Average sizes of mixed character sums

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(joint work with Max Xu)

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Deterministic versus random behavior

Many problems in analytic number theory concern the behavior of families of arithmetic sums, such as the family

$$\chi \mapsto \sum_{1 \leq n \leq x} \chi(n)$$

indexed by Dirichlet characters χ modulo a prime r, for some set of x. Defining properties of χ are *multiplicativity*

$$\chi(mn) = \chi(m)\chi(n), \qquad \chi(1) = 1, \qquad \chi(0) = 0,$$

and *periodicity*

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Deterministic versus random behavior (cont'd)

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with $f(p)$ randomly (iid) drawn from $S^1 \subset \mathbb{C}$ for each prime p. The advantage of random multiplicative functions (rmf) is that

$$\mathbb{E}_f f(m)\overline{f}(n) = \mathbf{1}_{m=n}$$

(orthogonality) holds for all $m, n \geq 1$, whereas (by periodicity)

$$\mathbb{E}_{\chi \bmod r} \chi(m)\overline{\chi}(n) = \mathbf{1}_{m=n}$$

holds only in ranges such as $1 \leq m, n < r$.

Mixed character sums

Fix a smooth function $w\colon \mathbb{R} \to \mathbb{R}$, supported on $[0,1]$, with $\int_0^1 w(t)^2\,dt > 0$. We consider the *mixed character sum*

$$S(\chi,\theta;x) := \sum_{n\in\mathbb{Z}} \chi(n)e(n\theta)w(n/x) = \sum_{1\leq n\leq x} \chi(n)e(n\theta)w(n/x),$$

featuring a multiplicative character χ mod r and an additive character $e(n\theta) := \exp(2\pi i n\theta)$.

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Question

Fix $\theta \in \mathbb{R}$. Assume $1 \le x \le r$. How does $S(\chi,\theta;x)$ behave as χ mod r varies?

[Harper 2023] (building on [Harper 2020]) implies, for $\theta \in \mathbb{Q}$,

$$\mathbb{E}_{\chi \bmod r}|S(\chi,\theta;x)| = O(x^{1/2}/(\log\log\min(x,r/x))^{1/4}) = o(x^{1/2})$$

if $\min(x,r/x) \to \infty$, even for piecewise continuous w. I will discuss joint work with Max Xu (2024) concerning $\theta \notin \mathbb{Q}$.

Mixed character sums (rmf model)

For random multiplicative f let

$$S^{\sharp}(f,\theta;x) := \sum_{1\leq n\leq x} f(n)e(n\theta).$$

Fix $\theta \in \mathbb{R}$. How does $S^{\sharp}(f,\theta;x)$ behave as f varies?

Theorem (Harper 2020)

If $\theta \in \mathbb{Q}$ and $x \to \infty$, then $\mathbb{E}_f|S^{\sharp}(f,\theta;x)| = o(x^{1/2})$.

Theorem (Soundararajan–Xu 2023)

Suppose $\|q\theta\| := \min_{a\in\mathbb{Z}}|q\theta - a| \gg \exp(-q^{1/50})$ for all $q \in \mathbb{N}$.[a] Then as $x \to \infty$, the random variable $S^{\sharp}(f,\theta;x)/x^{1/2}$ converges in distribution to the standard complex Gaussian $\mathcal{CN}(0,1)$. Moreover, $\mathbb{E}_f|S^{\sharp}(f,\theta;x)| \sim cx^{1/2}$ $(c > 0)$.

[a] This is satisfied for most $\theta \in \mathbb{R}$, including π, e, and any algebraic irrational θ. For most $\theta \in \mathbb{R}$, we have $\|q\theta\| \gg q^{-1-\epsilon}$ for all $q \in \mathbb{N}$.

Mixed character sums (deterministic)

Fix a smooth function $w\colon \mathbb{R} \to \mathbb{R}$, supported on $[0, 1]$, with $\int_0^1 w(t)^2\,dt > 0$. For characters χ mod r let

$$S(\chi, \theta; x) := \sum_{n\in\mathbb{Z}} \chi(n)e(n\theta)w(n/x) = \sum_{1\le n\le x} \chi(n)e(n\theta)w(n/x).$$

Fix $\theta \in \mathbb{R}$. Assume $1 \le x \le r$.

Theorem (Harper 2023)

If $\theta \in \mathbb{Q}$, then $\mathbb{E}_{\chi \bmod r}|S(\chi, \theta; x)| = o(x^{1/2})$ as $\min(x, r/x) \to \infty$, even for piecewise continuous w.

