A cubic fourfold with sextic dual

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joint work with Tim Browning and Ritabrata Munshi

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Abstract

I will report on joint work in progress with Tim Browning and Ritabrata Munshi, concerning the use of the circle method to count rational points on a particular cubic hypersurface in six variables.

This problem lies at the square-root barrier, and I will try to put it in context as time permits.

Background: Sums of Powers

Let $r_k(n) = \#\{\boldsymbol{x} \in \mathbb{Z}_{\geq 0}^k : \sum_{i=1}^k x_i^k = n\}$ for $k \in \{2, 3\}$.

Then for the first moment, regardless of k:

$$\sum_{n \leq y} r_k(n) = \#\{\boldsymbol{x} \in \mathbb{Z}_{\geq 0}^k : \sum_{i=1}^k x_i^k \leq y\} \sim \alpha_k (y^{1/k})^k = \alpha_k y.$$

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But for the second moment, there is a difference in behavior:

$$\alpha_k^2 y \lesssim \sum_{n \leq y} r_k(n)^2 \sim \begin{cases} \beta_2 y \log y & \text{if } k = 2 \\ \beta_3 y & \text{conjectured if } k = 3 \text{ (cf. Hooley).} \end{cases}$$

(Here $\alpha_k, \beta_k > 0$ are constants.)

The Case of Squares

Let $r_k(n) = \#\{\boldsymbol{x} \in \mathbb{Z}_{\geq 0}^k : \sum_{i=1}^k x_i^k = n\}$ for $k \in \{2, 3\}$.

In fact, $r_2(n)$ is supported on a small set (on which it is typically large):

$$\#\{n \leq y : r_2(n) \neq 0\} \sim \frac{c_{\text{Landau–Ramanujan}}\, y}{(\log y)^{1/2}}.$$

This follows from [Landau 1908], using the multiplicative structure of r_2.

Density of Support

Let $r_k(n) = \#\{\boldsymbol{x} \in \mathbb{Z}_{\geq 0}^k : \sum_{i=1}^k x_i^k = n\}$ for $k \in \{2, 3\}$.

In particular, for squares:

$$\text{density}(\text{Supp } r_2) = 0.$$

In contrast, for cubes:

Conjecture [Cf. Hooley 1986; cf. Erdős Problem #940]

$\text{density}(\text{Supp } r_3) > 0.$

(Deshouillers et al. have a precise prediction.)

Infinitude of Solutions for Cubes

Conjecture [Heath-Brown 1992]

If $n \not\equiv \pm 4 \bmod 9$,

$$\#\{(a, b, c) \in \mathbb{Z}^3 : a^3 + b^3 + c^3 = n\} = \infty.$$

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Infinitude is subtle compared to elliptic curves or Markoff-type surfaces, where group actions tend to produce new points for free.

One could also ask for Zariski density of the set of integral points, to eliminate cheap solutions like $t^3 + (-t)^3 + 1^3 = 1$.

Known Rational and Integral Points

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If $n \not\equiv \pm 4 \bmod 9$,

$$\#\{(a,b,c) \in \mathbb{Z}^3 : a^3 + b^3 + c^3 = n\} = \infty.$$

- [Booker–Sutherland]: LHS ≥ 1 for $n = 33, 42$.
- [Ryley, Verebrusov, Mahler, Segre, Lehmer, Beukers, Hassett–Tschinkel, et al.]: True in $\mathbb{Q}$ and $\mathbb{Z}[n^{1/3}]$, even for Zariski density.

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- [Wooley 1995+]: Showed that

  $$\#\{n \leq y : r_3(n) \neq 0\} \gg y^{0.917},$$

  in fact with a, b, c smooth (and positive).

The Second Moment

[Vaughan 1986+] showed that:

$$M(y) := \sum_{n \le y} r_3(n)^2 \ll \frac{y^{7/6}}{(\log y)^{\frac{5}{2}-\varepsilon}}.$$

Let $f(\mathbf{x}) = \sum_{i=1}^{3} x_i^3$. Then $M(y)$ is a point count

$$M(y) = \#\{(\mathbf{x}, \mathbf{x}') \in y^{1/3}\Omega : f(\mathbf{x}) = f(\mathbf{x}')\}$$

in the dilation of a certain region $\Omega \subseteq [-1, 1]^6$ by $y^{1/3}$.

