Integral points on Markoff type cubic surfaces

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Basic Question

Let $M \in \mathbb{Z}[x, y, z]$ denote the *Markoff polynomial* given by

$$M = x^2 + y^2 + z^2 - xyz.$$

For which integers a can we find a triple $(x_0, y_0, z_0) \in \mathbb{Z}^3$ such that

$$V_a : M(x_0, y_0, z_0) = a \,? \qquad (1)$$

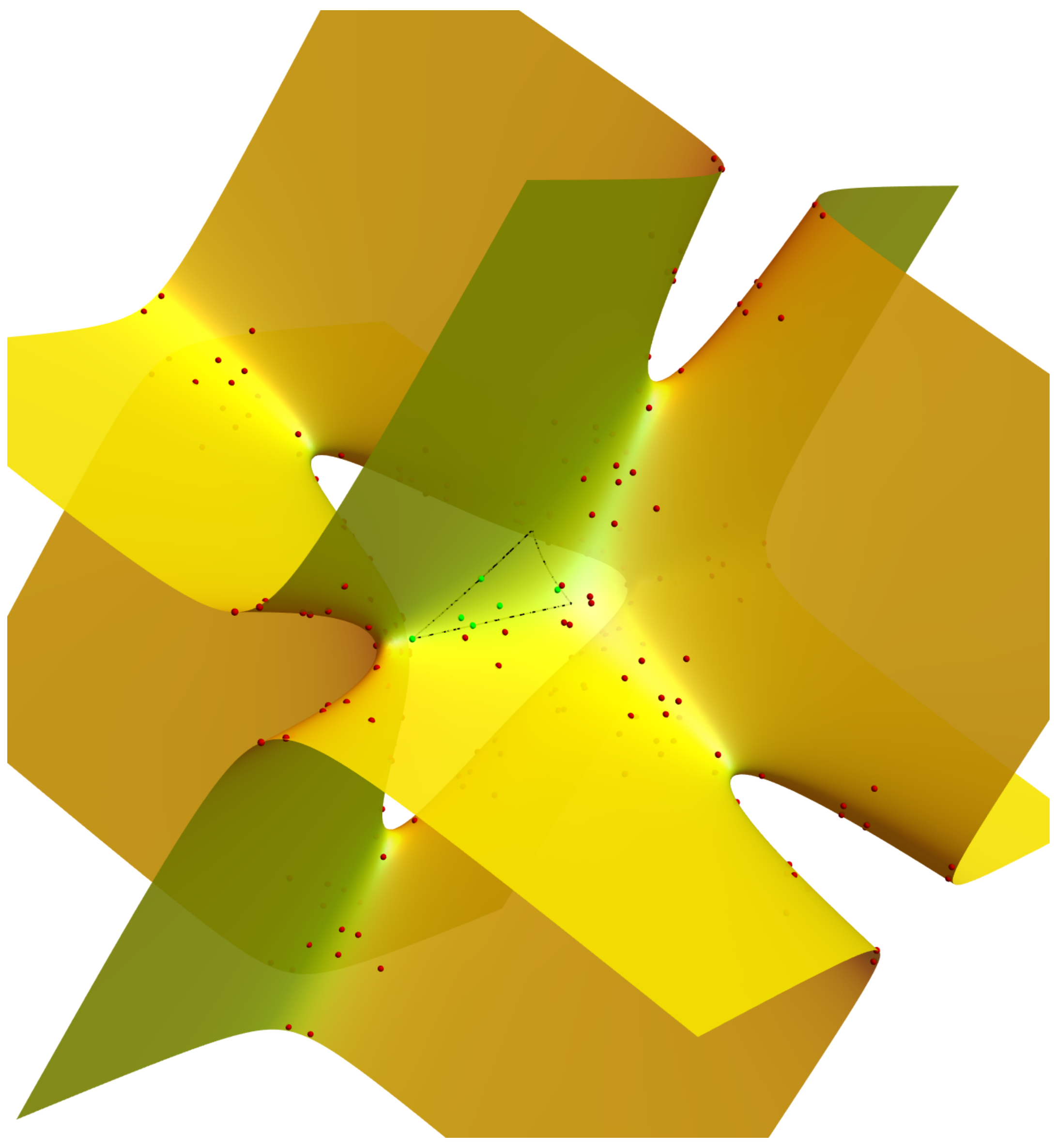

Figure 1. Integer points on V_a for $a = 3685$†.

