COHEN–LENSTRA HEURISTICS: INFORMAL NOTES

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Abstract. Following Smith [8], except in some of the definitions, details, and appendices. I’m not sure yet where things break down for real quadratic fields.

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1. Introduction

Roughly speaking, the algebraic input relies on three principles:

(1) Linear algebra and combinatorics over $\mathbb{F}_2$, including (but not limited to) the torsion class pairing below, and the notion of minimality (Definition 4.12 and Appendix D).\footnote{As Bjorn Poonen once said, “Your success in life is determined by how much linear algebra you know.”}

(2) Class field theory, including (but not limited to) representing class group characters of $K$ using Galois subextensions of $H_K/K$, which are actually Galois over $\mathbb{Q}$; and also calculating local Artin symbols. (See Propositions 2.1 and 2.5.)

(3) “Dihedral-like” Galois extensions (with restricted ramification) over $\mathbb{Q}$ can be “parameterized” using suitable $\mathbb{Q}$-cocycles. (See Propositions 3.3 and 3.7.)

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The third might be the most significant, because \( \mathbb{Q} \)-cocycles can be easily manipulated as we vary the quadratic field data encoded in “dihedral-like” groups. It can be motivated in at least two ways: extending characters over \( K \) to cocycles over \( \mathbb{Q} \) (surjectivity of inflation-restriction map), or the class-Selmer analogy (where Selmer groups are already defined over \( \mathbb{Q} \)). The challenge is then finding nontrivial relations in families: Theorems 4.15 and TBD.

2. Computing the torsion class pairing

The first principle extracts \( 2^{k+1} \)-ranks from the torsion class pairing. The left kernel of

\[
\text{Cl}[2] \times \text{Cl}[2] \xrightarrow{(x,\psi)\mapsto \psi(x)} \mu_2 = \pm 1
\]

is \( 2 \text{Cl}[4] \) (giving the 4-rank of Cl), because \( \psi(x)^2 = \psi(x^2) \) (left kernel detects whether \( x \) is a square). Generally, to find the \( 2^{k+1} \)-rank using \( \mathbb{F}_2 \)-linear algebra, consider the pairing

\[
2^{k-1} \text{Cl}[2^k] \times 2^{k-1} \text{Cl}[2^k] \xrightarrow{(2^{k-1}x,2^{k-1}\psi)\mapsto 2^{k-1}\psi(x)} \pm 1
\]

(easily check well-defined) with left kernel \( 2^k \text{Cl}[2^{k+1}] \). This should all be classical.

2.1. Global computation: general case. Fix \( k \geq 1 \) and let \( K/\mathbb{Q} \) be any number field.

**Proposition 2.1.** For \( u \in 2^{k-1}\text{Cl}_K[2^k] \) and \( v \in 2^{k-1}\text{Cl}_K[2^k] \), the above natural pairing

\[
2^{k-1}\text{Cl}_K[2^k] \times 2^{k-1}\text{Cl}_K[2^k] \rightarrow \mu_2
\]

is given by the Artin symbol formula

\[
\langle u, v \rangle := \psi_k(v) = \text{rec}_{L/K}(v) \in \text{Gal}(L/K)[2] \rightarrow \mu_2,
\]

where we have chosen \( \psi_k \in \text{Cl}_K[2^k] \) with \( u = 2^{k-1}\psi_k \), and where \( L = L(\psi_k) := H_K^{\text{rec}(\ker \psi_k)} \) is the fixed field of \( \ker \psi_k \leq \text{Cl}_K \) acting on the Hilbert class field \( H_K/K \).

*Remark 2.2.* Implicit in the identification \( \text{Gal}(L/K)[2] \rightarrow \mathbb{F}_2 \) (usually an isomorphism, unless \( \psi_k \) is the trivial character) is the fact that \( L/K \) is cyclic of order \( \# \text{im} \psi_k \mid 2^k \).

**Proof.** Use global Hilbert class field theory. The composite map

\[
\phi_k : \text{Gal}(H_K/K) \xrightarrow{\text{rec}^{-1} : \cong} \text{Cl}_K \xrightarrow{\psi_k} \mu_{2^k}
\]

has kernel \( \text{Gal}(H_K/L) \) (by definition of \( L/K \)), so it induces an injection of quotients:

\[
\phi_k : \text{Gal}(L/K) \xrightarrow{\text{rec}^{-1}_{L/K} : \cong} \text{Cl}_K / \ker \psi_k \rightarrow \mu_{2^k}.
\]

In particular, \( L/K \) is cyclic, and \( \phi_k(\text{rec}_{L/K}(v)) = \psi_k(v) \in 2^{k-1}\mu_{2^k} = \mu_2 \), so \( \langle u, v \rangle := \psi_k(v) \in \mu_2 \) has the same order as \( \text{rec}_{L/K}(v) \in \text{Gal}(L/K)[2] \rightarrow \mu_2 \). Now \( \text{Aut}(\mu_2 \cong \mathbb{Z}/2) = 1 \) allows the desired identification \( \psi_k(v) = \text{rec}_{L/K}(v) \in \mu_2 \). \( \square \)

*Remark 2.3.* The permissible fields \( L/K \) are precisely the degree \( 2 \leq k \) cyclic unramified extensions over \( K \) containing \( H_K^{\text{ker}u} \) (an unramified quadratic extension of \( K \); these have been studied more explicitly in classical genus theory and subsequent work).

**Proposition 2.4.** Every unramified \( 2 \)-abelian extension of a quadratic field \( K/\mathbb{Q} \) is Galois.

**Proof.** \( H_K^+/\mathbb{Q} \) is Galois by maximality. To prove that every subgroup of \( \text{Gal}(H_K^+/K) \) is normal in \( \text{Gal}(H_K^+/\mathbb{Q}) \), use Artin reciprocity and the fact that \( \sigma(I)I \in P_K^+ \) for \( I \in I_K \). \( \square \)
2.2. Local computation: quadratic case. Now specialize and simplify locally.

Proposition 2.5. In the notation of Proposition 2.4, if $K/Q$ is quadratic, then

1. $L/Q$ is dihedral Galois, and
2. for any prime $p \mid \Delta_K$ with $pO_K = p^2$, every decomposition field over $Q_p$ is abelian
   with Galois group $C_2$ or $C_2^2$. The character
   \[
   \psi \text{rec}^{-1}_{L/K} : \text{Gal}(L/K) \to \mu_2
   \]
   restricts on the abelian decomposition group $D_p$ to
   \[
   \chi|_{G_{K_p}} : D_p \to \mu_2
   \]
   for some local unramified quadratic or trivial character $\chi : G_{Q_p} \to \mu_2$ over $Q_p$.

Furthermore, $\text{rec}_{L/K}(p) = (\chi, b)_p = \text{inv}_p(\chi \cup \chi_b)$ for any uniformizer $b$ of $Q_p$.

Remark 2.6. Here $(\chi, b)_p$ is understood to mean $\text{Disc}_{Q_p}[\chi]$ of ideal classes in $\mathcal{O}_K$.

Proof that $L/Q$ is dihedral Galois. Since $L/K$ is a cyclic unramified extension of $K$, it is
Galois over $Q$ by Proposition 2.4. To prove $L/Q$ dihedral, use Artin reciprocity together
with the fact that $\text{Gal}(K/Q)$ inverts ideal classes in $\mathcal{O}_K$.

Proof of local restriction. Given $L/K$, choose primes $q/p/p$ with $p \mid \Delta_K$ (i.e. $p$ ramified
in $K$, so $pO_K = p^2$). Note that $L_q/K_p$ is (cyclic and) unramified. On the other hand, if
$F$ denotes the maximal unramified sub-extension of $L$ over $Q_p$ (so $F/Q_p$ is cyclic) then
$L_q/F$ is totally ramified by local structure theory. So $L_q/FK_p$ is both unramified and
totally ramified, hence trivial. Furthermore, $F$ and $K_p$ must be linearly disjoint over $Q_p$, so
$L_q/Q_p$ is abelian with Galois group $\text{Gal}(F/Q_p) \times \text{Gal}(K_p/Q_p)$. But $F$ must then be at
most quadratic, because $C_2^2$ (Klein four group) is the only non-cyclic abelian subgroup
of $\text{Gal}(L/Q)$, the dihedral group of size $2[L:K]$.

