Nonlinear Continuum Theory
of Smectic A Liquid Crystals

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1 Introduction

Smectic A is a liquid crystalline phase in which not only some degree of orientational order is present, there is also some degree of positional order. In the smectic A phase, molecules organize themselves into layers. Within each layer the molecules diffuse freely just as in ordinary liquids. We can think of smectic A liquid crystals as being a one-dimensional crystal in the direction normal to the layers, and a two dimensional liquid within the layers. In this sense smectic A liquid crystals provide the simplest example of crystalline liquids. Notice that nematics do not qualify as being crystalline since they do not have any positional order.

What makes the dynamics of smectic A unique is the possibility of permeation – the velocity of the molecules and the velocity of the layers at the same spatial location may not be the same. As a result, molecules may permeate through the layers [7]. What makes this permeation process unique is that the permeating agent and the underlying porous medium are made of the same thing – the constituent molecules. Indeed this unique permeation process is responsible for many of the unusual behavior observed in the dynamics and phase transition of smectic A liquid crystals [4].

The purpose of this paper is to establish a general nonlinear continuum theory for smectic A liquid crystals applicable to situations with large deformation and non-trivial flows. This can be viewed as the analog for smectic A of the Ericksen–Leslie theory for nematics [3, 8, 5, 11]. In the end we will see that it is actually simpler than the Ericksen-Leslie theory. As usual it consists of two parts: The hydrostatic or elasticity theory and the hydrodynamics. In the process of developing this theory, we will clarify a number of issues including: compressibility, the difference between elastic stress and molecular field, permeation and the dynamics of layers.

The first continuum theory for smectic A was proposed by de Gennes [7]. Later Martin, Parodi and Pershan established a general framework for developing hydrodynamic theories [12] which in principle is applicable to liquids, liquid crystals, and in particular to smectic A. This theory had a profound influence on the study of hydrodynamics of condensed matter [2]. Its application to smectic A liquid crystals is nicely summarized in the second edition of

de Gennes' well-known book [8] by de Gennes and Prost. Besides being able to successfully isolate the relevant modes in the dynamics smectic A and explain early scattering experiments [8], it also paved the way for studying hydrodynamic flows of smectic A [9].

Despite all its success, this theory is fundamentally linear and is restricted to small perturbations of the trivial planar structure. Consequently it is not applicable to commonly observed textures such as the focal conic domains. At a conceptual level, this theory does not distinguish between the elastic stress and the force acting on the directors—the permeation force. This is a major drawback since, as we discussed earlier, permeation is by far the most novel feature of the dynamics of smectic A.

An attempt was made in [10] to construct the general nonlinear elasticity theory. Among other things, [10] contains a nice discussion on selecting the correct free energy functional. This will be reviewed in the beginning of §2. However, it seems that the calculations in §3 of [10] and the subsequent discussions contain serious problems. We will come back to this at the end of §2.

Since this theory is meant for describing macroscopic variations, microscopic effects such as disclination loops will not be considered. In principle there is no difficulty to combine the present theory with Ericksen's new theory with a variable degree of orientation to establish a continuum theory capable of handling the defects. But the resulting theory is bound to be much more complicated. It seems wise to first explore the consequences of this simpler theory before proceeding to the much more complicated one.

The present theory does not assume that the layers are equally spaced. This will give rise to a second pressure term coming from the compressibility of the layers. In general, layer compression is very small compared with layer bending. This motivates us in the end to consider the incompressible limit for which the equations simplify. A novel Lagrangian multiplier appears to accompany the constraint of equal layer spacing.

Throughout this paper, we will use the summation convention.

2 Hydrostatic Theory

2.1 The free energy.

We briefly review the main argument in [10] leading to the free energy functional (2.9).

The smectic A phase is characterized by two types of order present in the system. As in the nematic phase it has the (uniaxial) orientational order: molecules tend to line up with each other. In a macroscopic theory we can describe the local orientation of the molecules by a director field $\boldsymbol{n}$ which gives the prefered direction of orientation. In addition it also has the translational order that molecules tend to position themselves on almost equally spaced layers. We will parameterize these layers by the iso-surfaces of a scalar function φ. In a continuum theory, the layer spacing becomes zero. In reality, the layer spacing is on the order of the molecular length. A key property that distinguishes smectic A from smectic C is that for smectic A, the molecules prefer to lie normal to the layers. This can be rephrased as saying that in the free energy functional, there should be a term reflecting the fact that there is a deep potential well at $\boldsymbol{n} = \nabla\varphi$. The simplest form of such a term is

$$(2.1) \qquad f_1 = \tfrac{1}{2}(\boldsymbol{n} - \nabla\varphi)^T B(\boldsymbol{n} - \nabla\varphi)$$

where $B = B_\perp I + (B_\parallel - B_\perp)\, \boldsymbol{n} \otimes \boldsymbol{n}$ is a rank 2 tensor consistent with the uniaxial symmetry of the material. Augmenting this by the lowest order terms in the distortion free energy

$$(2.2) \qquad f_2 = \tfrac{K_1}{2}(\nabla \cdot \boldsymbol{n})^2 + \tfrac{K_2}{2}(\boldsymbol{n} \cdot \nabla \times \boldsymbol{n})^2 + \tfrac{K_3}{2}(\boldsymbol{n} \times \nabla \times \boldsymbol{n})^2,$$

we obtain the total free energy

$$(2.3) \qquad \begin{aligned} f = f_1 + f_2 \;&=\; \tfrac{1}{2}(\boldsymbol{n} - \nabla\varphi)^T B(\boldsymbol{n} - \nabla\varphi) + \tfrac{K_1}{2}(\nabla \cdot \boldsymbol{n})^2 \\ &+\; \tfrac{K_2}{2}(\boldsymbol{n} \cdot \nabla \times \boldsymbol{n})^2 + \tfrac{K_3}{2}(\boldsymbol{n} \times \nabla \times \boldsymbol{n})^2 \end{aligned}$$

We should think of $B_\perp$ and $B_\parallel$ as being comparable in magnitude, and K_1, K_2, K_3 as being comparable in magnitude.

Not all of these terms are important in a macroscopic theory. Let us now isolate those terms which only manifest on variations over a microscopic length scale such as the size

of the disclination core. First we ask when tilt (deviation of $\boldsymbol{n}$ from $\frac{\nabla\varphi}{|\nabla\varphi|}$) can occur. Let $\delta\boldsymbol{n} = \boldsymbol{n} - \frac{\nabla\varphi}{|\nabla\varphi|}$. Since

$$(2.4) \qquad f_1 = \frac{B_{\|}}{2}(\boldsymbol{n} \cdot \nabla\varphi - 1)^2 + \frac{B_{\perp}}{2}|P_{\mathbf{n}}(\nabla\varphi)|^2$$

where the projection operator $P_{\mathbf{n}}$ is defined by $P_{\mathbf{n}}(\alpha) = \alpha - (\alpha \cdot \boldsymbol{n})\,\boldsymbol{n}$ for $\alpha \in R^3$, for tilt to occur, the energy cost of tilting must at least be compensated by the energy gain in distortion:

$$(2.5) \qquad B_{\perp}|\delta\boldsymbol{n}|^2 \approx K|\nabla\delta\boldsymbol{n}|^2.$$

