Conditionally around the square-root barrier for cubes

Victor Wang

Princeton University
Advised by Peter Sarnak

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Overview

Let $F(\boldsymbol{x}) := x_1^3 + \cdots + x_6^3$. This talk centers around Diophantine equations and L-functions, especially

1. $F(\boldsymbol{x}) = 0$ over $\mathbb{Z}$, as well as
2. $F(\boldsymbol{x}) = \boldsymbol{c} \cdot \boldsymbol{x} = 0$ over $\mathbb{F}_p$ (as $\boldsymbol{c}, p$ vary), and
3. the associated Hasse–Weil L-functions $L(s, V_{\boldsymbol{c}})$ (over $\Delta(\boldsymbol{c}) \neq 0$).

Problem (Many authors)

Estimate the number of integral solutions to $F(\boldsymbol{x}) = 0$ *in expanding boxes or other regions.*

Remark (Many authors)

This problem is closely tied to the statistics of *sums of* 3 *cubes*.

The usual randomness heuristic (via level sets)

Let $s := 6$. For $K \subset \mathbb{R}^s$ nice (cpt, semi-alg), $X \to \infty$, and $a \in \mathbb{Z}$, let $N_{F-a,K}(X) := \#\{\boldsymbol{x} \in \mathbb{Z}^s \cap XK : F = a\}$.

Example

Say $K = [-1,1]^s$. Then $XK = [-X,X]^s$, and

$$F(\mathbb{Z}^s \cap XK) \ll X^3 \quad \text{(since } F = x_1^3 + \cdots + x_s^3 \text{ is cubic).}$$

So $N_{F-a,K}(X)$ is $\asymp X^{s-3}$ *on avg* (in ℓ^1) over $a \ll X^3$.

Hardy–Littlewood ("randomness") prediction for $F = 0$:

$$N_{F,K}(X) \approx X^{s-3} \prod_{v \leq \infty} \sigma_v.$$

Randomness and structure (for $F := x_1^3 + \cdots + x_6^3$)

Hooley '86a: HL ("randomness") prediction misses triv. sol's ($x_i + x_j = 0$ in pairs); maybe the truth is HLH?

Conjecture (HLH)

For any nice $K \subset \mathbb{R}^6$,

$$N_{F,K}(X) = c_{\mathrm{HL},F,K} \cdot X^3 + \#\{\text{triv. } \boldsymbol{x} \in \mathbb{Z}^6 \cap XK\} + o(X^3).$$

Remark (Around the square-root barrier)

1. The full HLH lies beyond the classical $\circ$-method (according to square-root "pointwise" minor arc considerations).
2. But the δ-method opens the door to progress on HLH, by harmonically decomposing the true minor arc contribution in a "dual" fashion.

What's known towards HLH?

1. Hua '38: $N_{F,K}(X) \ll X^{7/2+\epsilon}$ (by Cauchy b/w structure and randomness in $4, 8$ vars, resp.).
2. Vaughan '86+: " " $\ll X^{7/2}(\log X)^{\epsilon-5/2}$ (by new source of randomness).
3. Hooley '86+: " " $\ll X^{3+\epsilon}$, under Hypo HW ($\approx$ modularity $+$ GRH for the Hasse–Weil L-functions $L(s, V_{\boldsymbol{c}})$).

Hooley used an "upper-bound precursor" to the δ-method.

The δ-method

Proposition (δ-method: Kloosterman '26, Duke–Friedlander–Iwaniec '93, Heath-Brown '96)

$$N_{F,K}(X) \approx\approx \mathbb{E}_{\boldsymbol{c} \ll X^{1/2}} \mathbb{E}_{n \leq X^{3/2}} [n^{-1} S_{\boldsymbol{c}}(n)] =: \star$$

($\boldsymbol{c} \in \mathbb{Z}^6$), where $\approx\approx$ means I may be lying a bit, and

$$S_{\boldsymbol{c}}(n) := \sideset{}{'}\sum_{a \bmod n} \sum_{\boldsymbol{x} \in (\mathbb{Z}/n)^6} e_n(aF(\boldsymbol{x}) + \boldsymbol{c} \cdot \boldsymbol{x}).$$

($e_n(t) := e^{2\pi i t/n}$) (Don't worry about the "$'$"; it means $a \perp n$)

Remark

Here $\boldsymbol{c} = 0$ captures major arcs (roughly speaking), producing HL but not full HLH. And $\boldsymbol{c} \neq 0$ captures...

