Special subvarieties over finite and infinite fields

Victor Wang

Princeton University
Advised by Peter Sarnak

L-function and Stratification FRG Grad Seminar, April 2022

Point counts over finite fields

We'll take a shamelessly classical point of view when possible. (This should make things more accessible. Also, a classical point of view may have not just historical but also future value.)

- Let $k = \mathbb{F}_q$ be a finite field of characteristic $p > 0$.
- Let $F(\boldsymbol{x}) = F(x_1, \ldots, x_s)$ be a (nonzero) homogeneous polynomial in s variables over k.
- Let $X = V(F) := \{F = 0\} \subseteq \mathbb{P}^{s-1}$ be the projective zero locus of F over k. We'll focus on varieties of this form, i.e. *projective hypersurfaces*.[1]
- Let $X(k)$ denote the set of k-points of X, i.e. points $[\boldsymbol{x}] = [x_1 : \cdots : x_s] \in \mathbb{P}^{s-1}(k)$ with $F(\boldsymbol{x}) = 0$.
- Let $E(X) := \#X(k) - \#\mathbb{P}^{\dim X}(k)$ (here $\dim X = s - 2$).

---

[1] Note that one can relate *affine* hypersurfaces, e.g. the circle $x^2 + y^2 = 1$ in $\mathbb{A}^2$, to projective hypersurfaces, e.g. $x^2 + y^2 = z^2$ in $\mathbb{P}^2$.

The Naive Riemann Hypothesis

Before talking about special subvarieties ("structure"), let's discuss randomness (at least over finite fields).

- Probabilistically one might expect "$E(X) \ll q^{(\dim X)/2}$" (maybe up to ϵ), for instance by a "square-root cancellation heuristic" for certain exponential sums.[2]

But q is fixed; it's easier to first discuss a technically cleaner question involving a *sequence* of point counts.

- For $r \geq 1$, let $X(\mathbb{F}_{q^r})$ denote the set of $\mathbb{F}_{q^r}$-points of X. Similarly, let $E(X_{\mathbb{F}_{q^r}}) := \#X(\mathbb{F}_{q^r}) - \#\mathbb{P}^{\dim X}(\mathbb{F}_{q^r})$.
- Say X is *error-good* (or *satisfies the Naive Riemann Hypothesis*) if there exists a real number $C > 0$ such that $|E(X_{\mathbb{F}_{q^r}})| \leq C \cdot (q^r)^{(\dim X)/2}$ holds for all $r \geq 1$.

[2]For instance, if $k = \mathbb{F}_p$, the sum $\sum_{a \in \mu_d} \sum_{\boldsymbol{x} \in \mathbb{F}_p^s} e_p(aF(\boldsymbol{x}))$, where $d = \deg F$ and $\mu_d = \{a \in \mathbb{F}_p : a^d = 1\}$.

Technical background

The Weil conjectures (specialized to the case of a smooth projective hypersurface) imply that if $X = V(F) \subseteq \mathbb{P}_k^{s-1}$ is *smooth* (i.e. *non-singular*),[3] then it is error-good.[4]

Definition

X is *singular* at a point $[\boldsymbol{x}] \in X(\bar{k})$ if the gradient ∇F (and F) vanishes at $\boldsymbol{x} = (x_1, \ldots, x_s)$.

The locus of singular hypersurfaces has codimension 1 among projective hypersurfaces.[5] (Explicitly: X is singular if and only if the *discriminant* of F vanishes.)

[3] i.e. "looks algebraically like a smooth manifold"

[4] This is an amazing result, but as we'll see later, it's far from an optimal criterion for error-goodness in general!

[5] Heuristic: $F = \nabla F = 0$ is a system of $s + 1$ equations in s variables, so having a solution is a nontrivial event.

Technical background (cont'd)

Let's say a bit more about the point counts $\#X(\mathbb{F}_{q^r})$ for $r \geq 1$.