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Conjecture [Hooley 1986; cf. Manin–Peyre, Vaughan–Wooley]

$$M(y) \sim (c_1 + c_2)y,$$

where $c_1 = \prod_{v \le \infty} \sigma_v$ (product of local densities) and c_2 accounts for $\mathbf{x}' \in S_3\mathbf{x}$ (linear solutions).

Remark: Hooley's Conjecture implies density(Supp r_3) > 0.

Conditional Results and Function Field Results

Let $f(\boldsymbol{x}) = \sum_{i=1}^{3} x_i^3$. Then $M(y)$ is a point count

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- [Hooley 1986, Heath-Brown 1998]: $M(y) \ll y^{1+\varepsilon}$ under GRH (for certain geometric Hasse–Weil L-functions; those associated to $\sum_{i=1}^{6} x_i^3 = \sum_{i=1}^{6} b_i x_i = 0$ where $\boldsymbol{b} \in \mathbb{Z}^6$).

Conditional Results and Function Field Results

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- [W. 2021]: Hooley's Conjecture $M(y) \sim (c_1 + c_2)y$ holds under GRH, the Ratios Conjecture (of Conrey et al.), and the Square-free Sieve Conjecture (SFSC).
- Over $\mathbb{F}_q[t]$, GRH and SFSC are known by Deligne and Poonen, respectively. Thus removed by [Glas–Hochfilzer] and [Browning–Glas–W.]. In particular, $M(y) \ll y^{1+\varepsilon}$ is unconditionally known in $\mathbb{F}_q[t]$ if $\mathrm{char}(\mathbb{F}_q) \neq 3$. Ratios remains open.

The Circle Method: Setup

Hooley, Heath-Brown, et al. use the circle method for the hypersurface $\sum_{i=1}^{6} x_i^3 = 0$.

In general, let $F \in \mathbb{Z}[\mathbf{x}]$ be a cubic form in 6 variables. Let

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The basic identity underlying the circle method ($e(t) := e^{2\pi i t}$):

$$N_F(B) = \int_{\mathbb{R}/\mathbb{Z}} \sum_{\boldsymbol{x} \in [-B,B]^6} e(\theta F(\boldsymbol{x}))\, d\theta.$$

For most $\theta \in \mathbb{R}/\mathbb{Z}$, we expect $\sum_{\boldsymbol{x}} \ll B^{3+\varepsilon}$. Thus we need cancellation over θ to prove $N_F(B) \sim cB^3(\log B)^a$. This is the *square-root barrier*, first surpassed by [Kloosterman 1926] for quadratic forms in 4 variables.

Cancellation over Minor Arcs

Let $F \in \mathbb{Z}[\boldsymbol{x}]$ be a cubic form in 6 variables. Let

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- We will use the method of [Kloosterman 1926, Hooley, DFI, Heath-Brown, Getz, Tran, et al.].

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- We will use the method of [Kloosterman 1926, Hooley, DFI, Heath-Brown, Getz, Tran, et al.].
- [Brüdern–Wooley 2019] have another approach for a particular cubic in 8 variables.
- Brüdern and Wooley have various additional works on certain systems of equations.
- See also [Leung–Pandey 2026], which is morally closely related to a pair of quadratic equations in 8 variables.

The Main Theorem

Conditionally, $N_F(B) \sim cB^3$ for $F = \sum_{i=1}^{6} x_i^3$. Note that by Weyl differencing,

$$\{\textstyle\sum_{i=1}^{6} x_i^3 = 0\} \cong_{\mathbb{Q}} \{\textstyle\sum_{i=1}^{3}(3h_i y_i^2 + h_i^3) = 0\},$$

which is a family of quadrics.

Unconditionally, using the circle method:

Theorem [Browning–Munshi–W.]

