Log K3 and zeta statistics

Victor Wang
(based on work of many authors)

IST Austria

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What do the following have in common?

(1) Varieties of commuting matrices.

(2) Markov-type surfaces $x^2 + y^2 + z^2 - xyz = k$.

(3) Moment, ratio, and zero statistics of L-function families.

(4) The singular dP4 surface $x_0^2 + x_0x_3 + x_2x_4 = x_1x_3 - x_2^2 = 0$, possibly blown up at singularities $\mathbf{A}_3 + \mathbf{A}_1$, or (repeatedly) at intersections of lines and/or exceptional divisors.

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All exhibit symmetry, and connect to representation theory.

- Modulo conjugation, (1) is a character variety of a torus.
- (2) is a *relative character variety* of a punctured torus.[1]
- (3) carries a monodromy representation (Deligne).
- (4) is a *solv-variety* (Derenthal–Loughran).
- (3) and (4) involve interesting compactifications. Over $\mathbb{F}_q(t)$, homological stability might hold (cf. Boyer et al.).

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Integer points on log K3 surfaces

The Markov-type surface $x^2 + y^2 + z^2 - xyz = k$ is *log Calabi–Yau*. We are interested in solutions $(x, y, z) \in \mathbb{Z}^3$.

- Heuristically, expect only $O_k((\log B)^2)$ solutions with $\max(|x|, |y|, |z|) \leq B$, as $B \to \infty$. More generally, see conjectures of [Browning–Wilsch 2024].
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- Such Diophantine equations lie at the boundary between heuristic solubility and paucity. Any integer solutions only barely exist (on average)!
- Another, infamous, example of a log K3 surface is the *sum of 3 cubes* problem $x^3 + y^3 + z^3 = k$. For $k = 42$ the only known solution [Booker–Sutherland 2019] is

$$(-80538738812075974)^3 + (80435758145817515)^3 + (126021232$$

- These problems test the limits of our understanding.
- They are directly adjacent to undecidable problems. ($\exists$ undecidable quartic equations over $\mathbb{Z}$.)

Solubility of Markov-type surfaces

The polynomial $M = x^2 + y^2 + z^2 - xyz$ is fixed by a group $\Gamma \subseteq \mathrm{Aut}(M)$, where Γ is formed by S_3, sign changes ± 1, and Vieta involutions $(x, y, z) \mapsto (x, y, xy - z)$. Let $h_M(k)$ be the number of Γ-orbits of the set $\{(x, y, z) \in \mathbb{Z}^3 : M = k\}$.

Theorem (Ghosh–Sarnak 2017)

We have $h_M(k) \to \infty$ along a density 1 of admissible[a] $k \in \mathbb{Z}$. In particular, the integral Hasse principle holds for almost all $k \in \mathbb{Z}$.

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Theorem (Mishra 2024; lower bound is new)

Fix $\epsilon > 0$. The inequality $(\log|k|)^{2-\epsilon} \leq h_M(k) \leq (\log|k|)^{2+\epsilon}$ holds for a density 1 of admissible $k \in \mathbb{Z}$.

(Upper bound $\Leftarrow$ Ghosh–Sarnak + Markov's inequality.)

For all $k < 0$ (and for all "generic" $k \geq 5$), Ghosh–Sarnak construct a fundamental domain $\mathcal{F}_k$ for the action of Γ on $\{(x,y,z) \in \mathbb{Z}^3 : x^2+y^2+z^2-xyz=k\}$. Let $r_M(k)$ be the number of points in a well-chosen region $\mathcal{F}'_k \subseteq \mathcal{F}_k$.

- For Ghosh–Sarnak, $\mathcal{F}'_k$ satisfies $|x| \asymp |yz| \asymp |k|^{1/2}$ and $|z| \leq |k|^{\epsilon}$. Real density of solutions: $\sigma_\infty(k) \asymp \epsilon \log|k|$.
- For Mishra, $\mathcal{F}'_k$ is part of a $\mathbb{G}_m^2$-torus $|xyz| \asymp |k|$, with $|k|^{\delta} \leq |x/y| \leq |k|^{-\delta}|z| \leq |k|^{2\delta}$. Here $\sigma_\infty(k) \asymp (\log|k|)^2$.

