Stochastic PDEs in Turbulence Theory

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Abstract

This paper reviews the recent progress on stochastic PDEs arising from different aspects of the turbulence theory including the stochastic Navier-Stokes equation, stochastic Burgers equation and stochastic passive scalar and passive vector equations. Issues discussed include the existence of invariant measures, scaling of the structure functions, asymptotic behavior of the probability density functions, dissipative anomaly, etc.

1 Introduction

The problem of hydrodynamic turbulence is a well-known notoriously difficult problem, often viewed as the last unsolved problem in classical physics [2, 11, 27, 41]. The present paper does not contain a solution to that problem. Worse than that, in spite of many years of hard work with contributions from some of the best-known names in physics and mathematics of this century, the Mountain Everest [3] stands as high as it was almost sixty years ago at the time of Kolmogorov. The repeated experimental and numerical confirmation of Kolmogorov's predictions as well as Landau's objections adds all the more mystery to this old and resilient subject. In the meantime, Kolmogorov's notion of cascade has permeated through several different branches of science.

One thing people realized from this experience is that we should not view turbulence as an isolated problem. Rather it is an example of a wide variety of non-equilibrium processes exhibiting cascades, i.e. a non-trivial flux of energy across scales. Secondly it is important

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to build intermediate steps on simpler problems so that different aspects of the problem can be understood separately. It is this philosophy that motivated the recent flourish of activity on stochastic Burgers equation and stochastic passive scalar equation. It is also with this philosophy that the present paper is written. The problems to be discussed are:

1. Stochastic Navier-Stokes equation

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \nu\Delta\boldsymbol{u} + \boldsymbol{f}, \\ \nabla\cdot\boldsymbol{u} &= 0. \end{aligned} \tag{1}$$

2. Stochastic Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} + f. \tag{2}$$

3. Stochastic passive scalar and passive vector equations

$$\frac{\partial T}{\partial t} + (\boldsymbol{u}\cdot\nabla)T = \kappa\Delta T + f, \tag{3}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} = \kappa\Delta\boldsymbol{B} + \boldsymbol{f} \tag{4}$$

In (2) (3) $f(x,t)$ is taken to be a zero-mean, Gaussian, statistically homogeneous, and white-in-time random process with covariance

$$\langle f(x,t)f(y,s)\rangle = 2B(x-y)\delta(t-s) \tag{5}$$

where B is smooth function. $\boldsymbol{f}(\boldsymbol{x},t)$ is similarly defined. We will associate a scale L over which B decays. L will be the integral scale in the problem. Another important scale is the dissipation scale denoted by ℓ_d which is defined in terms of ν or κ. $\boldsymbol{u}$ in (3) and (4) is a random velocity field whose statistics will be specified later in Section 4.

It should be emphasized that we are mostly interested in systems involving a dissipation mechanism. Therefore energy has to be supplied (here through the forcing) to maintain a statistical steady state. This excludes equilibrium states that can be constructed for Hamiltonian systems such as the nonlinear Schrödinger [8] and the incompressible Euler equations [13].

The second emphasis in this paper is on a new look at the issues in stochastic PDEs. A large amount of work has been done in this subject. Much of it takes the point of view that a stochastic PDE is a stochastic ODE in Banach space [16, 17]. A lot has been learned from this viewpoint. However, the presence of infinitely many degrees of spatial freedom gives rise to many important new phenomena that are unique to stochastic PDEs. Examples include space-time scaling, universality in local spatial structures, etc. Such characteristics are of special interest in this paper. More precisely, on the physical side, we are interested in the following issues:

(P1). Existence of an inertial range across which energy or some other quantity flows but is not dissipated. If the flux is going toward the small scales, it is called a direct cascade. If the flux is going toward the large scales, it is called an inverse cascade.

(P2). Effective behavior as well as fluctuations at large spatial and temporal scales. For example a often asked question for the passive scalar convection is whether the process is diffusive or super-diffusive at large scales.

(P3). Small scale characteristics. A standard way of measuring the small scale characteristics in a field $\boldsymbol{w}$ is to consider the structure function:

$$S_p(r) = \langle |\boldsymbol{w}(\boldsymbol{x}+\boldsymbol{r}, t) - \boldsymbol{w}(\boldsymbol{x}, t)|^p \rangle, \tag{6}$$

where $\langle \cdot \rangle$ denotes ensemble (statistical) average, $r = |\boldsymbol{r}|$. The field $\boldsymbol{w}$ is going to be $\boldsymbol{u}$ in (1), u in (2), T in (3) and $\boldsymbol{B}$ in (4). For small r, we typically have

$$S_p(r) \approx C_p r^{\alpha_p}. \tag{7}$$

α_p measures the spatial regularity of the field $\boldsymbol{w}$. For example, if the field $\boldsymbol{w}$ is smooth, we have $\alpha_p = p$. For Wiener process, we have $\alpha_p = \frac{1}{2}p$. These are examples of locally self-similar processes for which α_p is a linear function of p. In the case when α_p is a nonlinear function of p, we say that the field exhibits "anomalous scaling," or "multi-fractal" behavior. In such a case, $\boldsymbol{w}$ has a spectrum of singular behavior, or Hólder exponents.

(P4). A sometimes related but usually more refined question is the asymptotic behavior of PDFs (probability density functions) of certain quantities such as u or $\partial u/\partial x$ in the stochastic Burgers equation, or T and $\delta T(\boldsymbol{r}, \boldsymbol{x}, t) = T(\boldsymbol{x}+\boldsymbol{r}, t) - T(\boldsymbol{x}, t)$ in the stochastic passive scalar problem.

On the mathematical side, we are interested in the following questions:

(M1). Existence, uniqueness of statistical steady state or the invariant measure, say μ_ν or μ_κ, and their convergence as $\nu \to 0$ or $\kappa \to 0$. If μ_ν or μ_κ does converge, say to μ_0, then the central object of interest in connection with the questions in (P1-P4) is μ_0. These invariant measures also define the ensemble averages used in (6).

(M2). Characterization of the regularity and singularity behavior of the sample paths supported on the measure μ_0. In particular, the scaling exponents α_p defined in (7) are consequences of the singular structures in the solutions.

In many cases the existence of an invariant measure is guaranteed by the well-known Krylov-Bogoliubov Theorem which states that a dynamical system on a compact state space always admits an invariant measure. This applies also to PDEs whose solution operator is compact. This is the case for many parabolic equations [17]. The disadvantage of this approach is that it fails to reveal any insight on the behavior of the solutions supported on the invariant measure and mechanisms for uniqueness or non-uniqueness of the invariant measure.