Let $F(\boldsymbol{x}, \boldsymbol{y}) := \sum_{i=1}^{3} x_i y_i^2$. Then

$$N_F(B) \sim \frac{\sigma_\infty}{\zeta(3)} B^3 \log B,$$

where

$$\sigma_\infty = \lim_{\epsilon \to 0} (2\epsilon)^{-1} \operatorname{vol}\{(\boldsymbol{x}, \boldsymbol{y}) \in [-1,1]^6 : |F(\boldsymbol{x}, \boldsymbol{y})| \leq \epsilon\}.$$

Literature Comparison: Dual Varieties

Let $V = \{F = 0\}$ and let $V^\vee$ be its *dual variety*: the image of V under the Gauss map $\boldsymbol{x} \mapsto \nabla F(\boldsymbol{x})$. Roughly, $\deg(V^\vee)$ is a global curvature measure of V.

1. [Batyrev–Tschinkel 1998, Salberger 1998]: Toric varieties like the Perazzo primal $x_1x_2x_3 = y_1y_2y_3$. This is self-dual; $\deg(V^\vee) = 3$.
2. [Blomer–Brüdern–Salberger 2014]: The cubic equation $x_1y_2y_3 + x_2y_1y_3 + x_3y_1y_2 = 0$. Self-dual; $\deg(V^\vee) = 3$.
3. [Schmidt 1995, Derenthal 2025]: The cubic equation $\det(S) = 0$ in symmetric 3×3 matrices. Dual to the Veronese surface; $\deg(V^\vee) = 4$.

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3. [Schmidt 1995, Derenthal 2025]: The cubic equation $\det(S) = 0$ in symmetric 3×3 matrices. Dual to the Veronese surface; $\deg(V^\vee) = 4$.
4. **Our case:** For $F = \sum_{i=1}^{3} x_i y_i^2$, the variety $V^\vee$ is a sextic hypersurface; $\deg(V^\vee) = 6$.
5. Smooth case: For $F = \sum_{i=1}^{6} x_i^3$, we have $\deg(V^\vee) = 48$.

The Circle Method: Poisson Summation

Let $F = \sum_{i=1}^{3} x_i y_i^2$. Break $N_F(B) = \int_{\theta \in \mathbb{R}/\mathbb{Z}} (\cdots)\, d\theta$ into arcs $|r\theta - a| \leq 1/B^{3/2}$, where $1 \leq a \leq r \leq B^{3/2}$ and $\gcd(a, r) = 1$. Applying Poisson summation (following Kloosterman/Hooley; rigorously via DFI + Heath-Brown), we get

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$$N_F(B) = B^3 \sum_{r \leq B^{3/2}} \sum_{\boldsymbol{m}, \boldsymbol{n} \ll B^{1/2}} \frac{1}{r^6} S_r(\boldsymbol{m}, \boldsymbol{n}) I_r(\boldsymbol{m}, \boldsymbol{n})$$

where $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{Z}^3$. The exponential sum S_r:

$$S_r(\boldsymbol{m}, \boldsymbol{n}) = \sum_{a \bmod r}^{\star} \sum_{\boldsymbol{x}, \boldsymbol{y} \bmod r} e\left(\frac{aF(\boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{m} \cdot \boldsymbol{x} + \boldsymbol{n} \cdot \boldsymbol{y}}{r}\right),$$

with Kloosterman's $\sum_a$. The integral I_r (writing $\theta = \frac{a}{r} + \frac{\beta}{B^3}$):

$$\int_{|\beta| \leq \frac{B^{3/2}}{r}} \int_{(\boldsymbol{x}, \boldsymbol{y}) \in [-1,1]^6} e\left(\beta F(\boldsymbol{x}, \boldsymbol{y}) - \frac{B\boldsymbol{m} \cdot \boldsymbol{x} + B\boldsymbol{n} \cdot \boldsymbol{y}}{r}\right) d\boldsymbol{x}\, d\boldsymbol{y}\, d\beta.$$

The Discriminant $D(\boldsymbol{m}, \boldsymbol{n})$

The dual variety $V^\vee$ associated to $F = \sum_{i=1}^{3} x_i y_i^2$ is cut out by

$$D(\boldsymbol{m}, \boldsymbol{n}) := 2\sum_{i=1}^{3} m_i^2 n_i^4 - \left(\sum_{i=1}^{3} m_i n_i^2\right)^2.$$

