1. Linear spaces, relationships, building blocks, and size

**Definition 1.1** (Scalars: field, ring, noncommutative versions). Common notation: $F$ or $k$ for field ($k$ from German), $R$ for ring, $A$ for algebra, and $D$ for division ring.

**Definition 1.2** (Linear spaces: vector space, module, vectors, subspaces, quotients). Linearity is relative: it means linear over given scalars or coefficients.

**Definition 1.3** (Functions: linear maps, kernel, image, ⋆jections, isomorphisms).

**Definition 1.4** (Internal relationships: span, generation, dependence, independence).

**Definition 1.5** (Building blocks: direct sums, internal vs. abstract, complementary subspace, free module, basis). The symbol for direct sum is $\oplus$.

**Theorem 1.6** (Gaussian elimination). Let $V$ be a finitely-generated vector space. Then every subspace has a complement; hence $V$ is free by induction. Furthermore, we can define the size, i.e. dimension, of $V$, and show that it’s well-defined, independent of any choices.

**Problem 1.7** (Earthquake). Slightly perturb (“move” or “change”) the standard basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$. Prove that the $n$ perturbed vectors are still linearly independent.

**Problem 1.8** (Radon’s theorem). Prove that any set of $d + 2$ points in $\mathbb{R}^d$ can be split into two disjoint sets with intersecting convex hulls. Is $d + 2$ sharp?

**Problem 1.9** (Polynomial Thue). Let $F$ be a field, $M \in F[x]$ a non-zero polynomial of degree $d \geq 0$, and $C \in F[x]$ a polynomial relatively prime to $M$. Prove that there exist $A, B \in F[x]$ of degree at most $\frac{d}{2}$ such that $\frac{A(x) - C(x)B(x)}{M(x)} \in F[x]$.

**Problem 1.10** (Implicitization problem). Prove that two univariate polynomials $f(T), g(T)$ over a field $k$ are algebraically dependent. What if $f, g$ are replaced with rational maps?

**Problem 1.11**. Let $L: V \to W$ be a $k$-linear map of finite-dimensional $k$-vector spaces. Identify $L$ with a $\dim_k W \times \dim_k V$ matrix over $k$. What choices does it depend on?

**Theorem 1.12** (Fundamental theorem of linear algebra). With $L$ as above, find the sum of rank $L := \dim \ker L$ and null $L := \dim \ker L$. Can you classify linear maps $V \to W$?

**Question 1.13**. Should the fundamental theorem of linear algebra really be earlier above?

**Problem 1.14** (Heard from Allen Liu). Let $A$ be a (rectangular) matrix with all entries 0 or 1. If $p$ is a prime such that $p > 4(\text{rank}_{\mathbb{F}_p} A)^2$, then show that $\text{rank}_{\mathbb{Q}} A < 2 \text{rank}_{\mathbb{F}_p} A$. 


2. Linear operators: self-maps of a space

A \( k \)-linear \textbf{endomorphism} (self-map) \( T : V \to V \) of a single given \( k \)-vector space \( V \) is often called a \textbf{linear operator}. The \textit{identity} operator \( 1 : V \to V \) (or \( I \)) is defined by \( 1v = v \).

An operator \( T \) \textit{acts} or \textit{operates} on \( V \), making \( V \) a discrete \textbf{dynamical system}. People love to \textit{iterate} and study \( T^n \). For \( n = 0 \), the correct polynomial definition is \( T^0 = 1 \).

**Theorem 2.1.** Assume \( \dim_k V < \infty \). Prove that \( T \) is algebraic over \( k \), meaning \( p(T) = 0 \) for some polynomial \( p \in k[x] \).

**Definition 2.2** (Building blocks: \textbf{invariant} direct sums, indecomposable subspaces).

Any indecomposable \( T \)-invariant space \( V \) is “\( T \)-isomorphic” to \( k[X]/(f) \) for some prime power \( f \in k[X] \), where \( T \) acts on \( k[X]/(f) \) as multiplication by \( X \). What does this imply if \( V \) is decomposable? What if \( k \) is algebraically closed, like \( \mathbb{C} \)?

