Multiplicative structure in additive problems

Victor Wang
(based on work of many authors)

IST Austria and IoM Academia Sinica

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Some Diophantine statistics at the boundary

Consider values of a degree-d polynomial $\mathbb{Z}^d \to \mathbb{Z}$. For $d = 2$,

$$\#\{|n| \le x : \exists |a|, |b| \le x^{1/2},\ a^2 + b^2 = n\} \sim \frac{c_{\text{Landau–Ramanujan}}\, x}{(\log x)^{1/2}}$$

by [Landau 1908], and if $\delta := 1 - \frac{1+\log\log 2}{\log 2}$ then by [Ford 2008]

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But is there a (homogeneous or non-cylindrical, to avoid cheap examples like $P = a + b^2 + c^3$) cubic $P \in \mathbb{Z}[a, b, c]$ for which

$$\#\{|n| \le x : \exists |a|, |b|, |c| \le x^{1/3},\ P(a, b, c) = n\} \asymp x?$$

This is expected for $P \in \{a^3 + b^3 + c^3, a^2 + b^2 + c^2 - abc\}$. For larger variables, $a^3 + b^3 + c^3 = n$ is soluble affinely in $n \in \mathbb{F}_q[t]$ for most q [Vaserstein 1991], and $a^2 + b^2 + c^2 - abc = n$ is soluble in $a, b, c \ll |n|^{1/2}$ for a density 1 of $n \in \mathbb{Z}$ satisfying necessary local conditions [Ghosh–Sarnak 2017].

Cubic Weyl sums (and diagonal cubic equations)

Let $S(\alpha, B) := \sum_{|x| \le B} e^{2\pi i \alpha x^3}$, or a smoothed variant.

- Let $\gcd(a, r) = 1$, where $B \le r \le B^2$. Classically, by AA or B process, $S(\alpha, B) \ll_\epsilon B^{3/4+\epsilon}$ for $|r\alpha - a| \le 1/r$.
- By BAAB [Heath-Brown 2009], if $r = r_1 r_2 r_3 \asymp B^{3/2}$ where $r_1 \asymp B^{3/7}$, $r_2 \asymp B^{5/14}$, and $r_3 \asymp B^{5/7}$ are pairwise coprime with r_3 square-free, then $S(\alpha, B) \ll_\epsilon B^{5/7+\epsilon}$.
- By singularity bounds [Sawin 2024], if $|r| \asymp B^{3/2}$ and $\mathbb{F}_q[t]$ has characteristic $\gg_\epsilon 1$, then $S^{\mathbb{F}_q[t]}(\alpha, B) \ll_{q,\epsilon} B^{5/8+\epsilon}$.

Let $M_s(B) := \#\{|x_i| \le B : \sum_{i=1}^{s} x_i^3 = 0\} = \int_0^1 S(\alpha, B)^s \, d\alpha$.

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Let $M_s(B) := \#\{|x_i| \le B : \sum_{i=1}^s x_i^3 = 0\} = \int_0^1 S(\alpha, B)^s \, d\alpha$.

- $\#\{x, y, z \ll B : x^3 + y^3 + z^3 = n\} \ll_\epsilon B^{1+\epsilon}$.
- By [Heath-Brown 1997], $M_4(B) = cB^2 + O_\epsilon(B^{4/3+\epsilon})$.
- By [Vaughan 1986, 2020], $M_6(B) \ll_\epsilon B^{7/2}(\log B)^{\epsilon-5/2}$. Proving $\exists \delta > 0$ with $M_6(B) \ll B^{7/2-\delta}$ is equivalent to proving $\exists \delta > 0$ with $\Pr[|S(\alpha, B)| \ge B^{3/4-\delta}] \ll B^{-1-\delta}$.
- Sharper bounds on M_6 exist for smooth x_i (Wooley), or under GRH (Hooley et al.), or in $\mathbb{F}_q[t]$ (Glas–Hochfilzer).

Let $M_s(B) := \#\{|x_i| \le B : \sum_{i=1}^s x_i^3 = 0\}$ for $s \ge 4$. In $\mathbb{F}_q[t]$ of characteristic ≥ 5, dimension and Bézout-type degree bounds for morphism spaces imply $M_s^{\mathbb{F}_q[t]}(B) - cB^2 \mathbf{1}_{s=4} \ll B^{s-3+\log_q 27}$. (Dimension estimates are proven inductively via [Mori 1979]'s bend-and-break. See [Harris–Roth–Starr 2004, Coskun–Starr 2009, Beheshti–Lehmann–Riedl–Tanimoto 2023, Glas 2025]. An asymptotic would then require finer information about the morphism spaces.)

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- By [Hooley 1986, Heath-Brown 1998], $M_6(B) \ll_\epsilon B^{3+\epsilon}$ under automorphy and GRH for Hasse–Weil L-functions of $\sum_{i=1}^{6} x_i^3 = \sum_{i=1}^{6} c_i x_i = 0$. By [Glas–Hochfilzer 2022], this is unconditional in $\mathbb{F}_q[t]$ of characteristic $\neq 3$.
- By [W. 2021], $M_6(B) \sim (\nu_1 + \nu_2)B^3$ under automorphy, GRH, and the Ratios and Square-free Sieve Conjectures. By [Browning–Glas–W. 2024], in $\mathbb{F}_q[t]$ of characteristic ≥ 5 the Ratios Conjecture suffices.
- Statistical results on $x^3 + y^3 + z^3 = n$ follow.