Now
\[
D_p := \text{Gal}(L_q/K_p) \xrightarrow{\text{res}} \text{Gal}(F/Q_p)
\]
is at most order two. Yet $F/Q_p$ is unramified by definition, so $\psi \text{rec}^{-1}_{L/K}$ indeed restricts on
$D_p$ to a local character $\chi|_{G_{K_p}}$, with $\chi$ defined over $F/Q_p$ with the desired properties.

Proof of Artin symbol calculation. Fix $b \in Q_p$ with $v_p(b) = 1$, so $b/p$, a unit, must be a norm
in $F/Q_p$ (an unramified local extension). Then $\text{rec}_{L/K}(p)$ is trivial if and only if $L_q = K_p$
if and only if $F = Q_p$ if and only if $p \in N_{F}(F^*)$ if and only if $b \in N_{F}(F^*)$ if and only if
$\text{inv}_p(\chi \cup \chi_b) = 0$. But $\text{rec}_{L/K}(p)$ (killed by squaring) and $\text{inv}_p(\chi \cup \chi_b) = (\chi, b)_p$ (a quadratic
Hilbert symbol\footnote{Better proof of $[L_q : K_p] \leq 2$ using Artin reciprocity: $p$ must split into at most $[L : K]/2$ primes (so each/the decomposition group has size at most 2) because it has order at most 2 in $I_K/P_KN_{L/K}(L^*)$.}) are both $\mathbb{Z}/2$-valued, so they must coincide.

3. Relating characters and cocycles

Definition 3.1. Let $K/Q$ be quadratic with character
\[
\delta_K : G_Q \to \text{Gal}(K/Q) = 1 \in \text{End}(\mathbb{Z}/2),
\]
and set $M_K := \mathbb{Q}_2/\mathbb{Z}_2$ with Galois action $g \in G_Q$. Let $\iota_K : M_K \to \mathbb{Q}_2/\mathbb{Z}_2$
be the forgetful map (an identity of abelian groups).
3.1. **Global extension of characters.** We now use additive notation for characters.

**Proposition 3.3 (Cf. Proposition 2.7).** Define \( M = M_K \) as above. The cocycle group
\[
\overline{\text{Cl}}_K[2^k] := Z^1_{\text{cts}}(\text{Gal}(K^{ur}/\mathbb{Q}), M[2^k])
\]
surjects onto \( \hat{\text{Cl}}_K[2^k] = \text{Hom}_{\text{cts}}(\text{Gal}(K^{ur}/K), 2^{-k}\mathbb{Z}/\mathbb{Z}) \), via restriction of cocycles. Consequently, for \( k \geq 1 \), the image of \( 2^{k-1}\overline{\text{Cl}}_K[2^k] \) under \( \overline{\text{Cl}}_K[2] \to \hat{\text{Cl}}_K[2] \) is \( 2^{k-1}\hat{\text{Cl}}_K[2^k] \).

**Remark 3.4.** Smith explicitly extends \( K \)-characters to \( \mathbb{Q} \)-objects. It may be instructive to work this out later. See crossed homomorphism or MSE: motivating inhomogeneous cochains (esp. Mariano answer about section interpretation) for inspiration.

Here is another perspective.

**Proof.** For \( 1 \leq k \leq \infty \), consider the inflation-restriction exact sequence
\[
0 \to H^1(G/N, M[2^k]_N) \xrightarrow{\inf} H^1(G, M[2^k]) \xrightarrow{\text{res}} H^1(N, M[2^k])^G/N \to H^2(G/N, M[2^k]_N) \xrightarrow{\inf} H^2(G, M[2^k])
\]
with \( G = \text{Gal}(H_K/Q) \) and \( N = \text{Gal}(H_K/K) \cong \text{Cl}_K \). Since \( N \) and \( G/N \) act trivially on \( M \) and \( H^1(N, M[2^k]) \), resp., and \( G/N = \text{Gal}(K/Q) = \pm 1 \) is cyclic, the sequence simplifies to
\[
0 \to M[2^k]/2M[2^k] \to H^1(G, M[2^k]) \to \text{Hom}(N, M[2^k]) \to M[2] \to H^2(G, M[2^k]).
\]
One can abstractly conclude \( 2^{k-1}H^1(G, M[2^k]) \to \text{Hom}(N, M[2^k]) \) for \( k \geq 2 \) (this is also true for \( k = 1 \): any quadratic character on \( N \) lifts to a Klein four character on \( G \)), but in fact, Smith explicitly proves that the restriction map is surjective for \( M[2^k] \) \footnote{It should also be possible to show (through a computation likely boiling down to Smith’s argument) that the transgression (boundary) map \cite[Proposition 1.6.6, p. 65]{smith} is zero.}

**Remark 3.5.** To see why \( G/N \) acts trivially on \( H^1(N, M[2^k]) \), recall that \( G \) acts on \( Z^1(N, \cdot) \) by sending \( n \mapsto a_n \) to \( gng^{-1} \mapsto ga_n \). For \( a \in H^1(N, M[2^k]) = \text{Hom}(N, M[2^k]) \), Artin reciprocity over \( K \) gives \( a_{qg^{-1}} = ga_n \), since \( a_{n^{-1}} = -a_n \), and \( g \) acts on \( \text{Cl}_K \) by \( \delta_K(g) \). Of course, Smith’s proof crucially relies on this “dihedral-like” structure as well.

**Remark 3.6.** Consider the class-Selmer analogy (which Smith says Fouvry–Klöners used earlier): the Selmer groups involve \( H^1(G_\mathbb{Q}, \cdot) \) by definition, perhaps motivating the above \( H^1(G_\mathbb{Q}, \cdot) \) extension of the dual class group. Alternative motivation: \( \mathbb{Q} \)-cocycles can be added over varying ground fields \( K/\mathbb{Q} \), while \( K \)-characters maybe cannot (I’m not sure yet).

3.2. **Local restriction of cocycles.** We want to express Proposition 2.5 using cocycles.

**Proposition 3.7.** In Proposition 2.3, suppose the character \( \psi_K : \text{Gal}(L/K) \to 2^{-k}\mathbb{Z}/\mathbb{Z} \) extends to a cocycle \( \phi_K : \text{Gal}(L/\mathbb{Q}) \to M_K[2^k] \). Then the local restriction \( \phi_K|_{\text{Gal}(L_p/\mathbb{Q}_p)} \) is

1. a quadratic character extending \( \psi_K|_{D_p} = \chi|_{G_{K_p}} \), where \( D_p = \text{Gal}(L_\mathbb{Q}/K_p) \);

2. the sum of \( \chi \) with one of the two characters of \( \text{Gal}(K_p/\mathbb{Q}_p) \), say \( \chi' \).

Furthermore, \( (\chi', b)_p = 0 \) and \( \text{rec}_{L/K}(p) = (\chi, b)_p = (\phi_K, b)_p = \text{inv}_p(\phi_K \cup \chi_b) \)
for any uniformizer \( b \) of \( \mathbb{Q}_p \), as long as \( b \in N_{K_p/\mathbb{Q}_p}(K_p^x) \).

\[\text{Cl} \]
Remark 3.8. The appearance of quadratic “$\eta$” in $(\eta, b)_p$ is shorthand for the discriminant of the at most quadratic field of definition of $\eta$. In particular, $(\phi_k, b)_p = (\chi, b)_p + (\chi', b)_p$.

Remark 3.9. Later on, $b$ will be the norm of an ideal $w(K)$ depending on $K$, such that $w(K) \in 2\text{Cl}_K[4]$. In particular, $w(K) = \beta I^2$ for some element $\beta \in K^\times$ and fractional ideal $I$. But $N(I^2) = N(I)^2$ is the norm of $N(I) \in K^\times$, so $b$ is the norm of the element $\beta N(I) \in K^\times$. So $w(K) \in 2\text{Cl}_K[4]$ will give us $b \in N_{K/Q}(K^\times)$ for free, even as $K$ varies.

Proof. The global inflation-restriction sequence (see Proposition [3.3]) restricts down to
\[
0 \to H^1(\text{Gal}(K_p/Q_p), M[2^k]) \overset{\text{inf}}{\to} H^1(\text{Gal}(L_q/Q_p), M[2^k]) \overset{\text{res}}{\to} H^1(\text{Gal}(L_q/K_p), M[2^k]) \to H^2(\text{Gal}(K_p/Q_p), M[2^k]).
\]

Claim: everything is defined over $M[2]$, i.e. the inclusion $M[2] \to M[2^k]$ defines an isomorphism of inflation-restriction sequences. Proof: compute for the left and right $H^1$ terms and the $H^2$ term, perhaps using cyclic Tate cohomology. Then use the 5-lemma.