To put the issue about an invariant measure in a different perspective, let us consider the stochastic ODE

$$dx = b(x)\,dt + \sigma(x)\,dW, \tag{8}$$

where W is the Wiener process, b and σ are smooth functions. Its invariant measure, if exists, is a measure on R^1 with density $\rho(\cdot)$ satisfying

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\Big(\sigma^2(x)\rho\Big) - \frac{\partial}{\partial x}(b(x)\rho) = 0, \tag{9}$$

Take the example of the Ornstein-Ulenbeck process for which $b(x) = -x, \sigma(x) = 1$, we have from (9)

$$\rho(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}. \tag{10}$$

We see that finding the invariant measure for stochastic ODEs is equivalent to solving an elliptic equation on R^n where n is the dimension of the state space for the stochastic ODE. There is also a formal analogy of this elliptic equation for stochastic PDEs. It is loosely referred to as the Hopf's equation which is an equation satified by the functional PDF of the random field described by the stochastic PDE. It is an (degenerate) elliptic equation in function spaces. For (1)-(3), the Hopf's equations are respectively:

$$\begin{aligned}\frac{\partial}{\partial t}Z[\boldsymbol{u}(\cdot,t)] &= -\int_{R^d}\frac{\delta}{\delta u_\alpha}((-u_\beta\nabla_\beta u_\alpha + \nu\Delta u_\alpha)Z[\boldsymbol{u}(\cdot,t)])d^d\boldsymbol{x} \\ &\quad + \int_{R^d\times R^d} B_{\alpha\beta}(\boldsymbol{x}-\boldsymbol{x}')\frac{\delta^2}{\delta u_\alpha \delta u'_\beta}Z[\boldsymbol{u}(\cdot,t)]d^d\boldsymbol{x}d^d\boldsymbol{x}',\end{aligned} \tag{11}$$

where $u_\alpha = u_\alpha(\boldsymbol{x},t)$, $u'_\alpha = u_\alpha(\boldsymbol{x}',t)$, $\alpha,\beta = 1,\dots,d$, $Z[\boldsymbol{u}(\cdot,t)]$ is the functional PDF of $\boldsymbol{u}(\cdot,t)$,

$$\begin{aligned}\frac{\partial}{\partial t}Z[u(\cdot,t)] &= -\int_{R^1}\frac{\delta}{\delta u}((-u\frac{\partial u}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2})Z[u(\cdot,t)])dx \\ &\quad + \int_{R\times R} B(x-x')\frac{\delta^2}{\delta u \delta u'}Z[u(\cdot,t)]dxdx'\end{aligned} \tag{12}$$

where $u = u(x,t)$, $u' = u(x',t)$, $Z[u(\cdot,t)]$ is the functional PDF of $u(\cdot,t)$,

$$
\begin{aligned}
\frac{\partial}{\partial t} Z[T(\cdot,t)] &= -\int_{R^d} \frac{\delta}{\delta T}(\kappa \Delta T + C_{\alpha\beta}(0)\nabla_\alpha \nabla_\beta T) Z[T(\cdot,t)]) d^d\boldsymbol{x} \\
&\quad + \int_{R^d \times R^d} \frac{\delta^2}{\delta T \delta T'}((C_{\alpha\beta}(\boldsymbol{x}-\boldsymbol{x}')\nabla_\alpha T \nabla_\beta T') Z[T(\cdot,t)]) d^d\boldsymbol{x} d^d\boldsymbol{x}' \\
&\quad + \int_{R^d \times R^d} B_{\alpha\beta}(\boldsymbol{x}-\boldsymbol{x}') \frac{\delta^2}{\delta T \delta T'} Z[T(\cdot,t)] d^d\boldsymbol{x} d^d\boldsymbol{x}'
\end{aligned} \tag{13}
$$

where $T = T(\boldsymbol{x},t)$, $T' = T(\boldsymbol{x}',t)$, $Z[T(\cdot,t)]$ is the functional PDF of $T(\cdot,t)$. Clearly the amount of noise added to the system controls the strength of ellipticity in Hopf's equations.

Before ending this introduction, we should emphasize the fact that (2-4) are of interest to a wide variety of physical problems other than hydrodynamic turbulence. For instance, (3) is often used to model flow in porous media, diffusion of tracer particles and pollutants, etc [38]. (4) is also referred to as the kinematic dynamo problem [51]. (2) is one of the canonical examples in non-equilibrium statistical physics. As such, it describes the statistical mechanics of strings in a random potential. The string is assumed to be directed, i.e. there exists a time axis such that the realizations of the string can be viewed as (random) graphs over this time axis. Vortex lines in high temperature superconductors [7], charge density waves [25], directed polymers and stochastic interfaces in $1+1$ dimensional SOS models [36] are all examples of such strings. To see this connection, define the partition function for the configurations of the strings over the time interval [0, t] assuming that they are pinned at time t at location $\boldsymbol{x}$:

$$
Z(\boldsymbol{x},t) = \left\langle \exp\left(-\beta \int_0^t \left(\tfrac{1}{2}|\dot{\boldsymbol{\xi}}(\tau)|^2 + V(\boldsymbol{\xi}(\tau),\tau) \right) d\tau \right) \Big| \boldsymbol{\xi}(t) = \boldsymbol{x} \right\rangle,
$$

where $\beta = 1/kT$, k is the Boltzmann constant, T is the temperature. The first term in the exponent is the elastic energy and the second term is the potential energy with V being the random potential. The associated free energy $\varphi = kT \log Z$ satisfies

$$
\frac{\partial \varphi}{\partial t} + \tfrac{1}{2}|\nabla \varphi|^2 = kT \Delta \varphi + V. \tag{14}
$$

In one dimension, let $u = \partial\varphi/\partial x$, then we have

$$
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + \frac{\partial V}{\partial x},
$$

with $\nu = kT$. (14) is the well-known KPZ equation [32]. In this context, the question of interest is the effective large scale behavior in (14).

2 Stochastic Navier-Stokes Equation

Concerning the problem of hydrodynamic turbulence, the most solid and important theoretical result in turbulence theory is still Kolmogorov's 4/5 law stating that, under the condition of homogeneity and isotropy, the third order structure function for the longitudinal component of the velocity field $u_\|$ satisfies

$$\langle (u_\|(\boldsymbol{x}+\boldsymbol{r},t) - u_\|(\boldsymbol{x},t))^3\rangle \sim -\frac{4}{5}\,\bar\varepsilon\,|\boldsymbol{r}| \tag{15}$$

for $\ell_d \ll |\boldsymbol{r}| \ll L$, where $u_\|(\boldsymbol{x},t) = \boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{r}/|\boldsymbol{r}|$, $u_\|(\boldsymbol{x}+\boldsymbol{r},t) = \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t)\cdot\boldsymbol{r}/|\boldsymbol{r}|$, and ℓ_d is a dissipation length scale. $\bar\varepsilon = \langle \nu|\nabla\boldsymbol{u}|^2\rangle$ is the average rate of energy dissipation. One of the basic assumptions in turbulence theory is that under a fixed external condition such as forcing or boundary condition, $\bar\varepsilon$ stays finite in the limit $\nu\to 0$. Hence the limiting invariant measure, if it exists, supports singular Euler flows. This was pointed out by Onsager [42] in 1949. In fact Onsager conjectured that the solutions of the 3D incompressible Euler's equation conserve energy if they are spatially Hölder continuous with exponent larger than 1/3, and cease to conserve energy if the exponent is less than 1/3. The first half of this statement was proved in [14] in its sharp form formulated in terms of Besov spaces. For simplicity of presentation, we will assume periodic boundary condition on the domain D.

