1-color-avoiding paths, special tournaments, and incidence geometry

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Background: Ramsey argument of Erdős–Szekeres

- **Definition**
  The transitive tournament of size $N$ is the directed graph on $N$ vertices numbered 1, $\ldots$, $N$ with a directed edge $v_i \rightarrow v_j$ for each pair $i < j$.

- **Theorem (Cf. Erdős–Szekeres 1935)**
  Any 2-coloring of the edges of the transitive tournament of size $N$ contains a monochromatic directed path of length at least $\sqrt{N}$.
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![Diagram of a transitive tournament with vertices 1 to 5 and directed edges between them. Some edges are red and some are dashed blue, indicating a possible 2-coloring.](image-url)
Proof: Record and pairs problem

Record: assign vertex $i$ the pair of positive integers $(R_i, B_i)$ where $R_i$ (resp. $B_i$) is the length of the longest red (resp. blue) path in the graph that ends at vertex $i$.

Claim

Every vertex is assigned a different ordered pair.

Proof.

Suppose the edge $i \to j$ is red. Then $R_j > R_i$.

Now since each of the $N$ vertices is assigned a distinct ordered pair, at least one must have a coordinate of size at least $\sqrt{N}$. 
Moving on to three colors

Easy generalization: with $k$ colors, longest monochromatic (1-color-\textit{using}) path is $N^{1/k}$, with same proof. Harder question:

- **Question (Loh 2015)**

  Must any 3-coloring of the edges of the transitive tournament of size $N$ have a 1-color-avoiding directed path of length at least $N^{2/3}$?

  - Cannot \textit{guarantee} longer than $\sim N^{2/3}$.
  - “Trivial” lower bound: $N^{1/2}$ from normal Erdős–Szekeres (red-green or blue).
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Must any 3-coloring of the edges of the transitive tournament of size \( N \) have a 1-color-avoiding directed path of length at least \( N^{2/3} \)?

▶ Cannot \textit{guarantee} longer than \( \sim N^{2/3} \).

▶ “Trivial” lower bound: \( N^{1/2} \) from normal Erdős–Szekeres (red-green or blue).

▶ Idea: Record the following lengths: longest blue-\textit{avoiding} path \( x_i = RG_i \), green-\textit{avoiding} path \( y_i = RB_i \), and red-\textit{avoiding} path \( z_i = GB_i \), ending at vertex \( i \).
### Triples problem

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#### Proposition-Definition (Ordered set, Loh 2015)

The list of triples \(L_1 = (x_1, y_1, z_1), \ldots, L_N = (x_N, y_N, z_N)\) is ordered, meaning that for \(i < j\), difference \(L_j - L_i\) has at least 2 positive coordinates.

- Suppose all 1-color-avoiding paths have length at most \(n\), so all coordinates are at most \(n\), so \(L_i \in [n]^3\) for all \(i\).

#### Question (Loh 2015)

Must an ordered set of triples \(S \subseteq [n]^3\) contain at most \(n^{3/2}\) points?

- Would imply \(N^{2/3}\) bound for tournaments question.
- Exist examples with \(\sim n^{3/2}\) points.
- “Trivial” upper bound: at most \(n^2\) points.
Take an ordered set of triples
\[ L_1 = (x_1, y_1, z_1), \ldots, L_N = (x_N, y_N, z_N) \] in \([n]^3\).

Loh 2015: ordered sets are \textit{slice-increasing}: on a coordinate-slice (say \(x\) fixed), the points are increasing in the other two coordinates (i.e. \(y, z\)).

