Some examples of symmetry

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(based on work of many authors)

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What do the following have in common?

(1) Varieties of commuting matrices.

(2) Markov-type surfaces $x^2 + y^2 + z^2 - xyz = k$.

(3) Moment, ratio, and zero statistics of L-function families.

(4) The singular dP4 surface $x_0^2 + x_0x_3 + x_2x_4 = x_1x_3 - x_2^2 = 0$, possibly blown up at singularities $\mathbf{A}_3 + \mathbf{A}_1$, or (repeatedly) at intersections of lines and/or exceptional divisors.

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All exhibit symmetry, and connect to representation theory.

- Modulo conjugation, (1) is a character variety of a torus.
- (2) is a *relative character variety* of a punctured torus.[1]
- (3) carries a monodromy representation (Deligne).
- (4) is a *solv-variety* (Derenthal–Loughran).
- (3) and (4) involve interesting compactifications. Over $\mathbb{F}_q(t)$, homological stability might hold (cf. Boyer et al.).

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Solubility of Markov-type surfaces

The polynomial $M = x^2 + y^2 + z^2 - xyz$ is fixed by a group $\Gamma \subseteq \mathrm{Aut}(M)$, where Γ is formed by S_3, sign changes ± 1, and Vieta involutions $(x, y, z) \mapsto (x, y, xy - z)$. Let $h_M(k)$ be the number of Γ-orbits of the set $\{(x, y, z) \in \mathbb{Z}^3 : M = k\}$.

Theorem (Ghosh–Sarnak 2017)

We have $h_M(k) \to \infty$ along a density 1 of admissible[a] $k \in \mathbb{Z}$. In particular, the Hasse principle holds for almost all $k \in \mathbb{Z}$.

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Theorem (Mishra 2024; lower bound is new)

Fix $\epsilon > 0$. The inequality $(\log |k|)^{2-\epsilon} \leq h_M(k) \leq (\log |k|)^{2+\epsilon}$ holds for a density 1 of admissible $k \in \mathbb{Z}$.

(Upper bound $\Leftarrow$ Ghosh–Sarnak + Markov's inequality.)

For all $k < 0$ (and for all "generic" $k \geq 5$), Ghosh–Sarnak construct a fundamental domain $\mathcal{F}_k$ for the action of Γ on $\{(x, y, z) \in \mathbb{Z}^3 : x^2 + y^2 + z^2 - xyz = k\}$. Let $r_M(k)$ be the number of points in a well-chosen region $\mathcal{F}'_k \subseteq \mathcal{F}_k$.

- For Ghosh–Sarnak, $\mathcal{F}'_k$ satisfies $|x| \asymp |yz| \asymp |k|^{1/2}$ and $|z| \leq |k|^{\epsilon}$. Real density of solutions: $\sigma_\infty(k) \asymp \epsilon \log |k|$.
- For Mishra, $\mathcal{F}'_k$ is part of a $\mathbb{G}_m^2$-torus $|xyz| \asymp |k|$, with $|k|^\delta \leq |x/y| \leq |k|^{-\delta}|z| \leq |k|^{2\delta}$. Here $\sigma_\infty(k) \asymp (\log |k|)^2$.

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One then expands and upper-bounds an *arithmetic variance*

$$Var(K, A) := \sum_{k \leq K} (r_M(k) - r_M^{loc}(k; A))^2.$$

- The sum $\sum_{k \leq K} r_M(k)^2$ counts solutions in a region to $x^2 + y^2 + z^2 - xyz = u^2 + v^2 + w^2 - uvw$.
- Here $r_M^{loc}(k; A)$ is roughly a truncated L-function at 1. Ghosh–Sarnak (resp. Mishra) use a multiplicative (resp. additive) truncation.
- Some of this generalizes to sums of three cubes.

Let $L(s, \boldsymbol{c})$ be the L-function of $V_{\boldsymbol{c}} : x_1^3 + \cdots + x_6^3 = \boldsymbol{c} \cdot \boldsymbol{x} = 0$, where $\boldsymbol{c} = (c_1, \ldots, c_6) \in \mathbb{F}_q[t]^6$, with $\gcd(q, 6) = 1$ and $\Delta(\boldsymbol{c}) := \operatorname{disc}(V_{\boldsymbol{c}}) = \prod (c_1^{3/2} \pm c_2^{3/2} \pm \cdots \pm c_6^{3/2}) \neq 0$.