- (From now on, let $\ell := 2$ and assume $p \neq 2$. This is for technical reasons that you should ignore today.)
- For $i \in [0, 2\dim X]$, one can define (using ℓ-adic étale cohomology) certain multisets $\mathcal{E}^i(X)$ of complex numbers α (Grothendieck et al.) with $|\alpha| \leq q^{i/2}$ (Deligne).
- In terms of these complex numbers, we have a linear recurrence $\#X(\mathbb{F}_{q^r}) = \sum_{i=0}^{2\dim X} (-1)^i \sum_{\alpha \in \mathcal{E}^i(X)} \alpha^r$. A version of the linear recurrence was first proved by Dwork '60, by concrete (and rather explicit) p-adic methods.[6]
- Also $\#\mathbb{P}^{\dim X}(\mathbb{F}_{q^r}) = 1 + q^r + (q^2)^r + \cdots + (q^{\dim X})^r$, and $E(X_{\mathbb{F}_{q^r}}) = \#X(\mathbb{F}_{q^r}) - \#\mathbb{P}^{\dim X}(\mathbb{F}_{q^r})$ is the difference.
- If X is a *smooth* projective hypersurface, then $\mathcal{E}^i(X) = \mathcal{E}^i(\mathbb{P}^{\dim X})$ for all $i \neq \dim X$, so X is error-good.

[6] See e.g. https://terrytao.wordpress.com/2014/05/13/.

Technical background (cont'd^2)

Recall that $X = V(F) \subseteq \mathbb{P}_k^{s-1}$. Building on Dwork and Deligne, one can show that $|\mathcal{E}^i(X)| \ll_{s,\deg F} 1$ (Bombieri, Katz, et al.).

Remark

When F is diagonal with $p \nmid d = \deg F$, the multiset $\mathcal{E}^{\dim X}(X) \setminus \mathcal{E}^{\dim X}(\mathbb{P}^{\dim X})$ has a Fourier-analytic interpretation.[a] This was (at least part of) Weil's original motivation for the Weil conjectures (Weil '49).

[a] For instance, the cardinality of the multiset is $\#\{\boldsymbol{a} \in \{1, 2, \ldots, d-1\}^s : a_1 + \cdots + a_s \equiv 0 \bmod d\}$.

In any case, in general, using the recurrence for $\#X(\mathbb{F}_{q^r})$ and the bound $|\mathcal{E}^i(X)| \ll_{s,\deg F} 1$, one can prove the following:

1. X is error-good if and only if it is error-good with a constant depending only on $s, \deg F$.

Technical background (cont'd^3)

Recall that we say X is *error-good* if $E(X_{\mathbb{F}_{q^r}}) \ll (q^r)^{(\dim X)/2}$ as $r \to \infty$. Using the recurrence for $\#X(\mathbb{F}_{q^r})$ and the bound $|\mathcal{E}^i(X)| \ll_{s,\deg F} 1$, one can prove, by *amplification* arguments, the following (see e.g. W. '22):

1. X is error-good if and only if it is error-good with a constant depending only on $s, \deg F$.
2. At least if X has isolated singularities,[7] then X is error-good if and only if it is *potentially* error-good.[8]

(2) also uses some technical results of Skorobogatov '92 or Katz '91. (1) is useful in applications, and (2) can be useful in proofs.

---

[7] For instance, this is the case if X is an arbitrary hyperplane section of a smooth projective hypersurface; we'll encounter this case later.

[8] We say X is *potentially* error-good if there exists a finite field extension k'/k such that $X_{k'}$ is error-good; this is a statement about $E(X_{\mathbb{F}_{q^r}})$ for $r \equiv 0 \bmod [k' : k]$.

Dichotomy for quadrics

Recall that $X = V(F) \subseteq \mathbb{P}_k^{s-1}$, where $k = \mathbb{F}_q$ and $F \neq 0$.

Proposition (Quadric dichotomy)

Say $\deg F = 2$ *and* $p \neq 2$. *Then* X *is error-bad*[a] *if and only if* $2 \mid \operatorname{rank}(F) \leq s-1$, *in which case* X *contains a* $\lfloor s/2 \rfloor$*-plane.*

[a] i.e. not error-good

Proof.