Here K is a quantity comparable to K_1, K_2, K_3. This can only happen when the length scale of the variation is approximately $\ell = (\frac{K}{B_{\perp}})^{1/2}$ which is a molecular length. Therefore if we are interested in macroscopic variations outside the defect core, we can neglect the tilt and set

$$(2.6) \qquad \boldsymbol{n} = \frac{\nabla\varphi}{|\nabla\varphi|} \ .$$

(2.6) implies that

$$(2.7) \qquad \boldsymbol{n} \cdot \nabla \times \boldsymbol{n} \equiv 0$$

i.e., twist is prohibited outside the defect core. To see the effect of the bending term, let $\beta = |\nabla\varphi| - 1$. Then it is easy to see that

$$(2.8) \qquad \frac{\frac{1}{2}\,K_3\,|\boldsymbol{n} \times \nabla \times \boldsymbol{n}|^2}{\frac{1}{2}\,B_{\|}\,(\boldsymbol{n} \cdot \nabla\varphi - 1)^2} \ \leq \ \frac{K_3\,|\nabla\beta|^2}{B_{\|}\ \beta^2}$$

This ratio is only appreciable for variations over length scales comparable to $\ell = (\frac{K_3}{B_{\|}})^{1/2}$. Therefore the bending term in the Frank energy can also be safely neglected in a macroscopic theory.

Finally we obtain the free energy functional we will work with:

$$(2.9) \qquad f(\varphi) = \tfrac{1}{2}B(|\nabla\varphi| - 1)^2 + \tfrac{1}{2}K(\nabla \cdot \boldsymbol{n})^2$$

with $\boldsymbol{n}$ given by (2.6), and $B = B_{\|}$, $K = K_1$. Although $\ell = (\frac{K}{B_{\|}})^{1/2}$ is also a microscopic length, the first term, the compression energy, cannot be neglected since in many important situations the layers can be compressed on a macroscopic scale. Typically, the compression of the layers is very small.

2.2 Analog of the molecular field - the permeation force.

In contrast to ordinary elasticity of solids, there are two types of deformation we have to consider to construct the elasticity theory for liquid crystals. One is the deformation of the director field while keeping fixed the center of gravity of the molecules. The other is the displacement of the molecules while keeping fixed the director field. As a result there are two kinds of forces present: deformation of the former type gives rise to the so-called molecular field; deformation of the latter type gives rise to the more conventional elastic stress. The terminology "molecular field" originates from magnetism where it is used (in a mean field theory) to denote the average forces from other spins on the spin at a given location. Similarly in nematics, it is used to denote the forces acting on the director field. For smectic A the director field is specified by the location of the smectic layers, so it is more appropriate to discuss the forces acting on layers. Since tangential forces do not change the actual position of the layers, only the normal force is important. This is the permeation force.

To calculate this force, we compute the variation of the free energy with respect to the layer position, without deforming the material. Let $\alpha = \nabla \cdot \boldsymbol{n}$, $\beta = |\nabla \varphi| - 1$, $G(\alpha, \beta) = \frac{K}{2} \alpha^2 + \frac{B}{2} \beta^2$. Then $f(\varphi) = G(\alpha, \beta)$. Let $F(\varphi) = \int f(\varphi) \, d^3x$. First we note that:

$$(2.10) \qquad \delta|\nabla \varphi| = \frac{1}{|\nabla \varphi|} \nabla \varphi \cdot \nabla \delta \varphi, \qquad \delta \boldsymbol{n} = \frac{\nabla \delta \varphi}{|\nabla \varphi|} - \frac{(\nabla \varphi \cdot \nabla \delta \varphi)}{|\nabla \varphi|^3} \nabla \varphi$$

Neglecting surface terms, we have

$$
\begin{aligned}
(2.11) \quad \delta F &= \int \left\{ G_\alpha \, \delta(\nabla \cdot \boldsymbol{n}) + G_\beta \, \delta|\nabla \varphi| \right\} d^3x \\
&= \int \left\{ -\nabla G_\alpha \cdot \left(\frac{\nabla \delta \varphi}{|\nabla \varphi|} - \frac{\nabla \varphi \cdot \nabla \delta \varphi}{|\nabla \varphi|} \nabla \varphi \right) + G_\beta \frac{\nabla \varphi}{|\nabla \varphi|} \cdot \nabla \delta \varphi \right\} d^3x \\
&= \int \left\{ \nabla \cdot \frac{\nabla G_\alpha}{|\nabla \varphi|} - \nabla \cdot \left(\frac{(\nabla G_\alpha \cdot \nabla \varphi)}{|\nabla \varphi|^3} \nabla \varphi \right) - \nabla \cdot \left(G_\beta \frac{\nabla \varphi}{|\nabla \varphi|} \right) \right\} \delta \varphi \, d^3x.
\end{aligned}
$$

Hence the permeation force g is

$$(2.12) \qquad g = -\frac{\delta F}{\delta \varphi} = -\nabla \cdot \boldsymbol{\tau}$$

$$
\begin{aligned}
(2.13) \qquad \boldsymbol{\tau} &= -G_\beta \boldsymbol{n} + \frac{\nabla G_\alpha}{|\nabla \varphi|} - \frac{\nabla G_\alpha \cdot \boldsymbol{n}}{|\nabla \varphi|} \boldsymbol{n} \\
&= -G_\beta \boldsymbol{n} + \frac{1}{|\nabla \varphi|} P_{\mathbf{n}}(\nabla G_\alpha)
\end{aligned}
$$

$$= -B(|\nabla\varphi| - 1)\,\boldsymbol{n} + \frac{K}{|\nabla\varphi|}\,P_{\mathbf{n}}\Big(\nabla(\nabla\cdot\boldsymbol{n})\Big)$$

Integrating g over a volume D inside the smectic, we get

$$(2.14) \qquad\qquad \int_D g\,d^3x = -\int_{\partial D} \boldsymbol{\tau}\cdot\,d\boldsymbol{s}$$

We can interpret (2.11) and (2.14) as saying that the net effect of the permeation force is the creation and destruction of layers at the boundary, and $-\boldsymbol{\tau}$ can be thought of as being the force acting at the boundary.

Although [10] started with the same free energy functional (2.9), the calculation and discussions proceed very differently from ours. Among other things, [10] discusses separate notions of the permeation force and the molecular field.

2.3 Elastic stress

By definition, this is the variation of the free energy with respect to the deformation of the material, keeping the layer positions fixed.