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One then expands and upper-bounds an *arithmetic variance*

$$Var(K, A) := \sum_{k \leq K} (r_M(k) - r_M^{loc}(k; A))^2.$$

- The sum $\sum_{k \leq K} r_M(k)^2$ counts solutions in a region to $x^2 + y^2 + z^2 - xyz = u^2 + v^2 + w^2 - uvw$.
- Here $r_M^{loc}(k; A)$ is roughly a truncated L-function at 1. Ghosh–Sarnak (resp. Mishra) use a multiplicative (resp. additive) truncation.
- Some of this generalizes to sums of three cubes.

Let $L(s, \boldsymbol{c})$ be the L-function of $V_{\boldsymbol{c}} : x_1^3 + \cdots + x_6^3 = \boldsymbol{c} \cdot \boldsymbol{x} = 0$, where $\boldsymbol{c} = (c_1, \ldots, c_6) \in \mathbb{F}_q[t]^6$, with $\gcd(q, 6) = 1$ and $\Delta(\boldsymbol{c}) := \mathrm{disc}(V_{\boldsymbol{c}}) = \prod (c_1^{3/2} \pm c_2^{3/2} \pm \cdots \pm c_6^{3/2}) \neq 0$.

Theorem (Browning–Glas–W. 2024)

Assume sufficient progress on moments of $\frac{1}{L(s,\boldsymbol{c})}$ for $\Delta(\boldsymbol{c}) \neq 0$. Then $x^3 + y^3 + z^3 = n$ is soluble in elements $x, y, z \in \mathbb{F}_q[t]$ of degree $\sim \frac{1}{3} \deg n$ for a density 1 of elements $n \in \mathbb{F}_q[t]$.

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Builds on ideas of many authors, such as the following:

- Ghosh–Sarnak, Diaconu (log-K3 variance analysis),
- Kloosterman, Hooley 1986, Heath-Brown,
- Beauville (quadric bundles over $\mathbb{P}^2$), Getz, Tran,
- Rubinstein–Sarnak (Chebyshev's bias via prime squares),
- Deligne (GRH), Hooley 1994 (singular cubics),
- Huang ($\approx \mathbb{Q}$-points), Busé–Jouanolou ($\Delta \in (f, (f')^2)$),
- Bhargava (Ekedahl sieve), Poonen (square-free sieve),
- Kisin (local constancy of L-factors).

What kind of progress on L-functions?

Let $2 \nmid q$. Let $\mu(r)$ be the Möbius function over $\mathbb{F}_q[t]$, and let $\chi_m(r) = \left(\frac{r}{m}\right)$ be the Jacobi symbol over $\mathbb{F}_q[t]$.

Theorem (Bergström–Diaconu–Petersen–Westerland, Miller–Patzt–Petersen–Randal-Williams, W. 2024)

If $1 \leq M = 2g + 1$ *and* $1 \leq R \leq \alpha M$, *and* $q \ggg_\alpha 1$, *then*

$$\frac{\sum_{|m|=q^M} \sum_{|r|=q^R} \mu(r)\chi_m(r)}{q^M q^{R/2}} \ll q^{-0.001M+O(1)},$$

where the sums over m *and* r *run through square-free, monic* $m, r \in \mathbb{F}_q[t]$ *with* $\deg m = M$ *and* $\deg r = R$, *respectively.*

Theorem (Same papers; new for $q = p \equiv 1 \bmod 4$)

The set $\{m : L(\frac{1}{2}, \chi_m) = 0\}$ *has upper density* $o_{q\to\infty}(1)$.

Homological stability approach

Recall that $M = 2g+1$. By the Lefschetz trace formula,

$$\frac{\sum_{|m|=q^M}\sum_{|r|=q^R}\mu(r)\chi_m(r)}{q^M q^{R/2}} = \sum_{k\geq 0}(-1)^k \operatorname{tr}(Fr_q, H_k(X_M, Sym^R V_M)),$$

where $X_M = \mathrm{UConf}_M(\mathbb{A}^1) = \{m : \mathrm{disc}(m) \neq 0\}$, and where V_M is a local system on X_M with fibers $H^1_c(\{y^2 = m(t)\}, \overline{\mathbb{Q}}_\ell(\frac{1}{2}))$ of dimension $2g$ over $m \in X_M$. Captures zeros of $L(s, \chi_m)$.