Theorem (Wang–Xu 2024)

Suppose $\|q\theta\| := \min_{a\in\mathbb{Z}}|q\theta - a| \gg \exp(-q^{1/4})$ for all $q \in \mathbb{N}$. If $x \ggg 1$, then $x^{1/2} \ll \mathbb{E}_{\chi \bmod r}|S(\chi, \theta; x)| \ll x^{1/2}$.

Second moment

For $1 \leq x \leq r$, orthogonality over $\{\chi \bmod r\}$ implies that

$$\mathbb{E}_\chi \Big| \sum_{1 \leq n \leq x} \chi(n) e(n\theta) w(n/x) \Big|^2 = \sum_{1 \leq n \leq \min(x, r-1)} w(n/x)^2$$
$$\sim x \int_0^1 w(t)^2 \, dt \asymp x,$$

provided that x is sufficiently large (in terms of w).

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provided that x is sufficiently large (in terms of w). Thus

$$\mathbb{E}_{\chi \bmod r} |S(\chi, \theta; x)| = \mathbb{E}_\chi \Big| \sum_{1 \leq n \leq x} \chi(n) e(n\theta) w(n/x) \Big| \ll x^{1/2}$$

by Cauchy–Schwarz over $\{\chi \bmod r\}$. Thus the desired upper bound in [Wang–Xu 2024] holds without any Diophantine condition on $\theta \in \mathbb{R}$. The lower bound is the interesting part.

Fourth moment

By Hölder's inequality,

$$(\mathbb{E}_\chi |S(\chi,\theta;x)|)^2 (\mathbb{E}_\chi |S(\chi,\theta;x)|^4) \geq (\mathbb{E}_\chi |S(\chi,\theta;x)|^2)^3 \gg x^3,$$

so the desired lower bound $\mathbb{E}_\chi |S(\chi,\theta;x)| \gg x^{1/2}$ will follow if we can show that

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If $x \leq r^{1/2}$, then orthogonality over χ gives (for some smooth weight W, which is not important)

$$\begin{aligned}\mathbb{E}_\chi|S(\chi,\theta;x)|^4 &= \sum_{\substack{1\leq m_1,m_2,n_1,n_2\leq x\\ m_1m_1=n_1n_2}} e((m_1+m_2-n_1-n_2)\theta)W \\ &= \mathbb{E}_f|S(f,\theta;x)|^4 \ll x^2,\end{aligned}$$

by the methods of [Soundararajan–Xu 2023]. (Parameterize solutions; combinatorially decompose into geometric series.)

If $x \geq r^{1/2}$, then $m_1 m_1 \equiv n_1 n_2 \bmod r$ is no longer equivalent to $m_1 m_2 = n_1 n_2$. Thus, we choose not to directly compute the fourth moment as we did for $x \leq r^{1/2}$. Instead, we study a dual problem, with r/x replacing x.

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$$f_{r,\chi}(n) := \chi(n) e(\frac{kn}{r}), \quad f_\infty(n) := w(\frac{n}{x}) e(n\theta').$$

Then $S(\chi, \theta; x)$ may be written as

$$\sum_{n \in \mathbb{Z}} \chi(n) e(n\theta) w(\frac{n}{x}) = \sum_{n \in \mathbb{Z}} f_{r,\chi}(n) f_\infty(n) = \sum_{m \in \mathbb{Z}} \hat{f}_{r,\chi}(\frac{m}{r}) \hat{f}_\infty(\frac{m}{r})$$

by Poisson summation in $(\mathbb{Z}/r\mathbb{Z}) \times \mathbb{R}$, where

$$\hat{f}_{r,\chi}(\frac{m}{r}) = \frac{1}{r} \sum_{a \in \mathbb{Z}/r\mathbb{Z}} \chi(a) e\left(\frac{(k+m)a}{r}\right)$$

and $\hat{f}_\infty(\frac{m}{r}) = \int_{\mathbb{R}} w(\frac{t}{x}) e((\theta' - \frac{m}{r})t) dt$.