Let us assume that the material undergoes an infinitesimal deformation, but the relative position of the layers remains fixed:

$$(2.15) \qquad\qquad \boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{u}(\boldsymbol{r}), \qquad\quad \varphi'(\boldsymbol{r}') = \varphi(\boldsymbol{r}).$$

Neglecting terms of order $O(\boldsymbol{u}^2)$ and higher, we get

$$(2.16) \qquad\qquad \frac{\partial r'_k}{\partial r_j} = \delta_{kj} + \frac{\partial u_k}{\partial r_j}, \quad \frac{\partial r_j}{\partial r'_k} = \delta_{kj} - \frac{\partial u_j}{\partial r_k}.$$

Straightforward computation gives (again neglecting higher order terms):

$$(2.17) \qquad\qquad \frac{\partial\varphi'}{\partial r'_j} = \frac{\partial\varphi}{\partial r_j} - \frac{\partial\varphi}{\partial r_k}\frac{\partial u_k}{\partial r_j}, \quad \text{or} \quad \nabla'\varphi = (I - \nabla\boldsymbol{u})\nabla\varphi$$

where $(\nabla\boldsymbol{u})_{i,j} = \dfrac{\partial u_j}{\partial r_i}$, and

$$(2.18) \qquad\qquad |\nabla'\varphi'| = |\nabla\varphi|(1 - \boldsymbol{n}^T\nabla\boldsymbol{u}\,\boldsymbol{n}).$$

Hence

$$(2.19) \qquad\qquad \delta|\nabla\varphi| = -|\nabla\varphi|\boldsymbol{n}^T\nabla\boldsymbol{u}\,\boldsymbol{n}.$$

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Similarly,

$$(2.20) \qquad n_i' = \frac{1}{|\nabla'\varphi'|}\frac{\partial\varphi'}{\partial r_i'} = \frac{1}{|\nabla\varphi|}\left(\frac{\partial\varphi}{\partial r_i} - \frac{\partial\varphi}{\partial r_l}\frac{\partial u_l}{\partial r_i} + \frac{1}{|\nabla\varphi|^2}\frac{\partial\varphi}{\partial r_i}\frac{\partial\varphi}{\partial r_j}\frac{\partial\varphi}{\partial r_k}\frac{\partial u_k}{\partial r_j}\right)$$

or

$$(2.21) \qquad \boldsymbol{n}' = \boldsymbol{n} - \nabla\boldsymbol{u}\,\boldsymbol{n} + (\boldsymbol{n}^T\nabla\boldsymbol{u}\,\boldsymbol{n})\,\boldsymbol{n}.$$

Since

$$(2.22) \qquad \frac{\partial n_i'}{\partial r_i'} = \frac{\partial n_i'}{\partial r_m}\left(\delta_{mi} - \frac{\partial u_m}{\partial r_i}\right),$$

we get

$$(2.23) \qquad \delta(\nabla\cdot\boldsymbol{n}) = \nabla\cdot\left\{-\nabla\boldsymbol{u}\,\boldsymbol{n} + (\boldsymbol{n}^T\nabla\boldsymbol{u}\,\boldsymbol{n})\,\boldsymbol{n}\right\} - Tr(\nabla\boldsymbol{n}\nabla\boldsymbol{u})$$

Therefore we have, neglecting surface terms

$$(2.24) \qquad \begin{aligned}
\delta F &= \int\left\{G_\beta\delta|\nabla\varphi| + G_\alpha\delta(\nabla\cdot\boldsymbol{n})\right\}d^3\boldsymbol{r}\\
&= \int\left\{-G_\beta|\nabla\varphi|\boldsymbol{n}^T\nabla\boldsymbol{u}\,\boldsymbol{n}\right.\\
&\qquad\qquad + \nabla G_\alpha^T\,\boldsymbol{u}\,\boldsymbol{n} - (\nabla G_\alpha\cdot\boldsymbol{n})\boldsymbol{n}^T\nabla\boldsymbol{u}\,\boldsymbol{n})\\
&\qquad\qquad \left. - G_\alpha\,Tr(\nabla\boldsymbol{n}\nabla\boldsymbol{u})\right\}d^3\boldsymbol{r}\\
&= \int Tr(\sigma^T\nabla\boldsymbol{u})\,d^3\boldsymbol{r}
\end{aligned}$$

where

$$(2.25) \qquad \begin{aligned}
\sigma &= -\left(G_\beta|\nabla\varphi| + (\nabla G_\alpha\cdot\boldsymbol{n})\right)\boldsymbol{n}\otimes\boldsymbol{n}\\
&\qquad + \nabla G_\alpha\otimes\boldsymbol{n} - G_\alpha(\nabla\boldsymbol{n})^T\\
&= |\nabla\varphi|\boldsymbol{\tau}\otimes\boldsymbol{n} - K(\nabla\cdot\boldsymbol{n})(\nabla\boldsymbol{n})^T.
\end{aligned}$$

or

$$(2.26) \qquad \begin{aligned}
\sigma_{i,k} &= -\left(B(|\nabla\varphi| - 1)|\nabla\varphi| + K\frac{\partial(\nabla\cdot\boldsymbol{n})}{\partial r_j}\,n_j\right)n_i\,n_k\\
&\quad + K\frac{\partial(\nabla\cdot\boldsymbol{n})}{\partial r_i}\,n_k - K(\nabla\cdot\boldsymbol{n})\frac{\partial n_i}{\partial r_k}
\end{aligned}$$

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3   Hydrodynamic Theory

In general there are several important steps in the derivation of hydrodynamic equations.

1. Selecting the hydrodynamic variables. These are the physical quantities which relax infinitely slowly in the small wavenumber limit. Generally in condensed matter systems there are two classes of hydrodynamic variables: the conserved quantities such as the mass density ρ, momentum density $\rho\boldsymbol{u}$ and energy density E; and the variables associated with the spontaneously broken symmetry [2]. For the present problem, this is the φ. Strictly speaking, degree of orientation [6] is *not* a hydrodynamic variable since it relaxes to its equilibrium value on a faster time scale.

2. Writing down the conservation laws and identifying the reversible part of the currents. For fluids the standard conservation laws are:

$$\rho_t + \nabla \cdot (\rho\boldsymbol{u}) = 0,$$

$$(\rho\boldsymbol{u})_t + \nabla \cdot (\rho\boldsymbol{u} \otimes \boldsymbol{u} + \sigma) = 0,$$

$$E_t + \nabla \cdot J_E = 0,$$

$$(\rho S)_t + \nabla \cdot J_S = \frac{D}{T},$$

where S is the entropy per unit mass, T is the temperature, σ is the rank 2 stress tensor, J_E and J_S are the energy and entropy currents respectively, D/T is the entropy production term. Second law of thermodynamics states that D vanishes for reversible processes and D should be positive for irreversible processes.

3. Modelling the irreversible part of the currents through force-flux relation – the constitutive relation. Since we are dealing with small wavenumber perturbations, it is usually enough to retain the linear part of this relation. The real task then, is to isolate the independent phenomenological coefficients using symmetry considerations.

This is the basic strategy that we will follow. We found that in step 2 it is much easier to work with free energy than entropy. This seems to be true in general for nonlinear theories. In contrast to Ericksen's approach for nematics [5, 6, 3] in which the dynamic equations for the director field was obtained from postulated conservation equations for some generalized

momentum, we obtain the dynamic equation for the layer variable φ through the constitutive relation.

3.1 Conservation laws and entropy sources.

As we mentioned earlier, the basic variables are: density ρ, velocity $\boldsymbol{u}$, total energy density E and the layer variable φ. $E = \frac{1}{2}\rho|\boldsymbol{u}|^2 + \rho e$, where e is the internal energy per unit mass. We will write the equation for φ as

$$\dot{\varphi} + X = 0$$

Of course this does not give any information unless we specify X. Mass conservation gives

(3.1) $$\rho_t + \nabla \cdot (\rho\,\boldsymbol{u}) = 0$$

or

(3.2) $$\dot{\rho} + \rho\,(\nabla \cdot \boldsymbol{u}) = 0$$

Here and in the following, we will use the upper dot to denote material derivative: $\dot{f} = f_t + (\boldsymbol{u} \cdot \nabla)\,f$. Momentum conservation defines the stress tensor σ

(3.3) $$\rho\dot{\boldsymbol{u}} = \nabla \cdot \sigma \, , \qquad (\text{or} \quad \rho\dot{\boldsymbol{u}}_\alpha = \partial_\beta\,\sigma_{\beta\alpha})$$

We will neglect external body forces. These effects can be easily incorporated. As it stands, (3.3) does not define the stress tensor uniquely. This issue was discussed in [12].

