MOP 2018: LINEAR ALGEBRA (06/04, K)

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1. Linear spaces, relationships, building blocks, and size

Definition 1.1 (Scalars: field, ring, noncommutative versions)**.** Common notation: F or k for field (k from German), R for ring, A for algebra, and D for division ring.

Definition 1.2 (Linear spaces: vector space, module, vectors, subspaces, quotients)**.** *Linearity* is relative: it means *linear over* given scalars or coefficients.

Definition 1.3 (Functions: linear maps, kernel, image, $\star$jections, isomorphisms)**.**

Definition 1.4 (Internal relationships: span, generation, dependence, independence)**.**

Definition 1.5 (Building blocks: direct sums, internal vs. abstract, complementary subspace, free module, basis)**.** The symbol for direct sum is $\oplus$.

Theorem 1.6 (Gaussian elimination)**.** *Let V be a finitely-generated vector space. Then every subspace has a complement; hence V is free by induction. Furthermore, we can define the size, i.e.* dimension*, of V, and show that it's well-defined, independent of any choices.*

Problem 1.7 (Earthquake)**.** Slightly *perturb* ("move" or "change") the standard basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$. Prove that the n perturbed vectors are still linearly independent.

Problem 1.8 (Radon's theorem)**.** Prove that any set of $d+2$ points in $\mathbb{R}^d$ can be split into two disjoint sets with intersecting convex hulls. Is $d+2$ sharp?

Problem 1.9 (Polynomial Thue: USA TSTST 2014/4)**.** Let F be a field, $M \in F[x]$ a nonzero polynomial of degree $d \geq 0$, and $C \in F[x]$ a polynomial relatively prime to M. Prove that there exist $A, B \in F[x]$ of degree at most $\frac{d}{2}$ such that $\frac{A(x)-C(x)B(x)}{M(x)} \in F[x]$.

Problem 1.10 (Implicitization problem)**.** Prove that two univariate polynomials $f(T), g(T)$ over a field k are algebraically dependent. What if f, g are replaced with rational maps?

Problem 1.11. Let $L\colon V \to W$ be a k-linear map of finite-dimensional k-vector spaces. Identify L with a $\dim_k W \times \dim_k V$ matrix over k. What choices does it depend on?

Theorem 1.12 (Fundamental theorem of linear algebra)**.** *With L as above, find the sum of* $\operatorname{rank} L := \dim \operatorname{im} L$ *and* $\operatorname{null} L := \dim \ker L$*. Can you classify linear maps $V \to W$?*

Question 1.13. Should the fundamental theorem of linear algebra really be earlier above?

Problem 1.14 (Heard from Allen Liu)**.** Let A be a (rectangular) matrix with all entries 0 or 1. If p is a prime such that $p > 4(\operatorname{rank}_{\mathbb{F}_p} A)^2$, then show that $\operatorname{rank}_{\mathbb{Q}} A < 2 \operatorname{rank}_{\mathbb{F}_p} A$.

2. Linear operators: self-maps of a space

A k-linear *endomorphism* (self-map) $T: V \to V$ of a single given k-vector space V is often called a linear *operator*. The *identity* operator $\mathbf{1}: V \to V$ (or I) is defined by $\mathbf{1}v = v$.

An operator T *acts* or *operates* on V, making V a discrete *dynamical system*. People love to *iterate* and study T^n.[1] For $n = 0$, the correct polynomial definition is $T^0 = \mathbf{1}$.

Theorem 2.1. *Assume* $\dim_k V < \infty$. *Prove that T is* algebraic *over k, meaning $p(T) = 0$ for some polynomial $p \in k[x]$.*

Definition 2.2 (Building blocks: invariant direct sums, indecomposable subspaces)**.**

Theorem 2.3 (Rational canonical form)**.** *Let T act on a finite-dimensional vector space V over k. Any indecomposable T-invariant space V is "T-isomorphic" to $k[X]/(f)$ for some* prime power *$f \in k[X]$, where T acts on $k[X]/(f)$ as multiplication by X. What does this imply if V is decomposable? What if k is algebraically closed, like $\mathbb{C}$?*

Problem 2.4 (Real orthogonal representations of $\mathbb{Z}/4$)**.** Let T denote multiplication by $i = \sqrt{-1}$ on $V = \mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$, with $F = \mathbb{R}$. Work out the rational canonical form for V. Turn all this on its head to *define* $\mathbb{C}$ using 2×2 matrices over $\mathbb{R}$.