The highlighted triangular set is the fundamental set for the action of a suitable group Γ on the set of integral solutions; see [GS22] for more details.

The Integral Hasse Principle

Note that if (1) has a solution for some $a \in \mathbb{Z}$, then it has a solution modulo every prime power. Hence, a necessary condition on $a \in \mathbb{Z}$ for the existence of a solution is that the equation

$$x^2 + y^2 + z^2 - xyz = a \bmod p^n,$$

is solvable for all primes p and positive integers $n \geq 1$. In [GS22], Ghosh and Sarnak proved that the above condition is true if and only if $a \not\equiv 3 \pmod 4$ and $a \not\equiv \pm 3 \pmod 9$. These values of $a \in \mathbb{Z}$ are called *admissible*. We remark that

$$\sum_{\substack{|a| \leq A, \\ a \text{ admissible}}} 1 \sim \frac{7}{12} A$$

as $A \to \infty$.

The Problem

Is the property *admissible* sufficient for the existence of a solution to the equation (1)? In general, the answer is **NO**. For example, $a = 46$ is admissible, but (1) has no integer solutions.

One can ask how often the property *admissible* is sufficient. More precisely, we define the set

$$\mathcal{E} := \{a \in \mathbb{Z} : a \text{ is admissible but (1) has no solution}\}.$$

For $A > 0$, we set

$$\mathcal{E}(A) := \{|a| \leq A : a \in \mathcal{E}\}. \qquad (2)$$

Goal: Study the asymptotics for $\#\mathcal{E}(A)$, the size of the set $\mathcal{E}(A)$, as $A \to \infty$.

Previous Results

In [GS22], Ghosh and Sarnak prove the lower bound

$$\#\mathcal{E}(A) \gg \frac{\sqrt{A}}{(\log A)^{1/2}}.$$

In particular, there are infinitely many admissible $a \in \mathbb{Z}$ such that (1) has no integral solutions. For the upper bound, they prove that

$$\frac{\#\mathcal{E}(A)}{A} \to 0 \text{ as } A \to \infty.$$

This demonstrates that for 100% of integers $a \in \mathbb{Z}$, the property of being admissible is sufficient for the existence of a solution to (1).

Main Result

We improved the asymptotic upper bound on the size of the set $\mathcal{E}(A)$. We have that

$$\#\mathcal{E}(A) \ll_\epsilon \frac{A}{(\log A)^{2-\epsilon}}, \qquad (3)$$

for all $A \geq 2$. As a consequence of this improved bound, we can deduce that the property of being admissible is sufficient for the existence of a solution to (1) for 100% of the elements in certain sequences in $\mathbb{Z}$.

We now state an accessible consequence of the above bound:

$$\lim_{X \to \infty} \frac{\#\{p \leq X : p \text{ prime}, p - 1 = M(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{Z}^3\}}{\#\{p \leq X : p \text{ prime}\}} = \frac{2}{3}.$$

We highlight that the numerical evidence suggests that the bound (3) is not optimal. We present one such computation from [GS22]. Let $\mathcal{E}_+(A)$ denote the set of admissible $5 \leq a \leq A$ such that (1) has no solution.