"Pf".

Idea ("Kloosterman method") is to treat classical major and minor arcs uniformly (using Poisson summation[a]), and average over a mod n.

$$N_{F,K}(X) \approx\approx \sum_{n \le X^{3/2}} \frac{1}{nX^{3/2}} \sum_{a \bmod n}' \sum_{\boldsymbol{x} \ll X} e_n(aF(\boldsymbol{x})) \quad (\circ\text{-method})$$

$$\approx\approx \sum_{n \le X^{3/2}} \frac{1}{nX^{3/2}} \mathbb{E}_{\boldsymbol{c} \ll n/X}[S_{\boldsymbol{c}}(n)] \quad (\text{"complexity" } n/X)$$

$$\approx\approx \mathbb{E}_{n \le X^{3/2}} \mathbb{E}_{\boldsymbol{c} \ll X^{1/2}}[n^{-1} S_{\boldsymbol{c}}(n)] = \star.$$

Idea': In gen'l (for $n \gg X$ large), $\sum'_{a \bmod n} \sum_{\boldsymbol{x} \ll X} e_n(aF(\boldsymbol{x}))$ is incomplete mod n, but still a wt'd avg of the complete sums $S_{\boldsymbol{c}}(n)$, if we sample over enough $\boldsymbol{c}$'s (Nyquist–Shannon). □

[a] with $\boldsymbol{c} = 0$ "purely probabilistic", and $\boldsymbol{c} \neq 0$ subtler

The $S_{\boldsymbol{c}}(n)$'s relate to $\mathcal{V}_{\boldsymbol{c}} := \{[\boldsymbol{x}] \in \mathbb{P}^5 : F(\boldsymbol{x}) = \boldsymbol{c} \cdot \boldsymbol{x} = 0\}$.
Fact: $\exists$ disc poly $\Delta \in \mathbb{Z}[\boldsymbol{c}]$ measuring singularities of $\mathcal{V}_{\boldsymbol{c}}$.

Lemma (Hooley)

If $\Delta(\boldsymbol{c}) \neq 0$, then $\widetilde{S}_{\boldsymbol{c}}(n) := n^{-7/2} S_{\boldsymbol{c}}(n)$ look (to 1st order) like the coeffs $\mu_{\boldsymbol{c}}(n)$ of $1/L(s, V_{\boldsymbol{c}})$ ($V_{\boldsymbol{c}} := (\mathcal{V}_{\boldsymbol{c}})_{\mathbb{Q}}$).

Partial proof sketch.

Here F is homog (& a is summed), so $S_{\boldsymbol{c}}(n)$ is multiplicative.
Locally: If $p \nmid \boldsymbol{c}$, then $\widetilde{S}_{\boldsymbol{c}}(p) = \widetilde{E}_{\boldsymbol{c}}(p) + O(p^{-1/2})$, where $\widetilde{E}_{\boldsymbol{c}}(p) := p^{-3/2}[\#\mathcal{V}_{\boldsymbol{c}}(\mathbb{F}_p) - \#\mathbb{P}^3(\mathbb{F}_p)]$. Now use LTF. $\square$

Exercise (Cf. Hooley, "$\underline{\underline{2}}\times$-Kloosterman")

"Assume" $\forall \boldsymbol{c}, n, N$: $\Delta(\boldsymbol{c}) \neq 0$, $\widetilde{S}_{\boldsymbol{c}}(n) = \mu_{\boldsymbol{c}}(n)$, $\sum_{n \leq N} \mu_{\boldsymbol{c}}(n) \ll \|\boldsymbol{c}\|^{\epsilon} N^{1/2+\epsilon}$. Then $\star \ll X^{3+\epsilon}$.

By coincidence, the "double Kloosterman" misses HLH by ϵ.

Theorem (Hooley '86+/Heath-Brown '98)

$N_{F,K}(X) \ll_\epsilon X^{3+\epsilon}$, under Hypo HW ($\approx$ modularity + GRH) for $L(s, V_{\boldsymbol{c}})$'s (over $\Delta(\boldsymbol{c}) \neq 0$).[a]

[a] A large-sieve hypo would suffice (W.). It's open! But $\exists$ uncond. apps to $x^2 + y^3 + z^3$ (W., via Brüdern '91 + Duke–Kowalski '00 + Wiles et al).

Theorem (W.)