Theorem 2.1 [14]. *Let $\boldsymbol{u} = (u_1,u_2,u_3) \in L^3([0,T],B_3^{\alpha,\infty}(D))\cap C([0,T],L^2(D))$ be a weak solution of the 3D incompressible Euler's equation, i.e.*

$$-\int_0^T\int_D u_\alpha(\boldsymbol{x},t)\frac{\partial}{\partial t}\psi_\alpha(\boldsymbol{x},t)d^3\boldsymbol{x}dt - \int_D u_\alpha(\boldsymbol{x},0)\psi_\alpha(\boldsymbol{x},0)d^3\boldsymbol{x}$$
$$-\int_0^T\int_D u_\alpha(\boldsymbol{x},t)u_\beta(\boldsymbol{x},t)\nabla_\alpha\psi_\beta(\boldsymbol{x},t)d^3\boldsymbol{x}dt - \int_0^T\int_D p(\boldsymbol{x},t)\nabla_\alpha\psi_\alpha(\boldsymbol{x},t)d^3\boldsymbol{x}dt = 0,$$

for every test function $\boldsymbol{\psi} = (\psi_1,\psi_2,\psi_3)\in C^\infty(D\times[0,T))$ with compact support. If $\alpha > \frac{1}{3}$, then

$$\int_D |\boldsymbol{u}(\boldsymbol{x},t)|^2 d^3\boldsymbol{x} = \int_D |\boldsymbol{u}(\boldsymbol{x},0)|^2 d^3\boldsymbol{x}, \quad \textit{for}\ \ t\in[0,T).$$

Besov space is the natural setting for formulating this result since its definition closely resembles the definition of structure functions except that the ensemble average is replaced by the spatial average. In fact, in more physical terms, Theorem 2.1 states that if

$$\left(\int_D |u(\boldsymbol{x}+\boldsymbol{r},t) - u(\boldsymbol{x},t)|^3 d^3\boldsymbol{x}\right)^{1/3} \le C|\boldsymbol{r}|^\alpha, \tag{16}$$

for $\alpha > 1/3$, then $\bar\varepsilon = 0$. This is the deterministic analog of Kolmogorov's 4/5 law. It is clear from this that Theorem 2.1 is sharp. Furthermore, Theorem 2.1 suggests that in the inviscid limit, turbulent fields live in a space close to $B_3^{1/3,\infty}$.

Next by assuming scale-independence for the skewness factor $S_3(r)/S_2(r)^{3/2}$, Kolmogorov obtained from (16) a prediction for $S_2(r)$:

$$S_2(r) \sim C_2 \bar{\varepsilon}^{2/3} r^{2/3}, \qquad \ell_d \ll r \ll L. \tag{17}$$

Translating to Fourier space, we obtain the better known 5/3 law:

$$E(k) = C_K \bar{\varepsilon}^{2/3} k^{-5/3} \tag{18}$$

where $E(k)$ is the energy spectrum, $E(k) = d\mathcal{E}/dk$, $\mathcal{E}(k) = \int_{|\boldsymbol{k}|\le k} \int_{R^3} \langle \hat{\boldsymbol{u}}(\boldsymbol{k}) \cdot \hat{\boldsymbol{u}}(\boldsymbol{q}) \rangle d^3\boldsymbol{q} d^3\boldsymbol{k}$.

Kolmogorov's theory was immediately challenged by Landau who remarked that due to the intermittent nature of dissipation and the influence by the large scales, there cannot be universal relations such as (18) for all turbulent flows. This ultimately led to the proposal that turbulent fields are multi-fractal in the sense that the function α_p should be a highly nonlinear function. We refer to [27] for discussions on phenomenological multi-fractal models.

An alternative picture for accommodating intermittency was put forward by Barenblatt and Chorin [3, 4, 12]. Noting that Kolmogorov's picture was based on complete self-similarity, they argue that typical systems in statistical mechanics exhibit complete self-similarity only when they are well-described by the mean field theory. For problems such as turbulent flows for which fluctuations are thought to be important, incomplete self-similarity holds and they proceed to write down an ansatz for the structure functions based on incomplete self-similarity. The Barenblatt-Chorin theory predicts that the classical Kolmogorov theory holds in the vanishing viscosity limit, but is corrected at finite Reynolds numbers by terms depending on $\ln Re$.

Turning to the mathematical issues, the existence and uniqueness of an invariant measure is only proved so far for two dimension under stringent assumptions on the forcing. Assuming periodic boundary condition, if we write

$$\boldsymbol{f}(\boldsymbol{x}, t)dt = \sum_{\boldsymbol{k}} \sigma_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} d\boldsymbol{W}_{\boldsymbol{k}}(t), \tag{19}$$

where the $\{\boldsymbol{W}_{\boldsymbol{k}}(\cdot)\}$'s are independent Wiener processes, Flandoli and and Maslowski [26] proved the existence and uniqueness of the invariant measure under the assumption that there exist constants c and C such that

$$c|\boldsymbol{k}|^{-1/2} \le |\sigma_{\boldsymbol{k}}| \le C|\boldsymbol{k}|^{-3/8-\varepsilon} \tag{20}$$

for some $\varepsilon > 0$. Conditions of the type (20) are quite unnatural from a physical point of view. Intensive research is now going on to remove this condition. Other interesting aspects of the stochastic Navier-Stokes equation are discussed in [40].

3 Stochastic Burgers equation

3.1 Invariant Measures

The existence of an invariant measure for (2) is only understood so far on finite domains. Consider, for example, (2) on $[0, 2\pi]$ with periodic boundary condition. We will sometime identify the domain as S^1, the unit circle. The forcing function can be expressed as:

$$f(x,t)dt = \sum_k f_k(x) dW_k(t), \tag{21}$$

where the $\{W_k(\cdot)\}$'s are independent Wiener processes. We will assume that

$$f_k(\cdot) \in C^3(S^1), \qquad |\frac{\partial f_k}{\partial x}(x)| \leq \frac{C}{k^2}, \tag{22}$$

for all k. We will use $(\Omega, \mathcal{F}, P)$ to denote the probability space for the forcing functions f, and $\omega \in \Omega$ to denote a typical realization of the force. $\mathcal{F}_t$ denotes the σ-algebra generated by the forces up to time t.

When $\nu = 0$, (2) is understood in the weak sense with solutions satisfying the entropy condition. The precise definition for the random case is given in [19]. In this case, we write (2) as

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(u^2) = f. \tag{23}$$

A natural phase space for (23) is the Skorohod space D on S^1 which consists of functions admitting only discontinuities of the first kind [6]. Let $\mathcal{D}$ be the Borel σ-algebra on D. (23) can then be viewed as a Markov process on D with transition probability

$$P_t(u, A) = \int_\Omega \chi_A(u,\omega) P(d\omega) \tag{24}$$

where $u \in D$, $A \in \mathcal{D}$, and

$$\chi_A(u,\omega) = \begin{cases} 1 & \text{if } S_\omega(t)u \in A \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

Here $S_\omega(t)$ denotes the solution operator of (23) at time t with forcing ω.

Definition. *An invariant measure $\mu_0(du)$ of the Markov process (23) is a measure on $(D, \mathcal{D})$ satisfying*

$$\mu_0(A) = \int_D P_t(u, A)\mu_0(du) \tag{26}$$

for any $A \in \mathcal{D}$ and any $t > 0$.

Theorem 3.1 [19]. *(23) admits a unique invariant measure on the space D.*

The proof of Theorem 3.1 was based on the following variational characterization of weak solutions of (23) satisfying the entropy condition. Let $\xi : [t_1, t_2] \to R^1$ be a Lipschitz continuous curve. Define the action functional

$$
\begin{aligned}
\mathcal{A}_{t_1,t_2}(\xi) &= \int_{t_1}^{t_2} \Big(\tfrac{1}{2}(\dot{\xi}(s))^2 - \sum_k f_k(\xi(s))\dot{\xi}(s)(W_k(s) - W_k(t_1)) \Big) ds \\
&+ \sum_k f_k(\xi(t_2))(W_k(t_2) - W_k(t_1))
\end{aligned} \tag{27}
$$

Then we have for $\tau < t$,

$$
u(x,t) = \frac{\partial}{\partial x} \inf_{\xi(t)=x} \Big\{ \mathcal{A}_{\tau,t}(\xi) + \int_0^{\xi(\tau)} u(z,\tau) dz \Big\} \tag{28}
$$

Minimizers of the functional in (27) satisfy the following Euler-Lagrange equation:

$$
d\gamma(s) = v(s)ds, \qquad dv(s) = \sum_{k=1}^{\infty} f_k(\gamma(s))dW_k(s) \tag{29}
$$

From (28) one can easily verify the following recipe for constructing solutions of (23). Fix t. For values of x such that the minimizer to the functional (27) is unique, say $\xi(\cdot)$, then $u(\cdot,t)$ is continuous at x, and $u(x,t) = \dot{\xi}(t)$. On the other hand, for values of x such that the minimizers to the variational problem are not unique, then $u(\cdot,t)$ is discontinuous at x with $u(x+,t) = \inf_\alpha\{\dot{\xi}_\alpha(t)\}$ and $u(x-,t) = \sup_\alpha\{\dot{\xi}_\alpha(t)\}$ where $\{\xi_\alpha\}$ denotes the family of minimizers of (27).