**Problem 2.4** (Real orthogonal representations of \( \mathbb{Z}/4 \)). Let \( T \) denote multiplication by \( i = \sqrt{-1} \) on \( V = \mathbb{C} = \mathbb{R} \oplus i\mathbb{R} \), with \( F = \mathbb{R} \). Work out the rational canonical form for \( V \). Turn all this on its head to define \( \mathbb{C} \) using 2 \( \times \) 2 matrices over \( \mathbb{R} \).

**Problem 2.5** (Complex unitary representations of \( Q_8 \)). Let \( H = \mathbb{C} \oplus j\mathbb{C} \) be viewed as a right \( \mathbb{C} \)-vector space. Let \( I, J, K \) denote \textit{left} multiplication by \( i, j, k \), respectively. Extend the previous problem to define \( H \) using 2 \( \times \) 2 matrices over \( \mathbb{C} \).

**Definition 2.6** (Volume elements \textbf{coordinate-free determinant}). Assume \( \dim_k V < \infty \).

**Theorem 2.7.** \( T \) is invertible if and only if \( T \) is \( * \)-jective; if and only if \( \det(T) \neq 0 \). For \( \lambda \in k \) a constant, \( Tv = \lambda v \) has a \textbf{nonzero solution} \( v \in V \setminus \{0\} \) if and only if \( \det(\lambda I - T) \neq 0 \).

**Problem 2.8.** Let \( A = \mathbb{Z}[\sqrt{-5}] \) and \( I = (2, 1 + \sqrt{-5}) \) be the \textbf{ideal} consisting of \( A \)-linear combinations of \( 2, 1 + \sqrt{-5} \). Prove that \( I \) is not free over \( A \), but \( I \oplus I \) is.

**Theorem 2.9** (Cayley–Hamilton theorem). Assume \( \dim V < \infty \). Then \( T \) is a “root” of the polynomial \( p(x) = \det(xI - T) \), meaning \( p(T) = 0 \).

**Problem 2.10** (Observed by Noam Elkies). Use the Cayley–Hamilton theorem, when \( \dim V = 2 \), to compute the inverse of a 2 \( \times \) 2 matrix.

3. Projectivization

**Problem 3.1** (MOP 2010 first (and only) problem set). Let \( f(x) = (ax + b)/(cx + d) \) where \( c, d \) are not both zero. If \( f(x) = x \) has no real solutions but \( f^{2010}(x) = x \) has at least one real solution, show that \( f^{2010}(x) = x \) for all real \( x \) where \( f^{2010}(x) \) is defined.

4. Bilinear forms and inner products: geometries on a space

**Problem 4.1** (Robi Bhattacharjee; inspired by sphere packing?). Let \( e_1, \ldots, e_n \) be the standard basis of \( \mathbb{R}^n \). Let \( V \) be a subspace of \( \mathbb{R}^n \), and for each \( i \), choose \( v_i \in V \) such that the distance between \( v_i \) and \( e_i \) is as small as possible. Must \( v_1, v_2, \ldots, v_n \) span all of \( V \)?

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1For example, average behavior over time might “equidistribute” as in Weyl’s equidistribution theorem.

2But while \( \mathbb{H} = \langle I, J, K \rangle \mathbb{C} \) on the right, we only have \( \mathbb{H} = \mathbb{R}(I, J, K) \) on the left.
Problem 4.2 (Asked on MathOverflow). Fix \( a \in \mathbb{Z}^n \) with \( \gcd(a_1, \ldots, a_n) = 1 \), and let \( v_1, \ldots, v_{n-1} \) be an integer basis for the lattice \( L = \{ x \in \mathbb{Z}^n : a \cdot x = 0 \} \). Show that the fundamental parallelepiped \( P = \{ t_1 v_1 + \cdots + t_{n-1} v_{n-1} : t_i \in [0, 1) \} \) has volume equal to \( \|a\| \).

Theorem 4.3 (Euler’s rotation theorem). If \( T \) acts linearly on \( \mathbb{R}^3 \) with \( \langle Tv, Tw \rangle = \langle v, w \rangle \) for all \( v, w \in \mathbb{R}^3 \), then \( \det T \) is an eigenvalue of \( T \).

Theorem 4.4 (Frobenius). Up to isomorphism over \( \mathbb{R} \), the only finite-dimensional associative \( \mathbb{R} \)-algebras with division are \( \mathbb{R}, \mathbb{C}, \) and \( \mathbb{H} \).