Thm (Fermat, Landau, ...): The set $\{x^2+y^2 : x,y \in \mathbb{Z}\}$ has density 0 ~~barely~~ in $\mathbb{Z}$, but barely.
contains $\asymp \frac{N}{(\log N)^{\frac{1}{2}}}$ integers ~~[illegible]~~ $n \le N$. ~~are [illegible]~~

Let $r_2(n) = \#\{x,y \ge 0 : x^2+y^2=n\}$.

$[0,N]^2 \downarrow [0,2N]$

On average, $r_2(n) \sim c > 0$. But $\mathbb{E}_{n \le N}[r_2(n)^2] \to \infty$ as $N \to \infty$
$\left(\mathbb{E}_{n \le N}[r_2(n)] \approx c > 0\right)$.

Sol's are ~~sparse~~ rare, like in factoring ($xy=n$).
~~Let Similarly, if $r_3(n) = \#\{x,y,$~~

For 3 cubes, sometimes easy sols exist.
e.g. $(n+1)^3 - 2n^3 + (n-1)^3 = 6n$ (solution of height $\approx n$, not $n^{\frac{1}{3}}$)

But if $r_3(n) = \#\{x,y,z \ge 0 : x^3+y^3+z^3=n\}$,

$[0,N^{\frac{1}{3}}]^3 \downarrow [0,3N]$

again $\mathbb{E}_{n \le N}[r_3(n)] \approx c_1 > 0$.

Conj. (Folklore) $\mathbb{E}_{n \le N}[r_3(n)^2] < C_2$ and $\{x^3+y^3+z^3 : x,y,z \ge 0\}$ has pos. density.

Idea (2nd moment method):

Cauchy-Schwarz $\Rightarrow$ $\#\{x^3+y^3+z^3 \le N : x,y,z \ge 0\}$

$$\ge \frac{\left(\sum_{n \le N} r_3(n)\right)^2}{\sum_{n \le N} r_3(n)^2},$$

$$\gtrsim (C_1 N)^2/(C_2 N) \asymp N.$$

Hard b/c $\exists$ little structure.

Easier problems: (1) $x^3+y^3+z^3=n$ has ∞ many $\mathbb{Q}$-solutions. (Geometric proof/param.)

(2) $x^3+2y^3+z^3=n$ has ∞ many sols in $\mathbb{Z}[\frac{1}{6}]$ $\left(\frac{a}{6^k},\ a \in \mathbb{Z},\ k \ge 0\right)$ (Linear construction $a_i n + b_i$, $1 \le i \le 3$)

(3) Norm equation $x^3+2y^3+4z^3-6xyz=1$, say,

(Param: $x+y\sqrt[3]{2}+z\sqrt[3]{4} = (1+\sqrt[3]{2}+\sqrt[3]{4})^k$, (finite set))

Application of (3)

(Thue, Selmer) $\#\{(x,y) \in \mathbb{Z}^2 : x^3 + 2y^3 = 1\} < \infty$

Many finitely

Idea: Intersect (3) with $z=0$.

Thm (Browning – Glas – W., 2024)

Conj holds in $(\mathbb{Z}/p\mathbb{Z})[t]$ (a polynomials mod p, analogous to the set $\mathbb{Z}$)

under a Random Matrix Theory conjecture

("hidden symmetry" in the problem).

Proof uses modern form of Hardy – Littlewood circle method

(Fourier analysis (wave decomposition) + Algebraic geometry)

to establish biases + cancellation in waves

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