This construction is just a reformulation of the method of characteristics for weak solutions. It is a variational formulation of the backward characteristics defined in [15]. In particular, the Euler-Lagrange equation (29) is nothing but the equation for the characteristics of (23).

The variational formulation of the method of characteristics allows us to make a connection between weak solutions of the forced Burgers equation and the non-smooth invariant sets constructed in the Aubry-Mather theory [1, 39]. This is an extension to weak solutions of the classical Hamilton-Jacobi theory which establishes a connection between smooth solutions of the Hamilton-Jacobi equation and the smooth invariant tori for the corresponding Hamiltonian system [18, 31, 46].

Of particular interest to the construction of the invariant measure is a special class of minimizers called the one-sided minimizers (OSM).

Definition. *A piecewise C^1-curve $\xi : (-\infty, 0] \to S^1$ is a one-sided minimizer if for any Lipschitz continuous $\tilde{\xi} : (-\infty, 0] \to S^1$ such that $\tilde{\xi}(0) = \xi(0)$ and $\tilde{\xi} = \xi$ on $(-\infty, \tau]$ for some*

$\tau < 0$, we have

$$\mathcal{A}_{s,0}(\xi) \le \mathcal{A}_{s,0}(\tilde{\xi}), \tag{30}$$

for all $s \le \tau$. Similarly, we define one-sided minimizers on $(-\infty, t]$, for $t \in R^1$.

Next we ask the following question: Given (x,t), how many OSMs ξ exist such that $\xi(t) = x$? This question is answered by studying the intersection properties of OSMs. As a general fact in the calculus of variations, two different OSMs cannot intersect more than once. In other words, if ξ_1, ξ_2 are two OSMs such that there exist $t_1, t_2 \in R^1, t_1 \neq t_2$, such that $\xi_1(t_1) = \xi_2(t_1), \xi_1(t_2) = \xi_2(t_2)$, then $\xi_1 \equiv \xi_2$ on their common domain of definition [39]. However more is true for the random case. If ξ_1 and ξ_2 are two different OSMs, such that $\xi_1(s) = \xi_2(s)$ for some s, then neither ξ_1 nor ξ_2 can be extended as an OSM beyond the interval $(-\infty, s]$. This is because that in the random case, two OSMs always have an effective intersection at $t = -\infty$. The precise formulation of this property is given in [19].

These intersection properties have far-reaching consequences. Let us fix $t = 0$. By considering the image of all OSMs at $t = -1$, it is easy to see that the set

$$I_\omega = \{x \in S^1 : \text{there exist more than one OSM } \xi \text{ such that } \xi(0) = x\},$$

can at most be a countable set for almost all $\omega \in \Omega$. Therefore we can define:

$$u_\omega(x, 0) = \dot{\xi}(0), \tag{31}$$

where ξ is the OSM such that $\xi(0) = x$. $u_\omega(\cdot, 0)$ is a well-defined function in $L^\infty(S^1)$ for almost all ω.

Similar construction can be carried out for any $t \in R^1$ which defines $u_\omega(\cdot, t)$. Furthermore it is easy to conclude from the variational principle (28) that

$$u_\omega(\cdot, t) = S_\omega(t) u_\omega(\cdot, 0), \tag{32}$$

for $t > 0$. In other words, u_ω is a solution of (23). From this it is straightforward to check that the distribution of the mapping:

$$\Phi_0 : \omega \to u_\omega(\cdot, 0), \tag{33}$$

is an invariant measure for (23). If we define

$$\Phi_t : \omega \to u_\omega(\cdot, t), \tag{34}$$

then $\{\Phi_t\}$ satisfies the invariance principle:

$$\Phi_t(\omega) = S_\omega(t) \Phi_0(\omega) \tag{35}$$

Therefore we have

Theorem 3.2 [19]. *There exists a family of invariant mappings* $\{\Phi_t\} : \Omega \to D$ *satisfying (35). Furthermore the invariant measure* μ_0 *is the distribution of* Φ_0.

Theorem 3.2 states that the solutions supported by the invariant measure are functionals of the forcing. Uniqueness of the invariant measure follows from the fact that the OSMs are largely unique.

It is shown in [44] that for $\nu > 0$, (2) also admits a unique invariant mapping Φ_0^ν whose distribution is the unique invariant measure for (2), denoted by μ_ν. We have

Theorem 3.3 [19].

$$\Phi_0^\nu(\omega) \to \Phi_0(\omega), \tag{36}$$

in $L^1(S^1)$ *for almost every* $\omega \in \Omega$. *Consequently*

$$\mu_\nu \to \mu_0, \tag{37}$$

weakly.

We remark that (36), (37) are a different kind of statement from standard theorems on inviscid limits of (2) studied in the PDE literature [45, 47]. There we are given a sequence of initial data that converge in the inviscid limit, and we ask whether convergence still holds at later times. Here we are not given any initial data, and we proceed to establish convergence with the only information that the solutions are defined for all $t \in R^1$ in a special way using the OSMs. Consequently the techniques used to prove Theorem 3.3 are very different from the ones used in the PDE literature to study inviscid limits. See [19].

Our next task is to characterize the solutions supported on the invariant measure. This requires a non-degeneracy condition to the effect that the process (29) starting at any $x \in S^1$ is transitive on S^1. This condition is generically satisfied. However it is violated if the sum in (21) contains only one term. We refer to [19] for a detailed formulation and examination of this condition. Under this non-degeneracy condition, we have

Theorem 3.4 [19]. *For almost all* ω, u_ω *satisfies the following:*

(1). There exists a unique two-sided minimizer (TSM, defined below) $y_\omega : R^1 \to S^1$ *which is a characteristic of (23) associated with the solution* u_ω.

(2). There exists a unique so-called main shock $\gamma_\omega : R^1 \to S^1$, *which is a continuous shock curve defined for all* $t \in R^1$.

(3). The TSM is a hyperbolic trajectory for the dynamical systems (29).

(4). For any $t \in R^1$, *there exist global stable and unstable manifolds associated with* y_ω *at time* t, *denoted by* $W_\omega^s(t)$ *and* $W_\omega^u(t)$ *respectively, on the phase space* $S^1 \times R^1$.

(5). The graph of $u_\omega(\cdot, t)$ is a subset of $W^u_\omega(t)$.

As a corollary, we have that almost surely, u_ω is a piecewise smooth function.

Definition. *A piecewise C^1-curve $\xi : (-\infty, +\infty) \to S^1$ is a two-sided minimizer if for any Lipschitz continuous $\tilde{\xi} : (-\infty, +\infty) \to S^1$ such that $\tilde{\xi} = \xi$ away from a compact set, we have*

$$\mathcal{A}_{-s,s}(\xi) \leq \mathcal{A}_{-s,s}(\tilde{\xi}), \tag{38}$$

for large enough s.