Expect: generically $S_r(\boldsymbol{m}, \boldsymbol{n}) \ll r^{\frac{7}{2}+\varepsilon}$, where

$$S_r(\boldsymbol{m}, \boldsymbol{n}) = \sum_{a \bmod r}^{\star} \sum_{\boldsymbol{x},\boldsymbol{y} \bmod r} e\left(\frac{aF(\boldsymbol{x},\boldsymbol{y}) + \boldsymbol{m}\cdot\boldsymbol{x} + \boldsymbol{n}\cdot\boldsymbol{y}}{r}\right).$$

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Surprise: $S_p(\boldsymbol{m}, \boldsymbol{n}) = -p^3(1 + \sum_{1 \le i < j \le 3} (\frac{m_i m_j}{p})) \ll p^3$ unless $p \mid 6D(\boldsymbol{m}, \boldsymbol{n})$, in which case $S_p(\boldsymbol{m}, \boldsymbol{n}) \approx p^4$ on average.

(Our average-case is better than Hooley et al.'s, but worst-case more frequent. If F were $\sum_{i=1}^6 x_i^3$, then $S_p(\boldsymbol{b}) \ll p^{7/2}$ unless $p \mid \gcd(D(\boldsymbol{b}), \nabla D(\boldsymbol{b}))$; and $S_p(\boldsymbol{b}) \ll p^4$ otherwise.)

Decomposition of $N_F(B)$

We write

$$N_F(B) = B^3 \sum_{r \leq B^{3/2}} \sum_{\boldsymbol{m},\boldsymbol{n} \ll B^{1/2}} \frac{1}{r^6} S_r I_r(\boldsymbol{m}, \boldsymbol{n}) = B^3(\Sigma_0 + \Sigma_{sm} + \Sigma_{sing})$$

where

$$\Sigma_0 = \sum_{r \leq B^{3/2}} \frac{1}{r^6} S_r I_r(\mathbf{0}, \mathbf{0}),$$

$$\Sigma_{sm} = \sum_{r \leq B^{3/2}} \sum_{D(\boldsymbol{m},\boldsymbol{n}) \neq 0} \frac{1}{r^6} S_r I_r(\boldsymbol{m}, \boldsymbol{n}),$$

$$\Sigma_{sing} = \sum_{r \leq B^{3/2}} \sum_{\substack{D(\boldsymbol{m},\boldsymbol{n}) = 0 \\ (\boldsymbol{m},\boldsymbol{n}) \neq (\mathbf{0},\mathbf{0})}} \frac{1}{r^6} S_r I_r(\boldsymbol{m}, \boldsymbol{n}).$$

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We will prove that $\Sigma_0 \sim \frac{3\sigma_\infty}{4\zeta(3)} \log B$, $\Sigma_{sing} \sim \frac{\sigma_\infty}{4\zeta(3)} \log B$, and $\Sigma_{sm} = o(\log B)$. This will imply that $N_F(B) \sim \frac{\sigma_\infty}{\zeta(3)} B^3 \log B$.

The Central Term Σ_0

By definition, $\Sigma_0 = \sum_{r \leq B^{3/2}} \frac{1}{r^6} S_r I_r(\mathbf{0}, \mathbf{0})$.

Proposition

We have $\Sigma_0 = \frac{3\sigma_\infty}{4\zeta(3)} \log B + O(1)$.

Proof.

A calculation shows that:

- $S_p(\mathbf{0}, \mathbf{0}) = p^4 \left(1 - \frac{1}{p}\right)$,
- $S_{p^2}(\mathbf{0}, \mathbf{0}) = p^{11} \left(1 - \frac{1}{p}\right)$,
- $I_r(\mathbf{0}, \mathbf{0}) = \sigma_\infty + O(r/B^{3/2})$.

Thus $r = \square \leq B^{3/2}$ dominates the sum Σ_0, yielding a harmonic series that contributes $\log(B^{3/4})$. $\square$

The Singular Term Σ_{sing}

By definition, $\Sigma_{sing} = \sum_{r \leq B^{3/2}} \sum_{\substack{D(\boldsymbol{m},\boldsymbol{n})=0 \\ (\boldsymbol{m},\boldsymbol{n}) \neq (\mathbf{0},\mathbf{0})}} \frac{1}{r^6} S_r I_r(\boldsymbol{m}, \boldsymbol{n})$.