Let $t := \operatorname{rank}(F) \leq s$. Since $p \neq 2$, one can diagonalize F over $\mathbb{F}_q$. Thus $E(X(\mathbb{F}_{q^r})) = \pm (q^r)^{(s-t)/2} 1_{2|t} \cdot (q^r)^{(s-2)/2}$, by Fourier analysis as in the aforementioned (very readable) paper of Weil[a] for instance. Bias occurs if and only if $s - t \geq 1$ and $2 \mid t$. $\square$

[a] *Numbers of solutions of equations in finite fields* ('49)

Dichotomy for quadrics (cont'd)

Proposition (Quadric dichotomy)

Say $\deg F = 2$ *and* $p \neq 2$. *Then* X *is error-bad*[a] *if and only if* $2 \mid \operatorname{rank}(F) \leq s - 1$, *in which case* X *contains a* $\lfloor s/2 \rfloor$*-plane.*

[a] i.e. not error-good

Remark

Here $\operatorname{rank}(F) \leq s - 1$ if and only if X is singular. So if $2 \nmid s$, then the conditions "X error-bad" and "X singular" are *nearly equivalent* in some sense.

Remark

Here $\lfloor s/2 \rfloor$-planes are *special*. Note that if X is smooth, then X contains a $\lfloor (s-2)/2 \rfloor$-plane, but not a $\lfloor s/2 \rfloor$-plane.

General codimension-two results

So the "locus of error-bad quadrics" really can have codimension 1.[9] But in general one can do better.

- General perversity-based machinery of Fouvry–Katz '01 often allows one to prove square-root cancellation bounds away from inexplicit loci of codimension 2 (see e.g. Grimmelt–Sawin '21).
- By alternative means, based on "worst-case" results of Skorobogatov '92 or Katz '91 and "average-case" results of Lindner '20, I believe one can prove error-goodness for $\deg F \geq 3$ away from certain explicit loci of codimension 2, related to discriminants (see W. '22 for such results in the case when X is a hyperplane section of a smooth projective hypersurface of degree ≥ 3).

[9] Recall that by the Weil conjectures, "error-badness loci" always have codimension ≥ 1.

General codimension-two results (cont'd)

Remark

Although codimension-two results can be quite useful where codimension-one results fail (cf. Ekedahl's geometric sieve, and related work of Bhargava '14 and Bhargava–Shankar–Wang '16 on producing square-free values of certain polynomials), they still do not really provide a satisfactory explanation of when error-goodness fails, or why it fails when it does.

In general, one might hope to obtain satisfactory *dichotomies* similar in spirit to what we saw for quadrics. In what follows, we will sketch results along these lines for low-dimensional cubics.

Dichotomy for low-dimensional cubics

Recall that $X = V(F) \subseteq \mathbb{P}_k^{s-1}$, where $k = \mathbb{F}_q$ and $F \neq 0$.

Proposition (Warm-up)

Say $s = 3$ and $\deg F = 3$ (so X is a cubic plane curve). Suppose F is square-free. Then X is error-bad if and only if $X_{\overline{k}}$ contains a line.

Proof.

Factor F and use Lang–Weil. Note that $\lfloor (\deg F)/2 \rfloor = 1$. $\square$

One can also give a dichotomy statement for cubic surfaces (see W. '22, *Dichotomous point counts over finite fields*), but it has a different flavor (no longer of the "excess points caused perhaps by special subvarieties" sort), so let's skip it.

Dichotomy for low-dimensional cubics (cont'd)

Recall that $X = V(F) \subseteq \mathbb{P}_k^{s-1}$, where $k = \mathbb{F}_q$ and $F \neq 0$.