Next we have the energy balance in a material volume Ω (Ω moves with the fluid):

(3.4) $$\frac{d}{dt}\int_\Omega E d^3x = \int_{\partial\Omega} (\sigma\boldsymbol{u} + \dot{\varphi}\boldsymbol{\tau}) \cdot d\boldsymbol{s} - \int_{\partial\Omega} \boldsymbol{q} \cdot d\boldsymbol{s}$$

The first term on the right is the rate of work done at the boundary $\partial\Omega$ by the stress to the fluid in Ω. The second term is the rate of work done at the boundary to the layers: $\dot{\varphi}$ can be thought of as being the rate of displacement and $\boldsymbol{\tau}$ is the permeation force at the boundary $\partial\Omega$ applied to the layers in Ω. The third term is the amount of heat taken out of the volume Ω, $\boldsymbol{q}$ is the heat flux.

(3.4) can be written in a local form

(3.5) $$E_t + \nabla \cdot (E\boldsymbol{u} + \boldsymbol{q} - \sigma\boldsymbol{u} - \dot{\varphi}\,\boldsymbol{\tau}) = 0$$

Since

(3.6)
$$\left(\tfrac{1}{2}\rho|\boldsymbol{u}|^2\right)_t + \nabla \cdot \left(\tfrac{1}{2}\rho|\boldsymbol{u}|^2\,\boldsymbol{u} - \sigma\boldsymbol{u}\right) = -Tr\left(\sigma^T\,\nabla\boldsymbol{u}\right)$$

we get

(3.7)
$$\rho\dot{e} + \nabla \cdot \boldsymbol{q} - \nabla \cdot (\dot{\varphi}\,\boldsymbol{\tau}) = Tr\left(\sigma^T\,\nabla\boldsymbol{u}\right)$$

We will use the following form of the second law of thermodynamics:

(3.8)
$$\rho\,\dot{S} + \nabla \cdot \left(\frac{\boldsymbol{q}}{T}\right) \geq 0$$

where S is the entropy per unit mass, T is the absolute temperature. The Helmholtz free energy density is defined to be

(3.9)
$$F = e - T\,S$$

Using $(3.1 - 9)$, we have

(3.10)
$$\begin{aligned}
\rho\dot{F} &= \rho\left(\dot{e} - T\dot{S} - S\dot{T}\right) \\
&= Tr\left(\sigma^T\,\nabla\boldsymbol{u}\right) + \nabla \cdot (\dot{\varphi}\,\boldsymbol{\tau}) - \nabla \cdot \boldsymbol{q} - T\rho\dot{S} - \rho S\dot{T} \\
&\leq Tr\left(\sigma^T\,\nabla\boldsymbol{u}\right) + \nabla \cdot (\dot{\varphi}\,\boldsymbol{\tau}) - \frac{1}{T}\boldsymbol{q} \cdot \nabla T - \rho S\dot{T}
\end{aligned}$$

F is a function of ρ, T, α, β ($\alpha = \nabla \cdot \boldsymbol{n}$, $\beta = |\nabla\varphi|$). Hence

(3.11)
$$\dot{F} = F_T\,\dot{T} + F_\rho\,\dot{\rho} + F_\alpha\,\dot{\alpha} + F_\beta\,\dot{\beta}$$

A tedious but straightforward computation gives:

(3.12)
$$\begin{aligned}
\dot{\beta} &= \frac{1}{|\nabla\varphi|}\,\nabla\varphi \cdot (\nabla\varphi)^{\cdot} \\
\dot{\boldsymbol{n}} &= \frac{1}{|\nabla\varphi|}\,(\nabla\varphi)^{\cdot} - \frac{1}{|\nabla\varphi|^3}\left(\nabla\varphi \cdot (\nabla\varphi)^{\cdot}\right)\nabla\varphi \\
(\nabla \cdot \boldsymbol{n})^{\cdot} &= \nabla \cdot \dot{\boldsymbol{n}} - Tr\left(\nabla\boldsymbol{n}\nabla\boldsymbol{u}\right) \\
\nabla\dot{\varphi} &= (\nabla\varphi)^{\cdot} + \nabla\boldsymbol{u}\,\nabla\varphi
\end{aligned}$$

Using (3.11–12) and neglecting surface terms, we get from (3.10):

(3.13)
$$\begin{aligned}
Tr(\sigma^T\,\nabla\boldsymbol{u}) &- \tfrac{1}{T}\,\boldsymbol{q} \cdot \nabla T - \rho\dot{T}\,(S + F_T) - \rho\,F_\rho\dot{\rho} \\
&- \rho\,F_\beta\,\dot{\beta} - \rho\,F_\alpha\,\dot{\alpha} \geq 0
\end{aligned}$$

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Since $\dot{T}$ can be made arbitrary, we should have

(3.14)
$$S = -F_T \ .$$

Using (3.12), (3.13) becomes

(3.15)
$$\begin{aligned}
0 \ \leq \ & Tr(\sigma^T \ \nabla \boldsymbol{u}) - \tfrac{1}{T}\boldsymbol{q} \cdot \nabla T - \rho \, F_\rho \, \dot{\rho} - \rho \tfrac{F_\beta}{|\nabla\varphi|} \, \nabla\varphi \cdot (\nabla\varphi)^{\cdot} \\
& -\rho \, F_\alpha \nabla \cdot \dot{\boldsymbol{n}} + \rho \, F_\alpha \, Tr\,(\nabla \boldsymbol{n} \, \nabla \boldsymbol{u}) \\
= \ & Tr\left((\sigma^T + \rho \, F_\alpha \, \nabla \boldsymbol{n})\nabla \boldsymbol{u}\right) - \tfrac{1}{T} \, \boldsymbol{q} \cdot \nabla T + \rho^2 \, F_\rho(\nabla \cdot \boldsymbol{u}) \\
& -\frac{\rho F_\beta}{|\nabla\varphi|} \, \nabla\varphi \cdot \nabla\dot{\varphi} + \frac{\rho F_\beta}{|\nabla\varphi|} \, (\nabla\varphi)^T \, \nabla \boldsymbol{u} \, \nabla\varphi + \nabla(\rho \, F_\alpha) \cdot \dot{\boldsymbol{n}} \\
= \ & Tr\left((\sigma^T + \rho \, F_\alpha \, \nabla \boldsymbol{n} + \rho F_\beta |\nabla\varphi| \, \boldsymbol{n} \otimes \boldsymbol{n}) \, \nabla \boldsymbol{u}\right) - \tfrac{1}{T} \, \boldsymbol{q} \cdot \nabla T \\
& +\rho^2 \, F_\rho \, (\nabla \cdot \boldsymbol{u}) + \dot{\varphi}\nabla \cdot (\rho F_\beta \boldsymbol{n}) \\
& +\nabla(\rho F_\alpha) \cdot \left[\frac{1}{|\nabla\varphi|} \, (\nabla\varphi)^{\cdot} - \frac{1}{|\nabla\varphi|} \, \left(\boldsymbol{n} \cdot (\nabla\varphi)^{\cdot}\right)\boldsymbol{n}\right] \\
= \ & Tr\left[\left(\sigma^T + \rho F_\alpha \, \nabla \boldsymbol{n} + \rho F_\beta \, |\nabla\varphi| \, \boldsymbol{n} \otimes \boldsymbol{n} - \boldsymbol{n} \otimes \nabla(\rho F_\alpha)\right.\right. \\
& \left.\left. +(\nabla(\rho F_\alpha) \cdot \boldsymbol{n})\boldsymbol{n} \otimes \boldsymbol{n}\right)\nabla \boldsymbol{u}\right] - \tfrac{1}{T} \, \boldsymbol{q} \cdot \nabla T + \rho^2 F_\rho \, (\nabla \cdot \boldsymbol{u}) \\
& +\dot{\varphi}\nabla \cdot \left[\rho \, F_\beta \, \boldsymbol{n} - \frac{1}{|\nabla\varphi|} \, \nabla \, (\rho F_\alpha) + \frac{\nabla(\rho F_\alpha) \cdot \boldsymbol{n}}{|\nabla\varphi|} \, \boldsymbol{n}\right]
\end{aligned}$$