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We have $\Sigma_{sing} = \frac{\sigma_\infty}{4\zeta(3)} \log B + O(1)$.

Proof.

A calculation shows that if $D(\boldsymbol{m}, \boldsymbol{n}) = 0$, then $S_p(\boldsymbol{m}, \boldsymbol{n}) \approx p^4$ on average (a conductor-dropping phenomenon).

We cover the set $D(\boldsymbol{m}, \boldsymbol{n}) = 0$ by rank-3 lattices, mainly

$$\Lambda^\perp(\mathbf{t}) := \{(\boldsymbol{m}, \boldsymbol{n}) \in (\mathbb{Z}^3)^2 : \boldsymbol{m} \in \mathbf{t}^2\mathbb{Z},\ \boldsymbol{n} \cdot \mathbf{t} = 0\}$$

where $\mathbf{t} \ll B^{1/4}$. We then apply the Fourier-slice theorem on each plane to evaluate the sum over $(\boldsymbol{m}, \boldsymbol{n})$. □

Intuition: Divisors of y

Given a vector $\mathbf{t} \in \mathbb{Z}^3$, let $\mathbf{t}^2 = (t_1^2, t_2^2, t_3^2)$. Let

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This is a rank-3 lattice in $\{F = 0\}$, orthogonal to $\Lambda^{\perp}(\mathbf{t})$.

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Why is $\Sigma_0 : \Sigma_{sing} = \frac{3}{4} : \frac{1}{4}$ plausible? If $\gcd(y_1, y_2, y_3) = h$, then each dyadic range $h \asymp H$, for $1 \ll H \ll B$, should contribute

$$\asymp \sum_{h \asymp H} \frac{B^3 (B/h)^3}{B (B/h)^2} = \sum_{h \asymp H} \frac{B^3}{h} \asymp B^3$$

solutions $(\boldsymbol{x}, \boldsymbol{y})$ to $F = 0$. (Standard heuristic calculation.)

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- Solutions with $B^{3/4} \ll h \ll B$ are covered by $\Lambda(\mathbf{t})$ with $\mathbf{t} = \boldsymbol{y}/h \ll B/h \ll B^{1/4}$, captured by Σ_{sing}.
- More mysteriously, Σ_0 captures $h \ll B^{3/4}$. It seems that congruences $\boldsymbol{y} \equiv 0 \bmod h \Rightarrow F(\boldsymbol{x}, \boldsymbol{y}) \equiv 0 \bmod h^2$ are being detected by the moduli $r = h^2 \ll B^{3/2}$ in Σ_0.

The Smooth Term Σ_{sm} (Final Piece of Theorem)

By definition, $\Sigma_{sm} = \sum_{r \leq B^{3/2}} \sum_{D(\boldsymbol{m},\boldsymbol{n}) \neq 0} \frac{1}{r^6} S_r I_r(\boldsymbol{m}, \boldsymbol{n})$.

Proposition

$\Sigma_{sm} = O((\log B)^{\varepsilon})$.

Proof: Pretending $r \asymp B^{3/2}$ square-free and $I_r(\boldsymbol{m}, \boldsymbol{n}) = 1$, we look at two extreme cases.

Case 1: $\gcd(r, 6D) = 1$. Then $S_p = -p^3(1 + \sum_{i<j} (\frac{m_i m_j}{p}))$ so

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Case 1: $\gcd(r, 6D) = 1$. Then $S_p = -p^3(1 + \sum_{i<j}(\frac{m_i m_j}{p}))$ so

$$\Sigma_{sm} \approx \sum_{r \asymp B^{3/2}} \sum_{D(\boldsymbol{m},\boldsymbol{n}) \neq 0} \frac{1}{r^3} (\mu \star \mu\chi_{m_1 m_2} \star \mu\chi_{m_1 m_3} \star \mu\chi_{m_2 m_3})(r)$$

$$\ll \frac{(B^{1/2})^6}{(B^{3/2})^2} B^{\varepsilon} = B^{\varepsilon},$$

by the triangle inequality. By PNT on the Möbius factor μ and Heath-Brown's large sieve on the quadratic factors $\mu\chi_{m_i m_j}$, get cancellation over r. Thus improve B^{ε} to $1/(\log B)^A$.