Theorem (Cleanest case of a result in W. '22)

Say $s = 5$ and $\deg F = 3$ (so X is a cubic threefold in $\mathbb{P}^4$). Suppose X has isolated singularities.[a] *Then TFAE:*

1. *X is error-bad;*
2. *$X_{\overline{k}}$ contains a plane or a singular cubic scroll in $\mathbb{P}^4_{\overline{k}}$; and*
3. *there exist quadratic forms $(Q_1, Q_2) \in \overline{k}[x_1, \ldots, x_5]$ with a common singularity (i.e. "essentially in 4 variables"), and a homogeneous polynomial $A \in \overline{k}[x_1, \ldots, x_5]$, such that $AF \in (Q_1, Q_2)$ and $A \notin \sqrt{(Q_1, Q_2)}$.*[b]

[a] For instance, this holds if X is a hyperplane section of a smooth projective hypersurface; we'll encounter this case soon.

[b] I think it might be possible to take $\deg A \leq 1$, but have not checked.

Dichotomy for low-dimensional cubics (cont'd^2)

Remark

The statement was originally inspired by the "conic bundle" method of Bombieri–Swinnerton-Dyer '67 used to give the first proof of the (last of the) Weil conjectures for smooth X.
The actual proof ingredients include amplification (base change), the fact that (most) singular cubic hypersurfaces are rational (over $\overline{k}$ at least), the fact that $\lfloor (2 \cdot 3)/2 \rfloor = 3$, and some classification results over $\overline{k}$ (to distinguish, for instance, between cubic scrolls and cubic surfaces).

Question

Is there a more enlightening or more general proof? Can one avoid or minimize use of base change? Can one use the polynomial method?

Dichotomy for low-dimensional cubics (cont'd^3)

Corollary

Let G be a smooth cubic form in $m \in \{4, 6\}$ variables over a finite field k. Let $\boldsymbol{c} \in k^m \setminus \{0\}$. Then $V(G, \boldsymbol{c} \cdot \boldsymbol{x}) \subseteq \mathbb{P}_k^{m-1}$ is error-bad if and only if $V(G, \boldsymbol{c} \cdot \boldsymbol{x})_{\overline{k}}$ contains an $(m-2)/2$-plane or a singular cubic 2-scroll in $\mathbb{P}_{\overline{k}}^{m-1}$.

Example

Say $G = x_1^3 + \cdots + x_m^3$ and p is large. Then by a calculation (a singularity analysis involving, among other things, 3×3 Vandermonde determinants arising from diagonality), the phrase "or a singular cubic 2-scroll" is unnecessary for G. Also, the $(m-2)/2$-planes on $V(G)_{\overline{k}}$ are known to be cut out by systems of equations of the form "$x_i^3 + x_j^3 = 0$ in pairs". Thus a given $V(G, \boldsymbol{c} \cdot \boldsymbol{x})$ is error-bad if and only if "$c_i^3 = c_j^3$ in pairs".

Dichotomy for low-dimensional cubics (cont'd^4)

Question

Can the statements above be Manin-ized into results of the form $E(X(\mathbb{F}_{q^r})) = c_X \cdot (q^r)^2 + O((q^r)^{3/2})$, for some explicit constant c_X?

Remark

The answer is YES at least in some cases with $c_X = 1$. See W. '21 (*Isolating special solutions in the delta method: The case of a diagonal cubic equation in evenly many variables over* $\mathbb{Q}$), which applies these cases in a natural way to Manin-type conjectures for integral (or rational) points on cubic fourfolds such as $x_1^3 + \cdots + x_6^3 = 0$.

Food for thought

Question

To what extent do special subvarieties in the Manin conjectures correlate with special subvarieties over finite fields? For example, it would be interesting to determine whether the special quadratic locus

$$x_1 + x_2 + x_3 = x_4 + x_5 + x_6 = (x_1^2 + x_2^2 + x_3^2) - (x_4^2 + x_5^2 + x_6^2) = 0$$

on the 6-variable quartic $x_1^4 + x_2^4 + x_3^4 = x_4^4 + x_5^4 + x_6^4$ (which I believe was originally introduced by Wooley in a context related to the present FRG) remains special for hyperplane sections of the quartic over finite fields.