Fourier coefficients

We now estimate the Fourier coefficients $\hat{f}_{r,\chi}(\frac{m}{r})$ and $\hat{f}_\infty(\frac{m}{r})$. If $k+m \not\equiv 0 \bmod r$, then by standard properties of Gauss sums,

$$\hat{f}_{r,\chi}(\frac{m}{r}) = \frac{1}{r} \sum_{a \in \mathbb{Z}/r\mathbb{Z}} \chi(a) e\left(\frac{(k+m)a}{r}\right) = \chi(k+m)^{-1} \frac{C(\chi)}{r^{1/2}},$$

where $|C(\chi)| \leq 1$ and $C(\chi)$ depends only on χ. Moreover, integration by parts over $t \in \mathbb{R}$ gives

$$\hat{f}_\infty(\frac{m}{r}) = \int_{\mathbb{R}} w(\frac{t}{x}) e((\theta' - \frac{m}{r})t) dt \ll_A x\big(1 + \frac{x \max(|m|-1, 0)}{r}\big)^{-A}$$

for all $A \geq 0$, using smoothness of w.

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for all $A \geq 0$, using smoothness of w. Plugging this into $S(\chi,\theta;x) = \sum_{m\in\mathbb{Z}} \hat{f}_{r,\chi}(\frac{m}{r})\hat{f}_\infty(\frac{m}{r})$, we morally get

$$|S(\chi,\theta;x)| \approx |\sum_{\substack{|m|\leq 2+r/x \\ m\not\equiv -k \bmod r}} \frac{\chi(k+m)^{-1}}{r^{1/2}} x|.$$

Orthogonality after duality

We are essentially left with proving that

$$\mathbb{E}_\chi \Big| \sum_{\substack{|m| \le 2 + r/x \\ m \not\equiv -k \bmod r}} \frac{\chi(k+m)^{-1}}{r^{1/2}} x \Big|^4 \ll x^2.$$

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By orthogonality, LHS $= \frac{x^4}{r^2} \mathcal{N}_4(2 + r/x)$, where $\mathcal{N}_4(T)$ counts integer solutions

$$(m_1, m_2, n_1, n_2) \in \{|m| \le T : m \not\equiv -k \bmod r\}^4$$

to the congruence

$$(k + m_1)(k + m_2) \equiv (k + n_1)(k + n_2) \bmod r.$$

This congruence is equivalent to

$$k(m_1 + m_2 - n_1 - n_2) \equiv n_1 n_2 - m_1 m_2 \bmod r.$$

We want to prove $\mathcal{N}_4(T) \ll T^2$ for $3 \le T \le 2 + r^{1/2}$.

Write $S = m_1 + m_2 - n_1 - n_2$ and $P = n_1 n_2 - m_1 m_2$.

Lemma (Almost a parameterization of solutions)

There exists a linear map $\Phi\colon \mathbb{Z}^4 \to \mathbb{Z}^3$ such that if $S, P \in \mathbb{Z}$, then Φ maps the set $\mathcal{A}$ injectively into the set $\mathcal{B}$, where

$$\begin{aligned}\mathcal{A} &:= \{(m_1, m_2, n_1, n_2) \in \mathbb{Z}^4 : m_1 + m_2 - n_1 - n_2 = S,\\ &\qquad\qquad\qquad\qquad\qquad\qquad n_1 n_2 - m_1 m_2 = P\},\\ \mathcal{B} &:= \{(a, b, c) \in \mathbb{Z}^3 : ab + 2cS = S^2 - 4P\}.\end{aligned}$$

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Proof.

Let $\Phi(m_1, m_2, n_1, n_2) := (a, b, c)$ where

$$(a, b, c) := (n_1 - n_2 + m_1 - m_2, n_1 - n_2 - m_1 + m_2, m_1 + m_2).$$

Then $ab + c^2 = (c - S)^2 - 4P$. Therefore, Φ maps $\mathcal{A}$ into $\mathcal{B}$. Moreover, this map is injective, because the linear forms a, b, c, S are linearly independent over $\mathbb{Q}$. □

Fibering $\mathcal{N}_4(T)$ over (S, P)

We want to prove $\mathcal{N}_4(T) \ll T^2$ for $3 \leq T \leq 2 + r^{1/2}$, where $\mathcal{N}_4(T)$ counts certain solutions to the congruence

$$kS \equiv P \bmod r.$$

By the lemma, we have

$$\mathcal{N}_4(T) \leq \sum_{\substack{|S| \leq 4T,\ |P| \leq 2T^2 \\ kS \equiv P \bmod r}} N_{S,P}(T),$$

where

$$N_{S,P}(T) := \#\{a, b, c \ll T : ab + 2cS = S^2 - 4P\}.$$

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The equation $ab + 2cS = S^2 - 4P$ implies that

$$ab + 4P \equiv 0 \bmod S, \qquad ab + 4P \ll TS + S^2 \ll TS,$$

since $c \ll T$ and $S \ll T$. Therefore,

$$N_{S,P}(T) \leq \#\{a, b \ll T : S \mid ab + 4P, \quad ab + 4P \ll TS\}.$$

Lemma (Hyperbolic summation in a residue class)