Case 2: $r \mid 6D$. Then $S_p(\boldsymbol{m}, \boldsymbol{n}) \approx p^4$ on average. So

$$\Sigma_{sm} \approx \sum_{D(\boldsymbol{m},\boldsymbol{n}) \neq 0} \sum_{\substack{r \mid 6D(\boldsymbol{m},\boldsymbol{n}) \\ r \asymp B^{3/2}}} \frac{1}{r^2} \ll \frac{(B^{1/2})^6}{(B^{3/2})^2} \log B = \log B,$$

by divisor bounds of [Shiu, Nair–Tenenbaum, et al.] type, if we ignore the condition $r \asymp B^{3/2}$.

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by divisor bounds of [Shiu, Nair–Tenenbaum, et al.] type, if we ignore the condition $r \asymp B^{3/2}$. However, since $r \asymp B^{3/2}$, work of [Hooley, de la Bretèche and Tenenbaum, et al.] lets us replace $\log B$ with $(\log B)^{\varepsilon}$. Good enough for us.

(In order to remove the $(\log B)^{\varepsilon}$ term, one would want an asymptotic, or a sharp upper bound, for sums like

$$\sum_{\boldsymbol{m},\boldsymbol{n} \ll B^{1/2}} \sum_{r \asymp B^{3/2}} \mathbf{1}_{r \mid 6D(\boldsymbol{m},\boldsymbol{n}) \neq 0} \approx \sum_{\boldsymbol{m},\boldsymbol{n} \ll B^{1/2}} \sum_{a \asymp b \asymp B^{3/2}} \mathbf{1}_{D(\boldsymbol{m},\boldsymbol{n}) = \pm ab}.$$

The divisor $|D|/r$ is usually $\asymp r$, since $\deg D = 6$; it would be smaller if $\deg D \leq 5$. RHS is a divisor problem that might be interesting; over $\mathbb{F}_q[t]$ if necessary?)

Proof for Σ_{sm}: Hybrid Cases

By definition, $\Sigma_{sm} = \sum_{r \leq B^{3/2}} \sum_{D(\boldsymbol{m},\boldsymbol{n}) \neq 0} \frac{1}{r^6} S_r I_r(\boldsymbol{m}, \boldsymbol{n})$.

Proposition

$\Sigma_{sm} = O((\log B)^{\varepsilon})$.

Proof.

Pretend $r \asymp B^{3/2}$ square-free and $I_r(\boldsymbol{m}, \boldsymbol{n}) = 1$.

Write $r = r_1 r_2$, where $\gcd(r_1, 6D) = 1$ and $r_2 \mid 6D$. We use Hölder's inequality appropriately to separate the contributions from r_1 and r_2. □

Further Lemmas: Bounds on S_r

Lemma

Let $r \in \mathbb{N}$. Let $\{a, b\} = \prod_{p^j \| \gcd(a,b)} p^{2\lfloor j/2 \rfloor}$. Then

$$S_r(\boldsymbol{m}, \boldsymbol{n}) \ll r^{4+\varepsilon} \prod_{1 \le i \le 3} \{r, m_i\}^{1/2} \mathbf{1}_{\{r,m_i\}^{1/2} | n_i}.$$

Lemma

Let $r = p^e$ be a prime power such that $p \nmid 6 \nabla D(\boldsymbol{m}, \boldsymbol{n})$. Then

$$|S_r(\boldsymbol{m}, \boldsymbol{n})| \le r^4.$$

Lemma

Let r be square-full. Then $S_r(\boldsymbol{m}, \boldsymbol{n}) = 0$ unless $v_p(D(\boldsymbol{m}, \boldsymbol{n})) \ge v_p(r) - 1$ for all primes $p \mid r$.