The square-root barrier

Let $S(\alpha, B) := \sum_{|x| \le B} e^{2\pi i \alpha x^3}$, or a smoothed variant. Let

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In view of Plancherel's theorem and square-root heuristics, an asymptotic $M_6(B) \sim \nu B^3$ should reflect some cancellation over α on the minor arcs. We need formulas, not just bounds! In 1981, Vaughan observed that Poisson summation implies

$$S(\alpha, B) \ll_\epsilon r^{-1/3} B + r^{1/2+\epsilon} (1 + B^3 |\alpha - a/r|)^{1/2},$$

which for $B^{3/4} \le r \le B^{3/2}$ and $|r\alpha - a| \le 1/B^{3/2}$ recovers the bound $S(\alpha, B) \ll_\epsilon B^{3/4+\epsilon}$. [Hooley 1986, Heath-Brown 1998] obtained square-root cancellation over a mod r via Deligne's bounds and over r via GRH, for "good" square-free r, giving

$$M_6(B) \ll_\epsilon (B^{3/4+\epsilon})^{6-2} = B^{3+4\epsilon},$$

after accounting for small r, "bad" r, and square-full r.

More details on Hooley et al., say in $\mathbb{F}_q[t]$

Let $F(\boldsymbol{x}) := \sum_{i=1}^{6} x_i^3$ and $\gcd(q, 6) = 1$. Let $L(s, \boldsymbol{c})$ be the L-function of $V_{\boldsymbol{c}} : F(\boldsymbol{x}) = \boldsymbol{c} \cdot \boldsymbol{x} = 0$, where $\boldsymbol{c} \in \mathbb{F}_q[t]^6$ and $\Delta(\boldsymbol{c}) := \operatorname{disc}(V_{\boldsymbol{c}}) = \prod(c_1^{3/2} \pm c_2^{3/2} \pm \cdots \pm c_6^{3/2}) \neq 0$.

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$$M_6^{\mathbb{F}_q[t]}(B) = \sum_{\boldsymbol{c} \ll B^{1/2}} \sum_{r \ll B^{3/2}} S_{\boldsymbol{c}}(r) \cdot I_{\boldsymbol{c}}(r),$$

where $I_{\boldsymbol{c}}(r)$ is an oscillatory integral and $S_{\boldsymbol{c}}(r)$ is an exponential sum (for a suitable additive character ψ on $\mathbb{F}_q[[t^{-1}]]$)

$$S_{\boldsymbol{c}}(r) := \frac{1}{|r|^{7/2}} \sideset{}{'}\sum_{a \bmod r} \sum_{x_1,\dots,x_6 \bmod r} \psi\left(\frac{aF(\boldsymbol{x}) + \boldsymbol{c} \cdot \boldsymbol{x}}{r}\right),$$

which is multiplicative in r and behaves differently at primes $\pi \nmid \Delta(\boldsymbol{c})$ versus $\pi \mid \Delta(\boldsymbol{c})$. If $\pi \nmid \Delta(\boldsymbol{c})$ and $\Re(s) \geq \epsilon$, then

$$\sum_{l \geq 0} S_{\boldsymbol{c}}(\pi^l)\, |\pi|^{-ls} = (1 + O_\epsilon(|\pi|^{-s-1/2} + |\pi|^{-2s}))\, L_\pi(s,\boldsymbol{c})^{-1}.$$

Let $F(\boldsymbol{x}) := \sum_{i=1}^{6} x_i^3$ and $\gcd(q,6)=1$. Let $L(s,\boldsymbol{c})$ be the L-function of $V_{\boldsymbol{c}} : F(\boldsymbol{x}) = \boldsymbol{c}\cdot\boldsymbol{x} = 0$, where $\boldsymbol{c} \in \mathbb{F}_q[t]^6$ and $\Delta(\boldsymbol{c}) := \operatorname{disc}(V_{\boldsymbol{c}}) = \prod(c_1^{3/2} \pm c_2^{3/2} \pm \cdots \pm c_6^{3/2}) \neq 0$. Let $|r| := q^{\deg r}$. By Poisson summation a la Vaughan,

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$$\frac{\text{Igusa local zeta function}}{\text{Hasse–Weil local factor}} = \frac{\sum_{l \geq 0} S_{\boldsymbol{c}}(\pi^l)\,|\pi|^{-ls}}{L_\pi(s,\boldsymbol{c})^{-1}},$$

and $L(s,\boldsymbol{c})^{-1}$ itself. (Other ideas are required for $\Delta(\boldsymbol{c}) = 0$.)