Now, over $M[2] = 2^{-1}\mathbb{Z}/\mathbb{Z}$, all Galois actions are trivial, so $H^1 = Z^1 = \text{Hom}$. By Proposition [2.5] $\text{Gal}(L_q/Q_p) = \text{Gal}(F/Q_p) \times \text{Gal}(K_p/Q_p)$, so the $H^1$'s must form a split short exact sequence of characters. The splitting expresses $\phi_k|_{\text{Gal}(L_q/Q_p)}$ as the desired sum $\chi + \chi'$. Finally, $b \in N_{K_p/Q_p}(K^\times)$ implies $(\chi', b)_p = 0$, even if $\chi'$ is nontrivial. $\square$

4. Relating different ground fields

4.1. Defining families of objects.

Definition 4.1. Fix a quadratic field $K/\mathbb{Q}$ of discriminant $\Delta_K < 0$. Let $X_1, \ldots, X_d$ be pairwise disjoint sets of odd primes $p \nmid \Delta_K$. Let $X = X_{[d]}(K)$ denote the product $X_1 \times \cdots \times X_d$, with $i$th projection $\pi_i$ to $X_i$. As $x = x_{[d]} \in X$ varies, define the family of quadratic fields
\[
K(x) := \mathbb{Q}(\sqrt{\Delta_K \pi_1(x) \cdots \pi_d(x)}).
\]

Call this family simple if $p \; (\text{mod} \; 4)$ is constant for $p \in X_i$.

Remark 4.2. Simplicity requires the sign of the prime discriminant $p^* = (-1)^{(p-1)/2}p$ to be constant on each set $X_i$. This is natural when applying genus theory in families.

Definition 4.3. Let $X$ represent a simple family. Call $w_b$ a constant family of $2$-torsion elements if there exists a constant discriminant $\Delta_b \mid \Delta_K$ such that $w_b(x)$ is the image of
\[
(\Delta_b)^{\frac{1}{2}} := \prod_{p \in \text{Spec} \mathcal{O}_{K(x)}} \phi_{\mathbb{Z}_p}^{\frac{1}{2}}(\Delta_b) \in I_{K(x)}^4 \leq I_{K(x)}
\]
in $\overline{\text{Cl}}_{K(x)}[2] := I_{K(x)}^8/I_{K(x)}$, for all $x \in X$. Let the level be the largest integer $k \geq 1$ such that $w_b(x) \in 2^{k-1}\overline{\text{Cl}}_{K(x)}[2^k]$ for all $x \in X$.

Remark 4.4. Appendix [A] relates $\overline{\text{Cl}}_{K(x)}[2]$ to the actual 2-torsion group $\text{Cl}_{K(x)}[2]$.

Definition 4.5. Let $X$ represent a family. Call $w_a$ a constant family of characters if there exists a constant discriminant $\Delta_a \mid \Delta_K$ such that $w_a(x)$ is the image of
\[
\chi_{\Delta_a}: G_{\mathbb{Q}} \to 2^{-1}\mathbb{Z}/\mathbb{Z} = M_K[2] = M_{K(x)}[2]
\]
in $\overline{\text{Cl}}_{K(x)}[2] = \text{Hom}_{\text{cts}}(\text{Gal}(K(x)^ur/\mathbb{Q}), M_{K(x)}[2])$, for all $x \in X$. Let the level be the largest integer $k \geq 1$ such that $w_a(x) \in 2^{k-1}\overline{\text{Cl}}_{K(x)}[2^k]$ for all $x \in X$. 

Remark 4.6. Proposition 3.3 relates $\Cl_{K(x)}^\vee[2]$ to the actual dual 2-torsion group $\hat{\Cl}_{K(x)}[2]$.

To use Proposition 2.1 we need to witness the level of $w_a$ using elements of $\Cl_{K(x)}^\vee[2^k]$.

Definition 4.7. Call
$$\mathfrak{R}(X) = (\psi_1(x), \ldots, \psi_k(x))_{x \in X}$$
a set of raw cocycles (resp. cochains) if $\psi_j(x)$ is a $G_{Q}$-cocycle in $Z^1_{cts}(G_{Q}, M_{K(x)}[2^j])$ (resp. $G_{Q}$-cochain in $C^1_{cts}(G_{Q}, M_{K(x)}[2^j])$) for each $j \in [k(x)]$ and $x \in X$, such that $\psi_j(x) = 2\psi_{j+1}(x)$ for $j = 1, \ldots, k(x) - 1$. Let the level be the largest integer $k \geq 1$ such that $k(x) \geq k$ for all $x \in X$. If $w_a$ is a family of characters, say that $\mathfrak{R}(X)$ witnesses $w_a$ to level $\ell$ if it is a set of raw cocycles such that $\psi_1(x) = w_a(x)$ and $k \geq \ell$.

Remark 4.8. We do not require $\psi_j(x)$ to be defined over $\Gal(K(x)^{ur}/Q)$. That is OK for Proposition 3.3, a key combinatorial result. But ramification considerations will play a big role in the setup and proof of Theorem 4.15 due to the use of Proposition 3.3.

Definition 4.9. Call a $G_{Q}$-cocycle unramified over $L$ if it is defined over $\Gal(L^{ur}/Q)$.

Remark 4.10. A cocycle in $Z^1_{cts}(G_{Q}, M_{K}[2^k])$ is unramified over $K$ if and only if it lies in $\Cl_{K}^\vee[2^k]$. See Proposition 3.1 for how to think about fields of definition more precisely.

4.2. Raw cocycles: consistency and minimality.

Definition 4.11. If $\mathfrak{R}(X)$ is a set of raw cochains of level $k \geq d$, define the set map
$$\psi_d(X) := \sum_{x \in X} \iota_x \psi_d(x) : G_{Q} \to \mathbb{Q}/\mathbb{Z},$$
where $\iota_x$ is the forgetful map $\iota_{K(x)} : M_{K(x)} \to \mathbb{Q}/\mathbb{Z}$.

Definition 4.12. Let $X = X_{[d]}(K)$ represent a family. Call $\mathfrak{R}(X)$ minimal or oscillatory if it is a set of raw cocycles of level $k \geq d$, and the set map $\psi_d(X)$ is 0.

Remark 4.13. The subtlest part of the definition is $k \geq d$. Cf. Smith’s notion of consistency, which makes sense at level $k = 1$ for any $d$. When $k = d = 1$, the notions agree.

To appreciate minimality, and to formulate Theorem 4.15 below, we need to understand the combinatorics of restricted variation.

Definition 4.14. Let $X = X_{[d]}(K)$ represent a family, $S \subseteq [d]$ a set of variation indices, and $T = [d] - S$ the complementary set of fixed indices, with a choice of primes $y = (q_i)_{i \in T} \in \prod_{i \in T} X_i$. Let $\Delta_y$ denote the discriminant of the quadratic $K_y := \mathbb{Q}(\sqrt{\Delta_K \prod_{i \in T} q_i}) = K(y)$, and $X_S$ the product set $\prod_{i \in S} X_i$, representing the restricted family of fields
$$K_y(x_S) := \mathbb{Q}(\Delta_y^{1/2} \prod_{i \in S} p_i^{1/2}) = K(y \cup x_S)$$
for $x_S = (p_i)_{i \in S} \in \prod_{i \in S} X_i$. One can then define constant families, restricted levels, minimality, and so on with respect to the data $y, S$.

Constancy of families is stable under restriction, while level is nondecreasing, witnessing (of $w_a$ by raw cocycles) is stable, and minimality is stable (see Appendix D for details).

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Footnote: This means $w_a$ has level at least $\ell$, but possibly greater.
4.3. First half of main theorem. Let $X = X_d(K)$ represent a simple family of fields. Let $w_b$ denote a constant family of 2-torsion elements, and $w_a$ a constant family of characters. Assume the following conditions:

1. $w_b$ is of level at least $d$, where $d \geq 2$.
2. $|X_i| = 2$ for all $i \in [d]$, with a distinguished point $x_0 = (p_i)_{i \in [d]}$.
3. $\mathcal{R}(X)$ is a set of raw cocycles with $k(x) \geq d$ and $\psi_1(x) = w_a(x)$ for all $x \neq x_0$, such that $\psi_j(x)$ is unramified over $K(x)$ for all $j \in [d]$. (No condition at $x_0$.)
4. For every index $i \in [d]$ and complementary variation set $S = [d] - i$, the set $\mathcal{R}(X)$ is minimal with respect to the data $q_i, S$ for all $q_i \in X_i \setminus p_i$.