Suppose $1 \leq u, v \leq S \ll T$. *Then*

$$\sum_{\substack{a,b \ll T \\ (a,b) \equiv (u,v) \bmod S}} \mathbf{1}_{ab+4P \ll TS} \ll \frac{T}{S} \log(2 + \frac{T}{S}).$$

Proof idea.

Given a, we may accurately count integers $b \equiv v \bmod S$ in any interval of length $\min(T, TS/|a|) \gg S$, since $a \ll T$. □

For any $S \ll T$ with $S \neq 0$, the lemma implies

$$N_{S,P}(T) \leq \sum_{a,b \ll T} \mathbf{1}_{S|ab+4P} \mathbf{1}_{ab+4P \ll TS} \ll \frac{T}{|S|} \log(2+\frac{T}{|S|}) N(-4P, S),$$

where $N(d,q) := \#\{(a,b) \in (\mathbb{Z}/q\mathbb{Z})^2 : ab \equiv d \bmod q\}$.

We bound $N(d,q) := \#\{(a,b) \in (\mathbb{Z}/q\mathbb{Z})^2 : ab \equiv d \bmod q\}$.

Lemma (Counting residue classes)

Let $d \in \mathbb{Z}$ and $q \in \mathbb{N}$. Then $N(d,q) \le \tau(\gcd(d,q))q$, where $\tau(\cdot)$ is the divisor function.

Proof.

It suffices to prove the lemma when q is a prime power. Say $q = p^t$ and $\gcd(d,q) = p^m$. Then clearly $t \ge m \ge 0$. If $m = 0$, then

$$N(d,q) = \phi(q) \le q.$$

If $m = 1$, then $N(d,q) = 2\phi(q) + \mathbf{1}_{t=1} \le 2q$. If $m \ge 2$, then

$$N(d,q) = 2\phi(q) + p^2 N(d/p^2, q/p^2).$$

By induction on m, it follows that $N(d,q) \le (m+1)q$. □

Dyadic fibering over gcd

For any $S \ll T$ with $S \neq 0$, the lemma implies

$$\begin{aligned} N_{S,P}(T) &\ll \frac{T}{|S|} \log(2 + \frac{T}{|S|}) N(-4P, S) \\ &\ll T \log(2 + \frac{T}{|S|}) \tau(\gcd(P, S)), \end{aligned}$$

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$$\ll T \log(2 + \frac{T}{|S|}) \tau(\gcd(P, S)),$$

Upon writing $(S, P) = (gS', gP')$ with $g = \gcd(S, P) \geq 1$, and summing $\tau(g)$ over dyadic intervals $[G/2, G)$, we get (ignoring the $S = 0$ contribution, which is easy to deal with)

$$\mathcal{N}_4(T) \leq \sum_{\substack{|S| \leq 4T,\ |P| \leq 2T^2 \\ kS \equiv P \bmod r}} N_{S,P}(T)$$
$$\ll \sum_{\substack{G \in \{2,4,8,\ldots\} \\ G \ll T}} \sum_{\substack{S' \ll T/G,\ P' \ll T^2/G \\ kS' \equiv P' \bmod r}} T \log(2 + \frac{T}{|GS'|})(G \log G).$$

Lemma (Pigeonhole counting bound)

Assume $|q\theta - a| \gg \Upsilon(q)$ *for all* $(a, q) \in \mathbb{Z} \times \mathbb{N}$, *where* Υ *is a decreasing, nonnegative function. If* $\frac{r}{2} > M \geq N \geq 1$, *then*

$$\Upsilon\left(\frac{N}{\#\{(S', P') \in [1, N] \times [-M, M] : kS' \equiv P' \bmod r\}}\right) \ll \frac{M}{r}.$$

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Proof.