Further Lemmas: Bounds on I_r

The Hessian determinant of $F(\boldsymbol{x}, \boldsymbol{y})$ is $H_F(\boldsymbol{x}, \boldsymbol{y}) = y_1^2 y_2^2 y_3^2$. Work on a region Ω in $[-1, 1]^6$ for which $H_F(\boldsymbol{x}, \boldsymbol{y}) \gg 1$. Therefore:

$$|\nabla F(\boldsymbol{x}, \boldsymbol{y})| \geq \max(y_1^2, y_2^2, y_3^2) \gg 1$$

for all $(\boldsymbol{x}, \boldsymbol{y}) \in \Omega$. Although F is singular, this inequality acts as a suitable replacement for smoothness.

Lemma

Let $j, k \geq 0$ be integers. Then

$$r^j \frac{\partial^j}{\partial r^j} I_r(\boldsymbol{m}, \boldsymbol{n}) \ll_{j,k} \frac{\left(1 + \frac{B \max\{|\boldsymbol{m}|,|\boldsymbol{n}|\}}{r}\right)^{-2}}{\left(1 + \frac{\max\{|\boldsymbol{m}|,|\boldsymbol{n}|\}}{B^{1/2}}\right)^k \left(1 + \frac{|\hat{D}(\boldsymbol{m},\boldsymbol{n})| B \max\{|\boldsymbol{m}|,|\boldsymbol{n}|\}}{r}\right)^k},$$

where $\hat{D}(\boldsymbol{m}, \boldsymbol{n}) := \frac{D(\boldsymbol{m},\boldsymbol{n})}{(1+\max\{|\boldsymbol{m}|,|\boldsymbol{n}|\})^6} \ll 1.$

Further Examples: BTSC

Definition

Say *BTSC holds* for a cubic form $F \in \mathbb{Q}[x_1, \ldots, x_6]$ if there exists a nonzero $G \in \mathbb{Z}[b_1, \ldots, b_6]$ such that uniformly over vectors $\boldsymbol{b} \in \mathbb{Z}^6$ and primes $p \nmid G(\boldsymbol{b})$, we have $|S_p(\boldsymbol{b})| \ll p^3$.

BTSC holds for $F = \sum_{i=1}^{3} x_i y_i^2$, but not for $F = \sum_{i=1}^{6} x_i^3$.

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BTSC holds for $F = \sum_{i=1}^{3} x_i y_i^2$, but not for $F = \sum_{i=1}^{6} x_i^3$.

The dimension of the moduli space of cubic fourfolds over $\mathbb{C}$ is

$$\dim \mathrm{Sym}^3(\mathbb{C}^6) - \dim \mathrm{GL}_6(\mathbb{C}) = \binom{8}{3} - 6^2 = 20.$$

So there is a lot of room to explore.

Question

Classify cubic fourfolds F satisfying BTSC?

Other Examples Satisfying BTSC

Preliminary calculations indicate BTSC usually holds for:

1. Cubic fourfolds singular along a rational plane, such as $\sum_{i=1}^{3} x_i Q_i(\boldsymbol{y}) = C(\boldsymbol{y})$. LHS defines a *net of conics*.
2. Equations of pencils of quadrics: $x_1 Q_1(\boldsymbol{y}) = x_2 Q_2(\boldsymbol{y})$.
3. Trilinear equations in three pairs of variables.

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3. Trilinear equations in three pairs of variables.
4. $x(y^2 + xz) = u(v^2 + uw)$. Singular subscheme is 1-dim, deg 9. Dual is hypersurface of deg 5.
5. $x^2 r + y^2 s = xuv + y(u^2 + v^2)$. Sing is 1-dim, deg 8. Dual is hyper of deg 5.
6. $(x+v)ur + (y+u)vs = (x+y)xy$. Sing is 1-dim, deg 7. Dual is hyper of deg 5.
7. $\sum_{i=1}^{3} x_i y_i^2 = c_1 x_1 x_2 x_3 + c_2 y_1 y_2 y_3$, where $c_1, c_2 \in \mathbb{Q}$ are coefficients. If $c_1(c_1 c_2^2 - 4) \neq 0$, then sing is 1-dim, deg 6; dual seems a hyper of deg 6; and proof of BTSC uses either 6-nodal cubic threefolds, or Gauss and Salié sums.