The possibility of establishing hyperbolicity of TSMs in the random case comes from the following:

Basic Collision Lemma. *Assuming the non-degeneracy condition. Then there exists a constant p_0, depending only on the $\{f_k\}$'s, with the following property: Given an arbitrary pair of points $(x_1, x_2) \in [0,1]^2$ at $t = 0$ whose positions are $\mathcal{F}_0$-measurable,*

$$P\{x_1, x_2 \textit{ merge before } t = 1\} > p_0. \tag{39}$$

Heuristically two points merge before $t = 1$ if the forward characteristics emanating from them intersect before $t = 1$. This of course depends on the forces as well as the solution at $t = 0$. The lemma states that independent of the solution at $t = 0$, one can always find a set of forces with positive measure under which the two points merge.

The proof of the Basic Collision Lemma relies on PDE techniques and is given in the Appendix D of [19].

The Basic Collision Lemma provides the mechanism for the origin of the hyperbolicity. In particular, the uniqueness of the TSM and the main shock is a simple consequenc of the Basic Collision Lemma.

The regularity and structural properties described here are used in [20, 21, 22] to study the statistical behavior of the Burgers equation. A summary of these results is given below.

3.2 Statistical Theory

We now address the questions frequently asked in the physics literature regarding (2), building on the regularity results described in Section 3.1. Since we have established the existence of μ_0 which is the statistical steady state at the inviscid limit, we can restrict our attention to this case.

3.2.1 Structure functions

The fact that u_ω is piecewise smooth implies easily that

$$S_p(r) = \begin{cases} r^p\langle|\xi|^p\rangle + o(r^p) & \text{if } \ 0 \le p < 1, \\ r\rho\langle|s|^p\rangle + o(r) & \text{if } \ 1 < p, \end{cases} \tag{40}$$

where ρ is the number density of the shocks, which is finite from the results described earlier, s is the jump across the shocks, ξ is the regular part of the velocity gradient:

$$\frac{\partial}{\partial x}u(x,t) = \xi(x,t) + \sum_j s(y_j,t)\delta(x-y_j), \tag{41}$$

with $\xi(\cdot,t) \in L^1(S^1)$.

From (40), α_p is a linear function in p followed by a constant function. This is a reflection of the fact that as far as regularity is concerned, almost everywhere the solution is either Lipschitz continuous or discontinuous. The linear part in the graph of α_p probes the Lipschitz continuous part of the solution. The flat part probes the discontinuous part of the solution. Such a situation is sometimes referred to as a "bifractal".

3.2.2 Velocity gradient PDF

More difficult are the questions of PDFs for quantities such as u, $\partial u/\partial x$ and $\delta u(x,t) = u(x+y,t) - u(y,t)$. In particular, the PDF of $\xi = \partial u/\partial x$ (suitably defined), $Q(\xi)$, has attracted a lot of attention in recent years. In the inviscid limit, it is agreed that $Q(\xi)$ has the behavior

$$Q(\xi) \sim \begin{cases} C_-|\xi|^{-\alpha} & \text{as } \ \xi \to -\infty, \\ C_+\xi^\beta \mathrm{e}^{-\xi^3/(3B_1)} & \text{as } \ \xi \to +\infty. \end{cases} \tag{42}$$

where C_-, C_+ are constants, $B_1 = -B_{xx}(0)$. But many different values of α and β have been proposed (see [22, 33] for a review).

A priori, there is even an issue how to define Q. One can define for the inviscid limit the PDF for the velocity divided difference $(u(x+y,t) - u(y,t))/x$, $Q^\delta(\xi,x)$ and then take $x \to 0$. An alternative is to first define the PDF of $\partial u/\partial x$ for the viscous problem, call it $Q^\nu(\xi)$, and then take the limit as $\nu \to 0$:

$$\underline{Q}(\xi) = \lim_{\nu\to 0} Q^\nu(\xi), \qquad Q(\xi) = \lim_{\delta\to 0} Q^\delta(\xi,x). \tag{43}$$

While a fully rigorous proof was not constructed, [22] presented strong evidence that for the case studied here,

$$Q = \underline{Q}. \tag{44}$$

Below we will concentrate on Q.

Using calculus for functions of bounded variation [49], we can derive an equation for Q

$$\frac{\partial Q}{\partial t} = \xi Q + \frac{\partial}{\partial \xi}(\xi^2 Q) + B_1 \frac{\partial^2 Q}{\partial \xi^2} + F(\xi, t), \tag{45}$$

where

$$F(\xi, t) = \rho \int_{-\infty}^{0} sV(s, \xi, t) ds \leq 0. \tag{46}$$

Here $V(s, \xi, t) = V_+(s, \xi, t) + V_-(s, \xi, t)$, $V_\pm(s, \xi_\pm, t)$ are the PDFs of $(s(y,t), \xi_\pm(y,t) = \partial u(y\pm, t)/\partial x)$ conditional on y being a shock position. This equation should be compared with the equation for Q^ν:

$$\frac{\partial Q^\nu}{\partial t} = \xi Q^\nu + \frac{\partial}{\partial \xi}(\xi^2 Q^\nu) + B_1 \frac{\partial^2 Q^\nu}{\partial \xi^2} - \nu \frac{\partial}{\partial \xi}\Big(\Big\langle \frac{\partial^2 \xi}{\partial x^2}\Big|\xi\Big\rangle Q^\nu\Big), \tag{47}$$

where $\langle \partial^2 \xi / \partial x^2 | \xi \rangle$ is the ensemble-average of $\partial^2 \xi / \partial x^2$ conditional on ξ. We see that

$$F(\xi, t) = -\lim_{\nu \to 0} \nu \frac{\partial}{\partial \xi}\Big(\Big\langle \frac{\partial^2 \xi}{\partial x^2}\Big|\xi\Big\rangle Q^\nu\Big),$$

Even though we are primarily interested in statistical steady states, we have written down the master equations for the more general case including transients.

Integrating (45), we get

$$\frac{d}{dt} \int_{R^1} Q(\xi, t) d\xi = \langle \xi \rangle + \rho \langle s \rangle = 0. \tag{48}$$

Consequently

$$\lim_{\nu \to 0} \int_{R^1} \xi Q^\nu(\xi) d\xi \neq \int_{R^1} \xi Q(\xi) d\xi, \tag{49}$$

since the left hand side vanishes due to homogeneity.

Even though (45) is not a closed equation since the form of F is unknown, we can already obtain from it non-trivial information. As an example, we have

Theorem 3.5 [21, 22].

$$\lim_{|\xi| \to +\infty} |\xi|^3 Q(\xi, t) = 0, \tag{50}$$

i.e. Q decays faster than $|\xi|^{-3}$ as $\xi \to -\infty$.

This result rules out all the proposed value of α except that of [20] which gives $\alpha = 7/2$.

Theorem 3.5 is obtained by combining (45) together with the following:

(1). Take the first moment of (45) gives

$$\frac{d}{dt}\langle \xi \rangle = [\xi^3 Q]_{-\infty}^{+\infty} + \frac{\rho}{2}\left(\langle s\xi_- \rangle + \langle s\xi_+ \rangle\right).$$

(2). Along the shock, we have

$$\frac{d}{dt}(\rho\langle s\rangle) = -\frac{\rho}{2}(\langle s\xi_-\rangle + \langle s\xi_+\rangle),$$

which is a consequence of the equations for the dynamics of the shocks. Since formally

$$\int_{R^1} \xi F(\xi,t)d\xi = \frac{\rho}{2}\left(\langle s\xi_-\rangle + \langle s\xi_+\rangle\right),$$

the process of proving (3.2.2) also proves that $\xi F(\xi,t)$ is absolutely integrable on R^1 for all t.