Comparing with §2, we have $G = \rho F$. Let

(3.16)
$$\begin{aligned}
\sigma^e \ &= \ -\rho F_\alpha(\nabla \boldsymbol{n})^T - \rho F_\beta \, |\nabla\varphi| \, \boldsymbol{n} \otimes \boldsymbol{n} + \nabla(\rho F_\alpha) \otimes \boldsymbol{n} \\
& \quad -(\nabla(\rho F_\alpha) \cdot \boldsymbol{n})\boldsymbol{n} \otimes \boldsymbol{n} \\
&= \ -(G_\beta|\nabla\varphi| + \nabla G_\alpha \cdot \boldsymbol{n}) \, \boldsymbol{n} \otimes \boldsymbol{n} + \nabla G_\alpha \otimes \boldsymbol{n} - G_\alpha(\nabla \boldsymbol{n})^T \\
&= \ |\nabla\varphi|\boldsymbol{\tau} \otimes \boldsymbol{n} - K(\nabla \cdot \boldsymbol{n})(\nabla \boldsymbol{n})^T \\
g^I \ &= \ \nabla \cdot \left[G_\beta\boldsymbol{n} - \frac{\nabla G_\alpha}{|\nabla\varphi|} + \frac{\nabla G_\alpha \cdot \boldsymbol{n}}{|\nabla\varphi|} \, \boldsymbol{n}\right] = -\nabla \cdot \boldsymbol{\tau}
\end{aligned}$$

and

(3.17)
$$\sigma = -pI + \sigma^e + \sigma^d$$

where p is some kind of pressure. We can rewrite (3.15) as

(3.18)
$$Tr((\sigma^d)^T\nabla \boldsymbol{u}) + (\rho^2 F_\rho - p)\nabla \cdot \boldsymbol{u} - \tfrac{1}{T}\boldsymbol{q} \cdot \nabla T + \dot{\varphi} \, g^I \geq 0$$

In most cases bulk compression of the fluid can be neglected and the fluid can be treated as being incompressible,

$$(3.19) \qquad \nabla \cdot \boldsymbol{u} = 0.$$

If not, we choose p to be the thermodynamic pressure

$$(3.20) \qquad p = \rho^2 F_\rho.$$

The difference between the thermodynamic pressure and the isotropic part of the stress tensor will be accounted by the bulk viscosity. In the end, (3.18) reduces to

$$(3.21) \qquad Tr\left((\sigma^d)^T \nabla \boldsymbol{u}\right) - \tfrac{1}{T}\, \boldsymbol{q} \cdot \nabla T + \dot{\varphi}\, g^I \geq 0$$

Or

$$Tr\left((\sigma^d)^T \nabla \boldsymbol{u}\right) - \tfrac{1}{T}\, \boldsymbol{q} \cdot \nabla T - X\, g^I \geq 0$$

(3.16) is the same as (2.25) except that the displacement field in (2.25) is replaced by the velocity field in (3.16).

3.2 Angular momentum balance and the asymmetric part of the stress tensor.

The role of angular momentum balance is better viewed as a constitutive restriction rather than a new independent conservation law. For simple fluids that do not have intrinsic body torque, angular momentuem balance implies that the Cauchy stress tensor must be symmetric [1]. For more general situations, it can be used to further extract the reversible part of the stress tensor. In the case of nematics, angular momentum balance implies that the relative rotation between the directors and the fluids contributes directly to the energy dissipation (or entropy production), not the absolute rotation of the directors.

Let us take a control volume Ω whose unit outward normal we denote by $\boldsymbol{t}$. It experiences two sources of torques: (1) The stress exerts a torque on the fluid given by $\int_{\partial\Omega} \boldsymbol{r} \times (d\boldsymbol{s} : \sigma)$ where $d\boldsymbol{s} : \sigma$ is a vector whose jth component is $\sigma_{kj} t_k ds$ (2) The bending force exerts a torque on the directors given by $\int_{\partial\Omega} \boldsymbol{n} \times (G_\alpha d\boldsymbol{s})$. Therefore, angular momentum balance gives:

$$(3.22) \qquad \frac{d}{dt}\int_\Omega (\boldsymbol{r} \times \rho\boldsymbol{u})\, d^3r = \int_{\partial\Omega} \boldsymbol{r} \times (d\boldsymbol{s} : \sigma) + \int_{\partial\Omega} \boldsymbol{n} \times (G_\alpha d\boldsymbol{s})$$

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Using (3.3) and integrating by parts we get (assuming $\nabla \cdot \boldsymbol{u} = 0$)

$$(3.23) \qquad \int_\Omega \Gamma(\sigma)\, d^3r = \int_{\partial\Omega} \boldsymbol{n} \times (G_\alpha d\boldsymbol{s})$$

where $\Gamma(\sigma)$ is a vector representing the asymmetric part of the tensor Ω:

$$(3.24) \qquad \Gamma(\sigma) = (\sigma_{32} - \sigma_{23}, \quad \sigma_{13} - \sigma_{31}, \quad \sigma_{21} - \sigma_{12})$$

On the other hand, we have

$$(3.25) \qquad \begin{aligned} \int_\Omega \Gamma(\sigma^e)\, d^3r &= \int_\Omega \Gamma\left(\nabla G_\alpha \otimes \boldsymbol{n} - G_\alpha \left(\nabla \boldsymbol{n}\right)^T \right) d^3r \\ &= -\int_\Omega \nabla \times (G_\alpha\, \boldsymbol{n})\, d^3r = \int_{\partial\Omega} \boldsymbol{n} \times (G_\alpha d\boldsymbol{s}) \end{aligned}$$

Since $\Gamma(\sigma) = \Gamma(\sigma^e) + \Gamma(\sigma^d)$, we get

$$(3.26) \qquad \Gamma(\sigma^d) = 0$$

i.e., σ^d is a symmetric tensor. In other words, all asymmetric part of σ is contained in σ^e.

