1-color-avoiding paths, special tournaments, and incidence geometry

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SPUR 2016
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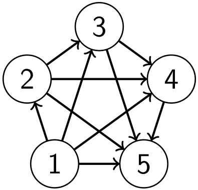
August 5, 2016

Background: Ramsey argument of Erdős–Szekeres

- Definition
  The transitive tournament of size N is the directed graph on N vertices numbered $1, \ldots, N$ with a directed edge $v_i \to v_j$ for each pair $i < j$.

- Theorem (Cf. Erdős–Szekeres 1935)
  Any 2-coloring of the edges of the transitive tournament of size N contains a monochromatic directed path of length at least $\sqrt{N}$.



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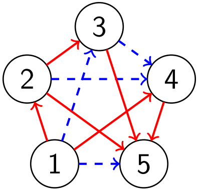

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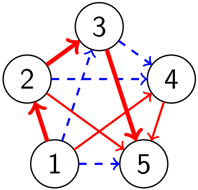
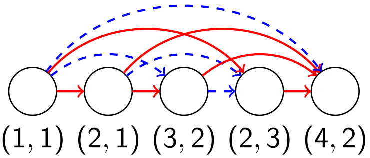


Proof: Record and pairs problem

Record: assign vertex i the pair of positive integers (R_i, B_i) where R_i (resp. B_i) is the length of the longest red (resp. blue) path in the graph that ends at vertex i.



Claim

Every vertex is assigned a different ordered pair.

Proof.

Suppose the edge $i \to j$ is red. Then $R_j > R_i$. □

Now since each of the N vertices is assigned a distinct ordered pair, at least one must have a coordinate of size at least $\sqrt{N}$.

Moving on to three colors

Easy generalization: with k colors, longest monochromatic (1-color-*using*) path is $N^{1/k}$, with same proof. Harder question:

- Question (Loh 2015)

  *Must any 3-coloring of the edges of the transitive tournament of size N have a 1-color-*avoiding *directed path of length at least $N^{2/3}$?*

  - Cannot *guarantee* longer than $\sim N^{2/3}$.
  - "Trivial" lower bound: $N^{1/2}$ from normal Erdős–Szekeres (red-green or blue).

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  - "Trivial" lower bound: $N^{1/2}$ from normal Erdős–Szekeres (red-green or blue).
  - Idea: Record the following lengths: longest blue-*avoiding* path $x_i = RG_i$, green-*avoiding* path $y_i = RB_i$, and red-*avoiding* path $z_i = GB_i$, ending at vertex i.

Triples problem

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- Proposition-Definition (Ordered set, Loh 2015)

  The list of triples $L_1 = (x_1, y_1, z_1), \ldots, L_N = (x_N, y_N, z_N)$ is ordered, *meaning that for $i < j$, difference $L_j - L_i$ has at least 2 positive coordinates.*

  - Suppose all 1-color-avoiding paths have length at most n, so all coordinates are at most n, so $L_i \in [n]^3$ for all i.

- Question (Loh 2015)

  Must an ordered set of triples $S \subseteq [n]^3$ contain at most $n^{3/2}$ points?

  - Would imply $N^{2/3}$ bound for tournaments question.
  - Exist examples with $\sim n^{3/2}$ points.
  - "Trivial" upper bound: at most n^2 points.

Triples in grids: slice-increasing observation

- Take an ordered set of triples $L_1 = (x_1, y_1, z_1), \ldots, L_N = (x_N, y_N, z_N)$ in $[n]^3$.
- Loh 2015: ordered sets are *slice-increasing*: on a *coordinate-slice* (say x fixed), the points are increasing in the other two coordinates (i.e. y, z).
- Corollary: for any x, y, there is at most one triple $(x, y, ?)$. This proves the "trivial bound" of $N \leq n^2$.

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- $n \times n$ grid view: for each i, fill in square $(x_i, y_i) \in [n]^2$ with the z-coordinate z_i. Leave other squares blank.
- Row and column labels are increasing. The squares containing a fixed label z must be increasing.