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- GRH controls $L(s, \boldsymbol{c})^{-1}$ for $\Re(s) > \frac{1}{2}$.
- The local errors are controlled in part by $\Delta(\boldsymbol{c})$, which measures the extent to which $V_{\boldsymbol{c}}$ is singular.
- Eventually, the bound $M_6(B) \ll_\epsilon B^{3+\epsilon}$ follows. Sources of ϵ include GRH, local errors, and dyadic summation over r.

Theorem (Glas–Hochfilzer 2022)

Unconditionally, $M_6^{\mathbb{F}_q[t]}(B) \ll_{q,\epsilon} B^{3+\epsilon}$.

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Theorem (Browning–Glas–W. 2024; unconditional)

The full main term $(\nu_1 + \nu_2)B^3$ *conjectured by Hooley, Manin, et al. arises from the locus* $\Delta(\boldsymbol{c}) = 0$ *in the circle method.*

Proof idea.

Whereas $S_{\boldsymbol{c}}(\pi) \ll 1$ for $\pi \nmid \Delta(\boldsymbol{c})$, it turns out that if $\Delta(\boldsymbol{c}) = 0$, then typically $S_{\boldsymbol{c}}(\pi) = |\pi|^{1/2} + O(1)$ for most primes π, thanks to delicate results of [Beauville 1977] on quadric surface and conic bundles. Since the main term $|\pi|^{1/2}$ is independent of $\boldsymbol{c}$ (a conductor-dropping phenomenon), we may eventually replace the integral $I_{\boldsymbol{c}}(r)$ with a suitable $\boldsymbol{c}$-average thereof. □

To prove $M_6^{\mathbb{F}_q[t]}(B) \ll B^3$, say, it remains to show that

$$\sum_{\substack{\boldsymbol{c} \ll B^{1/2} \\ \Delta(\boldsymbol{c}) \neq 0}} \sum_{r \ll B^{3/2}} S_{\boldsymbol{c}}(r) \cdot I_{\boldsymbol{c}}(r) \ll B^3.$$

We want to control the LHS using the Ratios Conjecture for $L(s, \boldsymbol{c})$. A beautiful symmetry observation of Glas–Hochfilzer shows that $I_{\boldsymbol{c}}(r)$ is constant on dyadic ranges $|r| = N$. Let

$$\Sigma(B, N_0, N_1) := \sum_{\substack{\boldsymbol{c} \ll B^{1/2} \\ \Delta(\boldsymbol{c}) \neq 0}} I_{\boldsymbol{c}}(t^{\log_q N}) \sum_{\substack{|r_0| = N_0 \\ r_0 \mid \Delta(\boldsymbol{c})^\infty}} S_{\boldsymbol{c}}(r_0) \sum_{\substack{|r_1| = N_1 \\ \gcd(r_1, \Delta(\boldsymbol{c})) = 1}} S_{\boldsymbol{c}}(r_1),$$

for $N_0 N_1 \ll B^{3/2}$.

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for $N_0 N_1 \ll B^{3/2}$. By Glas–Hochfilzer, $\Sigma(B, N_0, N_1) \ll_\epsilon B^{3+\epsilon}$. If $N_\infty := B^{3/2}/(N_0 N_1)$, we [Browning–Glas–W.] further prove

$$\Sigma(B, N_0, N_1) \ll B^3 N_0^{-\delta} N_\infty^{-\delta}.$$

When N_0 and N_∞ are tiny, we mainly use Ratios and other L-function techniques, but in general we also need other ideas, based on the size and factorization of $\Delta(\boldsymbol{c})$.

If for instance $N_0 \to \infty$, then we need a suitable geometric analog of the Sarnak–Xue Density Hypothesis to control failures of square-root cancellation in $\max_{|r_0|=N_0} |S_{\boldsymbol{c}}(r_0)|$. (These are like failures of a Ramanujan-type conjecture.)

Proposition (Browning–Glas–W.; unconditional)

Let $N_0 \ll B^{3/2}$. Let $\lambda > 0$. As $\boldsymbol{c} \ll B^{1/2}$ varies,

$$\Pr[\max_{|r_0|=N_0} |S_{\boldsymbol{c}}(r_0)| \geq \lambda \cdot N_0^{1/2-\delta}] \ll \lambda^{-2}.$$

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- This result would *fail* if we replaced $x_1^3 + \cdots + x_6^3$ with $x_1^2 + \cdots + x_6^2$; the fact $\deg F \geq 3$ is crucial.
- The proof uses results and ideas of Hooley (on square-root cancellation for mildly *singular* varieties), Busé–Jouanolou (putting discriminants in *square* ideals), Poonen (on large square divisors), and Ekedahl (on the geometric sieve).