3.3 Constitutive relations and phenomenological coefficients.

The final step is to relate the generalized fluxes σ^d, $\boldsymbol{q}$, $\dot{\varphi}$ to the generalized forces $\nabla \boldsymbol{u}$, $-\frac{1}{T}\nabla T$ and g^I. Since we are concerned with long wavelength variations in the system, we will restrict ourselves to a linear constitutive theory. The most general linear relations are:

$$(3.27) \qquad \begin{aligned} \sigma^d_{ij} &= A^1_{ijkl} \frac{\partial u_l}{\partial r_k} + B^1_{ijk} \frac{1}{T} \frac{\partial T}{\partial r_k} + C^1_{ij}\, g^I \\ q_i &= A^2_{ijk} \frac{\partial u_k}{\partial r_j} + B^2_{ij} \frac{1}{T} \frac{\partial T}{\partial r_j} + C^2_i\, g^I \\ -X &= A^3_{jk} \frac{\partial u_j}{\partial r_k} + B^3_i \frac{1}{T} \frac{\partial T}{\partial r_i} + C^3\, g^I \end{aligned}$$

Among the hydrodynamic varibles, $\boldsymbol{u}$ is odd under time reversal, ρ, E and φ are even. The irreversible part of their currents should have the same parity as the hydrodynamic variables under time reversal. Since $\nabla T, g^I$ are even under time reversal, it should not contribute to the irreversible part of the stress. Similarly since $\nabla \boldsymbol{u}$ is odd, it should not contribute

to $\boldsymbol{q}$ and $\dot{\varphi}$ which represent the irreversible part of the currents with respect to E and φ respectively. Therefore we must have

(3.28)
$$B^1_{ijk} = 0, \quad C^1_{ij} = 0, \quad A^2_{ijk} = 0, \quad A^3_{jk} = 0.$$

for all i, j, k. In principle there may still be reversible contributions to $\dot{\varphi}$. Therefore A^3_{jk} may not be zero. But dimensional consideration implies that A^3_{jk} is on the order of molecular length. Hence it is neglected. There cannot be reversible contributions to σ^d from ∇T and g^I since it would violate the $\boldsymbol{n} \to -\boldsymbol{n}$ symmetry.

The system we are dealing with has uniaxial symmetry with axis $\boldsymbol{n}$. The most general strain-stress relation consistent with uniaxial symmetry is [3, 8, 5]

(3.29)
$$\begin{aligned}
\sigma^d_{ij} &= \rho_1\,\delta_{ij}\,d_{ll} + \rho_2\,n_i\,n_j\,d_{ll} + \rho_3\,\delta_{ij}\,n_k\,n_l\,d_{kl} \\
&\quad + \mu_1\,n_k\,n_m\,d_{km}\,n_i\,n_j + \mu_4\,d_{ij} \\
&\quad + \mu_5\,n_i\,n_k\,d_{kj} + \mu_6\,n_j\,n_k\,d_{ki}
\end{aligned}$$

where $d_{km} = \frac{1}{2}(\frac{\partial u_k}{\partial r_m} + \frac{\partial u_m}{\partial r_k})$, or with $D = (d_{km})$

(3.30)
$$\begin{aligned}
\sigma^d &= \rho_1(\nabla \cdot \boldsymbol{u})\,I + \rho_2\,(\nabla \cdot \boldsymbol{u})\,\boldsymbol{n} \otimes \boldsymbol{n} + \rho_3\,(\boldsymbol{n}^T D\,\boldsymbol{n})\,I \\
&\quad + \mu_1\,(\boldsymbol{n}^T D\,\boldsymbol{n})\,\boldsymbol{n} \otimes \boldsymbol{n} + \mu_4\,D + \mu_5\,D\,\boldsymbol{n} \otimes \boldsymbol{n} + \mu_6\,\boldsymbol{n} \otimes D\,\boldsymbol{n}
\end{aligned}$$

Since σ^d is symmetric we must have $\mu_5 = \mu_6$. Similarly we have for $\boldsymbol{q}$ and $\dot{\varphi}$

(3.31)
$$\begin{aligned}
\boldsymbol{q} &= -(k_\perp\,I + (k_\parallel - k_\perp)\,\boldsymbol{n} \otimes \boldsymbol{n})\tfrac{1}{T}\,\nabla T + \mu\,g^I\,\boldsymbol{n} \\
-X &= -\mu'\,\frac{\boldsymbol{n} \cdot \nabla T}{T} + \lambda\,g^I
\end{aligned}$$

Onsager's relation implies that $\mu = \mu'$.

In most cases we can make the approximation that the fluid is incompressible. In such cases, the stress-strain relation (3.29-30) simplifies since the ρ_1, ρ_2 terms drop out, and the ρ_3 term can be lumped into the pressure term. We summarize here the hydrodynamic equations in this case:

(3.32)
$$\begin{aligned}
\rho\dot{\boldsymbol{u}} &= \nabla \cdot \sigma, \qquad \nabla \cdot \boldsymbol{u} = 0, \\
E_t &+ \nabla\,(E\,\boldsymbol{u} + \boldsymbol{q} - \sigma\boldsymbol{u} - \dot{\varphi}\,\boldsymbol{\tau}) = 0 \\
\dot{\varphi} &= -\mu\,\frac{\boldsymbol{n} \cdot \nabla T}{T} + \lambda\,g^I
\end{aligned}$$

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where

$$
\begin{aligned}
\sigma \;=\;& -p\,I + \sigma^e + \mu_1(\boldsymbol{n}^T D\,\boldsymbol{n})\,\boldsymbol{n}\otimes\boldsymbol{n} + \mu_4\,D \\
& +\mu_5\,(D\,\boldsymbol{n}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes D\,\boldsymbol{n}) \\
\boldsymbol{q} \;=\;& -(k_\perp\,I + (k_\parallel - k_\perp)\,\boldsymbol{n}\otimes\boldsymbol{n})\tfrac{1}{T}\,\nabla T + \mu\,g^I\,\boldsymbol{n} \\
g^I \;=\;& \nabla\cdot\Big(B(|\nabla\varphi|-1)\boldsymbol{n} - \tfrac{K}{|\nabla\varphi|}P_{\mathbf{n}}(\nabla(\nabla\cdot\boldsymbol{n}))\Big)
\end{aligned}
$$

(3.33)

In the isothermal approximation, this reduces to

(3.34)
$$
\begin{cases}
\rho\dot{\boldsymbol{u}} \;=\; \nabla\cdot\sigma \qquad \nabla\cdot\boldsymbol{u}=0 \\
\dot{\varphi} \;=\; \lambda\,g^I
\end{cases}
$$

3.4   Discussions.

Equation (3.34) is among the simplest systems of equations for nonlinear anisotropic fluids. This system of equations is actually simpler than the hydrodynamic equations for nematic liquid crystals [3, 8]. One interesting feature is the nature of the equation for φ: It is a fourth order nonlinear PDE coupled to the momentum equation through the convective terms.

Let us now return to the general systems (3.32-33). Denote by $\boldsymbol{v}(\boldsymbol{x},t)$ the velocity of the layers at $(\boldsymbol{x},t)$, i.e. φ satisfies

(3.35)
$$
\varphi_t + (\boldsymbol{v}\cdot\nabla)\varphi = 0.
$$

Then we can rewrite the last equation in (3.32) as

(3.36) $\qquad |\nabla\varphi|(\boldsymbol{u}-\boldsymbol{v})\cdot\boldsymbol{n} = -\mu\dfrac{\boldsymbol{n}\cdot\nabla T}{T} + \lambda\nabla\cdot\Big(B(|\nabla\varphi|-1)\boldsymbol{n} - \tfrac{K}{|\nabla\varphi|}P_{\mathbf{n}}(\nabla(\nabla\cdot\boldsymbol{n}))\Big)$

$(\boldsymbol{u}-\boldsymbol{v})\cdot\boldsymbol{n}$ is the permeation velocity. (3.36) states that there are three different ways to drive the permeation. The first is an example of thermal-mechanical effect: temperature gradient can drive permeation. The second is layer compression: molecules permeate from the more dilated side of the layer to the more compressed side. The third is layer bending: molecules permeate through curved layers.