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- The first few homology groups $H_k = H_c^{2M-k}$ are the most significant.[2] By Deligne, $\operatorname{tr}(Fr_q, H_k) \leq q^{-k/2} \dim H_k$. It is also known that $\dim H_k \leq 2^M \binom{R+2g-1}{R} \leq 2^M 2^{R+2g-1}$. Thus $\dim H_k \leq 2^{(2+\alpha)M}$, since $1 \leq R \leq \alpha M$.
- By geometric series, $\sum_{k+1>\delta M}$ is negligible if $q^{\delta/2} \geq 3^{2+\alpha}$.
- We are left with estimating $\sum_{k+1\leq\delta M}$. Want a stability isomorphism $H_k(X_M, Sym^R V_M) \to H_k(X_{M+2}, Sym^R V_{M+2})$.

[2] "Obstructions to cancellation". For instance, $H_0 = (Sym^R V_M)_{\pi_1(X_M)}$.

No useful map $X_M \to X_{M+2}$. So BDPW compactify a quotient of X_M into a space X'_M. Want gluing/stabilization maps $\sigma\colon X'_M \to X'_{M+2}$ and $\tau\colon V_M \to \sigma^* V_{M+2}$.

- Roughly, if $m(t) \in X_M$, then $\sigma(m(t)) \in X'_{M+2}$ could be a *stable map* $\mathbb{P}^1 \cup_{\{\infty\}\sim\{0\}} \mathbb{A}^1 \to \mathbb{P}^1$ extending $m(t)$.
- There are many possible maps σ. The collection of possible σ has a *braided monoidal structure*.
- In general, $\sigma_1\sigma_2 \neq \sigma_2\sigma_1$. This is different than maps like multiplication in $\mathbb{F}_q[t]$. (More on this later…)

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- Here σ maps X'_M into the boundary $\partial X'_{M+2} \subseteq X'_{M+2}$. The map τ on H^1_c is induced by proper base change ("deformation retraction onto the boundary") and a collapse map $\sigma(X'_M) \to X'_M$.
- By log geometry, $H_k(X'_M, Sym^R V_M) = H_k(X_M, Sym^R V_M)$. A map $\sigma \colon H_k(X_M, Sym^R V_M) \to H_k(X_{M+2}, Sym^R V_{M+2})$ thus arises via σ and τ. It can be checked to agree with a topological version of the map over $\mathbb{C}$.

The local system V_M is symplectic, by Poincaré duality on H^1_c. So $(V_M)^{\otimes R}$ breaks up into symplectic pieces $(V_M)_\lambda$ associated to partitions λ of R, with $\lambda = (R)$ giving $Sym^R V_M$.

- The map $\sigma\colon H_k(X_M, (V_M)_\lambda) \to H_k(X_{M+2}, (V_{M+2})_\lambda)$ is an isomorphism for $k+1 \leq \delta M$, for all R and λ (MPPRW).
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- In general the stable H_k are nonzero, leading to arithmetic main terms (matched to H_k by a limiting process).
- Using monodromy $\rho\colon \pi_1(X_M) \twoheadrightarrow Q_M \subseteq Sp(V_M)$ over $\mathbb{C}$, MPPRW build $\ker(H_k(X_M, (V_M)_\lambda) \to H_k(BQ_M, (V_M)_\lambda))$ out of $H_{k+1-k'}(X_{M-M'}, (V_M)_\lambda)$ for various pairs (k', M') with $k' \geq 2$ (by surjectivity of ρ) and $M' \ll k'$.
- If $le(\lambda) \ggg k+1$, then $le(\lambda) - 2M' \geq le(\lambda) - O(k) > 0$, so the restriction $(V_M)_\lambda|_{Sp(V_{M-M'})}$ has no trivial piece, by branching rules for restriction between symplectic groups. By vanishing results for $H_k(Q_M, \cdot)$ (Borel et al.), we get $H_k(X_M, (V_M)_\lambda) = 0$ via induction on $k \leq \delta M - 1$.
- The case $le(\lambda) \ll k+1 \leq \delta M$ relies on stable branching, and on a relative version of the argument above.

Analytic approach to extracting main terms

Idea: The stable traces can be evaluated at any $M \ggg k+1$ of our choice. In particular, we can enlarge M so that $|\lambda|$ lies within the available sieve-theoretic *level of distribution* from analytic number theory. (Already observed by BDPW.)