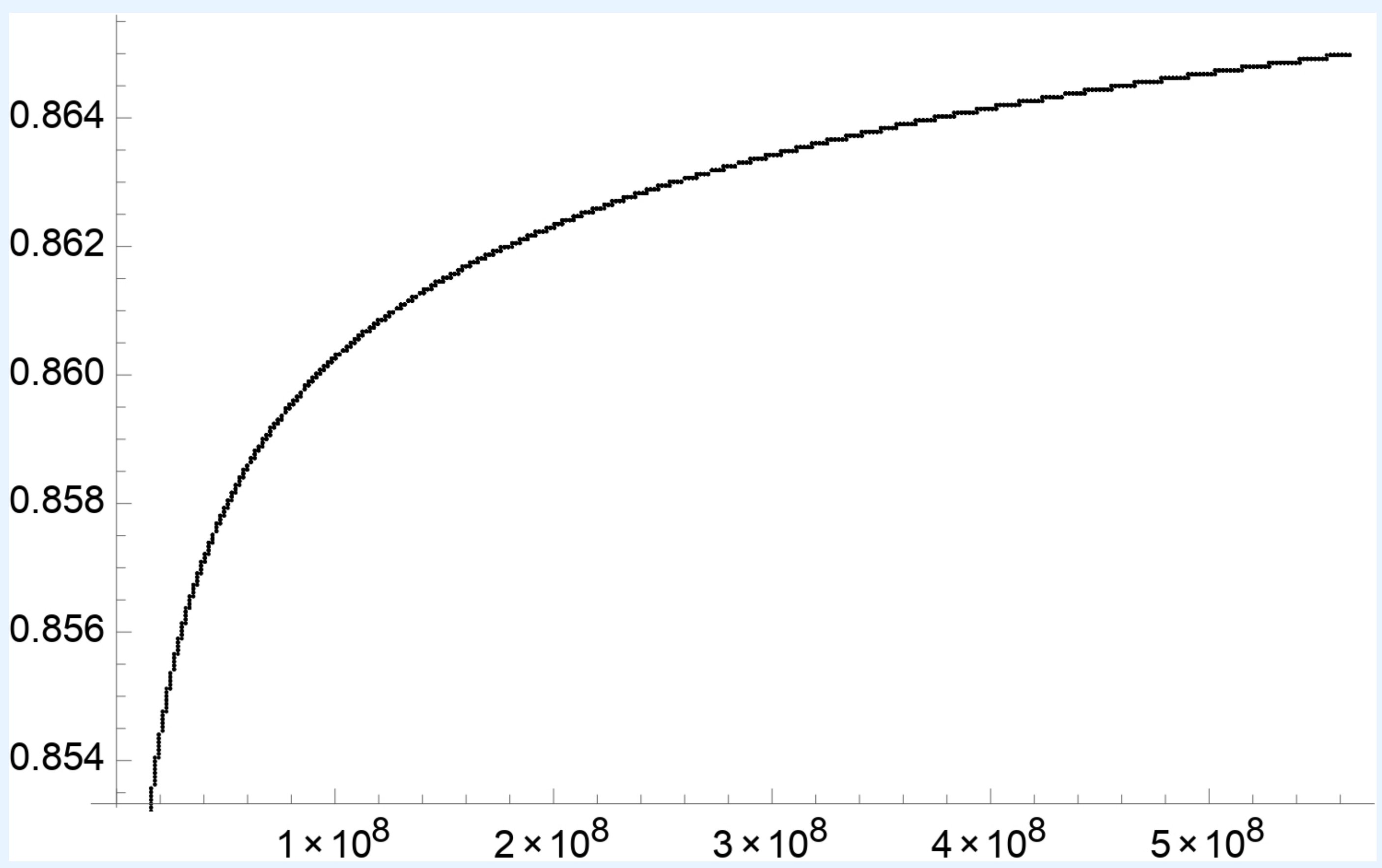

Figure 2. Plot of $\log \#\mathcal{E}_+(A) / \log(7A/12)$†.

The above plot suggests that $\mathcal{E}_+(A)$ grows like A^θ with $\theta \approx 0.887$.

Acknowledgements

I am grateful to Prof. Tim Browning for hosting me in his research group over the summer. I extend special thanks to Victor Wang for proposing this problem, engaging in numerous insightful discussions, offering valuable suggestions, meticulously reviewing earlier drafts, and writing the proof of a crucial estimate needed for the proof. I also wish to thank the entire Browning group for their support throughout my internship.

†Both figures are from the paper [GS22].

References

[GS22] Amit Ghosh and Peter Sarnak. Integral points on Markoff type cubic surfaces. *Invent. Math.*, 229(2):689–749, 2022.

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