The Ratios Conjecture is used as follows. By Poincaré duality and a large sieve, we have a very good approximation

$$\sum_{\gcd(r_1,\Delta(\boldsymbol{c}))=1} S_{\boldsymbol{c}}(r_1)|r_1|^{-s} \approx \frac{1}{L(s,\boldsymbol{c})P(s)}$$

on average over $\boldsymbol{c}$, where $P(s) = \zeta_{\mathbb{F}_q[t]}(2s)L(s+\frac{1}{2},\{F=0\})$. Multiplying the $(0,2)$ case of the Ratios Conjecture ($\sum \frac{1}{LL}$),

$$\mathbb{E}_{\substack{\boldsymbol{c} \ll B^{1/2} \\ \Delta(\boldsymbol{c}) \neq 0}}\left[\frac{1}{\prod_{j=1}^{2} P(s_j)L(s_j,\boldsymbol{c})}\right] = A(s_1,s_2)(\zeta_{\mathbb{F}_q[t]}(s_1+s_2)+O(B^{-3\beta})),$$

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by $N_1^{s_1+s_2} \ll (B^{3/2})^{\Re(s_1+s_2-1)}N_1$, and averaging over the lines $\Re(s_i) = \frac{1}{2}+\beta$, we get (by a contour shifting argument)

$$\mathbb{E}_{\substack{\boldsymbol{c}\ll B^{1/2}\\ \Delta(\boldsymbol{c})\neq 0}}\left|\sum_{\substack{|r_1|=N_1\\ \gcd(r_1,\Delta(\boldsymbol{c}))=1}} S_{\boldsymbol{c}}(r_1)\right|^2 \ll N_1.$$

This is perfect (log-free) square-root cancellation!

Theorem (Browning–Glas–W. 2024)

Assume the Ratios Conjecture for $\sum_{\boldsymbol{c}\in\mathbb{F}_q[t]^6:\,\deg(c_i)\leq Z} \frac{1}{L(s_1,\boldsymbol{c})L(s_2,\boldsymbol{c})}$ *for* $\Delta(\boldsymbol{c}) \neq 0$*, as* $Z \to \infty$*. Then* $M_6^{\mathbb{F}_q[t]}(B) \ll_q B^3$*. Moreover,* $x^3 + y^3 + z^3 = n$ *is soluble in monic elements* $x, y, z \in \mathbb{F}_q[t]$ *of degree exactly* $\frac{1}{3}\deg n$ *for a positive density of* $n \in \mathbb{F}_q[t]$*.*

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The proof combines biases and cancellations "beyond the Weil conjectures" (loosely speaking):

- Local biases on the boundary $\Delta = 0$. This uses [Beauville].
- Local cancellations near the boundary. Here there are significant traces of boundary biases, handled by an eclectic collection of inputs (including [Huang, Hooley, Busé–Jouanolou, Poonen, Ekedahl]).
- Global cancellations when Δ is close to square-free and maximal size. This uses Ratios, Poincaré duality, and a large sieve.

Theorem (Browning–Glas–W. 2024)

Assume sufficient progress on moments of $\frac{1}{L(s,\boldsymbol{c})}$ *for* $\Delta(\boldsymbol{c}) \neq 0$. *Then* $M_6^{\mathbb{F}_q[t]}(B) \sim (\nu_1 + \nu_2)B^3$. *Moreover,* $x^3 + y^3 + z^3 = n$ *is soluble in elements* $x, y, z \in \mathbb{F}_q[t]$ *of degree* $\sim \frac{1}{3} \deg n$ *for a density* 1 *of elements* $n \in \mathbb{F}_q[t]$.

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The proof builds on ideas of many authors, e.g. the following:

- Ghosh–Sarnak, Diaconu (log-K3 variance analysis),
- Kloosterman, Hooley 1986, Heath-Brown,
- Beauville (quadric bundles over $\mathbb{P}^2$), Getz, Tran,
- Rubinstein–Sarnak (Chebyshev's bias via prime squares),
- Deligne (GRH), Hooley 1994 (singular cubics),
- Huang ($\approx \mathbb{Q}$-points), Busé–Jouanolou ($\Delta \in (f, (f')^2)$),
- Bhargava (Ekedahl sieve), Poonen (square-free sieve),
- Kisin (local constancy of L-factors).

A deformation along the square-root barrier

Browning, Munshi, and I (joint work in progress) seek to use the [Duke–Friedlander–Iwaniec 1993, Heath-Brown 1996] version of the circle method to prove an unconditional asymptotic over $\mathbb{Q}$ for the singular 6-variable homogeneous cubic equation

$$x_1 y_1^2 + x_2 y_2^2 + x_3 y_3^2 = 0$$

of *Perazzo* type (singular along the plane $y_1 = y_2 = y_3 = 0$).

A deformation along the square-root barrier

Browning, Munshi, and I (joint work in progress) seek to use the [Duke–Friedlander–Iwaniec 1993, Heath-Brown 1996] version of the circle method to prove an unconditional asymptotic over $\mathbb{Q}$ for the singular 6-variable homogeneous cubic equation

$$x_1 y_1^2 + x_2 y_2^2 + x_3 y_3^2 = 0$$

of *Perazzo* type (singular along the plane $y_1 = y_2 = y_3 = 0$). This can be viewed as a deformation of $\sum_{i=1}^{6} x_i^3 = 0$. Whereas L-functions turn out to be less important here, certain divisor problems play a more prominent role (which we handle via Hooley Δ-functions and a uniform, multivariate-polynomial Nair–Tenenbaum bound of [de la Bretèche–Tenenbaum 2024]). Sums like $\sum_{\boldsymbol{m},\boldsymbol{n} \ll T} \Delta(G(\boldsymbol{m},\boldsymbol{n}))$ turn out to be useful, where $G := 2\sum_{i=1}^{3} m_i^2 n_i^4 - (\sum_{i=1}^{3} m_i n_i^2)^2$ is dual to $\sum_{i=1}^{3} x_i y_i^2$ with respect to the pairing $(\boldsymbol{m},\boldsymbol{n}) \cdot (\boldsymbol{x},\boldsymbol{y}) = \sum_{i=1}^{3}(m_i x_i + n_i y_i)$.