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For instance, using Grothendieck–Lefschetz, and our earlier bounds on (unstable) traces to justify manipulations,

$$\begin{aligned}
&\sum_{k\geq 0}(-1)^k \lim_{M\to\infty} \operatorname{tr}(Fr_q, H_k(X_M, Sym^R V_M)) \\
&= \lim_{M\to\infty} \frac{\sum_{|m|=q^M}\sum_{|r|=q^R}\mu(r)\chi_m(r)}{q^M q^{R/2}} \\
&= \sum_{|r|=q^R} \frac{\mu(r)}{q^{R/2}} \lim_{M\to\infty} \frac{\sum_{|m|=q^M}\chi_m(r)}{q^M} = 0,
\end{aligned}$$

since $r \neq 1$. However, this doesn't prove the deeper fact that $\lim_{M\to\infty} \operatorname{tr}(Fr_q, H_k(X_M, Sym^R V_M)) = 0$ for $R > 4k$ (BDPW).

More general geometric families

Definition

Let $s, n, \delta \in \mathbb{N}$. Let $P \in k[t][x_1, \dots, x_s]$ be square-free. Let $k[t]_n$ denote the set of polynomials in $k[t]$ of degree n. Define the *Poonen space* $X = X_{n,\delta} \subseteq \mathbb{A}_k^{(n+1)s}$ so that for k-algebras R,

$$X(R) = \{(f_1, \dots, f_s) \in R[t]_n^s : P(f_1, \dots, f_s) \in R^\times \cdot \mathrm{UConf}_\delta(R)\}.$$

For $s = 1$, $P(f) = f$: $X(R) = \{f \in R[t]_n : \mathrm{disc}(f) \in R^\times\}$, which is essentially the setting of all the classical geometric families (CFKRS et al.), such as $L(\cdot, \chi_f)$.

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For general s and P, the space X is natural when studying, for instance, the family of L-functions $L(\cdot, \chi_{P(f_1,\ldots,f_s)})$.

Theorem (Poonen 2003)

Let $k = \mathbb{F}_q$. Then $\lim_{n\to\infty} \frac{\#X_{n,n\deg P}(\mathbb{F}_q)}{q^{(n+1)s}}$ exists. Power-saving error term $O_{P,q}(q^{-(n+1)s\eta})$ where $\eta \asymp_P 1/\mathrm{char}(\mathbb{F}_q)$.

Aside: Another motivation for sieve study

Theorem (Elkies)

Let $F(A, B, C)$ be the ternary cubic form

$$9A^3+9A^2B+3A^2C+3AB^2-6ABC+3AC^2+3B^3+3B^2C+BC^2+C^3.$$

$\mathbb{P}^2 \dashrightarrow V_{\mathbb{P}^3}(x^3+y^3+z^3+w^3)$, $[A:B:C] \mapsto [F(A,B,C) : F(-A,B,-C) : F(-A,-B,C) : F(A,-B,-C)]$ *is birational.*

Corollary

$\#\{[-T,T]^4 : x^3+y^3+z^3+w^3 = 0 \neq x+y+z+w\} \ll_\epsilon T^{1+\epsilon}$ *(weak Manin conjecture) is equivalent to (1). Also, (2)⇒(1).*

1. $\#\{[1,N]^3 : \gcd(a,b,c)=1,\ \gcd(a,3b^2+c^2)\gcd(b,3a^2+c^2)\gcd(c,3a^2+b^2) \asymp G\}/N^3 \ll_\epsilon N^\epsilon/G.$
2. $\#\{[1,N]^3 : \gcd(a,b,c) = 1,\ \mathrm{sq}(abc(3b^2+c^2)(3a^2+b^2+c^2)) \asymp G^2\}/N^3 \ll_\epsilon N^\epsilon/G.$

Something probably doable

Question

Can sieve-theoretic topology methods (such as those of [Das–Tosteson 2024]) compute the stable homology of some Poonen spaces $X_{n,n\deg P}(\overline{\mathbb{F}}_q)$ or $X_{n,n\deg P}(\mathbb{C})$ with polynomials $P(f) \neq f$ such as $P(f) = f^2 - 1$ or $P(f_1, f_2) = (f_1^2 - f_2^2)f_1 f_2$, or more generally products of linear things?

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The recent progress on quadratic Dirichlet L-functions is monodromy-theoretic rather than sieve-theoretic. However, it might be helpful to know the "answer" (stable homology) in some cases, in order to guide further progress on square-free values and statistics of geometric L-functions.