3.2.3 Asymptotics for the statistical stationary state

In statistical steady state, (45) becomes

(51) $$\xi Q + \frac{d}{d\xi}(\xi^2 Q) + B_1 \frac{d^2 Q}{d\xi^2} + F(\xi) = 0.$$

This is a second order ODE with an inhomogeneous term F. We can write the general solutions of (51) as

(52) $$Q = Q_s + C_1 Q_1 + C_2 Q_2,$$

where Q_1 and Q_2 are two linearly independent solutions of the homogeneous equation, and Q_s is a particular solution. For example, we can take:

(53) $$Q_1(\xi) = \xi \mathrm{e}^{-\Lambda},$$

(54) $$Q_2(\xi) = 1 - \frac{\xi \mathrm{e}^{-\Lambda}}{B_1} \int_{-\infty}^{\xi} \xi' \mathrm{e}^{\Lambda'} d\xi',$$

(55) $$Q_s(\xi) = \frac{1}{B_1} \int_{-\infty}^{\xi} \xi' F(\xi') d\xi' - \frac{\xi \mathrm{e}^{-\Lambda}}{B_1} \int_{-\infty}^{\xi} \mathrm{e}^{\Lambda'} G(\xi') d\xi',$$

where

(56) $$\Lambda = \frac{\xi^3}{3B_1} \qquad G(\xi) = F(\xi) + \frac{\xi}{B_1} \int_{-\infty}^{\xi} d\xi' \ \xi' F(\xi').$$

3.2.4 Bounds from realizability constraints

Using the realizability constraint $Q \in L^1(R^1)$, $Q \geq 0$, we can show that

(57) $$Q = Q_s,$$

and

(58) $$\lim_{\xi \to +\infty} \xi^{-2} \mathrm{e}^{\Lambda} F(\xi) = 0.$$

Indeed, starting from $Q = Q_s + C_1 Q_1 + C_2 Q_2$, we have:

(1). $Q_s \in L^1(R^1)$, $Q_2 \in L^1(R^1)$, but $Q_1 \notin L^1(R^1)$. Therefore C_1 must be zero.

(2). As $|\xi| \to +\infty$, $|Q_2| > |Q_s|$, but $Q_2 > 0$ as $\xi \to -\infty$, and $Q_2 < 0$ as $\xi \to +\infty$. Therefore C_2 must be zero.

(3). As $\xi \to +\infty$, $Q_s \geq 0$, *iff* (58) holds.

These statements can be established by evaluating Q_2, Q_s using Laplace's method. We arrive at:

$$Q(\xi) \sim \begin{cases} |\xi|^{-3} \displaystyle\int_{-\infty}^{\xi} \xi' F(\xi') d\xi' & \text{as } \xi \to -\infty, \\ C_+ \xi \mathrm{e}^{-\Lambda} & \text{as } \xi \to +\infty. \end{cases} \tag{59}$$

Once again, we obtain (50). Furthermore we get $\beta = 1$ in (42).

3.2.5 The exponent 7/2

Let $W(s, \xi_+, \xi_-, x, t)$ be the PDF of

$$(u(y+x,t) - u(y-x,t), \xi(y+x,t), \xi(y-x,t)),$$

conditional on y being a shock location. Then

$$V_\pm(s, \xi_\pm, t) = \int_{R^1} W(s, \xi_+, \xi_-, 0, t) d\xi_\mp. \tag{60}$$

W satisfies an equation of the form [22]

$$\frac{\partial W}{\partial t} = \mathcal{O} W + S, \tag{61}$$

where $\mathcal{O}$ is a differential operator in x, s, $\xi_\pm$, S is a source term accounting for shock creation and collision. Information on the source term S can be obtained using local analysis around shock creation and collision points. Upon using this information in (60), we get:

$$F(\xi) \sim C|\xi|^{-5/2} \qquad \text{as } \xi \to -\infty, \tag{62}$$

Therefore

$$Q(\xi) \sim |\xi|^{-3} \int_{-\infty}^{\xi} \xi' F(\xi') d\xi' = C_- |\xi|^{-7/2} \qquad \text{as } \xi \to -\infty. \tag{63}$$

which confirms the prediction of [20].

The analysis in [22] that accomplishes this last step is quite involved. The result of that is a confirmation of the geometric picture proposed in [20], namely the leading order contribution to the left tail of Q comes from the boundary of the set of the shocks, here the points of shock creation. This geometric picture may have interesting consequences in higher dimension. The analysis in [22] is also a success in working with the master equation without making closure assumptions. In a sense closure in [22] is achieved through dimension reduction. The PDF of ξ is first expressed in terms of the lower dimensional dissipative structures, here the shocks. The PDF for the shocks and the shock environment is further expressed in terms of the singular structures on the shocks, namely the points of shock creation and collision, which are then amenable to local analysis. Clearly this approach should be applicable to high dimensions and should yield more interesting results.

4 Stochastic Passive Scalar and Passive Vector Equations

4.1 The stochastic passive scalar equation

Turning now to (3). There are two main questions that have been studied in the literature. The first is the effective behavior at large scales, particularly when $\boldsymbol{u}$ contains energy at large scales. We will not discuss this problem here. Instead we refer to the extensive review article [38]. For issues with regard to the behavior for the average of temperature, including for the case of white-in-time velocity fields, we refer to [38]. Here we will be mainly interested in higher order statistics.

The second question concerns the local characteristics in the temperature field T, such as the scaling of the structure functions. In this context, the most thoroughly studied problem is the so-called Kraichnan's model [34, 35] in which $\boldsymbol{u}$ is a Gaussian random process with zero-mean and covariance

$$\langle u_\alpha(\boldsymbol{x},t)u_\beta(\boldsymbol{y},s)\rangle = 2C_{\alpha\beta}(\boldsymbol{x}-\boldsymbol{y})\delta(t-s), \tag{64}$$

The white-in-time nature of the velocity field allows us to significantly simplify the analytical treatment, but is sometimes a poor approximation to the physical situation. For incompressible velocity fields, the tensor C is assumed to satisfy

$$\nabla \cdot C = 0. \tag{65}$$

Following [2, 41], we will take

$$C_{\alpha\beta}(\boldsymbol{x}) = F(r)\delta_{\alpha\beta} + \nabla_\alpha\nabla_\beta G(r), \tag{66}$$

where $r = |\boldsymbol{x}|$,

$$F(r) = D_0 \int_{R^d} \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}(k_0^2+k^2)^{-(d+\xi)/2}\frac{d^d\boldsymbol{k}}{(2\pi)^d}, \tag{67}$$

and $-\Delta G = F$. Here d is the dimension, $\xi \in (0,2)$ is a parameter, D_0 is a constant, k_0 is an infrared cut-off parameter.