**Theorem 4.15** ([8, Theorem 2.8(1)]). Above, $\mathcal{R}(X)$ can be modified at $x_0$ so that

1. $\mathcal{R}(X)$ witnesses $w_a$ to level $d$;
2. $\psi_j(x_0)$ is unramified over $K(x_0)$ for all $j \in [d]$; and
3. $\psi_d(X)$ is a quadratic $G_Q$-character defined over $\prod_{x \in X} K(x)^{ur}$.

Furthermore,

$$\sum_{x \in X_d} \langle w_a(x), w_b(x) \rangle = 0,$$

where the pairing $\langle -, - \rangle$ is induced by the torsion class pairing computed in Proposition 2.1.

**Remark 4.16.** The sum is independent of the witness $\mathcal{R}(X)$. Can the theorem be strengthened (e.g. smaller sums)? Or can it be weakened (e.g. larger sums) with an easier proof?

**Proof.** Whenever $k(x) \geq d$, Proposition 2.1 says

$$\langle w_a(x), w_b(x) \rangle = \psi_d(x)(w_b(x)) = \text{rec}_{L(\psi_d(x))/K(x)}(w_b(x)) \in 2^{-1}\mathbb{Z}/\mathbb{Z},$$

where $L(\psi_d(x))$ is the fixed field of $\ker \psi_d(x)|_{G_K(x)}$ acting on the Hilbert class field $H_{K(x)}/K(x)$. Since $w_b$ is a constant family of 2-torsion elements, there is a constant discriminant $\Delta_b | \Delta_K$ such that $w_b(x) = (\Delta_b)^{\frac{1}{2}} \pmod{I_K(x)}$, the ideal square root taking place in $I_K(x)$. As the relevant Artin symbol at $w_b(x)$ is $\mathbb{F}_2$-valued, we can ignore any squares in $\Delta_b$. In other words, let $b = \Delta_b$ if $\Delta_b$ is odd, and $b = \Delta_b/4$ otherwise. Then $b$ is squarefree, and equal to the norm of $w_b(x)$, up to a rational square. **Since $w_b$ is of level $d \geq 2$, the ideal class of $w_b(x)$ is a square, so $b \in N_{K(x)/K(x)}(K(x)^\times)$** by the remark following Proposition 3.7. By Propositions 2.5 and 3.7 applied to primes $p | b$ of the form $pO_{K(x)} = p(x)^2$, we find

$$\langle w_a(x), w_b(x) \rangle = \sum_{p | b} \text{rec}_{L(\psi_d(x))/K(x)}(p(x)) = \sum_{p | b} (\psi_d(x), b)_p.$$

We can at last modify $\mathcal{R}(X)$ at $x_0$. Provisionally define a 1-cochain

$$\psi_d(x_0) = -\lambda_{x_0}^{-1} \sum_{x \neq x_0} t_x \psi_d(x) : G_Q \to M_K(x_0),$$

which is in fact a cocycle by Proposition 2.3.2. Although this is a continuous 1-cocycle $G_Q \to M_K(x_0)[2^d]$, it may be ramified. For now, multiplying by $2^{d-1}$ gives $\psi_1(x_0) = w_a(x_0)$ by constancy of $w_a$ and oddness of the number of summation indices $x \neq x_0$.

We now study ramification. Minimality with respect to $q_i, [d] - i$ for $q_i \neq p_i$ implies

$$\psi_{d-1}(x_0) = 2^d \psi_d(x_0) = \sum_{x \neq x_0 : \pi(x) = p_i} t_{x_0}^{-1} t_x \psi_{d-1}(x),$$

defined over $K(x)^{ur}$. 
for each index $i \in [d]$. If $q_i \in X_i \setminus p_i$, then $\psi_{d-1}(x_i)$ is unramified at $q_i$, since $K(x)^{\text{ur}}/K(x)/\mathbb{Q}$ is unramified at $q_i$ for $x \in X$ such that $\pi_i(x) = p_i$. By Lemma B.3 it follows that $\psi_{d-1}(x_0)$ is defined over $K(x_0)^{\text{ur}}$. Let $L_0/K(x_0)/\mathbb{Q}$ be the smallest Galois extension $E/\mathbb{Q}$ containing $K(x_0)$ such that $\psi_{d-1}(x_0)$ can be defined over $\text{Gal}(E/\mathbb{Q})$ (see Proposition C.1).

Letting $\psi$ denote the provisional choice of $\psi_{d}(x_0)$, Lemma C.3 furnishes $c \in \mathbb{Q}^\times$ such that the cocycle $\psi + \chi_c$ is defined over a Galois tower $L/L_0/K(x_0)/\mathbb{Q}$ with $L/L_0$ quadratic and $L/K(x_0)$ unramified. Redefine $\psi_{d}(x_0) := \psi + \chi_c$, now inside $\mathbb{Q}^{\text{ur}}(K(x_0))[2^d]$; since $2\chi_c = 0$, this definition preserves $\psi_{d-1}(x_0)$ and lower, including $\psi_1(x_0) = w_a(x_0)$.

With this new definition,

$$\sum_{x \in X} \psi_{d}(x) = \chi_c + 0 = \chi_c.$$ 

So

$$\sum_{x \in X_{(d)}} \langle w_a(x), w_b(x) \rangle = \sum_{x \in X_{(d)}} \sum_{p \mid b} \langle \psi_{d}(x), b \rangle_p = \sum_{p \mid b} \langle \chi_c, b \rangle_p = \sum_{p \mid b} \langle c, b \rangle_p.$$ 

By Hilbert reciprocity, $\sum_{p \in \text{Spec} \mathbb{Z}} \langle c, b \rangle_p = 0$, so the previous sum vanishes if and only if

$$\sum_{p \mid b} \langle c, b \rangle_p = 0.$$

In fact, each term vanishes! Fix $p \nmid b$. For convenience, replace $c$ with the discriminant of $\mathbb{Q}(\sqrt{c})/\mathbb{Q}$. If $p \nmid c$, then $b$, a unit, must be a norm in the unramified local extension $\mathbb{Q}_p(\sqrt{c})/\mathbb{Q}_p$, so $\langle c, b \rangle_p = 0$.

Now suppose $p \mid c$; we will uniformly treat odd and even $p$. Recall from earlier that $b \in N_{K(x)/\mathbb{Q}}(K(x)^{\times})$ for all $x \in X$, so $\langle \Delta_{K(x)}, b \rangle_p = 0$. Since $X$ is a simple family, the 2-part $\Delta_2 \in \{ -4, \pm 8 \}$ of the discriminant of $K(x)$ is constant as $x \in X$ varies. Since $\chi_c$ is defined over $\prod K(x)^{\text{ur}}$, Proposition B.2 says the prime discriminant $\Delta_p$ of $c$ must lie in the prime discriminant factorization of $K(x)$ for some $x \in X$, even if $p = 2$. So $\mathbb{Q}(\sqrt{c\Delta_{K(x)}})$ is unramified at $p$, even if $p = 2^7$. As in the $p \nmid c$ case, we get $\langle c\Delta_{K(x)}, b \rangle_p = 0$. Finally,

$$\langle c, b \rangle_p = \langle \Delta_{K(x)}, b \rangle_p + \langle c\Delta_{K(x)}, b \rangle_p = 0 + 0 = 0$$

by bilinearity of the quadratic Hilbert symbol, as desired. \hfill \square

Remark 4.17. On the Selmer side, Smith’s proof of [8, Theorem 2.9] seems easier, without need for anything like Lemma C.3. If we weakened Theorem 4.15 by doubling the sizes of the sums, I imagine we would have a correspondingly easier proof here, but I may be missing the bigger picture (either in terms of analytic input, or class-Selmer analogy).

Remark 4.18. We can say more about $L_0 \leq K(x)^{\text{ur}}$. Since $\psi_{d-1}(x_0)$ kills $G_{L_0}$, the restricted character kernel $G_{F_0} := \ker \psi_{d-1}(x_0)|_{G_{K(x_0)}}$ contains $G_{L_0}$, so $F_0 \leq L_0$, so $F_0 \leq K(x)^{\text{ur}}$. But $M_{K(x_0)}[2^d-1]$ cyclic implies $F_0/K(x_0)$ cyclic Galois, so $F_0 \leq H^+_L$. By Proposition 2.4 $F_0/\mathbb{Q}$ is Galois. Now Corollary C.2 says $\psi_{d-1}(x_0)$ is defined over $F_0$, so $F_0 = L_0$.