Complex Betti bounds

Let $s, n, \delta \in \mathbb{N}$. Let $P \in k[t][x_1, \ldots, x_s]$ be square-free. Let $k[t]_n$ denote the set of polynomials in $k[t]$ of degree n. Define the *Poonen space* $X = X_{n,\delta} \subseteq \mathbb{A}_k^{(n+1)s}$ so that for k-algebras R,

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Proposition (W. 2025+)

Let $k = \mathbb{C}$. Let $\deg_{tot} P := \deg P + \deg_t P \geq 1$. For every local system $\mathcal{L}$ on X of finite-dimensional vector spaces over $\mathbb{C}$,

$$\sum_{i \geq 0} \dim H^i(X, \mathcal{L}) \leq \exp(O_{\deg_{tot} P}(ns)) \operatorname{rank} \mathcal{L}.$$

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Proof.

X is smooth, connected, and cut out in $\mathbb{A}_k^{1+(n+1)s} \times \mathrm{UConf}_\delta(k)$ by equations like "degree $1 + \deg P$ poly $=$ linear in e_i"... $\square$

Lemma

Let $k = \mathbb{C}$. *Let* $m, n, r, d \geq 1$ *be integers. Let* $g_1, \ldots, g_r \in k[\boldsymbol{x}] + k[\boldsymbol{e}] = k[x_1, \ldots, x_m] + k[e_1, \ldots, e_n]$ *such that* $\deg_{\boldsymbol{x}}(g_j) \leq d$ *and* $\deg_{\boldsymbol{e}}(g_j) \leq 1$ *for all* $1 \leq j \leq r$. *Let* $W \subseteq k^m \times \mathrm{UConf}_n(k) \subseteq k^{m+n}$ *be the complex subvariety* $g_1 = \cdots = g_r = 0$, *where* e_i *is the coordinate on* $\mathrm{UConf}_n(k) \subseteq k^n$ *corresponding to the* i*th elementary symmetric polynomial under the covering map* $\mathrm{PConf}_n(k) \to \mathrm{UConf}_n(k)$. *Assume* W *is smooth and connected. Then*

$$\sum_{i \geq 0} \dim H^i(W, \mathcal{L}) \leq \exp(O_d(m + n + r)) \operatorname{rank} \mathcal{L}.$$

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Proof.

W is affine. Slicing induction as in [Katz 2001], by Artin vanishing and affine weak Lefschetz, reduces us to bounding $\chi(W, \mathcal{L}) = \chi(W) \operatorname{rank} \mathcal{L}$. But $\chi(W) = \frac{\chi(V)}{n!} \ll \frac{\chi(V)}{(n/e)^n} \ldots$ $\square$

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$$|\chi(V)| \leq \exp(O_d(m + n + r)) \cdot n^n.$$

Proof.

V is smooth, so $\chi(V) = \chi_c(V)$. But $V \subseteq k^{m+n} \subseteq k^{(m+2n)+n}$ is the zero locus of $r + 2n$ polynomials: $g_j(\boldsymbol{x}, \boldsymbol{e})$ for $1 \leq j \leq r$, and $v_i - \prod_{l \neq i}(z_l - z_i)$ and $v_i w_i - 1$ for $1 \leq i \leq n$... $\square$

Proof.

…Use the following lemma with $n_2 = d_2 = n$. $\square$

Lemma (Adolphson–Sperber 1987; "Dwork theory")

Let k be an algebraically closed field. Let ℓ be a prime number invertible in k. Let $n_1, n_2, r, d_1, d_2 \geq 1$ be integers. Let $f_1, \ldots, f_r \in k[\boldsymbol{x}] + k[\boldsymbol{y}] = k[x_1, \ldots, x_{n_1}] + k[y_1, \ldots, y_{n_2}]$ such that $\deg_{\boldsymbol{x}}(f_j) \leq d_1$ and $\deg_{\boldsymbol{y}}(f_j) \leq d_2$ for all $1 \leq j \leq r$. Then

$$|\chi_c(\operatorname{Spec}(k[\boldsymbol{x}, \boldsymbol{y}]/(f_1, \ldots, f_r)))| \leq \exp(O(n_1 + n_2 + r)) d_1^{n_1} d_2^{n_2}.$$

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Proof.