Comparison with other 6-variable cubic forms

Given a hypersurface $V = \{F = 0\}$, let $V^\vee$ be its *dual variety*: the image of V under the Gauss map $\boldsymbol{x} \mapsto \nabla F(\boldsymbol{x})$. Roughly, $\deg(V^\vee)$ is a global curvature measure of V.

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1. Via multiplicative harmonic analysis [Batyrev–Tschinkel 1998] or torsors [Salberger 1998], Manin's conjecture is proven for toric varieties, including the *Perazzo primal* $x_1x_2x_3 = y_1y_2y_3$. This is self-dual, so $\deg(V^\vee) = 3$.
2. Via torsors and lattices, [Blomer–Brüdern–Salberger 2014] treated $x_1y_2y_3 + x_2y_1y_3 + x_3y_1y_2 = 0$ (think $\sum_{i=1}^3 \frac{x_i}{y_i} = 0$). This is self-dual, so $\deg(V^\vee) = 3$.
3. Via a geometric reduction to [Schmidt 1995], [Derenthal 2025] treated $\det(S) = 0$ in symmetric 3×3 matrices S. This is dual to the Veronese surface, so $\deg(V^\vee) = 4$.
4. Our $\sum_{i=1}^3 x_i y_i^2 = 0$ is dual to a degree-6 hypersurface.
5. For $\sum_{i=1}^6 x_i^3 = 0$, we have $\deg(V^\vee) = \deg(\Delta) = 48$.

Direct approach?

Our main goal in [Browning–Munshi–W. 2025+] is to break the square-root barrier for a specific cubic form, unconditionally, *in the circle method*. However, it is worth pointing out that

$$\sum_{i=1}^{3} x_i y_i^2 = 0$$

has the structure of both a conic bundle and a linear bundle, and thus could potentially yield to other (difficult) methods.

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has the structure of both a conic bundle and a linear bundle, and thus could potentially yield to other (difficult) methods. [Le Boudec 2015], using uniform bounds on conics by [Browning–Swarbrick Jones 2014], showed that

$$\#\{(\boldsymbol{x},\boldsymbol{y}) \in (\mathbb{Z}^3_{\neq 0})^2_{\mathrm{prim}} : |\boldsymbol{x}|^2|\boldsymbol{y}| \leq B,\ \sum_{i=1}^{3} x_i y_i^2 = 0\} \asymp B \log B.$$

[Browning–Heath-Brown 2018] proved Manin's conjecture for $\sum_{i=1}^{4} x_i y_i^2 = 0$ in $(\mathbb{P}^3)^2$; [Dehnert 2019] treats $\sum_{i=1}^{8} x_i y_i^3 = 0$. They combine lattice methods with the circle method.

Better than square-root cancellation or not

Fix $F \in \mathbb{Z}[x_1, \ldots, x_s]$, irreducible over $\mathbb{C}$. Let

$$T_{\boldsymbol{c}}(p) := \frac{1}{p^{(s-1)/2}} \sum_{F(\boldsymbol{x}) \equiv 0 \bmod p} e^{2\pi i \boldsymbol{c} \cdot \boldsymbol{x}/p}.$$

Then $\mathbb{E}_{\boldsymbol{c} \in \mathbb{F}_p^s}[|T_{\boldsymbol{c}}(p)|^2] = \frac{1}{p^{s-1}} \#\{F(\boldsymbol{x}) \equiv 0 \bmod p\} \sim 1$.

- Call T *focused* if there exists a nonzero $G \in \mathbb{Z}[c_1, \ldots, c_s]$ such that $\mathbb{E}_{\boldsymbol{c} \in \mathbb{F}_p^s}[|T_{\boldsymbol{c}}(p)|^2 \mathbf{1}_{G(\boldsymbol{c})=0}] \sim 1$.
- Call T *balanced* if for every nonzero $G \in \mathbb{Z}[c_1, \ldots, c_s]$, we have $\mathbb{E}_{\boldsymbol{c} \in \mathbb{F}_p^s}[|T_{\boldsymbol{c}}(p)|^2 \mathbf{1}_{G(\boldsymbol{c})=0}] \ll_G p^{-1/2}$.

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T is focused if $F = \sum_{i=1}^3 x_i y_i^2$, and balanced if $F = \sum_{i=1}^6 x_i^3$. In the former case, primes $p \mid G$ are more significant than $p \nmid G$. In the latter case, $p \nmid \Delta$ are more significant than $p \mid \Delta$. (Inspired in part by correspondence with Hooley, [Katz 1989] defined an *A-number*. If $A = 0$, then $\mathbb{E}_{\boldsymbol{c} \in \mathbb{F}_p^s}[|T_{\boldsymbol{c}}(p)|] \ll p^{-1/2}$, so T is focused by [Fouvry–Katz 2001]. If $A \neq 0$, then at least if $F = 0$ is smooth then T is balanced by [Fouvry–Katz 2001].)