	3		4
3		4	
	1		2
1		2	

(tight example for $n = 4$; generalizes to large n)

Ordered induced matchings

- Row and column labels are increasing. The squares containing a fixed label z must be increasing.
- Suppose for $i \in [n]$, the label $z = i$ appears a_i times. Goal: bound number of labeled squares, $a_1 + a_2 + \cdots + a_n$.
- Since row and column labels are increasing, the labels $z = i$ form the increasing main diagonal of an otherwise "blocked" $a_i \times a_i$ grid (Loh 2015: "ordered induced matching").
- Example for $n = 3$. The x's are "blocked" as part of the grid for $z = 1$; the y's for $z = 3$. (The x,y squares must be empty.)

2	y	3
x		1
1	3	xy

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- Loh 2015: the "ordered induced matching" property *alone* is enough to get a bound of $\sim n^2/e^{\log^*(n)}$, but *cannot alone* beat the bound $\sim n^2/e^{\sqrt{\log(n)}}$ (Behrend construction).

Sum of squares of slice-counts

- Natural to consider a_i^2 "blocked" squares.
- Does $a_1^2 + a_2^2 + \cdots + a_n^2 \leq n^2$ always hold?

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Here $a_1^2 + a_2^2 + a_3^2 = 2^2 + 1^2 + 2^2 = 9 = n^2$.

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- If one only remembers the slice-increasing condition, then no:

			2		4
					1
			1	4	
2		4			
		1		3	
1	4				

- This example is slice-increasing, but it turns out not to be an ordered set of triples.

Back to tournaments: Color

- Color: given any ordered set of triples $L_1 = (x_1, y_1, z_1), \ldots,$ $L_N = (x_N, y_N, z_N)$, for $i < j$, the difference $L_j - L_i$ has at least two positive coordinates:
  - $(+, +, \leqslant 0)$
  - $(+, \leqslant 0, +)$
  - $(\leqslant 0, +, +)$
  - $(+, +, +)$

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  - $(+, +, \leqslant 0)$ R
  - $(+, \leqslant 0, +)$ G
  - $(\leqslant 0, +, +)$ B
  - $(+, +, +)$???

RGBK-tournaments

- Definition

  An RGBK-tournament of size N is a four-coloring of the transitive tournament of size N with colors R, G, B, and K.

  - We'll think of K as a "wild color" and try to find an RGK-, RBK-, or GBK-path of length at least $N^{2/3}$.
  - It's not to hard to show that this is equivalent to the original RGB-tournament problem.

Back to tournaments: Color

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  - $(+, +, \leqslant 0)$ R
  - $(+, \leqslant 0, +)$ G
  - $(\leqslant 0, +, +)$ B
  - $(+, +, +)$ K

Color $\circ$ Record

- What we've done so far:
  - Record reduces the RGBK-tournament problem to the triples problem.
  - Color reduces the triples problem to the RGBK-tournament problem.
- This means that it is sufficient to prove the result for tournaments in the image of Color $\circ$ Record.

Geometric tournaments

- Definition
  Call an RGBK-tournament geometric if it is the image of some ordered set under Color.
  - Take a geometric torunament that comes from some ordered set of triples $L_1 = (x_1, y_1, z_1), \dots, L_N = (x_N, y_N, z_N)$.
    - Suppose the edges $v_i \to v_j$ and $v_j \to v_k$ are R-colored.
    - This means that $z_i \geq z_j \geq z_k$.
    - This in turn implies that the $v_i \to v_k$ is R-colored.
- Proposition-Definition (2016)
  For a set of colors $\mathcal{C}$, a tournament is $\mathcal{C}$-transitive if for every $i < j < k$ with $v_i \to v_j$ and $v_j \to v_k$ both $\mathcal{C}$-colored, so is $v_i \to v_k$. Geometric tournaments are exactly the tournaments that are R-, G-, B-, RGK-, RBK-, *and* GBK-*transitive.*

Gallai decomposition

- In the special case where a geometric tournament has no K-colored edges, this constraint becomes much simpler.
- A K-free geometric tournament is exactly one which is R-, G-, and B-transitive and has no trichromatic triangles.