Roughly: Assume standard NT conjectures on L-functions (e.g. Hypo HW + RMT-type predictions) and "unlikely" divisors ("$p^2 \mid \Delta(\boldsymbol{c})$").
Then $N_{F,K}(X) \ll X^3$, and in fact HLH Conj. holds for a large class of regions K.[a]

[a] This has nice applications to sums of 3 cubes (Diaconu '19 + ϵ).

More precisely:

Theorem (W.)

Assume standard NT conj's on

- $L(s, V_{\boldsymbol{c}}), L(s, V_{\boldsymbol{c}}, \bigwedge^2), L(s, V(F))$ *(Hypo HW2 + Ratios Conj's + Krasner*[a]*), and*
- *"unlikely" divisors (Square-free Sieve Conjecture for* $\Delta(\boldsymbol{c})$*).*

Then for any nice $K \subset \mathbb{R}^6$ *w/* $K \cap \operatorname{hess} F = \emptyset$*,*[b] *we have* $N_{F,K}(X) \ll X^3$*, & in fact HLH Conj. holds. (Actual hypo's for former are cleaner than those for latter.)*

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[a] "effective version of Kisin's thesis (*Local constancy in p-adic families of Galois representations*)"

[b] This could probably be removed with enough work, but is mild enough for our main qualitative needs.

Glossary for hypo's

1. Hypo HW2: Similar in spirit to Hooley's Hypo HW.
2. Ratios Conj's: Give predictions of Random Matrix Theory (RMT) type for mean values of $1/L(s, V_{\boldsymbol{c}})$ and $1/L(s_1, V_{\boldsymbol{c}})L(s_2, V_{\boldsymbol{c}})$ over families of $\boldsymbol{c}$'s.[1]
3. Krasner: Need $L_p(s, V_{\boldsymbol{c}})$ to only depend on $\boldsymbol{c} \bmod p\Delta(\boldsymbol{c})^{1000}$ (cf. Kisin's thesis).
4. SFSC: Need (for $Z \geq 1$, $P \leq Z^3$)

$$\Pr\left[\boldsymbol{c} \in [-Z, Z]^6 : \exists\, p \in [P, 2P] \text{ with } p^2 \mid \Delta(\boldsymbol{c})\right] \ll P^{-\delta}.$$

[1] How does $\boldsymbol{c} \mapsto L(s, V_{\boldsymbol{c}})$ behave on average? RMT predictions originated for L-zeros "in the bulk" from Montgomery–Dyson, and "near 1/2" from Katz–Sarnak. CFKRS (2005) developed *full main term* predictions for L-powers, and CFZ (2008) for L-ratios.

Proof hint.

We want to bound/estimate (via δ-method)

$$N_{F,K}(X) \approx\approx \mathbb{E}_{\boldsymbol{c} \ll X^{1/2}} \mathbb{E}_{n \leq X^{3/2}} [n^{-1} S_{\boldsymbol{c}}(n)] =: \star.$$

Exponent numerics over various loci (if $d = 3$, $s = 6$):

$$\underbrace{s - d}_{\boldsymbol{c}=0,\ n \text{ small}} = \underbrace{\frac{s}{2} + \cancel{O(\epsilon)}}_{\Delta(\boldsymbol{c})=0,\ n \text{ large}} = \underbrace{\frac{d}{4}(s - \underline{\underline{2}}) + \cancel{O(4\epsilon)}}_{\Delta(\boldsymbol{c})\neq 0}$$
$$= 3 + \cancel{O(5\epsilon)}.$$

Main terms of HLH: $\Delta(\boldsymbol{c}) = 0$ (key: $S_{\boldsymbol{c}}(n)$ is biased for special $\boldsymbol{c}$'s). Conditional/hardest part: $\Delta(\boldsymbol{c}) \neq 0$ (which “factors” into certain mean-value and pointwise estimates over $\boldsymbol{c}$). □

Remark (Some more details)

There are maybe 5 sources of ϵ in Hooley/Heath-Brown, incl. what I'll call "Special", "Generic", & "Bad$p$".
The locus $\Delta(\boldsymbol{c}) = 0$ in $\star$ *unconditionally* produces the conj'd main term $c_{\mathrm{HLH}} \cdot X^3$. This resolves "Special".
The remaining sum (over $\Delta(\boldsymbol{c}) \neq 0$) is *conditionally*

$$\approx\approx \sum_{\text{finite set}} (\text{typically } O(1))^{a} \times (\text{RMT-type sum}).$$

- To prove "typical-$O(1)$" (*under* SFSC), re: "Bad$p$", need partial results towards a conjectural dichotomy/$\mathbb{F}_p$.
- Each "RMT-type sum" is $0 + O(X^{3-\delta})$ (*under* Ratios), improving on GRH bound $O_\epsilon(X^{3+\epsilon})$ (cf. "Generic").