In the case of nematics, cross effects analogous to the μ-term does not appear since it would also violate the $\boldsymbol{n}\to-\boldsymbol{n}$ symmetry.

4 Incompressible Limit

First we must clarify that compressibility of the fluid and compressibility of the layers are separate issues. Bulk compression of the fluid gives rise to the conventional hydrodynamic pressure, which acts as a Lagrange multiplier for incompressible fluids

$$(4.1) \qquad \nabla \cdot \boldsymbol{u} = 0.$$

Compression of the layers gives rise to the terms associated with the compression modulus B in the elastic stress. In constrast to the hydrodynamic pressure force which is isotropic, this force acts only in the direction normal to the layers. The constraint associated with the incompressibility of the layers is

$$(4.2) \qquad |\nabla \varphi| = 1$$

and the corresponding Lagrange multiplier will be denoted by ξ. The free energy now becomes

$$(4.3) \qquad \tilde{G}(\varphi) = \frac{K}{2} |\nabla \cdot \boldsymbol{n}|^2 + \xi(|\nabla \varphi| - 1)$$

Therefore in this case to compute the permeation force, elastic stresses etc., we just have to replace G_β in (2.13) by ξ:

$$(4.4) \qquad \begin{aligned} \boldsymbol{\tau} &= -\xi \boldsymbol{n} + \nabla G_\alpha - (\nabla G_\alpha \cdot \boldsymbol{n}) \, \boldsymbol{n} \\ \sigma^e &= -(\xi + \nabla G_\alpha \cdot \boldsymbol{n}) \, \boldsymbol{n} \otimes \boldsymbol{n} + \nabla G_\alpha \otimes \boldsymbol{n} - G_\alpha (\nabla \boldsymbol{n})^T. \end{aligned}$$

Or if we replace $\xi + \nabla G_\alpha \cdot \boldsymbol{n}$ by ξ we get

$$(4.5) \qquad \begin{aligned} \boldsymbol{\tau} &= -\xi \boldsymbol{n} + \nabla G_\alpha \\ \sigma^e &= -\xi \boldsymbol{n} \otimes \boldsymbol{n} + \nabla G_\alpha \otimes \boldsymbol{n} - G_\alpha (\nabla \boldsymbol{n})^T = \boldsymbol{\tau} \otimes \boldsymbol{n} - G_\alpha (\nabla \boldsymbol{n})^T \end{aligned}$$

Similarly for the hydrodynamic theory, we have the following inequality from the second law of the thermodynamics

$$(4.6) \qquad \begin{aligned} & Tr\left[(\sigma^T + G_\alpha \nabla \boldsymbol{n} + \boldsymbol{n} \otimes \nabla G_\alpha) \nabla \boldsymbol{u}\right] \\ & -\frac{1}{T} \, \boldsymbol{q} \cdot \nabla T + \rho^2 \, F_\rho \, (\nabla \cdot \boldsymbol{u}) - \dot{\varphi} \nabla G_\alpha \geq 0 \end{aligned}$$

The left hand side of (4.6) still contains other reversible sources besides the pressure: Since $|\nabla\varphi| = 1$, we have $\nabla\varphi \cdot (\nabla\varphi)^{\cdot} = 0$. Therefore adding $-\xi\nabla\varphi \cdot (\nabla\varphi)^{\cdot} \equiv 0$ to the left hand side of (4.6) does not affect the inequality. We obtain then

$$
\begin{aligned}
Tr\Big[(\sigma^T + G_\alpha\nabla\boldsymbol{n} + \boldsymbol{n} \otimes \nabla G_\alpha + \xi\boldsymbol{n} \otimes \boldsymbol{n})\nabla\boldsymbol{u}\Big] \\
-\tfrac{1}{T}\,\boldsymbol{q}\cdot\nabla T + \rho^2\,F_\rho\nabla\cdot\boldsymbol{u} + \dot\varphi\nabla\cdot(\xi\boldsymbol{n} - \nabla G_\alpha) \ge 0
\end{aligned}
$$

(4.7)

Now the hydrodynamic theory proceeds in the same way as in the case when the layers are compressible. We will only write down the final equations for the case when the process is isothermal:

$$
\begin{aligned}
\rho_t \; + \; \nabla\cdot(\rho\boldsymbol{u}) &= 0 \\
\rho\dot{\boldsymbol{u}} &= \nabla\cdot(-p\,I + \sigma^e + \sigma^d) \\
\dot\varphi &= \lambda\nabla\cdot(\xi\boldsymbol{n} - \nabla G_\alpha) = \lambda(\nabla\cdot(\xi\nabla\varphi) - K\Delta^2\varphi)
\end{aligned}
$$

(4.8)

with σ^e, σ^d given by (4.5) (3.30) respectively. The Lagrange multiplier is determined from the constraint $|\nabla\varphi| = 1$. For the case when both the fluid and the layers are incompressible, the full system becomes:

$$
\begin{aligned}
\rho\dot{\boldsymbol{u}} &= \nabla(-pI + \sigma^d + \sigma^e)\,, & \nabla\cdot\boldsymbol{u} &= 0 \\
\dot\varphi &= \lambda(\nabla\cdot(\xi\nabla\varphi) - K\Delta^2\varphi)\,, & |\nabla\varphi| &= 1
\end{aligned}
$$

(4.9)

where

$$
\begin{aligned}
\sigma^d &= \mu_1\,(\boldsymbol{n}^T D\boldsymbol{n})\,\boldsymbol{n}\otimes\boldsymbol{n} + \mu_4\,D + \mu_5\,(D\,\boldsymbol{n}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes D\,\boldsymbol{n}) \\
\sigma^e &= -\xi\,\boldsymbol{n}\otimes\boldsymbol{n} + K\nabla(\nabla\cdot\boldsymbol{n})\otimes\boldsymbol{n} - K(\nabla\cdot\boldsymbol{n})\,\nabla^2\varphi
\end{aligned}
$$

(4.10)

4.1   Derivation of the incompressible limit equations using asymptotic analysis

The equations obtained above can also be derived from the equations for compressible layers using systematic asymptotic analysis in the limit as $B \to \infty$. We will present this derivation for the isothermal case where the algebra is slightly easier.