Writing $C_{\alpha\beta}(\boldsymbol{x}) = C_{\alpha\beta}(0) - c_{\alpha\beta}(\boldsymbol{x})$, it can be easily seen that $C_{\alpha\beta}(0) = \bar{D}k_0^{-\xi}\delta_{\alpha\beta}$ for some $\bar{D} \propto D_0$ independent of k_0 (thus $C_{\alpha\beta}(0)$ is divergent as $k_0 \to 0$), and

$$c_{\alpha\beta}(\boldsymbol{x}) = D\left((d+\xi-1)\delta_{\alpha\beta} - \xi\frac{x_\alpha x_\beta}{r^2}\right) r^\xi + O(r^2 k_0^2), \tag{68}$$

for some $D \propto D_0$ independent of k_0. Thus ξ is a measure the smoothness of the velocity field: Roughly speaking, when $\xi = 0$ the velocity field is spatially independent; when $\xi = 2$ the velocity field is spatially Lipschitz continuous. Note that, since $C_{\alpha\beta}(\boldsymbol{x}) \sim 0$ for $r \gg \ell_0 = k_0^{-1}$, we have

$$c_{\alpha\beta}(\boldsymbol{x}) \sim \bar{D}\,\ell_0^\xi\,\delta_{\alpha\beta}, \qquad r \gg \ell_0. \tag{69}$$

However, in the sequel we will be mostly interested in the range $r \ll \ell_0$.

Existence of an invariant measure for $\kappa \geq 0$ and the convergence as $\kappa \to 0$ are established in [23]. Below we will study the statistical properties associated with these invariant measures.

4.1.1 Master equation and the dissipative anomaly

Assume that the initial temperature profile, $T_0(\boldsymbol{x}) = T(\boldsymbol{x},0)$ is statistically isotropic. Define $\theta(\boldsymbol{x},\boldsymbol{x}',t) = T(\boldsymbol{x},t) - T(\boldsymbol{x}',t)$ and let $Z^\kappa(\theta, |\boldsymbol{x}-\boldsymbol{x}'|, t)$ be the PDF of $\theta(\boldsymbol{x},\boldsymbol{x}',t)$ (Z^κ depends on $|\boldsymbol{x}-\boldsymbol{x}'|$ only by isotropy). Then Z^κ satisfies [9]

$$\frac{\partial Z^\kappa}{\partial t} = \frac{2}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\eta\frac{\partial Z^\kappa}{\partial r}\right) + 2b(r)\frac{\partial^2 Z^\kappa}{\partial \theta^2} - \frac{\partial}{\partial\theta}(H^\kappa Z^\kappa), \tag{70}$$

where $b(r) = B(0) - B(r), \eta(r) = D(d-1)r^\xi$, and

$$H^\kappa(\theta, r, t) = \kappa\langle(\Delta - \Delta')\theta(\boldsymbol{x},\boldsymbol{x}',t)|\theta\rangle. \tag{71}$$

This is the analog of (47) for the Burgers equation. We are interested in the limit

$$Z(\theta, r, t) = \lim_{\kappa\to 0} Z^\kappa(\theta, r, t). \tag{72}$$

In particular, we are interested in the limit of the dissipation term: $\partial(H^\kappa Z^\kappa)/\partial\theta$. This term does not vanish in the limit as $\kappa \to 0$, giving rise to the dissipative anomaly [23]. In contrast, it can be shown that the κ term in the equations for the correlation functions can be neglected in this limit [24].

4.1.2 Equations for the correlation functions

Instead of working with (70), recent progress hinges on the fact that a closed set of equations for the single time correlation functions can be derived for Kraichnan's passive scalar model.

Consider

$$F_n(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n, t) = \langle T(\boldsymbol{x}_1, t) \cdots T(\boldsymbol{x}_n, t)\rangle \tag{73}$$

$F_{2n+1} = 0$ by symmetry. F_{2n} satisfies

$$\begin{aligned} \frac{\partial}{\partial t} F_{2n}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2n}, t) &= \sum_{j,k=1}^{2n} C_{\alpha\beta}(\boldsymbol{x}_j - \boldsymbol{x}_k) \nabla^j_\alpha \nabla^k_\beta F_{2n}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2n}, t) \\ &+ 2 \sum_{\substack{j,k=1 \\ j<k}}^{2n} B(\boldsymbol{x}_j - \boldsymbol{x}_k) F_{2n-2}(\boldsymbol{x}_1 \underset{\widehat{j}\ \widehat{k}}{\cdots} \boldsymbol{x}_{2n}, t). \end{aligned} \tag{74}$$

At statistical steady state, these equations reduces to

$$\sum_{j,k=1}^{2n} C_{\alpha\beta}(\boldsymbol{x}_j - \boldsymbol{x}_k) \nabla^j_\alpha \nabla^k_\beta F_{2n}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{2n}) = -2 \sum_{\substack{j,k=1 \\ j<k}}^{2n} B(\boldsymbol{x}_j - \boldsymbol{x}_k) F_{2n-2}(\boldsymbol{x}_1 \underset{\widehat{j}\ \widehat{k}}{\cdots} \boldsymbol{x}_{2n}). \tag{75}$$

Evaluation of the structures functions $S_{2n}(r) = \langle (T(\boldsymbol{x} + \boldsymbol{r}, t) - T(\boldsymbol{x}, t))^{2n}\rangle$ can in principle be carried out once the solution of (75) is known. However, solving these equations is a very difficult task and so far only perturbative methods have been successful in some regimes.

A simple dimensional argument or balancing $(\boldsymbol{u}\nabla)T$ with f suggests that a Kolmogorov-like theory would predict normal scaling exponents $\alpha_p = \frac{2-\xi}{2} p$. We will see below that this is not true for $p > 2$.

4.1.3 Zero modes

Our next task is to analyze the behavior of F_{2n} for small $|\boldsymbol{x}_j - \boldsymbol{x}_k|$. We will restrict ourselves to translation invariant solutions.

From (75), F_{2n} has contributions from the inhomogeneous part F_{2n-2}, and the homogeneous part, which at small distances are solutions of

$$\mathcal{M}^{sc}_{2n} \tilde{F}_{2n} = \sum_{j,k=1}^{2n} d_{\alpha\beta}(\boldsymbol{x}_j - \boldsymbol{x}_k) \nabla^j_\alpha \nabla^k_\beta \tilde{F}_{2n} = 0 \tag{76}$$

where

$$d_{\alpha\beta}(\boldsymbol{x}) = D\left((d + \xi - 1)\delta_{\alpha\beta} - \xi \frac{x_\alpha x_\beta}{r^2}\right) r^\xi \tag{77}$$

(76) is obtained as the scaling limit of the homogeneous equation associated with (75). The solutions of (76) are called zero modes [10, 29]. This is an important concept that characterizes the origin of the anomalous scaling.

As an example, let us study the behavior of the 2-point function at $d = 3$. F_2 satisfies

$$-\frac{2}{r^2}\frac{\partial}{\partial r}\Big(Dr^{2+\xi}\frac{\partial F_2}{\partial r}\Big) = B(r). \tag{78}$$

It is easy to obtain from this equation that

$$F_2(r) = C_0 - C_1 r^{2-\xi} + \cdots \tag{79}$$

where C_0, C_1 are constants depending on B and the neglected terms are of higher order. Hence we have for S_2

$$S_2(r) = C_2 r^{2-\xi} + \cdots \tag{80}$$

This implies that S_2 obeys normal scaling. Note that the constant C_2 is in general non-universal.