[a] needs proof; loosely resembles Sarnak–Xue "density philosophy"

A sample pointwise ingredient

Among other things, we need partial results[2] toward a conjectural dichotomy/$\mathbb{F}_p$, amusingly parallel to HLH:

Conjecture (Randomness vs. structure over $\mathbb{F}_p$)

If $p \geq 100$ and $\boldsymbol{c} \in \mathbb{F}_p^6$ with $|\#\mathcal{V}_{\boldsymbol{c}}(\mathbb{F}_p) - \#\mathbb{P}^3(\mathbb{F}_p)| \geq 10^{10} p^{3/2}$, then $\mathcal{V}_{\boldsymbol{c}} \bmod p$ contains a plane (i.e. $c_i^3 = c_j^3$ in pairs).

Remark

R. Kloosterman told me that in the nodal case, a char. 0 analog of a stronger conj. holds (w/ Hodge-theoretic proof). Lindner '20 proved partial results towards the "stronger conjecture".

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[2] proven using "worst-case" results of Skorobogatov '92 (or Katz '91) and "average-case" results of Lindner '20 (or Debarre–Laface–Roulleau '17)

A sample mean-value ingredient

Over $\Delta(\boldsymbol{c}) \neq 0$, the reciprocal L-functions $1/L(s, V_{\boldsymbol{c}})$ are the main players. The Ratios Conjectures imply e.g. the following:

Conjecture (R2', roughly)

For certain holomorphic $f(s)$, e.g. e^{s^2}, we have

$$\mathbb{E}'_{\boldsymbol{c} \ll X^{1/2}} \left| \int_{(\sigma)} ds \, \frac{\zeta(2s)^{-1} L(s+1/2, V(F))^{-1}}{L(s, V_{\boldsymbol{c}})} \cdot f(s) N^s \right|^2 \ll_f N$$

($\sigma > 1/2$; $1 \ll N \ll X^{3/2}$).

- There are no $\log N$ or $\log X$ factors on the RHS! Such factors are determined by the "symmetry type" of the underlying family of L-functions.
- This is enough "RMT input" for $N_{F,K}(X) \ll X^3$.

More on mean values (Cancellation over $\boldsymbol{c}$)

Also, for some $\delta > 0$, one expects the following:

Conjecture (R1, roughly)

$$\mathbb{E}'_{\boldsymbol{c} \ll X^{1/2}} \left[\frac{1}{L(s, V_{\boldsymbol{c}})} - \underbrace{\zeta(2s) L(s+1/2, V(F))}_{\text{polar factors}} A_F(s) \right] \ll_{\sigma,t} X^{-\delta}$$

(over $\Delta(\boldsymbol{c}) \neq 0$) (for $X \geq 1$; $s = \sigma + it$; $\sigma > 1/2$)
Here $A_F(s) \ll 1$ for $\Re(s) \geq 1/2 - \delta$.

Remark

For $N_{F,K}(X) \ll X^3$, we only use (R2'). But for HLH (which requires "cancellation over $\boldsymbol{c}$"), we need a "slight adelic perturbation" of (R1).

Applications to sums of 3 cubes

Let $g := x^3 + y^3 + z^3$.

Question (Integral Hasse principle)

Is every *admissible*[a] integer a represented by g (over $\mathbb{Z}$)?

[a] i.e. locally represented; i.e. $\not\equiv \pm 4 \bmod 9$

Theorem (S. Diaconu '19 + ϵ)

Say, $\forall$ nice $K \subset \mathbb{R}^6$, HLH holds. Then 100% *Hasse holds.*

Theorem (W.)

Assume standard NT conj's on L-functions (e.g. Hypo HW + "RMT") & "unlikely" divisors ("$p^2 \mid \Delta(\boldsymbol{c})$"). Then 100% *(resp. $> 0\%$) of admiss. ints lie in $g(\mathbb{Z}^3)$ (resp. $g(\mathbb{Z}^3_{>0})$).*