By pigeonhole, there exists $(q, d) \in [1, N] \times [-2M, 2M]$ such that $kq \equiv d \bmod r$ and $q \leq \frac{N}{\#\{(S', P') \in [1,N] \times [-M,M] : kS' \equiv P' \bmod r\}}$. For such a pair (q, d), we have $kq = d + ra$ for some $a \in \mathbb{Z}$. But by definition of k, we have $|r\theta - k| < 1$. Therefore,

$$|qr\theta - ra| \leq |qr\theta - kq| + |kq - ra| < q + |d| \leq N + 2M \leq 3M,$$

whence $|q\theta - a| \leq 3M/r$. Yet by assumption, $|q\theta - a| \gg \Upsilon(q)$. Since $\Upsilon(q)$ is decreasing, the lemma follows. □

Applying the lemma

If $\frac{r}{2} > M \geq N \geq 1$ and $\Upsilon(q) = \exp(-q^{1/3})$, then

$$\#\{S' \ll N,\ P' \ll M : kS' \equiv P' \bmod r\} \ll \frac{N}{(\log(2 + r/M))^3}$$

by the lemma; this is also trivially true if $M \asymp r$.

Applying the lemma

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by the lemma; this is also trivially true if $M \asymp r$. Thus

$$\mathcal{N}_4(T) \ll \sum_{\substack{G \in \{2,4,8,\ldots\} \\ G \ll T}} \sum_{\substack{S' \ll T/G,\ P' \ll T^2/G \\ kS' \equiv P' \bmod r}} T \log(2 + \frac{T}{|GS'|})(G \log G)$$

$$\ll \sum_{\substack{G,N \in \{2,4,8,\ldots\} \\ GN \ll T}} T \log(2 + \frac{T}{|GN|})(G \log G) \frac{N}{(\log(2 + rG/T^2))^3}$$

$$\ll \sum_{\substack{G,N \in \{2,4,8,\ldots\} \\ GN \ll T}} T(\frac{T}{|GN|})^{0.1}(G \log G) \frac{N}{(\log G)^3} \ll T^2$$

for $3 \leq T \leq 2 + r^{1/2}$, by summing over N and then over G.

Final moments

We thus obtain the following result:

Theorem (Wang–Xu 2024)

Suppose $\|q\theta\| := \min_{a\in\mathbb{Z}}|q\theta - a| \gg \exp(-q^{1/4})$ *for all* $q \in \mathbb{N}$. *If* $x \ggg 1$, *then* $\mathbb{E}_{\chi \bmod r}|S(\chi,\theta;x)|^b \asymp x^{b/2}$ *for all* $0 \le b \le 4$.

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(Setting of the theorem: Fix a smooth function $w\colon \mathbb{R} \to \mathbb{R}$, supported on $[0,1]$, with $\int_0^1 w(t)^2\,dt > 0$. Let

$$S(\chi,\theta;x) := \sum_{n\in\mathbb{Z}} \chi(n)e(n\theta)w(n/x) = \sum_{1\le n\le x} \chi(n)e(n\theta)w(n/x),$$

Fix $\theta \in \mathbb{R}$. Assume $1 \le x \le r$.)

Some interesting behavior

Shala used work of Matomäki (Diophantine approximation with prime denominators), the Burgess bound, and properties of Gauss sums, to prove the following result:

Theorem (Shala 2024)

There is a sequence of prime $r \to \infty$ such that the distribution of $\frac{1}{\sqrt{r}} \sum_{1 \le n \le r} \chi(n) e(n\sqrt{2})$ tends to the uniform distribution on the unit circle. (In particular, not Gaussian!)

(Thanks to Bober, Klurman, and Shala for informing us of this result.)

Some questions

1. Can one remove the smoothness assumption on w in [Wang–Xu 2024]? If w is the indicator function of an interval, no longer have convenient decay in $\hat{f}_\infty(\frac{m}{r})$.
2. What is the threshold between rational/irrational θ for having $\mathbb{E}_{\chi \bmod r}|S(\chi,\theta;x)| = o(\sqrt{x})$? Maybe already an interesting question for $\theta \approx 0$?
3. Can one compute e.g. sixth moment $\mathbb{E}_{\chi \bmod r}|S(\chi,\theta;x)|^6$ for $x \leq r^\epsilon$? This equals the rmf moment $\mathbb{E}_f|S(f,\theta;x)|^6$.
4. Can one compute e.g. fourth moment $\mathbb{E}_{\chi \bmod r}|S(\chi,\theta;x)|^4$ for $x \geq r^{1-\epsilon}$? In particular, how does the fourth moment depend on θ and r?
5. What if we also average over r (not necessarily prime)? What are the moments/distribution of $S(\chi,\theta;x)$ then?