Corollary: for any \(x, y\), there is at most one triple \((x, y, ?)\). This proves the “trivial bound” of \(N \leq n^2\).
Triples in grids: slice-increasing observation

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- Corollary: for any \(x, y\), there is at most one triple \((x, y, ?)\). This proves the “trivial bound” of \(N \leq n^2\).
- \(n \times n\) grid view: for each \(i\), fill in square \((x_i, y_i) \in [n]^2\) with the \(z\)-coordinate \(z_i\). Leave other squares blank.
- Row and column labels are increasing. The squares containing a fixed label \(z\) must be increasing.

\[
\begin{array}{ccc}
3 & 4 & \\
3 & 4 & \\
1 & 2 & \\
1 & 2 & \\
\end{array}
\] (tight example for \(n = 4\); generalizes to large \(n\))
Ordered induced matchings

- Row and column labels are increasing. The squares containing a fixed label $z$ must be increasing.
- Suppose for $i \in [n]$, the label $z = i$ appears $a_i$ times. Goal: bound number of labeled squares, $a_1 + a_2 + \cdots + a_n$.
- Since row and column labels are increasing, the labels $z = i$ form the increasing main diagonal of an otherwise “blocked” $a_i \times a_i$ grid (Loh 2015: “ordered induced matching”).
- Example for $n = 3$. The x’s are “blocked” as part of the grid for $z = 1$; the y’s for $z = 3$. (The x,y squares must be empty.)

<table>
<thead>
<tr>
<th></th>
<th>y</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>1</td>
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<td>xy</td>
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<tbody>
<tr>
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<td></td>
<td></td>
<td>1</td>
</tr>
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<td>3</td>
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- Loh 2015: the “ordered induced matching” property alone is enough to get a bound of $\sim n^2/e^{\log^*(n)}$, but cannot alone beat the bound $\sim n^2/e^{\sqrt{\log(n)}}$ (Behrend construction).
Sum of squares of slice-counts

- Natural to consider $a_i^2$ “blocked” squares.
- Does $a_1^2 + a_2^2 + \cdots + a_n^2 \leq n^2$ always hold?

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Here $a_1^2 + a_2^2 + a_3^2 = 2^2 + 1^2 + 2^2 = 9 = n^2$. 
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- Does $a_1^2 + a_2^2 + \cdots + a_n^2 \leq n^2$ always hold?

\[
\begin{array}{ccc}
2 & y & 3 \\
x & 1 & \\
1 & 3 & xy
\end{array}
\]

Here $a_1^2 + a_2^2 + a_3^2 = 2^2 + 1^2 + 2^2 = 9 = n^2$.

- If one only remembers the slice-increasing condition, then no:

\[
\begin{array}{ccc}
 & 2 & 4 \\
 & 1 & \\
1 & 4 & \\
2 & 4 & \\
 & 1 & 3 \\
1 & 4 & \\
\end{array}
\]

- This example is slice-increasing, but it turns out not to be an ordered set of triples.
Back to tournaments: Color

- Color: given any ordered set of triples $L_1 = (x_1, y_1, z_1), \ldots, L_N = (x_N, y_N, z_N)$, for $i < j$, the difference $L_j - L_i$ has at least two positive coordinates:
  - $(+, +, \leq 0)$
  - $(+, \leq 0, +)$
  - $(\leq 0, +, +)$
  - $(+, +, +)$
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  - \((+, +, \leq 0)\) R
  - \((+, \leq 0, +)\) G
  - \((\leq 0, +, +)\) B
  - \((+, +, +)\) ???
Definition
An RGBK-tournament of size $N$ is a four-coloring of the transitive tournament of size $N$ with colors R, G, B, and K.

- We’ll think of K as a “wild color” and try to find an RGK-, RBK-, or GBK-path of length at least $N^{2/3}$.
- It’s not to hard to show that this is equivalent to the original RGB-tournament problem.
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- Color: given any ordered set of triples $L_1 = (x_1, y_1, z_1), \ldots, L_N = (x_N, y_N, z_N)$, for $i < j$, the difference $L_j - L_i$ has at least two positive coordinates:
  - $(+, +, \leq 0)$ R
  - $(+, \leq 0, +)$ G
  - $(\leq 0, +, +)$ B
  - $(+, +, +)$ K
What we’ve done so far:

- Record reduces the RGBK-tournament problem to the triples problem.
- Color reduces the triples problem to the RGBK-tournament problem.