LHS $\leq \exp(O(n_1 + n_2 + r)) \cdot (n_1 + n_2)! \operatorname{vol}(S_{d_1,d_2})$, where S_{d_1,d_2} is the convex hull of $\mathbf{0}, d_1 e_1, \dots, d_1 e_{n_1}, d_2 e_{n_1+1}, \dots, d_2 e_{n_1+n_2}$ in $\mathbb{R}^{n_1+n_2}$. By a diagonal re-scaling of $\mathbb{R}^{n_1+n_2}$, we have $\operatorname{vol}(S_{d_1,d_2}) = d_1^{n_1} d_2^{n_2} \operatorname{vol}(S_{1,1}) = \frac{d_1^{n_1} d_2^{n_2}}{(n_1+n_2)!}$. (Cf. [Weil 1949].) $\square$

An affine stabilization map

Let $P \in k[x] \subseteq k[t][x]$. Define $X = X_{n,\delta} \subseteq \mathbb{A}_k^{n+1}$ so that

$$X(R) = \{f \in R[t] : \deg(f) = n,\ P(f) \in R^\times \cdot \mathrm{UConf}_\delta(R)\}.$$

Lemma (W. 2025+)

Let k be a field. Let $m, n \geq 1$ be integers. Let

$$j \in (1 + t \cdot k[t]) \cap (k^\times \cdot \mathrm{UConf}_m(k^\times)).$$

Let $f \in X_{n, n \deg P}$. Then $j(\epsilon t) f(t) \in X_{m+n, (m+n) \deg P}$ for all ϵ in a punctured neighborhood of $0 \in \mathbb{A}^1$.

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Proof.

If $k = \mathbb{C}$, use Rouché's theorem. In general, use discriminants, Gauss' lemma, and Newton polygons, over the valued field $\overline{k((\epsilon))}$, and use a change of variables $t \mapsto r := \epsilon t$. $\square$

A projective stabilization map

Same argument gives a nicer result in the following setting: Let $P \in k[x_1, \ldots, x_s]$ be homogeneous. Let $f^{rev}(t) := t^{\deg f} f(\frac{1}{t})$. Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a}', \boldsymbol{b}', \boldsymbol{\lambda} \in (k^\times)^s$. Let $X_n^{\boldsymbol{a},\boldsymbol{b},\boldsymbol{\lambda}}(R)$ be

$$\begin{aligned}\{(f_1, \ldots, f_s) \in R[t]_n^s : {} & [f_1^{rev}(0) : \cdots : f_s^{rev}(0)] = [a_1 : \cdots : a_s] \in \mathbb{P}^{s-1}, \\ & [f_1(0) : \cdots : f_s(0)] = [b_1 : \cdots : b_s] \in \mathbb{P}^{s-1}, \\ & P(\lambda_1 f_1, \ldots, \lambda_s f_s) \in R^\times \cdot \mathrm{UConf}_{n \deg P}(R^\times)\}.\end{aligned}$$

Lemma (W. 2025+)

If $(j_1, \ldots, j_s) \in X_m^{\boldsymbol{a}',\boldsymbol{b}',\boldsymbol{a}\boldsymbol{\lambda}/\boldsymbol{b}'}$ and $(f_1, \ldots, f_s) \in X_n^{\boldsymbol{a},\boldsymbol{b},\boldsymbol{\lambda}}$, then $(j_1(\epsilon t) f_1(t), \ldots, j_s(\epsilon t) f_s(t)) \in X_{m+n}^{\boldsymbol{a}'\boldsymbol{a},\boldsymbol{b}'\boldsymbol{b},\boldsymbol{\lambda}/\boldsymbol{b}'}$ for all ϵ in a punctured neighborhood of $0 \in \mathbb{A}^1$.

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This suggests that $X_\bullet^{\mathbf{1},\mathbf{1},\boldsymbol{\lambda}}(\mathbb{C})$ is a braided monoid (with braiding $t \mapsto t/\epsilon$ as $\epsilon \to 0$), and $C_*(X_\bullet^{\boldsymbol{a},\boldsymbol{b},\mathbf{1}}(\mathbb{C}))$ is a bi-module over $(X_\bullet^{\mathbf{1},\mathbf{1},\boldsymbol{a}}(\mathbb{C}), X_\bullet^{\mathbf{1},\mathbf{1},\boldsymbol{b}}(\mathbb{C}))$. Can we build cells over $(C_*(\widetilde{X}), \pi_1)$?

General approaches to consider

Say we're interested in a geometric family Y_M of L-functions (e.g. a Poonen space), possibly within a larger family X_M.