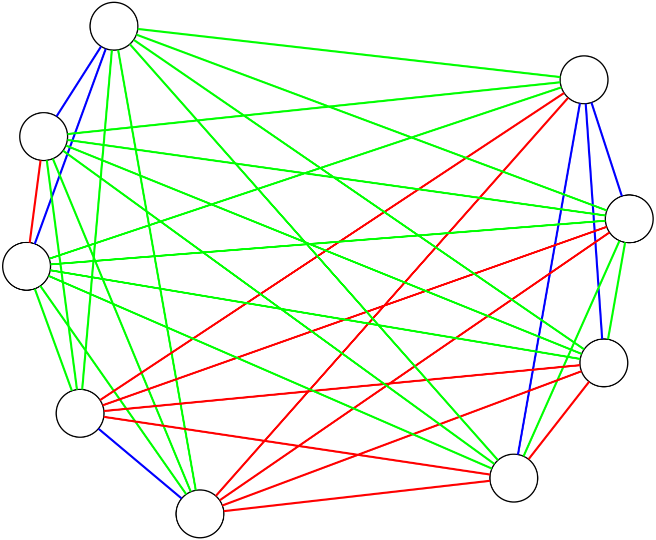
- **Definition**

  A Gallai 3-coloring of K_N is a 3-coloring of the edges of K_N such that no triangle is trichromatic.

- **Theorem (Gallai 1967)**

  For $N \geq 2$, a Gallai 3-coloring of K_N has a base decomposition, *meaning a vertex-partition into $m \geq 2$* strictly smaller nonempty *graphs $H_1, \ldots, H_m$, where the edges between two distinct blocks H_i, H_j use* at most one *of the colors* R, G, B, *and the edges between the various blocks $H_1, \ldots, H_m$* in total *use* at most two *of the colors* R, G, B.

Gallai decomposition, cont.



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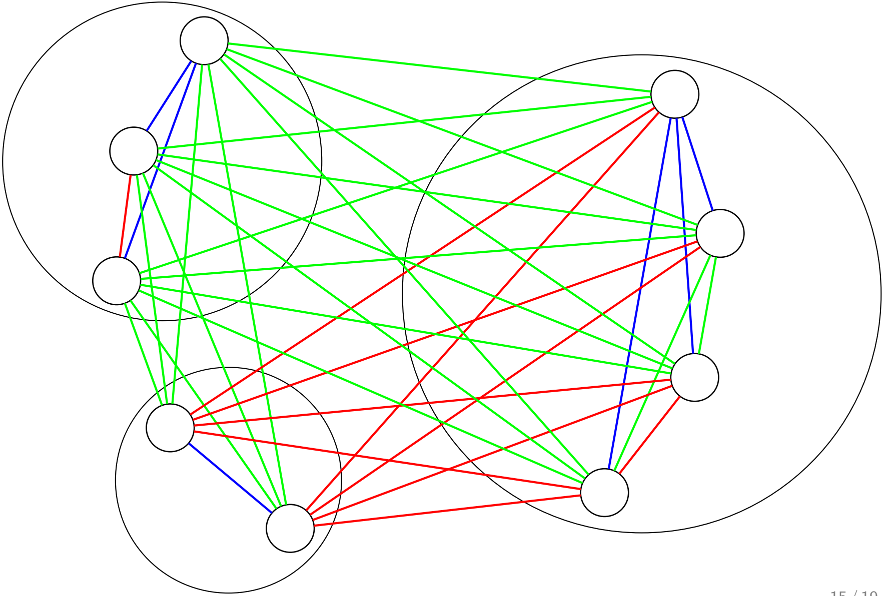