Theorem (Browning–Glas–W. 2024)

Assume sufficient progress on moments of $\frac{1}{L(s,\boldsymbol{c})}$ *for* $\Delta(\boldsymbol{c}) \neq 0$. *Then* $x^3 + y^3 + z^3 = n$ *is soluble in elements* $x, y, z \in \mathbb{F}_q[t]$ *of degree* $\sim \frac{1}{3} \deg n$ *for a density* 1 *of elements* $n \in \mathbb{F}_q[t]$.

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Builds on ideas of many authors, such as the following:

- Ghosh–Sarnak, Diaconu (log-K3 variance analysis),
- Kloosterman, Hooley 1986, Heath-Brown,
- Beauville (quadric bundles over $\mathbb{P}^2$), Getz, Tran,
- Rubinstein–Sarnak (Chebyshev's bias via prime squares),
- Deligne (GRH), Hooley 1994 (singular cubics),
- Huang ($\approx \mathbb{Q}$-points), Busé–Jouanolou ($\Delta \in (f, (f')^2)$),
- Bhargava (Ekedahl sieve), Poonen (square-free sieve).

What kind of progress on L-functions?

Let $2 \nmid q$. Let $\mu(r)$ be the Möbius function over $\mathbb{F}_q[t]$, and let $\chi_m(r) = \left(\frac{r}{m}\right)$ be the Jacobi symbol over $\mathbb{F}_q[t]$.

Theorem (Bergström–Diaconu–Petersen–Westerland, Miller–Patzt–Petersen–Randal-Williams, W. 2024)

If $1 \leq M = 2g+1$ and $1 \leq R \leq \alpha M$, and $q \ggg_\alpha 1$, then

$$\frac{\sum_{|m|=q^M} \sum_{|r|=q^R} \mu(r)\chi_m(r)}{q^M q^{R/2}} \ll q^{-0.001M+O(1)},$$

where the sums over m and r run through square-free, monic $m, r \in \mathbb{F}_q[t]$ with $\deg m = M$ and $\deg r = R$, respectively.

Theorem (Petersen et al.; new for $q = p \equiv 1 \bmod 4$)

The set $\{m : L(\frac{1}{2}, \chi_m) = 0\}$ has upper density $o_{q\to\infty}(1)$.

Homological stability approach

Recall that $M = 2g + 1$. By the Lefschetz trace formula,

$$\frac{\sum_{|m|=q^M} \sum_{|r|=q^R} \mu(r)\chi_m(r)}{q^M q^{R/2}} = \sum_{k \geq 0} (-1)^k \operatorname{tr}(Fr_q, H_k(X_M, Sym^R V_M)),$$

where $X_M \subseteq \mathbb{A}^M$ is the square-free locus, and where V_M is a local system on X_M with fibers $H^1_c(\{y^2 = m(t)\}, \overline{\mathbb{Q}}_\ell(\frac{1}{2}))$ of dimension $2g$ over $m \in X_M$. Captures zeros of $L(s, \chi_m)$.

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- The first few homology groups $H_k = H_c^{2M-k}$ are the most significant.[2] By Deligne, $\operatorname{tr}(Fr_q, H_k) \leq q^{-k/2} \dim H_k$. It is also known that $\dim H_k \leq 2^M \binom{R+2g-1}{R} \leq 2^M 2^{R+2g-1}$. Thus $\dim H_k \leq 2^{(2+\alpha)M}$, since $1 \leq R \leq \alpha M$.
- By geometric series, $\sum_{k+1>\delta M}$ is negligible if $q^{\delta/2} \geq 3^{2+\alpha}$.
- We are left with estimating $\sum_{k+1\leq\delta M}$. Want a stability isomorphism $H_k(X_M, Sym^R V_M) \to H_k(X_{M+2}, Sym^R V_{M+2})$.

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No useful map $X_M \to X_{M+2}$. So BDPW compactify a quotient of X_M into a space X'_M. Want gluing/stabilization maps $\sigma\colon X'_M \to X'_{M+2}$ and $\tau\colon V_M \to \sigma^* V_{M+2}$.

