestions.

1. The steady state case might be the easiest to look for convergence theorems to Euler plus Prandtl in the spirit of (43).
2. Any technique for proving (43) or (44) must distinguish the data (41) and (55).
3. Convergence to Euler may continue to hold after boundary layer separation, though may be at a slower rate.
4. One may also attempt to show that the scenario described in the beginning of Section 4 is not valid by considering u_0 monotonic in y.
5. Careful numerical results are needed to shed light on these questions.

6. Even though the present article is mostly concerned with incompressible NS, we cannot ignore the embarrassing situation that not much is understood for compressible NS, even for short times in one space dimension.

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Appendix: Inviscid Prandtl's Equation

It is of interest to consider further the "inviscid Prandtl" equation

$$\begin{cases} u_t + uu_x + vu_y + P_x = 0 \\ u_x + v_y = 0 \end{cases} \tag{61}$$

Assume for simplicity that $P_x = 0$. Consider the initial data such that u_0 is monotonically increasing for $0 \leq y \leq Y_0(x)$, and decreasing for $Y \geq Y_0(x)$, where Y_0 is assumed to be smooth. To construct a solution to (61), let us assume that there exists a $Y(x,t)$, such that u is increasing for $Y \leq Y(x,t)$ and decreasing for $y \geq Y(x,t)$. We can then use Crocco's transform separately for the two regions $I = \{y \leq Y\}$ and $II = \{y > Y\}$, and obtain

$$w_t + uw_x = 0 \text{ on I and II} \tag{62}$$

On $y = Y$, $\tilde{u}(x,t) = u(x, Y(x,t), t)$ satisfies

$$\tilde{u}_t + \tilde{u}\tilde{u}_x = 0 \tag{63}$$

This is a free boundary problem since I and II in the (x,u) plane can both be expressed on $\{0 \leq u \leq \tilde{u}\}$. This gives local existence for (61) for the type of data considered here. Clearly blow up may only happen when the free boundary $u = \tilde{u}(x,t)$ develops a singularity.

Equations of the type (61) are closely related to the ones considered in [1] and [7], although there P is given self-consistently by an integral expression of u.

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