1. One might try to generalize the cell-based inductive approach of MPPRW to Y_M or X_M, possibly introducing new ideas to build cells and set up the induction.

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Say we're interested in a geometric family Y_M of L-functions (e.g. a Poonen space), possibly within a larger family X_M.

1. One might try to generalize the cell-based inductive approach of MPPRW to Y_M or X_M, possibly introducing new ideas to build cells and set up the induction.
2. If the fact (due to BDPW + MPPRW) that
$$\mathrm{tr}(Fr_q, H_k(X_M, Sym^R V_M)) = 0 \tag{1}$$
for $M, R \ggg k+1$ is robust enough to hold more generally, then one might hope to express X_M as an average of families $Y_{M,a}$, indexed by some parameter a with $Y_{M,0} = Y_M$, and use fiber comparison methods of [Sawin, Acta 2024] to prove the same vanishing result for Y_M.
3. The recursive analytic methods of Soundararajan, Harper, Bui–Florea–Keating, et al., writing $1/L = \exp(-\log L)$, are currently the most flexible approach available.

Let $L(s, \boldsymbol{c})$ be the L-function of $V_{\boldsymbol{c}} : x_1^3 + \cdots + x_6^3 = \boldsymbol{c} \cdot \boldsymbol{x} = 0$, where $\boldsymbol{c} = (c_1, \ldots, c_6) \in \mathbb{F}_q[t]^6$, with $\gcd(q, 6) = 1$ and $\Delta(\boldsymbol{c}) := \operatorname{disc}(V_{\boldsymbol{c}}) = \prod (c_1^{3/2} \pm c_2^{3/2} \pm \cdots \pm c_6^{3/2}) \neq 0$.

Theorem (Browning–Glas–W. 2024)

Assume sufficient progress on moments of $\frac{1}{L(s,\boldsymbol{c})}$ *for* $\Delta(\boldsymbol{c}) \neq 0$. *Then* $x^3 + y^3 + z^3 = n$ *is soluble in elements* $x, y, z \in \mathbb{F}_q[t]$ *of degree* $\sim \frac{1}{3} \deg n$ *for a density* 1 *of elements* $n \in \mathbb{F}_q[t]$.

Builds on ideas of many authors, such as the following:

- Ghosh–Sarnak, Diaconu (log-K3 variance analysis),
- Kloosterman, Hooley 1986, Heath-Brown,
- Beauville (quadric bundles over $\mathbb{P}^2$), Getz, Tran,
- Rubinstein–Sarnak (Chebyshev's bias via prime squares),
- Deligne (GRH), Hooley 1994 (singular cubics),
- Huang ($\approx \mathbb{Q}$-points), Busé–Jouanolou ($\Delta \in (f, (f')^2)$),
- Bhargava (Ekedahl sieve), Poonen (square-free sieve),
- Kisin (local constancy of L-factors).

Variance analysis

We actually reduce everything to counting solutions to

$$\sum_{i=1}^{6} x_i^3 = 0$$

in certain regions of $\mathbb{F}_q[t]^6$. A homogeneous equation of degree d in $2d$ variables lies at the *square-root barrier*. The expected asymptotic (Manin et al.) often features *two main terms* of the same order of magnitude.

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The hyperplane sections

$$V_{\boldsymbol{c}} : x_1^3 + \cdots + x_6^3 = \boldsymbol{c} \cdot \boldsymbol{x} = 0,$$

for $\boldsymbol{c} = (c_1, \ldots, c_6) \in \mathbb{F}_q[t]^6$, and especially their L-functions, arise through the circle method and Fourier analysis. Analysis of these hyperplane sections branches out based on the vanishing, size, and divisibility of $\Delta(\boldsymbol{c})$.

A deformation along the square-root barrier

Browning, Munshi, and I (2025+) hope to use the smooth (Duke–Friedlander–Iwaniec) version of the *circle method* to prove an unconditional asymptotic over $\mathbb{Q}$ for the singular 6-variable homogeneous cubic equation

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named after *Perazzo*. This can be viewed as a deformation of a smooth 6-variable cubic like the Fermat

$$\sum_{i=1}^{6} x_i^3 = 0$$

considered in [Browning–Glas–W. 2024]. Whereas L-functions turn out to be less important here, certain divisor problems play a more prominent role (which we hope to handle via Hooley Δ-functions; cf. [de la Bretèche–Tenenbaum 2024]).