Let $\varepsilon = \frac{1}{B}$. We look for solutions of (3.34) in the form:

$$
\begin{aligned}
\varphi &= \varphi_0 \;+\; \varepsilon\,\varphi_1 \;+\; O\,(\varepsilon^2) \\
\boldsymbol{u} &= \boldsymbol{u}_0 \;+\; \varepsilon\,\boldsymbol{u}_1 \;+\; O\,(\varepsilon^2) \\
p &= p_0 \;+\; \varepsilon\,p_1 \;+\; O\,(\varepsilon^2)
\end{aligned}
$$

(4.11)

Substituting these into (), we find

$$
\begin{aligned}
\boldsymbol{n} &= \boldsymbol{n}_0 + \varepsilon\,\boldsymbol{n}_1 \\
\boldsymbol{\tau} &= \tfrac{1}{\varepsilon}\,\boldsymbol{\tau}_{-1} + \boldsymbol{\tau}_0 + O\left(\varepsilon\right) \\
\sigma^e &= \tfrac{1}{\varepsilon}\,\sigma^e_{-1} + \sigma^e_0 + O\left(\varepsilon\right)
\end{aligned}
$$

(4.12)

where

$$
\boldsymbol{n}_0 = \frac{\nabla\varphi_0}{|\nabla\varphi_0|}, \qquad\qquad \boldsymbol{n}_1 = \frac{\nabla\varphi_1}{|\nabla\varphi_0|} - \frac{\nabla\varphi_0 \cdot \nabla\varphi_1}{|\nabla\varphi_0|^3}\,\nabla\varphi_0
$$

$$
\boldsymbol{\tau}_{-1} = -(|\nabla\varphi_0| - 1)\,\boldsymbol{n}_0, \qquad \boldsymbol{\tau}_0 = -(|\nabla\varphi_0| - 1)\,\boldsymbol{n}_1 - \frac{\nabla\varphi_0 \cdot \nabla\varphi_1}{|\nabla\varphi_0|}\,\boldsymbol{n}_0
$$

$$
+\, K(\nabla(\nabla \cdot \boldsymbol{n}_0) - (\nabla(\nabla \cdot \boldsymbol{n}_0) \cdot \boldsymbol{n}_0)\,\boldsymbol{n}_0)
$$

$$
\sigma^e_{-1} = -|\nabla\varphi_0|\,(|\nabla\varphi_0| - 1)\,\boldsymbol{n}_0 \otimes \boldsymbol{n}_0
$$

$$
\sigma^e_0 = |\nabla\varphi_0|\,\boldsymbol{\tau}_0 \otimes \boldsymbol{n}_0 + \frac{\nabla\varphi_0 \cdot \nabla\varphi_1}{|\nabla\phi_0|}\,\boldsymbol{\tau}_{-1} \otimes \boldsymbol{n}_0
$$

$$
+|\nabla\varphi_0|\,\boldsymbol{\tau}_{-1} \otimes \boldsymbol{n}_1 - K\,(\nabla \cdot \boldsymbol{n}_0)\,\nabla\boldsymbol{n}_0^T
$$

Balancing $O(\tfrac{1}{\varepsilon})$ terms in the momentum equations, we get

$$
\nabla \cdot \sigma^e_{-1} = 0
$$

Integrating this along thin tubes of integral curves of $\boldsymbol{n}_0$, we conclude that $|\nabla\varphi_0|\,(|\nabla\varphi_0| - 1)\,\boldsymbol{n}_0$ is constant along the integral curves of $\boldsymbol{n}_0$. There are only two possibilities that this can happen: either $|\nabla\varphi_0| \equiv 1$ along the integral curve, or $\boldsymbol{n}_0$ is a constant vector along the integral curve. The latter implies that the integral curve is a straight line and $|\nabla\varphi_0|$ is also constant along the integral curve.

Let S be an iso-surface of φ_0. In a neighborhood Ω of S, let us define $g(\boldsymbol{x}) = \mathrm{dist}(\boldsymbol{x}, S)$. Assume that $\nabla\phi_0$ is not identically equal to 1 in Ω. Then since the integral curves of ∇g are also straight lines perpendicular to the tangent planes of S, there exists a function h, such that $g = h(\varphi_0)$. Differentiating we get

$$
|h'\left(\varphi_0\right)| = \frac{1}{|\nabla\varphi_0|}
$$

The left hand side is a function which is constant along iso-surfaces of φ_0. The function on the right hand side is constant along integral curves of $\nabla\varphi_0$. Therefore we conclude that

$$
|\nabla\varphi_0| \equiv C_0
$$

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in Ω where C_0 is a constant. Now the $O(\frac{1}{\varepsilon})$ terms in the φ-equation gives

$$\nabla \cdot \boldsymbol{\tau}_{-1} = 0$$

i.e.

$$(C_0 - 1)\, \nabla \cdot \frac{\nabla \varphi_0}{|\nabla \varphi_0|} = 0$$

If we allow the layers to be curved, $\nabla \cdot \frac{\nabla \varphi_0}{|\nabla \varphi_0|} \not\equiv 0$. Hence $C_0 = 1$. We conclude that

$$|\nabla \varphi_0| \equiv 1.$$

Let $\xi = \nabla \varphi_0 \cdot \nabla \varphi_1$, the expressions for the $O(1)$ terms simplify to

$$
\begin{aligned}
\boldsymbol{n}_0 &= \nabla \varphi_0, \\
\boldsymbol{\tau}_0 &= -\xi\, \boldsymbol{n}_0 + K\Big(\nabla\left(\nabla \cdot \boldsymbol{n}_0\right) - \left(\nabla\left(\nabla \cdot \boldsymbol{n}_0\right)\cdot \boldsymbol{n}_0\right)\boldsymbol{n}_0\Big) - K(\nabla \cdot \boldsymbol{n}_0)(\nabla\, \boldsymbol{n}_0)^T, \\
\sigma_0^e &= \boldsymbol{\tau}_0 \otimes \boldsymbol{n}_0.
\end{aligned}
$$

Balancing the $O(1)$ terms in the equations, we obtain (4.9).

4.2 Linearization around planar layers

To get more insight into the nature of the hydrodynamic equations for smectic A and to make a comparison with the results of de Gennes et. al. [8, 12], we study the dynamics of infinitesimal perturbations of the planar structure. It is enough to restrict ourselves to the two-dimensional situations where all the variables are independent of z. The unperturbed layers lie along the y-axis. Linearization around the state $u = 0,\, v = 0, p = 0,\, \varphi\left(x,\, y\right) = x$ gives us

(4.13)
$$
\begin{aligned}
\rho_0\, u_t + p_x &= -B\, \varphi_{xx} + K\, \varphi_{yyyy} + \nu \Delta u \\
\rho_0\, v_t + p_y &= \nu \Delta v \\
u_x + v_y &= 0 \\
\varphi_t + u &= -\lambda\left(B\, \varphi_{xx} - K\, \varphi_{yyyy}\right)
\end{aligned}
$$

This is the same as the equations (8.17) in [8] except that we take the fluid to be incompressible. Let us look for solutions of the type

$$\left(u,\, v,\, p,\, \varphi\right) = \left(\hat{u}_0,\, \hat{v}_0,\, \hat{p}_0,\, \hat{\varphi}_0\right) e^{\omega t + i l x + i k y}$$

It is easy to see that ω must satisfy

(4.14) $$\left[\omega + \frac{\nu}{\rho_0} \left(l^2 + k^2 \right) \right] \left(\omega + \lambda \widetilde{B} \right) \;=\; \widetilde{\mu} \, \widetilde{B}$$

where

(4.15) $$\widetilde{B} = B \, \ell^2 \;+\; K \, k^4, \qquad \widetilde{\mu} = -\frac{1}{\rho_0 \left(1 + \frac{l^2}{k^2} \right)}$$

The real part of the roots ω_1, ω_2 is always negative. Therefore the planar structure is linearly stable, as expected.

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