MOP 2018: COMPLEX NUMBERS AND GEOMETRY (06/11, K)

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Abstract. We seek to complement well-known existing resources on complex numbers in geometry, such as the exceptional online notes of Marko Radovanović and Yi Sun.

1. Complex yoga

Problem 1.1. Relate Möbius transformations, cross ratios on $\mathbb{CP}^1$, and generalized circles.

Problem 1.2 (Arc midpoint parameterization)**.** Let $A_1A_2\dots A_n$ be a cyclic n-gon on the unit circle with $n \geq 3$. Let $B_{i,i+1}$ be the midpoint of arc A_iA_{i+1}. If n is *odd*, show that there are exactly two tuples $(u_1,\dots,u_n)$ of phases such that $u_i^2 = a_i$ and $b_{i,i+1} = -u_iu_{i+1}$ for all i. What needs to be changed for n *even*?

Problem 1.3 (How to remember the intersection formula)**.** Let A, B, C, D lie on the unit circle. Take $B \to A$ and $D \to C$ in the intersection formula

$$AB \cap CD = \frac{ab(c+d) - cd(a+b)}{ab - cd}$$

to show that $AA \cap CC = \frac{2ac}{a+c}$. Also, what does $ab - cd = 0$ mean geometrically?

Problem 1.4 (ESL 2013 G6: "Not like a G6")**.** Let $ABCDEF$ be a non-degenerate cyclic hexagon with no two opposite sides parallel, and define $X = AB \cap DE$, $Y = BC \cap EF$, and $Z = CD \cap FA$. Prove that $XY : XZ = BE \sin|\angle B - \angle E| : AD \sin|\angle A - \angle D|$.

Problem 1.5 ("Diagonal polynomial" dependence; inspired by ISL 2007 G3)**.** Let $ABCDEF$ be a complete quadrilateral (formed by lines ACE, BDE, BCF, ADF) in the complex plane. Prove that $(z-a)(z-b)$, $(z-c)(z-d)$, and $(z-e)(z-f)$ are $\mathbb{R}$-linearly dependent in $\mathbb{C}[z]$. As an easy corollary, the midpoints of the diagonals of $ABCDEF$ are collinear.

Problem 1.6 (MOP 2007)**.** If $z \in \mathbb{C} \setminus \mathbb{R}$ but $(z^{2n+1} - 1)/(z^{2n} - z) \in \mathbb{R}$, show that $|z| = 1$.

2. Complex analysis

Problem 2.1. Let $f(x) = x^3 - ax^2 + bx - c$ be a cubic with a, b fixed positive reals and c a real variable. If f is maximal in c among f with all real roots, show that $\operatorname{Disc} f = 0$.

Problem 2.2 (ROM TST 2004/4, $n = 2$ case)**.** Let D be a closed disk in $\mathbb{C}$. If $z_1, z_2 \in D$, prove that there exists $z \in D$ such that $z^2 = z_1z_2$.

Problem 2.3. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a *polynomial* function.

(1) Prove that $f'(z_0) = 0$ at a point $z_0 \in \mathbb{C}$ if and only if all directional derivatives of $|f(z)|$ at z_0 vanish.
(2) Using a higher-order Taylor version of the previous first-order result, prove that f has a zero, thus establishing the *fundamental theorem of algebra*.
(3) Using the first-order result, give a geometric proof of the *Gauss–Lucas theorem*.

Question 2.4 (Asked by K in Polynomials class; see MathOverflow answer)**.** Can the intersection of the convex hulls of the level sets $\{z \in \mathbb{C} : f(z) = c\}$ of a polynomial $f\colon \mathbb{C} \to \mathbb{C}$, as c varies over $\mathbb{C}$, be strictly larger than the convex hull of $\{z \in \mathbb{C} : f'(z) = 0\}$?

Problem 2.5. A function f is called *harmonic* at a if $f(a) = \operatorname{Avg}_{|z-a|=r} f(z)$ for all disks $|z-a| \le r$ in the domain of f.

(1) Prove that $\arg z$ is harmonic in, say, the upper half plane $\Im z > 0$.
(2) Describe, geometrically, a harmonic function that is 0 on the diameter $(-1, 1)$ of the upper unit semicircle, and 1 on the upper half-circumference $\{|z| = 1\} \cap \{\Im z > 0\}$.
(3) Prove that $\log|z|$ is harmonic in, say, the upper half plane $\Im z > 0$.
(4) Modify the $=$ in the definition of harmonicity to either $\le$ or $\ge$, so that $\log|z|$ satisfies the inequality even in disks containing the origin.

3. Math154's favorite Olympiad transformation

Problem 3.1. Let P be outside the unit circle. Let a line ℓ through P intersect the unit circle at A, B. Let A', B' be the *reflections* of A, B over line OP.

(1) Compute P in terms of A, B', and the intersections $X, -X$ of OP with the unit circle.
(2) How are $A, A', X, -X$ related?
(3) Derive the formula for the intersection of two tangents to the unit circle.

Problem 3.2 (ARO 2011/11-8)**.** Let N be the midpoint of arc ABC on the circumcircle of triangle ABC. Let M be the midpoint of side AC, and let I_1, I_2 be the incenters of triangles ABM and CBM, respectively. Prove that points I_1, I_2, B, N lie on a circle.

4. When complex fails...

Problem 4.1. ...reorient yourself and complete the quote.

Problem 4.2 (IMO 2012/5)**.** Let ABC be a triangle with $\angle C = 90^\circ$, and let D be the foot of the C-altitude. Let X be a point in the interior of segment CD. Let K be the point on segment AX such that $BK = BC$. Similarly, let L be the point on segment BX such that $AL = AC$. Let M be the point of intersection of AL and BK. Show that $MK = ML$.

Problem 4.3 (USA Dec TST for IMO 2013: IMO 2012/5 Mockup)**.** Let ABC be a scalene triangle with $\angle C = 90^\circ$, and let D be the foot of the C-altitude. Define X, K, L as in IMO 2012/5. The circumcircle of triangle DKL intersects segment AB at a second point T (other than D). Prove that $\angle ACT = \angle BCT$.

5. What would you do?

Problem 5.1 (USA Jan TST for IMO 2012)**.** In cyclic quadrilateral $ABCD$, diagonals AC and BD intersect at P. Let E and F be the respective feet of the perpendiculars from P to lines AB and CD. Segments BF and CE meet at Q. Prove that lines PQ and EF are perpendicular to each other.

Problem 5.2 (IMO 2013/3)**.** Let the A-excircle of triangle ABC be tangent to side BC at point A_1. Define points B_1 on CA and C_1 on AB analogously. Suppose that the circumcenter of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that $\angle A = 90^\circ$.