- Roughly, if $m(t) \in X_M$, then $\sigma(m(t)) \in X'_{M+2}$ could be a *stable map* $\mathbb{P}^1 \cup_{\{\infty\}\sim\{0\}} \mathbb{A}^1 \to \mathbb{P}^1$ extending $m(t)$.
- There are many possible maps σ. The collection of possible σ has a *braided monoidal structure*.
- In general, $\sigma_1\sigma_2 \neq \sigma_2\sigma_1$. This is rather different than more familiar maps like multiplication in $\mathbb{F}_q[t]$.

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- In general, $\sigma_1\sigma_2 \neq \sigma_2\sigma_1$. This is rather different than more familiar maps like multiplication in $\mathbb{F}_q[t]$.
- Here σ maps X'_M into the boundary $\partial X'_{M+2} \subseteq X'_{M+2}$. We may restrict V_{M+2} to $\partial X'_{M+2}$ via the Galois groups of the generic points of X'_{M+2} and $\partial X'_{M+2}$. The map τ on H^1_c is induced by a collapse map $\sigma(X'_M) \to X'_M$.
- By log geometry, $H_k(X'_M, Sym^R V_M) = H_k(X_M, Sym^R V_M)$. A map $\sigma\colon H_k(X_M, Sym^R V_M) \to H_k(X_{M+2}, Sym^R V_{M+2})$ thus arises via σ and τ. It can be checked to agree with a topological version of the map over $\mathbb{C}$.

The local system V_M is symplectic, by Poincaré duality on H^1_c. So $(V_M)^{\otimes R}$ breaks up into symplectic pieces $(V_M)_\lambda$ associated to partitions λ of R, with $\lambda = (R)$ giving $Sym^R V_M$.

- The map $\sigma\colon H_k(X_M, (V_M)_\lambda) \to H_k(X_{M+2}, (V_{M+2})_\lambda)$ is an isomorphism for $k + 1 \leq \delta M$, for all R and λ (MPPRW).
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- In general the stable H_k are nonzero, leading to arithmetic main terms (matched to H_k by a limiting process).
- Using monodromy $\rho\colon \pi_1(X_M) \twoheadrightarrow Q_M \subseteq Sp(V_M)$, MPPRW build $\ker(H_k(X_M, (V_M)_\lambda) \to H_k(BQ_M, (V_M)_\lambda))$ out of $H_{k+1-k'}(X_{M-M'}, (V_M)_\lambda)$ for various pairs (k', M') with $k' \geq 2$ (by surjectivity of ρ) and $M' \ll k'$.
- If $le(\lambda) \ggg k+1$, then $le(\lambda) - 2M' \geq le(\lambda) - O(k) > 0$, so the restriction $(V_M)_\lambda|_{Sp(V_{M-M'})}$ has no trivial piece, by branching rules for restriction between symplectic groups. By vanishing results for $H_k(Q_M, \cdot)$ (Borel et al.), we get $H_k(X_M, (V_M)_\lambda) = 0$ via induction on $k \leq \delta M - 1$.
- The case $le(\lambda) \ll k+1 \leq \delta M$ relies on stable branching, and on a relative version of the argument above.

Counting with symmetry

Compactifications can also appear more directly in Diophantine problems. For example, let $G := \mathrm{Aff}(\mathbb{A}^1) = \{\left[\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right]\} \subseteq \mathrm{GL}_2$ be the *affine group* of $\mathbb{A}^1$ over $\mathbb{Q}$. Explicitly, the group law on $(a, b), (u, v) \in G$ is

$$(a, b) \cdot (u, v) = (au, av + b).$$

Theorem (W. '23)

Manin's conjecture holds for sufficiently split smooth equivariant compactifications X of G over $\mathbb{Q}$.