Let us now look at the 4-point function $F_4(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4)$ and let $\boldsymbol{x}_{jk} = \boldsymbol{x}_j - \boldsymbol{x}_k, j, k = 1, \cdots 4$. F_4 has contributions from the "Gaussian" part:

$$F_2(\boldsymbol{x}_{12})F_2(\boldsymbol{x}_{34}) + F_2(\boldsymbol{x}_{13})F_2(\boldsymbol{x}_{24}) + F_2(\boldsymbol{x}_{14})F_2(\boldsymbol{x}_{23}), \tag{81}$$

which give rise to normal scaling, as well as the contribution from the zero modes which may be the dominant contribution to S_4. To find the precise form of the zero modes is a very difficult task. Both [29] and [10] resorted to perturbation technique in either ξ or $1/d$. Here we follow [29] and write

$$\tilde{F} = E_0 + \xi G_0 + O(\xi^2), \tag{82}$$

where E_0 is the zero mode for the case when $\xi = 0$. Using the notations $\nabla_{12} = \nabla_{\boldsymbol{x}_{12}}$, etc, we can write down an equation for E_0:

$$-(\Delta_{12} + \Delta_{23} + \Delta_{34} - \nabla_{12}\cdot\nabla_{23} - \nabla_{23}\cdot\nabla_{34})E_0 = 0. \tag{83}$$

Substituting (83) to (76), we obtain an equation for G_0:

$$-\Delta_4 G_0 + \mathcal{L}E_0 = 0$$

where

$$\Delta_4 = \Delta_{\boldsymbol{x}_1} + \cdots + \Delta_{\boldsymbol{x}_4}, \quad \mathcal{L} = \sum_{j\neq k}\Big(\delta_{\alpha\beta} - \frac{1}{2}\frac{x_{jk}^{\alpha}x_{jk}^{\beta}}{|\boldsymbol{x}_{jk}|^2}\Big)\nabla_j^{\alpha}\nabla_k^{\beta} - \frac{1}{2}\Delta_4$$

The precise form of G_0 is quite complicated. We refer to [5, 29] for details of this calculation from which one obtains that $\tilde{F}$ must be homogeneous of degree

$$\alpha_4 = 4 - \frac{14}{5}\xi + O(\xi^2) = 2\alpha_2 - \rho_2, \tag{84}$$

where $\rho_2 = \frac{4}{5}\xi + O(\xi^2)$ is the anomalous exponent. Numerical results of Frisch et.al. [27] has found very good agreement with (84).

4.2 Passive vector equation: The kinematic dynamo problem

Under the same assumptions on the velocity field, the passively advected magnetic fields described by (4) can also be analyzed using similar methods. The general form of the covariance of $\boldsymbol{B}$ is given by [41]

$$\langle B_\alpha(\boldsymbol{x}+\boldsymbol{r},t)B_\beta(\boldsymbol{x},t)\rangle = G_1(r,t)\delta_{\alpha\beta} + G_2(r,t)\frac{r_\alpha r_\beta}{r^2}, \tag{85}$$

where $r = |\boldsymbol{r}|$ and the functions G_1 and G_2 are related by

$$\frac{\partial G_1}{\partial r} = -\frac{1}{r^2}\frac{\partial}{\partial r}(G_2 r^2) \tag{86}$$

Following [48], it is convenient to work with the trace of the correlation function tensor:

$$H(r,t) = 3G_1(r,t) + G_2(r,t). \tag{87}$$

G_1 and G_2 can then be found through

$$G_1(r,t) = \frac{H(r,t)}{2} - \frac{1}{2r^3}\int_0^r H(s,t)s^2 ds, \tag{88}$$

$$G_2(r,t) = -\frac{H(r,t)}{2} + \frac{3}{2r^3}\int_0^r H(s,t)s^2 ds. \tag{89}$$

In analogy with (78), a closed equation can be derived for H:

$$\begin{aligned}\frac{\partial H}{\partial t} &= \frac{2}{r^2}\frac{\partial}{\partial r}\Big((\kappa r^2 + Dr^{2+\xi})\frac{\partial H}{\partial r}\Big) + B_L(r) \\ &+ 2D\xi r^{\xi-2}\Big((1+2\xi)H + r\frac{\partial H}{\partial r} + \frac{\xi(\xi-2)}{r^3}\int_0^r H(s,t)s^2 ds\Big)\end{aligned} \tag{90}$$

where $B_L(r) = \int\langle \boldsymbol{f}(\boldsymbol{x},t)\cdot\boldsymbol{f}(0,0)\rangle dt$. (90) admits a steady state solution in the regime $\xi < 1$ when the dynamo effects are not present. To analyze this steady state solution in the limit of infinite Peclet number, it is convenient to introduce

$$\psi(r) = \frac{\sqrt{Dr^\xi}}{r}\int_0^r H(s)s^2 ds. \tag{91}$$

Then ψ satisfies

$$\frac{d^2\psi}{dr^2} - m(r)U(r)\psi = -\frac{m(r)\sqrt{Dr^\xi}}{r}\int_0^r B_L(s)s^2 ds, \tag{92}$$

for $r \ll \ell_0$, where

$$m(r) = \frac{1}{2Dr^\xi}, \qquad U(r) = \frac{D^2 r^{2\xi}(4 - 3\xi - \frac{3}{2}\xi^2)}{Dr^{2+\xi}}. \tag{93}$$

The zero modes for this problem can be found by solving the homogeneous equation. They are:

$$\tilde{F}_1(r) = r^{s_1}, \qquad \tilde{F}_2(r) = r^{s_2}, \tag{94}$$

where

$$s_{1,2} = \frac{1}{2} \pm \frac{3}{2}\sqrt{1 - \frac{1}{3}\xi(\xi+2)} \tag{95}$$

The general solutions of (92) is given by:

$$\psi = a\tilde{F}_1 + b\tilde{F}_2 - \frac{r^{s_1}}{2\sqrt{D}}\int_0^r \rho^{-2s_1}\int_0^\rho \rho_1^{-s_2-\frac{\xi}{2}}\int_0^{\rho_1} \rho_2^2 B_L(\rho_2)d\rho_2 d\rho_1 d\rho. \tag{96}$$

Clearly b must be zero in order to avoid having a $\tilde{F}_2$-type of singularity at small r. Matching with large scales, we get:

$$a = \frac{r^{s_1}}{2\sqrt{D}}\int_0^\infty \rho^{-2s_1}\int_0^\rho \rho_1^{-s_2-\frac{\xi}{2}}\int_0^{\rho_1} \rho_2^2 B_L(\rho_2)d\rho_2 d\rho_1 d\rho \tag{97}$$

Going back to H we find that the dominant contribution for $r \ll \ell_0$ comes from

$$H(r) \sim \frac{r^\gamma}{2D}\frac{L^{4-s_1-\frac{\xi}{2}}}{s_1 - s_2}\frac{s_1 + 1 - \frac{\xi}{2}}{s_1 - 1 - \frac{\xi}{2}}\int_0^\infty \rho^{3-s_1-\frac{\xi}{2}}B_L(\rho)d\rho \tag{98}$$

where

$$\gamma = -\frac{3+\xi}{2} + \frac{3}{2}\sqrt{1 - \frac{1}{3}\xi(\xi+2)} \sim -\xi - \frac{\xi^2}{3} + O(\xi^3) \tag{99}$$

Note in this case the second order correlation function does not obey the normal scaling.

5 Conclusion

In summary, we conclude that the study of stochastic PDEs in the style described in this paper is a very new, challenging and fruitful area of research. It is at the frontier of several different areas in mathematics as well as physics, ranging from PDE, large deviation theory, stochastic dynamical systems, to quantum field theory. It is at the foundation of many important areas in science and engineering. Fundamental understanding of these questions will likely come from and contribute to our understanding of analysis in infinite dimensional spaces, a recurring theme in this subject.

While a solution to the problem of hydrodynamic turbulence does not seem to be in sight, recent work on related problems of Burgers turbulence, stochastic passive scalar convection and 2D turbulence [50] gives renewed hope that definitive progress are being made.

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