Remark 4.19. We can say more about $L$ as well. Tracing through Lemma C.3 one sees $G_L := \ker \psi_{d}(x_0)|_{G_{L_0}}$. On the other hand, the character kernel $G_F := \ker \psi_{d}(x_0)|_{G_{K(x_0)}}$ lies in $G_{F_0}$, so $G_F = \ker \psi_{d}(x_0)|_{G_{F_0}}$. But $G_{F_0} = G_{L_0}$ from the previous remark, so $G_F = G_L$ and $F = L$. As before, $M_{K(x_0)}[2^d]$ cyclic implies $F \leq H^+_L$, so $L = F \leq H^+_K$. 

\footnote{I.e., “defined over a field unramified at”}

\footnote{For $p = 2$, the point is that $c\Delta_{K(x)}$ is $\Delta_2^2$ times a product of odd prime discriminants.}
Appendix A. Genus theory and the 2-class group

For $K/Q$ quadratic, let $\overline{Cl_K[2]}$ be the $\mathbb{F}_2$-vector subspace of $I_K/I_Q$ generated by the finite primes of $K$ ramified over $Q$. An easy computation gives a short exact sequence $I_Q \rightarrow I_K^\sigma \rightarrow \overline{Cl_K[2]}$, so $\overline{Cl_K[2]}$ can also be described as $I_K^\sigma/I_Q$. Define the map $\iota: \overline{Cl_K[2]} \rightarrow \overline{Cl_K[2]}$, where $\overline{Cl_K^+} := I_K/P_K^+$ denotes the narrow class group. For convenience, let $K_\infty$ denote the group of totally positive elements of $K^\times$. Write $2k^{-1}\overline{Cl_K[2]} := \iota^{-1}(2k^{-1}\overline{Cl_K^+[2^k]})$.

Proposition A.1. The map $\iota$ is surjective. Its kernel is isomorphic to $\mathbb{Z}/2$, generated by $(x)I_Q$, where $x$ is given uniquely up to unique $O_KQ^\times$-scalar by

- $\sqrt{\Delta_K}$ if $K/Q$ is imaginary;
- $\epsilon + \epsilon^{-1} \in \mathbb{Q}\sqrt{\Delta_K}$ if $K/Q$ is real with fundamental unit $\epsilon$ of norm $-1$; and
- $1 + \epsilon$ otherwise, if $K/Q$ is real with $N\epsilon = +1$, where $\epsilon$ is chosen to lie in $K_\infty^\times$.

Remark A.2. For a “dual” perspective, see Milovic’s thesis on (and slightly generalizing) the work of Fouvry–Kl¨uners. Early on it has a description mapping out of $Cl^+/2Cl^+$ (instead of mapping into $Cl^+/2Cl^+$) using Hilbert symbols and reciprocity.

Proof. The ideal norm $N = 1 + \sigma$ maps into $I_Q \leq P_K^+$, so an ideal $I \in I_K$ satisfies $I^2 \sim (1)$ if and only if $(1-\sigma)I = (x)$ for some $x \in K_\infty^\times$. In this case, $(Nx) = N(1-\sigma)I = (1)$, so $Nx = \pm 1$; total positivity forces $Nx = +1$. By Hilbert 90, $x = (1-\sigma)y$ for some $y \in K^\times$, so $(1-\sigma)(iy^{-1}) = (1)$, i.e. $iy^{-1} \in I_K^\sigma$. Since $x = y/\sigma y$ is totally positive, $y$ must be either totally positive or negative, so $(y)$ admits a totally positive generator. Hence $[I] = [iy^{-1}] \in [I_K^\sigma/2] = im \iota$, establishing surjectivity of $\iota$.

Remark A.3. We started with the equivalence $I^2 \sim (1) \iff \sigma(I) \sim I$. The latter is natural for generalization to cyclic extensions $K/Q$: see Klys or Emerton’s notes.

Remark A.4. For examples of $1 + \epsilon$ in the third case, see fundamental unit tables. For $d = 21$, we have $\epsilon = (5 + \sqrt{21})/2$, so $1 + \epsilon = (7 + \sqrt{21})/2$. For $d = 33$, we have $\epsilon = 23 + 4\sqrt{33}$, so $1 + \epsilon = 24 + 4\sqrt{33}$. In general, $N(1 + \epsilon) = 2 + a$ if $2\epsilon = a + b\sqrt{d}$ (where $a^2 - db^2 = 4$).

Appendix B. Results on ramification

Proposition B.1. Suppose $M/K$ is generated by two subextensions $E, F$. If $K = E \cap F$ and either $E$ or $F$ is finite Galois over $K$, then $E, F$ are linearly disjoint over $K$.

Proof. Say $E = K(\alpha)$ is finite Galois over $K$. The minimal polynomial $f$ of $\alpha$ over $K$ remains irreducible over $F$, because $K = E \cap F$. See MSE for further discussion.

The following results are used to control the ramification of fields and objects of interest.

Proposition B.2. For $X$ a family, $\prod_{x \in X} K(x)^w/E$ is unramified, where $E := \prod K(x)/Q$. If $F/Q$ is a quadratic subfield of $\prod K(x)^w$, then $\Delta_F$ is, up to a square, a product of prime discriminants in $P(X)$, the union of the prime discriminants of $\Delta_K(x)$ for $x \in X$.

Proof. $C/A$ and $D/B$ unramified implies $CD/AB$ unramified, so $\prod K(x)^w/E$ is unramified. Now use the structure of multiquadratic fields: $E$ lies in a linearly disjoint compositum (see Proposition B.1) of “prime discriminant fields” $\mathbb{Q}(\sqrt{\Delta_p})/Q$, where $\Delta_p \in \{-4, \pm 8\}$ (any two of which disjointly generate the third), and $\Delta_p = (-1)^{(p-1)/2}p$ if $p$ is odd.
• Let $E_{\text{gen}}$ be the smallest compositum of prime discriminant fields such that $E \leq E_{\text{gen}}$. Then the odd $\Delta_p$’s all lie in $P(X)$, while the $\Delta_2$’s in $E_{\text{gen}}$ either arise from $P(X)$ or a product of $\Delta_2$’s from $P(X)$, up to a square (e.g. $-4$ is $(+8)(-8)$ up to a square).
• $E$ is ramified precisely at primes dividing $\prod \Delta_{K(x)}$, i.e. the underlying primes of $P(X)$. Easily check that $E_{\text{gen}} \leq \prod K(x)^{ur}$.

The quadratic $F/Q$ lies in $\prod K(x)^{ur}$, so every prime discriminant $\Delta_q$ of $\Delta_F$ must either be in $P(X)$ or a product of $\Delta_2$’s from $P(X)$, up to a square. Otherwise, $F$ and $E$ would be linearly disjoint over $Q$ (again, see Proposition B.1), and $FE/E$ would be ramified at $q$. $\square$

**Lemma B.3.** For $X$ a simple family with $x_0 \in X$ distinguished, $\prod K(x)^{ur}/K(x_0)$ is unramified outside of $R := \bigcup_{i \in [d]} (X_i \setminus \pi_i(x_0))$. Consequently, $K(x_0)^{ur}$ is the maximal subextension of $\prod_{x \in X} K(x)^{ur}/Q$ unramified at every prime in $R$.

**Proof.** The first part of Proposition B.2 says $\prod K(x)^{ur}/E$ is unramified. Since $X$ is a simple family, the 2-part $\Delta_2 \in \{-4, \pm 8\}$ of $\Delta_{K(x)}$ is constant as $x \in X$ varies. Thus $K(x)/K(x_0)$ can only be ramified over odd primes $p \mid \Delta_{K(x)}$ with $p \nmid \Delta_{K(x_0)}$. This automatically excludes the primes $p \mid \Delta_K$. We are left with precisely the primes $p \in R$ as possibilities. In other words, $E/K(x_0)$ is unramified outside of $R$. Thus the whole tower $K(x)^{ur}/E/K(x_0)$ is unramified outside of $R$, proving the first part of the lemma.