Dyadic analysis

Let $G := 2\sum_{i=1}^{3} m_i^2 n_i^4 - (\sum_{i=1}^{3} m_i n_i^2)^2$ and $\boldsymbol{c} := (\boldsymbol{m}, \boldsymbol{n}) \in \mathbb{Z}^6$.
The circle method for $\sum_{i=1}^{3} x_i y_i^2 = 0$ leads to the sums

$$\Sigma(B, N_0, N_1) := \sum_{\substack{\boldsymbol{c} \ll B^{1/2} \\ G(\boldsymbol{c}) \neq 0}} I_{\boldsymbol{c}}(N_0 N_1) \sum_{\substack{r_0 \asymp N_0 \\ r_0 \mid G(\boldsymbol{c})^\infty}} S_{\boldsymbol{c}}(r_0) \sum_{\substack{r_1 \asymp N_1 \\ \gcd(r_1, G(\boldsymbol{c}))=1}} S_{\boldsymbol{c}}(r_1)$$

for $N_0 N_1 \ll B^{3/2}$. It is relatively straightforward to show that

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If $N_\infty := B^{3/2}/(N_0 N_1)$, we [Browning–Munshi–W.] prove

$$\Sigma(B, N_0, N_1) \ll_\epsilon B^3 (\log B)^\epsilon (\log N_1)^{-1/\epsilon} N_\infty^{-\delta}$$

for some absolute $\delta > 0$, where $(\log B)^\epsilon$ comes from using the Hooley Δ-function to control divisors in dyadic ranges $\ll N_0$, where $\log N_1$ comes from bounds on $\zeta(s)^{-1}$ for $\Re(s) \approx 1$, and where N_∞ comes from cancellation over arcs à la [Huang 2020]. Summing over $N_1, N_\infty \geq 1$ leaves $B^3(\log B)^\epsilon$.

Tentative theorem

Theorem (Browning–Munshi–W. 2025+)

Let $F(\boldsymbol{x}, \boldsymbol{y}) := \sum_{i=1}^{3} x_i y_i^2$. *Let* $w \colon (\mathbb{R}^3)^2 \to \mathbb{R}$ *be smooth, and supported on a compact subset of* $\mathbb{R}^3 \times (\mathbb{R}^\times)^3$. *Then*

$$\sum_{\substack{\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}^3 \\ F(\boldsymbol{x}, \boldsymbol{y}) = 0}} w\left(\frac{\boldsymbol{x}, \boldsymbol{y}}{B}\right) = c(w) B^3 \log B + O_{w,\epsilon}(B^3 (\log B)^\epsilon).$$

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$$\sum_{\substack{\boldsymbol{x},\boldsymbol{y}\in\mathbb{Z}^3 \\ F(\boldsymbol{x},\boldsymbol{y})=0}} w\left(\frac{\boldsymbol{x},\boldsymbol{y}}{B}\right) = c(w)B^3\log B + O_{w,\epsilon}(B^3(\log B)^\epsilon).$$

In order to remove the $(\log B)^\epsilon$ term, one would want an asymptotic (or a sharp upper bound) for sums like

$$\sum_{\boldsymbol{m},\boldsymbol{n}\ll B^{1/2}} \sum_{r_0\asymp B^{3/2}} \mathbf{1}_{r_0 \mid G(\boldsymbol{m},\boldsymbol{n})\neq 0} \approx \sum_{\boldsymbol{m},\boldsymbol{n}\ll B^{1/2}} \sum_{a\asymp b\asymp B^{3/2}} \mathbf{1}_{G(\boldsymbol{m},\boldsymbol{n})=\pm ab},$$

where $G := 2\sum_{i=1}^{3} m_i^2 n_i^4 - (\sum_{i=1}^{3} m_i n_i^2)^2$. The divisor $|G|/r_0$ is usually $\asymp r_0$, since $\deg G = 6$; it would be smaller if $\deg G \leq 5$. RHS is a divisor problem that might be interesting over $\mathbb{F}_q[t]$?

Recent progress on geometric L-functions

Let $2 \nmid q$. Let $\mu(r)$ be the Möbius function over $\mathbb{F}_q[t]$, and let $\chi_d(r) = (\frac{d}{r})$ be the Jacobi symbol over $\mathbb{F}_q[t]$.

Theorem (Bergström–Diaconu–Petersen–Westerland, Miller–Patzt–Petersen–Randal-Williams, W. 2024)

If $1 \leq D = 2g+1$ and $1 \leq R \leq \alpha D$, and $q \ggg_\alpha 1$, then

$$\frac{\sum_{|d|=q^D} \sum_{|r|=q^R} \mu(r)\chi_d(r)}{q^D q^{R/2}} \ll \frac{q^{O_\alpha(1)}}{q^{D/38.1}},$$

where the sums over d and r run through square-free, monic $d, r \in \mathbb{F}_q[t]$ with $\deg d = D$ and $\deg r = R$, respectively.