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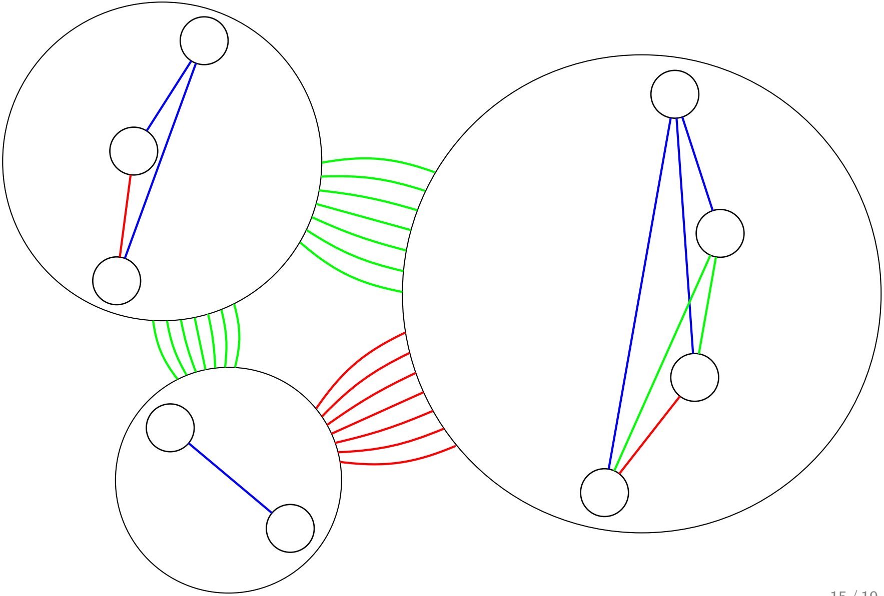

Proof of special case

- Theorem (2016)

  For any K-*free geometric tournament on* N *vertices, there exists an* RGK-, RBK-, *or* GBK-*path of length at least* $N^{2/3}$.

- Proof sketch.

  We prove the result by induction on N. Our tournament has a Gallai decomposition into some set of blocks.
  To find an RB-colored path, all we have to do is find a set of blocks such that all edges between them are RB-colored and find an RB-colored path in each of these blocks.
  We can do the latter by the inductive hypothesis and the former is a problem that only involves two colors, so is easier. □

Weighted Erdős–Szekeres

- Theorem (2016)

  Suppose we are given a 2-coloring of the transitive tournament of size N. Assign each vertex a pair of positive reals (R_i, B_i) and let R be the maximum possible sum of R_i over any R*-colored path. Define B similarly. Then $R \cdot B \geq \sum_{i=1}^{N} R_i \cdot B_i$.*

- Proof sketch.

  It's sufficient to prove for positive integer weights. We construct a 2-coloring of the transitive tournament on $\sum_{i=1}^{N} R_i \cdot B_i$ vertices by blowing up each vertex of our original 2-coloring by a 2-coloring of the transitive tournament on $R_i \cdot B_i$ vertices... □

Bonus: a problem of Erdős documented by Steele

Problem (Erdős 1973)

Given $x_1, \ldots, x_n$ distinct positive real numbers determine $\max_M \sum_{i \in M} x_i$ over all subsets $M \subseteq [n]$ of indices $i_1 < \cdots < i_k$ such that $x_{i_1}, \ldots, x_{i_k}$ is monotone.

Corollary (2016)

The maximum is at least $(\sum x_i^2)^{1/2}$.

Proof.

Construct a transitive RB-tournament on vertices $v_1, \ldots, v_n$, with $v_i \to v_j$ colored R if $x_i < x_j$, and B if $x_i > x_j$. Then monochromatic paths correspond to monotone subsequences, so Weighted Erdős–Szekeres, applied with equal weights (x_i, x_i) at vertex v_i, gives the desired result. □

Thanks for listening!

Special thanks to:

- Ben Yang
- Profs. David Jerison and Ankur Moitra
- Prof. Po-Shen Loh (CMU)
- Dr. Slava Gerovitch
- SPUR and the MIT Math Department