We then immediately get that $\prod K(x)^{ur}/K(x_0)^{ur}$ is unramified outside of $R$. Yet by definition of $K(x_0)^{ur}$, every subextension $E/K(x_0)^{ur}$ of $\prod K(x)^{ur}/K(x_0)^{ur}$ is ramified, hence ramified somewhere over $R$. So $K(x_0)^{ur}$ has the desired maximality property. $\square$

**Remark B.4.** Similarly, if $p^* \in P(X)$, then $\prod K(x)^{ur}/E/Q(\sqrt{p^*})$ is unramified at $p$.

**Lemma B.5.** Let $K = F(\sqrt{a})$ and $L = F(\sqrt{b})$ be two ramified quadratic extensions of local fields over $Q_p$. If $KL/L$ is unramified, then so is $F(\sqrt{ab})/F$.

**Proof.** $KL/F$ has $e = 2 \geq f$, so $F(\sqrt{ab})/F$ must be the maximal unramified extension. $\square$

**Appendix C. Fields of definition of cocycles**

**Proposition C.1.** Let $N$ be a $G_{Q}$-module, and let $\psi : G_{Q} \to N$ be a continuous 1-cocycle. Let $G_{L}$ be a normal open subgroup in the kernel of set map $\psi$. Then $\psi$ is defined over $L$.

**Proof.** Take $h \in G_{L}$ in $\psi(gh) = g\psi(h) + \psi(g)$ to get $\psi(gh) = \psi(g)$ for all $g \in G_{Q}$. Since $G_{L}$ is normal, $\psi$ induces a set map $\overline{\psi} : \text{Gal}(L/Q) = G_{Q}/G_{L} \to N$. Now $\overline{\psi}(gh) = g\overline{\psi}(h) + \overline{\psi}(g)$ for any $g, h \in G_{Q}$, so $\overline{\psi}(h)$ is independent of the coset representative $g \in \overline{G}$. Thus $\overline{\psi} : \text{Gal}(L/Q) \to N^{G_{L}}$ is a finite cocycle with $G_{Q}$-inflation $\psi$, as desired. $\square$

**Corollary C.2.** Take $K/Q$ Galois, and take $N$ on which $G_{K}$ acts trivially. Let $G_{L}$ be the kernel of the homomorphism $\psi|_{G_{K}} : G_{K} \to N$. If $L/Q$ is Galois, then $\psi$ is defined over $L$.

The following “quadratic twist” result is used in proving Theorem 1.15. For $X$ a simple family with $x_0 \in X$ distinguished, let $\psi$ be a cocycle $G_{Q} \to M_{K(x_0)}$ such that

1. $\psi$ is defined over $\prod_{x \in X} K(x)^{ur}$, and
2. $2\psi$ is defined over $L_0$, where $K(x_0) \leq L_0 \leq K(x_0)^{ur}$ and $L_0/Q$ is finite Galois.

**Lemma C.3.** In the setting above, if $\chi_c$ denotes the quadratic character of $Q(\sqrt{c})/Q$, then for any $c \in \overline{Q})^\times$, the twist $\psi + \chi_c$ is a cocycle defined over a Galois tower $L^c/L_0/Q$, with $L^c/L_0$ at most quadratic. Furthermore, there exists $c$ such that $L^c/K(x_0)$ is unramified, i.e.

$$\psi + \chi_c \in \overline{Q}^{L(x_0)}.$$
Proof of field of definition. Take $g \in G_{\mathbb{Q}}$ and $n \in G_{L_0}$. Then $2\psi(n) = 0$ by definition of $L_0$, so $g$ acts trivially on $\psi(n) \in 2^{-1}\mathbb{Z}/\mathbb{Z}$. Also, $n \in G_{L_0} \leq G_{K(x_0)}$, so $\delta_{K(x_0)}(n) = +1$. Thus

$$\psi(gng^{-1}) = \psi(g) + g\psi(n) + gn\psi(g^{-1}) = \psi(g) + \psi(n) + g\psi(g^{-1}) = \psi(n).$$

In particular, $G_L$, the subgroup of $G_{L_0}$ killed by $\psi$, is normal in $G_{\mathbb{Q}}$, so $L/\mathbb{Q}$ is Galois $\psi$ is defined over $L$ by Proposition C.1. The group $G_L$ is actually the kernel of $\psi|_{G_{L_0}} : G_{L_0} \to M_{K(x_0)}[2] = 2^{-1}\mathbb{Z}/\mathbb{Z}$ (a character), so $L/L_0$ is at most quadratic.

With the twist, $\psi + \chi_c$ is still a cocycle with $2(\psi + \chi_c) = 2\psi$. The previous paragraph, applied to $\psi + \chi_c$ instead of $\psi$, yields a Galois tower $L^c/L_0/\mathbb{Q}$ with $L^c/L_0$ quadratic. □

Proof of existence of twist. Suppose $L/K(x_0)$ is ramified along a tower of primes $\Omega/q/p/p$, with $q/p$ unramified but $\Omega/q$ ramified. Assumption (1) on $\psi$ says $L \leq \prod K(x)^{wr}$, so

- Lemma B.3 implies $p \in \bigcup_{i \in [d]}(X_i \setminus \pi_i(x_0))$, because $L/K(x_0)$ is ramified over $p$; while
- if we choose $x^p \in X$ with $p$ ramified in $K(x^p)$, say $\pi_i(x^p) = p$, and $\pi_j(x_0)$ for $j \in [d] - i$, then Lemma B.3 implies that $LK(x^p)/K(x^p)$ is unramified over $p$.

By the first point, $p/p$, hence $q/p$, is unramified. Let $p_0 = \pi_i(x_0)$ and $c = p^s p_0^*$, so $L(\sqrt{c}) = LK(x^p)$ by simplicity of $X$. Clearly no new primes of $\mathbb{Z}$ can ramify in $L^c$. If we show that $L^c/L_0$ is unramified over $p$, then the desired twist will exist by induction.

Now, $\psi|_{G_{L_0}}$ and $\chi_c|_{G_{L_0}}$ are both quadratic characters, with kernels $G_L$ and $G_{L_0(\sqrt{c})}$, respectively. If $L = L_0(\sqrt{\alpha})$, then the kernel of $\psi|_{G_{L_0}} + \chi_c|_{G_{L_0}}$ is $G_{L_0(\sqrt{\alpha c})}$, so $L^c = L_0(\sqrt{\alpha c})$, which is Galois over $\mathbb{Q}$ by the first half of the lemma. By the second point above, $LK(x^p)/K(x^p)$ is unramified over $p$, so $\wp_p(LK(x^p)/\mathbb{Q}) = 2$. Yet $\wp_p(L/L_0) = 2$ in the Galois tower $L(\sqrt{c})/L/L_0/\mathbb{Q}$, so $L(\sqrt{c})/L = LK(x^p)/L$ must be unramified over $p$. Now restrict attention to the biquadratic extension $L(\sqrt{c})/L_0$, all of which is Galois over $\mathbb{Q}$. Since $q/p$ is unramified, $q$ must ramify in $L_0(\sqrt{c})$. In the local Galois picture, $L(\sqrt{c})/L_0$ satisfies Lemma B.5 so $L_0(\sqrt{\alpha c})/L_0 = L^c/L_0$ is unramified over $p$. □

Remark C.4. I got stuck trying to prove this lemma while reading [8]; thanks to Alex for explaining the details to me, especially for the field of definition. Below is what Alex suggested for the twist proof; it differs a little from the proof above.

Once we have $q/p$ unramified, the normal subgroup $\text{Gal}(L/L_0) \cong \mathbb{Z}/2$ of $\text{Gal}(L/\mathbb{Q})$ must be the inertia group of every prime $\Omega/p$ of $L$. Since $\wp_p(LK(x^p)/\mathbb{Q}) = 2$, the inertia group of every prime of $L(\sqrt{c})$ over $\mathbb{Q}$ is also of size two. It also always lies in the pullback $\text{Gal}(L(\sqrt{c})/L_0)$ of $I_p(L/\mathbb{Q}) = \text{Gal}(L/L_0)$ under $\text{Gal}(L(\sqrt{c})/\mathbb{Q}) \to \text{Gal}(L/\mathbb{Q})$. But $\text{Gal}(L(\sqrt{c})/\mathbb{Q}) \to \text{Gal}(L/\mathbb{Q})$ induces a surjection, hence isomorphism, of the equally-sized inertia groups, with target $I_p(L/\mathbb{Q})$. Consequently, if $\sigma$ is a nontrivial inertia element in $\text{Gal}(L(\sqrt{c})/\mathbb{Q})$, then $\psi(\sigma) = \psi(\sigma|_L)$ is nonzero, or else $L$ would be equal to $L_0$ by definition of $L/L_0$. Similarly, $\chi_c(\sigma) = \chi_c(\sigma|_{K(x^p)}) \neq 0$. Thus $(\psi + \chi_c)(\sigma) = 2^{-1} + 2^{-1} = 0$, and $\psi + \chi_c$ kills the twisted inertia group $I_p(L^c/\mathbb{Q})$. So $L^c/K(x_0)$ must be unramified over $p$.