Theorem (Same papers; new for $q = p \equiv 1 \bmod 4$)

The set $\{d : L(\frac{1}{2}, \chi_d) = 0\}$ has upper density $o_{q\to\infty}(1)$.

First steps of [BDPW, MPPRW]

The proof begins as the Ratios Recipe does, using the functional equation to expand the product of L-functions in the numerator. But then instead of directly working with Dirichlet coefficients, one expands a given dyadic sum of coefficients in terms of L-function zeros, taking into account natural symmetries of orthogonal or symplectic type.

First steps of [BDPW, MPPRW]

The proof begins as the Ratios Recipe does, using the functional equation to expand the product of L-functions in the numerator. But then instead of directly working with Dirichlet coefficients, one expands a given dyadic sum of coefficients in terms of L-function zeros, taking into account natural symmetries of orthogonal or symplectic type. On each irreducible piece of the resulting decomposition, one uses the Grothendieck–Lefschetz trace formula to pass from a sum over $\mathbb{F}_q$ to a sum of traces of $Frob_q \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ on $\overline{\mathbb{F}}_q$-objects, the first few being the most important in view of Deligne's and Betti bounds. Ultimately, reduce to proving *uniform twisted homological stability*,

$$k + 1 \lll n \Rightarrow H_k(X_n, \mathbb{S}_\lambda^{Sp} V_n) \xrightarrow{\cong} H_k(X_{n+1}, \mathbb{S}_\lambda^{Sp} V_{n+1}),$$

where $X_n \subseteq \mathbb{C}^n$ consists of monic square-free polynomials of degree n, and rank $V_n \sim n \sim \log_q$-conductor of $L(s, \chi_d)$ for $\deg d = n$, and $\mathbb{S}_\lambda^{Sp} \subseteq V_n^{\otimes|\lambda|}$ is a symplectic Schur functor.

Statistical topology and homological stability

In a 2004 Abel interview, Singer predicted “a new subject of statistical topology” in which “rather than count the number of holes, Betti-numbers, etc., one will be more interested in the distribution of such objects on noncompact manifolds as one goes out to infinity” and “insights will come from condensed matter physics as to what, statistically, the topology might look like as one approaches infinity”.

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Some such limiting behavior may boil down to *monodromy*.

Examples of interesting families (before puncturing objects):

1. *Smooth hypersurfaces* of degree $2g \to \infty$ in $\mathbb{P}(g, 1, 1)$. There are $3g + 2$ affine parameters and $2g$ zeros, with homological stability established by [BDPW, MPPRW].
2. *Smooth hypersurfaces* of degree $e \to \infty$ in $\mathbb{P}^r$. Here, $\binom{e+r}{r} \sim \frac{e^r}{r!}$ parameters and $\frac{(e-1)^r-(-1)^r}{e/(e-1)} \sim e^r$ zeros.
3. *Smooth hypersurfaces* in $\mathbb{P}^1 \times V_{\mathbb{P}^5}(x_1^3 + \cdots + x_6^3)$ of bi-degree $(n, 1)$. Here, $\sim 6n$ parameters and $\sim 48n$ zeros.

Interesting local systems on a family X_n

Examples of interesting families (before puncturing objects):

1. *Smooth hypersurfaces* of degree $2g \to \infty$ in $\mathbb{P}(g,1,1)$. After completing the square, these are $y^2 = d(t)$, where $\deg d = 2g$. Stability holds by [BDPW, MPPRW].
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Let $f \in X_n$ and let Y_f be the variety indexed by f in the family. At least two natural kinds of local systems V_n exist.

- The local system $V_n = H_*(Y_f, \mathbb{Q})$ is relevant to the statistics of $L(s, f) \in \{L(s, \chi_d), L(s, \boldsymbol{c}), \dots\}$.
- If Y_f fibers over $t \in \mathbb{P}^1$, let $\nu \colon X_n' \to X_n$ be the natural S_N cover, where $N = \deg_t(\mathrm{disc}_{\mathbb{F}_q(t)}(Y_f))$. Then $V_n = \mathbb{Q}^{\nu^{-1}(f)}$ is relevant to divisor statistics, prime values, etc. of $\mathrm{disc}_{\mathbb{F}_q(t)}(Y_f) \in \{d(t), \Delta(\boldsymbol{c}), G(\boldsymbol{m}, \boldsymbol{n}), \dots\}$.

Ingredients in [BDPW, MPPRW] for a family X_n

1. Bounds on twisted Betti numbers of X_n, exponential in n.
2. Gluing maps $(X_m, *_m) \times (X_n, *_n) \to (X_{m+n}, *_{m+n})$ and $V_m \oplus V_n \to V_{m+n}$ of a semi-algebraic nature.
3. A loop in $(X_{m+n}, *_{m+n})$ that conjugates $\pi_1(X_m \times *_n)$ into $\pi_1(*_n \times X_m)$. (A braiding on the groups $\pi_1(X_\bullet)$.)