Problem 2.5 (Complex unitary representations of Q_8)**.** Let $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$ be viewed as a *right* $\mathbb{C}$-vector space. Let I, J, K denote *left* multiplication by i, j, k, respectively. Extend the previous problem to define $\mathbb{H}$ using 2×2 matrices over $\mathbb{C}$.[2]

Definition 2.6 (Volume elements, coordinate-free determinant)**.** Assume $\dim_k V < \infty$.

Theorem 2.7. *T is* invertible *if and only if T is $\star$-jective; if and only if* $\det(T) \neq 0$. *For $\lambda \in k$ a constant, $Tv = \lambda v$ has a* nonzero *solution $v \in V \setminus \{0\}$ if and only if* $\det(\lambda I - T) \neq 0$.

Problem 2.8. Let $A = \mathbb{Z}[\sqrt{-5}]$ and $I = (2, 1 + \sqrt{-5})$ be the *ideal* consisting of A-linear combinations of $2, 1 + \sqrt{-5}$. Prove that I is not free over A, but $I \oplus I$ is.

Theorem 2.9 (Cayley–Hamilton theorem)**.** *Assume* $\dim V < \infty$. *Then T is a "root" of the polynomial $p(x) = \det(xI - T)$, meaning $p(T) = 0$.*

Problem 2.10 (Observed by Noam Elkies)**.** Use the Cayley–Hamilton theorem, when $\dim V = 2$, to compute the inverse of a 2×2 matrix.

3. Projectivization

Problem 3.1 (MOP 2010, first (and only) problem set)**.** Let $f(x) = (ax+b)/(cx+d)$ where c, d are not both zero. If $f(x) = x$ has no real solutions but $f^{2010}(x) = x$ has at least one real solution, show that $f^{2010}(x) = x$ for all real x where $f^{2010}(x)$ is defined.

4. Bilinear forms and inner products: geometries on a space

Problem 4.1 (Robi Bhattacharjee, inspired by sphere packing?)**.** Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Let V be a subspace of $\mathbb{R}^n$, and for each i, choose $v_i \in V$ such that the distance between v_i and e_i is as small as possible. Must $v_1, v_2, \ldots, v_n$ span all of V?

[1] For example, average behavior over time might "equidistribute" as in Weyl's equidistribution theorem.

[2] But while $\mathbb{H} = \langle I, J, K \rangle \mathbb{C}$ on the right, we only have $\mathbb{H} = \mathbb{R}\langle I, J, K \rangle$ on the left.

Problem 4.2 (Asked on MathOverflow)**.** Fix $\mathbf{a} \in \mathbb{Z}^n$ with $\gcd(a_1, \dots, a_n) = 1$, and let $v_1, \dots, v_{n-1}$ be an integer basis for the lattice $L = \{\mathbf{x} \in \mathbb{Z}^n : \mathbf{a} \cdot \mathbf{x} = 0\}$. Show that the fundamental parallelotope $P = \{t_1 v_1 + \cdots + t_{n-1} v_{n-1} : t_i \in [0, 1)\}$ has volume equal to $\|\mathbf{a}\|$.

Theorem 4.3 (Euler's rotation theorem)**.** *If T acts linearly on $\mathbb{R}^3$ with $\langle Tv, Tw \rangle = \langle v, w \rangle$ for all $v, w \in \mathbb{R}^3$, then $\det T$ is an eigenvalue of T.*

Theorem 4.4 (Frobenius)**.** *Up to isomorphism over $\mathbb{R}$, the only finite-dimensional associative $\mathbb{R}$-algebras* with division *are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$.*