Appendix D. Results on minimality

First, minimality is stable under restriction. There might be a conceptual reason (dihedral intuition?). For now, see the formal argument below (downwards induction on $d$).
View the target of $\psi_d(X)$ as a trivial $G_\mathbb{Q}$-module, so $d\psi_d(X)$ measures how far $\psi_d(X)$ is from being a group homomorphism. Compute the coboundary:

$$d\psi_d(X)(g, h) = \sum_{x \in X} d_x\psi_d(x)(g, h)$$

$$= \sum_{x \in X} g_t^x\psi_d(x)h - t_x\psi_d(x)gh + t_x\psi_d(x)g$$

$$= \sum_{x \in X} g_t^x\psi_d(x)h - t_xg\psi_d(x)h = \sum_{x \in X} \psi_d(x)h - g\psi_d(x)h = \sum_{x : \delta_K(x)(g) = -1} \psi_{d-1}(x)h,$$

where the last step uses the $G_\mathbb{Q}$-action $g \mapsto \delta_K(x)(g) = \pm 1$, and $2\psi_d = \psi_{d-1}$.

**Remark D.1.** If $d = 1$, we automatically get 0 coboundary with no hypotheses on $\psi_d(X)$, because the $\psi_1(x)$ cocycles are actually characters (homomorphisms).

What does $\{ x : \delta_K(x)(g) = -1 \}$ look like? Let $E/\mathbb{Q}$ be the Galois extension defined by

$$G_E := \ker(g \mapsto (\delta_K(x)(g))_{x \in X}) = \bigcap_{x \in X} G_{\mathbb{Q}}(x).$$

**Observation D.2.** Fix $i \in [d]$ and $p_i \in X_i$, odd by definition. There exists $g = g_{i, p_i} \in G_\mathbb{Q}$, unique modulo $G_E$, such that $\delta_K(x)(g) = -1$ if and only if the $i$th component of $x$ is $p_i$. As $i, p_i$ vary, these elements generate $G_\mathbb{Q}/G_E = \text{Gal}(E/\mathbb{Q})$.

**Proof.** See Proposition B.2 and its proof, which places $E = \prod K(x)/\mathbb{Q}$ in $E_{\text{gen}}$. Adjoin $\sqrt{-4}$ for simplicity, and let $L/\mathbb{Q}$ be the resulting multiquadratic field of dimension $t$.

Choose $g \in \text{Gal}(L/\mathbb{Q}) = \mathbb{F}_2^t$ acting nontrivially on $\mathbb{Q}(\sqrt{\Delta_{p_i}})/\mathbb{Q}$, but trivially on the remaining $t - 1$ pieces $\mathbb{Q}(\sqrt{\Delta_{p_i}})/\mathbb{Q}$ of $L$, including $\mathbb{Q}(\sqrt{-4})/\mathbb{Q}$ for $p = 2$. Since $\Delta_{p_i} = \pm p_i$, the element $g$ acts nontrivially on $\sqrt{p_i}$ but trivially on $\sqrt{q}$ if $g \mid \Delta_K$ or $q \in X_1 \cup \cdots \cup X_d \setminus p_i$. Thus $\delta_K(x)(g) = -1$ if and only if $\pi_i(x) = p_i$. So $g$ induces the desired $g_{i, p_i} \in G_\mathbb{Q}$.

Two different $g_{i, p_i}$'s in $G_\mathbb{Q}$ agree under the map $g \mapsto (\delta_K(x)(g))_{x \in X}$, so their ratio lies in $G_E$ by definition. Therefore, $g_{i, p_i} \bmod G_E$ is unique.

Clearly $G_L \leq G_E$, so $E \leq L$. Take the explicit representatives $g_{i, p_i} = g \in \text{Gal}(L/\mathbb{Q})$ defined earlier. To show generation, pick $\sigma \in \text{Gal}(L/\mathbb{Q})$. Modulo the images of $g_{i, p_i}$ in $\text{Gal}(E/\mathbb{Q})$, we may assume $\sigma$ acts trivially on $\mathbb{Q}(\sqrt{\Delta_{p_i}})$ for all primes $p_i \in X_1 \cup \cdots \cup X_d$. Then $\sigma$ acts uniformly on the fields $K(x)$ as $x \in X$ varies. If the action is +1, then $\sigma \equiv 1 \bmod \text{Gal}(L/E)$ as desired. Otherwise, if the action is -1, then

$$\sigma \equiv \prod_{p_i \in X_1} g_{1, p_i} \bmod \text{Gal}(L/E),$$

because $\prod_{p_i \in X_1} \delta_K(x)(g_{1, p_i}) = -1$ for all $x \in X$, by construction of the $g_{1, p_i}$'s.

If $\psi_d(X_1^{[d]}(K)) = 0$, then $d\psi_d(X) = 0$, so $\psi_{d-1}(X_S(K_S)) = 0$ for any index $i \in [d]$ with complementary variation set $S = [d] - i$, and any singleton $y$ of $X_i$. In other words, $\mathfrak{R}(X)$ is minimal with respect to $y, S$. By induction, minimality is stable under restriction. In fact, we only needed that $\psi_d(X)$ was a cocycle (or homomorphism) to conclude that $\mathfrak{R}(X)$ is minimal with respect to all proper subsets. What is the best converse statement?

**Proposition D.3** (Cf. Proposition 2.5]). Let $X = X_{[d]}(K)$ represent a family.
(1) Assume \( \mathcal{R}(X) \) is a set of raw cocycles of level \( k \geq d \), such that \( \mathcal{R}(X) \) is minimal with respect to \( p_i, [d] - i \) for all \( i \in [d] \) and \( p_i \in X_i \). Then \( \psi_d(X) \) is a quadratic character. 

(2) Assume \( \mathcal{R}(X) \) is a set of raw cochains of level \( k \geq d \), such that \( \psi_d(x) \) is a cocycle for all \( x \neq x_0 \). Assume \( \mathcal{R}(X) \) is minimal with respect to \( p_i, [d] - i \) for all \( i \in [d] \) and \( p_i \in X_i \setminus \pi_i(x_0) \). If \( \psi_d(X) \) is a quadratic character, then \( \psi_d(x_0) \) is in fact a cocycle.

Remark D.4. Smith does not explicitly state the second version, but at least when \( \psi_d(X) = 0 \) (trivial character), it is used in proving Theorem 4.15.

Smith gives a binomial theorem proof. Here is another perspective.

Proof. In the first case, \( 2\psi_d(X) \) breaks up (in any number of ways) into sums of \( \psi_{d-1} \) terms, where each sum vanishes by the minimality assumptions. So \( 2\psi_d(X) = 0 \) and \( \psi_d(X) \) is a set map \( G_Q \to 2^{-1}\mathbb{Z}/\mathbb{Z} \). In the second case this is assumed.

Earlier, we computed the coboundary of \( \psi_d(X) \) at \((g, h) \in G_Q^2 \) to be 

\[
d\psi_d(X)(g, h) = \sum_{x \in X} \psi_d(x)h - g\psi_d(x)h = \sum_{x: \delta_K(x)(g) = -1} \psi_{d-1}(x)h
\]

in the first case. In the second case, a similar coboundary calculation shows that

\[
0 = d\psi_d(X)(g, h) = \sum_{x \in X} \psi_d(x)h - \psi_d(x)gh + \psi_d(x)g
\]

\[
= d\psi_d(x_0)(g, h) + \sum_{x \in X} \psi_d(x)h - g\psi_d(x)h
\]

\[
= d\psi_d(x_0)(g, h) + \sum_{x: \delta_K(x)(g) = -1} \psi_{d-1}(x)h.
\]

In both cases, we wish to show that

\[
\sum_{x: \delta_K(x)(g) = -1} \psi_{d-1}(x) = 0
\]

for all \( g \in G_Q \), or equivalently that

\[
\sum_{x \in X} \psi_d(x) = \sum_{x \in X} g\psi_d(x).
\]

Certainly, the first equality holds for any \( g_{i, p_i} \) from the previous observation. In the first case, that’s just minimality of \( \mathcal{R}(X) \) with respect to \( p_i, [d] - i \). In the second case, if \( p_i \) is exceptional for \( i \), then bundling up minimality with respect to \( q_i, [d] - i \) for \( q_i \neq p_i \), together with \( 2\psi_d(X) = 0 \), still recovers the desired first equality.