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4. Surjectivity of a monodromy representation $\pi_1(X_n) \to Q_n$, where $Q_n \subset \mathrm{GL}((V_n)_{*_n})$ is a classical arithmetic group.
5. High connectivity of two cell complexes, one associated with the sequence $\pi_1(X_n)$ and one with the sequence Q_n.
6. An efficient construction of $\mathbb{Q}[\pi_1(X_m)]$-module cells for Q_n (Galatius–Kupers–Randal-Williams), and for $(\widetilde{X}_n, \tilde{*}_n)$.

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7. Classical representation theory of the Zariski closure $\overline{Q}_n(\mathbb{C})$ (Cauchy, Schur, Weyl, Littlewood, et al.).
8. Automorphic representation theory of $\overline{Q}_n(\mathbb{R})$ (Matsushima, Borel, Vogan–Zuckerman, Franke, et al.).
9. Algebraic versions of the topological inputs 1 and 2.

Role of gluing and braiding for a family X_n

Stability of $H_k(X_n)$ is equivalent to $\frac{\partial}{\partial n} H_k(X_n) = 0$. If we induct, comparing first derivatives naturally leads to diagrams like

$$\begin{array}{ccc} \pi_1(X_n) & \xrightarrow{\times g_b} & \pi_1(X_{n+b}) \\ {\scriptstyle \times g_1}\big\downarrow & & \big\downarrow{\scriptstyle \times g_1} \\ \pi_1(X_{n+1}) & \xrightarrow{\times g_b} & \pi_1(X_{n+1+b} = X_{n+b+1}), \end{array}$$

which only commutes up to braiding. (The other diagram, with vertical maps $g_1\times$ in place of $\times g_1$, does commute. In practice, a stability proof requires both sorts of diagrams.)

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If X_n consisted of *ordered* configurations of n distinct points, then $\frac{|X_n(\mathbb{F}_q)|}{q^n} = \frac{q(q-1)\cdots(q-n+1)}{q^n} = 1 - \binom{n}{2}\frac{1}{q} + O_n(\frac{1}{q^2})$. The X_n glue without braiding. The $H_1(X_n)$ do not stabilize.

If we (algebraically) mod out X_n by S_n, then we get the monic square-free polynomials of degree n, whose fundamental groups are the braid groups, which glue with braiding. Stable $1 - \frac{1}{q}$.

Betti bounds for a class of geometric families

Let $P \in \mathbb{Z}[t, x_1, \ldots, x_s]$, e.g. $P \in \{d, \Delta(\boldsymbol{c}), G(\boldsymbol{m}, \boldsymbol{n})\}$. Let

$$X_n(R) := \{(f_1, \ldots, f_s) \in R[t]^s_{n,1} : P(f_1, \ldots, f_s) \in R[t]_{n \deg P, 1}, \\ \operatorname{disc}_t(P(f_1, \ldots, f_s)) \in R^\times\},$$

where $R[t]_{n,1}$ is the set of monic degree-n polynomials in $R[t]$.

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Proposition (W. 2025+)

For every local system $\mathcal{L}$ on $X_n(\mathbb{C})$ of $\mathbb{C}$-vector spaces,

$$\sum_{i \geq 0} \dim H_i(X_n(\mathbb{C}), \mathcal{L}) \leq \exp(O_{P,s}(n+1)) \operatorname{rank} \mathcal{L}.$$

Proof idea.

$$\frac{O(1)^{1+n\deg P + 2n \deg P} O(1+\deg P)^{(n+1)s + 2n\deg P} O(n \deg P)^{n \deg P} \text{ (Dwork theory)}}{(n \deg P)! \text{ (cover theory)}} \leq$$

$O_{P,s}(1)^{n+1} e^{n \deg P}$, by Stirling's formula. □

Gluing for homogeneous $P \in \mathbb{Z}[x_1, \ldots, x_s]$

Again, this includes the options $P \in \{d, \Delta(\boldsymbol{c}), G(\boldsymbol{m}, \boldsymbol{n})\}$. Let

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Lemma (W. 2025+)

Let $k = \overline{k}$. If $(j_1, \ldots, j_s) \in X_m(k)$ and $(f_1, \ldots, f_s) \in X_n(k)$, then $(j_1(t + \epsilon^{-1})f_1(t), \ldots, j_s(t + \epsilon^{-1})f_s(t)) \in X_{m+n}(k)$ for all ϵ in a punctured neighborhood of $0 \in k$.

Proof idea.

Rouché's theorem over $\mathbb{C}$, or v_ϵ analysis in general, produces a disjoint union of two ϵ^{-1}-separated sets of roots of $P(\cdots)$. □

(A braiding is $\approx$ given by a 180° rotation of t about $t + \frac{1}{2}\epsilon^{-1}$.)

Addendum to the talk

[Heath-Brown 1992] conjectured $\#\{a^3 + b^3 + c^3 = n\} = \infty$ if $n \not\equiv \pm 4 \bmod 9$. For further background on sums of cubes, see `https://en.wikipedia.org/wiki/Sums_of_three_cubes`.

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