This means that it is sufficient to prove the result for tournaments in the image of $\text{Color} \circ \text{Record}$. 
Geometric tournaments

Definition
Call an RGBK-tournament geometric if it is the image of some ordered set under Color.

Take a geometric tournament that comes from some ordered set of triples $L_1 = (x_1, y_1, z_1), \ldots, L_N = (x_N, y_N, z_N)$.

- Suppose the edges $v_i \to v_j$ and $v_j \to v_k$ are R-colored.
- This means that $z_i \geq z_j \geq z_k$.
- This in turn implies that the $v_i \to v_k$ is R-colored.

Proposition-Definition (2016)
For a set of colors $C$, a tournament is $C$-transitive if for every $i < j < k$ with $v_i \to v_j$ and $v_j \to v_k$ both $C$-colored, so is $v_i \to v_k$.
Geometric tournaments are exactly the tournaments that are R-, G-, B-, RGK-, RBK-, and GBK-transitive.
Gallai decomposition

- In the special case where a geometric tournament has no K-colored edges, this constraint becomes much simpler.
- A K-free geometric tournament is exactly one which is R-, G-, and B-transitive and has no trichromatic triangles.

Definition
A Gallai 3-coloring of $K_N$ is a 3-coloring of the edges of $K_N$ such that no triangle is trichromatic.

Theorem (Gallai 1967)
For $N \geq 2$, a Gallai 3-coloring of $K_N$ has a base decomposition, meaning a vertex-partition into $m \geq 2$ strictly smaller nonempty graphs $H_1, \ldots, H_m$, where the edges between two distinct blocks $H_i, H_j$ use at most one of the colors R, G, B, and the edges between the various blocks $H_1, \ldots, H_m$ in total use at most two of the colors R, G, B.
Gallai decomposition, cont.
Gallai decomposition, cont.
Gallai decomposition, cont.
Proof of special case

- **Theorem (2016)**

For any $K$-free geometric tournament on $N$ vertices, there exists an $RGK$-, $RBK$-, or $GBK$-path of length at least $N^{2/3}$.

- **Proof sketch.**

We prove the result by induction on $N$. Our tournament has a Gallai decomposition into some set of blocks. To find an RB-colored path, all we have to do is find a set of blocks such that all edges between them are RB-colored and find an RB-colored path in each of these blocks. We can do the latter by the inductive hypothesis and the former is a problem that only involves two colors, so is easier.
Theorem (2016)

Suppose we are given a 2-coloring of the transitive tournament of size \( N \). Assign each vertex a pair of positive reals \((R_i, B_i)\) and let \( R \) be the maximum possible sum of \( R_i \) over any \( R \)-colored path. Define \( B \) similarly. Then \( R \cdot B \geq \sum_{i=1}^{N} R_i \cdot B_i \).

Proof sketch.

It’s sufficient to prove for positive integer weights. We construct a 2-coloring of the transitive tournament on \( \sum_{i=1}^{N} R_i \cdot B_i \) vertices by blowing up each vertex of our original 2-coloring by a 2-coloring of the transitive tournament on \( R_i \cdot B_i \) vertices...
Problem (Erdős 1973)

Given \( x_1, \ldots, x_n \) distinct positive real numbers determine \( \max_M \sum_{i \in M} x_i \) over all subsets \( M \subseteq [n] \) of indices \( i_1 < \cdots < i_k \) such that \( x_{i_1}, \ldots, x_{i_k} \) is monotone.

Corollary (2016)

The maximum is at least \( \left( \sum x_i^2 \right)^{1/2} \).

Proof.

Construct a transitive RB-tournament on vertices \( v_1, \ldots, v_n \), with \( v_i \rightarrow v_j \) colored R if \( x_i < x_j \), and B if \( x_i > x_j \). Then monochromatic paths correspond to monotone subsequences, so Weighted Erdős–Szekeres, applied with equal weights \( (x_i, x_i) \) at vertex \( v_i \), gives the desired result. \( \square \)
Thanks for listening!

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