This builds on adelic harmonic analysis of Tanimoto–Tschinkel, who decomposed a point count on $G(\mathbb{Q})$ into the form

$$\sum_{\alpha \in \mathbb{Q}} \int_{t \in \mathbb{R}} (\text{decay factor}) \prod_{p \text{ prime}} (p\text{-adic integral})\, dt.$$

Tanimoto–Tschinkel decomposed the *height zeta function* $\mathsf{Z}(\boldsymbol{s}, g) = \sum_{\gamma \in G(\mathbb{Q})} H(\boldsymbol{s}, \gamma g)^{-1}$ into the form

$$\sum_{\alpha \in \mathbb{Q}} \int_{t \in \mathbb{R}} (\text{real integral}) \prod_{p \text{ prime}} (p\text{-adic integral})\, dt,$$

via the short exact sequence (of pointed sets)

$$1 \to \mathbb{Q} \backslash \mathbf{A}_{\mathbb{Q}} \xrightarrow{b \mapsto (1,b)} G(\mathbb{Q}) \backslash G(\mathbf{A}_{\mathbb{Q}}) \xrightarrow{\det \colon (a,b) \mapsto a} \mathbb{Q}^{\times} \backslash \mathbf{A}_{\mathbb{Q}}^{\times} \to 1,$$

using Fourier expansion on $b \in \mathbb{Q} \backslash \mathbf{A}_{\mathbb{Q}}$ followed by Mellin inversion on $a \in \mathbf{A}_{\mathbb{Q}}^{\times}$. The p-adic integral is, roughly,

$$\int_{G(\mathbb{Q}_p) \colon \alpha a_p \in \mathbb{Z}_p} H_p(\boldsymbol{s}, g_p)^{-1} e(-\alpha b_p \bmod \mathbb{Z}_p) |a_p|_p^{-it}\, dg_p.$$

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Both additive and multiplicative harmonics appear above; the sum over $\alpha \in \mathbb{Q}$ somehow reflects the non-abelian nature of G. The "central term" $\alpha = 0$, like $\boldsymbol{c} = \mathbf{0}$ in the δ-method, gives the main term in the Manin–Peyre conjecture.

Special divisors

Write $D := X \setminus G = \bigcup_{j \in J} D_j$, where the D_j are irreducible over $\mathbb{Q}$. Roughly, Tanimoto–Tschinkel handled the case where

$$\mathrm{ord}_{D_j}(a) < 0 \Rightarrow \mathrm{ord}_{D_j}(b) < \mathrm{ord}_{D_j}(a). \tag{1}$$

Condition (1) relates to positivity of K_X^{-1}. Similar conditions, with variables and degrees, are familiar in the circle method.

Proposition (W.)

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Definition (W.)

Given $j \in J$, call D_j *special* if $\max_{c \in \mathbb{Q}} \operatorname{ord}_{D_j}(b - c) = \operatorname{ord}_{D_j}(a)$.

When (1) fails we seem to need a new idea. Main culprit: pairs of special divisors (D_j, D_i) with $\operatorname{ord}_{D_j}(a) \operatorname{ord}_{D_i}(a) < 0$.

Suppose there are $k \geq 0$ special divisors with $\operatorname{ord}_{D_j}(a) < 0$, and $l \geq 0$ special divisors with $\operatorname{ord}_{D_j}(a) > 0$. Then the main issue, after new leading-order "bias" computations (in the spirit of Heath-Brown, Getz, Tran, et al.) relying on a new G-related source of local coordinates and cancellation in p-adic integrals, is to appropriately bound multiple Dirichlet series like

$$\sum_{\substack{\alpha = m_1 \cdots m_k / n_1 \cdots n_l: \\ \text{pairwise coprime } m_1, \ldots, m_k, n_1, \ldots, n_l \geq 1}} \frac{f(\alpha) e(c_0 \alpha)}{m_1^{\beta_1} \cdots m_k^{\beta_k}} \prod_{1 \leq j \leq l} \frac{e(-c_j \alpha \bmod \mathbb{Z}_{n_j})}{n_j^{\gamma_j}},$$

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for some $c_0, c_1, \ldots, c_l \in \mathbb{Q}$ and a transform $f : \mathbb{R}_{>0} \to \mathbb{C}$ of $H_\infty(\boldsymbol{s}, g)^{-1}$. We proceed by analytic NT methods:

- Additive reciprocity ($\prod_v \psi_v|_{\mathbb{Q}} = 1$) and Weyl-type inequalities in ranges with two or more large variables n_j.
- Local cancellations (via Poisson summation over the largest m_i) in ranges with only one large variable n_j.