Now, let \( U \) be the subset of \( G_Q \) for which either equality holds. Clearly \( U \) contains \( G_E \), and we have just shown \( g_{i, p_i} \in U \). But if \( g, g' \in U \), then

\[
\sum_{x \in X} (gg' - 1)\psi_d(x) = \sum_{x \in X} (gg' - g - g' + 1)\psi_d(x)
\]

\[
= \sum_{x \in X} (1 - g)(1 - g')\psi_d(x) = \sum_{x: \delta_K(x)(g') = -1} (1 - g)\psi_{d-1}(x).
\]
If \( g' \) is one of the \( g_{i,p} \), then the sum vanishes under the first hypothesis by minimality of \( \mathcal{R}(X) \) with respect to \( p, [d] - i \) and still under the second hypothesis by a minimality bundling argument, at least if the technical assumption

\[
\sum_{x \in X} \psi_{d-1}(x) = \sum_{x \in X} g \psi_{d-1}(x)
\]

holds. Assuming this, \( g, g' \in U \) implies \( gg' \in U \) whenever \( g' \in \{g_{i,p}\} \). So \( U = G_\mathbb{Q} \), since \( G_\mathbb{Q}/G_E \) is a finite group generated by the \( g_{i,p} \).

Under the second hypothesis, it remains to verify the technical assumption. Set

\[
L := \{ \ell \geq 0 : \sum_{x \in X} 2^\ell \psi_d(x) = \sum_{x \in X} g 2^\ell \psi_d(x) \quad \forall g \in G_\mathbb{Q} \}.
\]

Above, we proved \( 0 \in L \) as long as \( 1 \in L \). The same argument with \( \psi_1, \ldots, \psi_d \) doubled shows that \( 1 \in L \) as long as \( 2 \in L \). Generally, \( \ell \in L \) as long as \( \ell + 1 \in L \). But certainly \( d \in L \), so \( d - 1 \in L \), etc. and finally \( 0 \in L \). ∎

### Appendix E. Class group heuristics

For any set \( S \) of finite primes of \( K \), we have \( I_K^S = \mathbb{Z}^{[S]} \) canonically, while \( \text{rank}(U_K^S) = |S| + \text{rank}(U_K) \) (easiest proof uses finiteness of class group: \( \mathfrak{g}^h \) is principal for \( \mathfrak{g} \in S \)). If \( S \) is large enough (i.e. generates the class group), \( \text{Cl}(K) \xrightarrow{\sim} \text{coker}(U_K^S \to P_K^S \hookrightarrow I_K^S) \).

As \( K \) varies in a natural family (of global fields), want to understand distribution of \( \text{Cl}(K) \); for simplicity, let’s restrict attention to \( \text{Cl}_{p^\infty}(K) := \text{Cl}(K)[p^\infty] \) separately for each prime \( p \).

#### E.1. Random matrix formulation

Fix \( u := \text{rank}(U_K) \) and let \( n := |S| \to \infty \). For any \( K, S \), consider the map \( \iota := i_K^S : U_K^S \otimes \mathbb{Z}_p \to I_K^S \otimes \mathbb{Z}_p \); choosing bases on the left\(^{11} \) and right, we get a matrix \( A = A_K^S : \mathbb{Z}_p^{n+u} \to \mathbb{Z}_p^n \). By Smith normal form theory, the set of possible resulting matrices is precisely \( \{ A : \text{coker } A \cong \text{coker } \iota \} \). We want to know the resulting distribution on \( \text{coker } A \cong \text{coker } \iota \), at least as \( n \to \infty \).

The safest form of the Cohen–Lenstra heuristics roughly states:

**Conjecture E.1.** Let \( K/\mathbb{Q} \) vary among, say, degree \( d \) number fields with a given unit rank \( u := \text{rank} U_K \), containing no \( p \)th roots of unity. Then

\[
\mathbb{P}(\text{Cl}_{p^\infty}(K) \cong P) = \lim_{n \to \infty} \mathbb{P}(\text{coker } A_n \cong P) = |P|^{-u} |\text{Aut}(P)|^{-1} \prod_{k \geq 1} (1 - p^{-k-u})
\]

for any finite abelian \( p \)-group \( P \), where \( A_n : \mathbb{Z}_p^{n+u} \to \mathbb{Z}_p^n \) is a random matrix drawn with respect to Haar measure on \( M_{n,n+u}(\mathbb{Z}_p) \).

**Remark E.2.** The \( \mu_p \) assumption might be unnecessary sometimes, especially if \( p = 2 \).

**Remark E.3.** To understand the second equality, note that for large \( n \) and \( e \) with \( p^e P = 0 \), almost all maps \( (\mathbb{Z}/p^e)^n \to P \) are surjective, so there are around \( |\text{Aut}(P)|^{-1} |P|^n \) subgroups of \( (\mathbb{Z}/p^e)^n \), say—or better, open subgroups of \( \mathbb{Z}_p^n \)—with cokernel isomorphic to \( P \). But there are Haar measure ~ \( |P|^{-n-u} \) matrices \( \mathbb{Z}_p^{n+u} \to \mathbb{Z}_p^n \) with a prescribed image of index \( |P|^{12} \). So there should be Haar measure ~ \( |P|^{-u} |\text{Aut}(P)|^{-1} \) matrices with cokernel isomorphic to \( P \).

\(^{10}\)Minimality implies coboundary zero, which implies \( \sum \psi_{d-1}(x_S) = \sum g \psi_{d-1}(x_S) \).

\(^{11}\)mod torsion (if \( K \) contains \( p \)th roots of unity)

\(^{12}\)First show this for image contained in that prescribed subgroup, a la Ellenberg–Venkatesh–Westerland surjections perspective \cite{2, 3}, and then use inclusion-exclusion.
See Wood [10] for more details on the random matrix train of thought.

Remark E.4. If \( A \) has full rank, then \( \text{coker} \, A \) is unaffected up to isomorphism by small perturbations. For example, we can play around with Gaussian elimination on the perturbed Smith normal form of \( A \). Alternatively, we can even show that \( \text{im} \, A = \text{im} \, A' \) by noting that \( \text{im} \, A \) is finite-index, hence open (contains \( p^r \mathbb{Z}_p^n \) for large \( r \)), in \( \mathbb{Z}_p^n \), so \( A'e_i \approx Ae_i \) lies in \( \text{im} \, A \) for all \( i \). This means \( \text{im} \, A' \leq \text{im} \, A \), and vice versa.

Remark E.5. Why expect Cohen–Lenstra? Can we choose \( A = A_K^S \) equidistributed (or weaker, see [10])? Perhaps one can choose canonical bases on the left and right for which we have no known structure obstructing equidistribution. For example, the “canonical” choice \( I_K^S \otimes \mathbb{Z}_p = \mathbb{Z}_p^n \) is justified precisely because the matrices \( A \) should automatically equidistribute in \( \text{GL}_n(\mathbb{Z}_p)A \) as \( K \) varies. Similarly, even though there might not be a nice identification \( U_K^S \otimes \mathbb{Z}_p = \mathbb{Z}_p^{n+u} \), the ultimate distribution of \( A \) should be dense in \( A \text{GL}_{n+u}(\mathbb{Z}_p) \).

Remark E.6. What if instead, we modeled the inclusions \( \iota': P_K^S \otimes \mathbb{Z}_p \hookrightarrow I_K^S \otimes \mathbb{Z}_p \) by random matrices \( A': \mathbb{Z}_p^n \rightarrow \mathbb{Z}_p^n \)? (Requiring \( A' \) to be injective or not shouldn’t matter: almost all matrices \( \mathbb{Z}_p^n \rightarrow \mathbb{Z}_p^n \) are injective.) The resulting distribution \( \text{coker} \, A' \) (independent of \( u \)) would not match \( \text{coker} \, A \) (dependent on \( u \)) chosen above, except when \( u = 0 \). How do we rule out this alternative heuristic for \( u > 0 \)?

References


\footnote{In this case, one lazy way to see this is to vary \( S \) among sets of size \( n \gg h_K \), and as long as \( U_K \) is not always identified in a stupidly consistent way in \( \mathbb{Z}_p^{n+u} \), we should be OK.}