Conditional approaches to sums of cubes

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Sec 0: Intro

Example (3-var cubics soluble/$\mathbb{Z}$)

1. Covid: $(x+y+z)^3 = 100x + 10y + z$
   Pf: $\exists$ Zoomers (512)
2. Ghosh–Sarnak '17: $x^2 + y^2 + z^2 - xyz = b$ for 100% of admissible (locally rep'd) ints b
3. Let $g := x^3 + y^3 + z^3$
   Booker '19: $g = 33$
   Wooley '95+: $g = b$ for $\gg A^{0.917}$ ints $b \le A$ ($A \to \infty$)
   Hooley '86+: " " for $\gg_\epsilon A^{1-\epsilon}$ ints, under Hypo HW ($\approx$ modularity + GRH for Hasse–Weil L-fn's)

Theorem (W.)

Roughly: Assume standard NT conj's on L-fn's (e.g. Hypo HW + "RMT") & "unlikely" divisors ("$p^2 \mid \Delta(\boldsymbol{c})$")
Then 100% (resp. $> 0\%$) of admiss. ints b are sums of 3 cubes (resp. 3 cubes > 0)

Remark (Re: 100% Hasse)

For $5x^3 + 12y^3 + 9z^3$, $\exists$ Hasse failures (Cassels–Guy '66 $+ \epsilon$)

Thm pf hint.

$$d = 3, m = 6 \implies m - d = \frac{m}{2} + \cancel{O(\epsilon)} = \frac{d}{4}(m - \underline{\underline{2}}) + \cancel{O(4\epsilon)}$$
$$= 3 + \cancel{O(5\epsilon)}$$

+ Stats 101 & 102. $\square$

Stats 101: Zero/Level sets (Counting basics)

For $P = x_1^3 + \cdots + x_s^3$ ($s = 3, 6$), $K \subset \mathbb{R}^s$ nice (cpt, semi-alg), $X \to \infty$, let $N_{P-b,K}(X) := \#\{\boldsymbol{x} \in \mathbb{Z}^s \cap XK : P = b\}$ ($b \in \mathbb{Z}$)

Example

$K = [-1, 1]^s \implies XK = [-X, X]^s$,

$$\mathbb{Z}^s \cap XK \xrightarrow{P} \mathbb{Z}$$
$$\boldsymbol{x} \mapsto P \ll X^3.$$

So $N_{P-b,K}(X)$ is $\asymp X^{s-3}$ *on avg* (in ℓ^1) over $b \ll X^3$.

HL ("randomness") prediction: $N_{P-b,K}(X) \approx\approx X^{s-3} \prod_{v \le \infty} \sigma_v$

Stats 102: Doubling (Rags to riches)

Let $g := y_1^3 + y_2^3 + y_3^3$. From $\mathbb{Z}^3 \xrightarrow{g} \mathbb{Z}$, get (the 2nd moment map, or "fiber-wise square")

$$\mathbb{Z} \leftarrow \mathbb{Z}^3 \times_g \mathbb{Z}^3 = \{(\boldsymbol{y}, \boldsymbol{z}) \in (\mathbb{Z}^3)^2 : g(\boldsymbol{y}) = g(\boldsymbol{z})\}.$$

Here $g(\boldsymbol{y}) = g(\boldsymbol{z}) \iff F(\boldsymbol{y}, -\boldsymbol{z}) = 0$ ($F := x_1^3 + \cdots + x_6^3$).

Observation

Let $K = [-1, 1]^6$. If $N_{F,K}(X) \ll X^3$ ($X \to \infty$), then $> 0\%$ of $\mathbb{Z}$ lies in $g(\mathbb{Z}^3_{>0})$.

Proof.

C–S ineq (2nd mom't method) □

Hooley '86a: HL misses triv. sol's
(e.g. $x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = 0$). But:

Conjecture (HLH)

For any nice $K \subset \mathbb{R}^6$,

$$N_{F,K}(X) = c_{\mathrm{HL},F,K} \cdot X^3 + \#\{\text{triv. } \boldsymbol{x} \in \mathbb{Z}^6 \cap XK\} + o(X^3)$$

($X \to \infty$).

Theorem (S. Diaconu '19 + ϵ)

Say, $\forall$ nice $K \subset \mathbb{R}^6$, HLH holds. Then 100% *Hasse holds.*

Proof.

Variance analysis (for log C–Y's) (cf. Ghosh–Sarnak '17) □

Sec 3: What's known?

Hua '38: $N_{F,K}(X) \ll X^{7/2+\epsilon}$ (by Cauchy b/w structure and randomness).
Vaughan '86+: " " $\ll X^{7/2}(\log X)^{\epsilon-5/2}$ (by new source of randomness).
Hooley '86+: " " $\ll X^{3+\epsilon}$, under Hypo HW.

Remark

A large-sieve hypo[a] would suffice (W.).
(It's open! But)
$\exists$ uncond. apps to $x^2+y^3+z^3$ (W., via Brüdern '91 + Duke–Kowalski '00 + Wiles et al).

[a] a la Bombieri–Vinogradov

Proposition (δ-method: Kloosterman '26, Duke–Friedlander–Iwaniec '93, Heath-Brown '96)

$$N_{F,K}(X) \approx\approx \mathbb{E}_{\boldsymbol{c} \ll X^{1/2}} \mathbb{E}_{n \leq X^{3/2}} [n^{-1} S_{\boldsymbol{c}}(n)] =: \star$$

(Hooley '86: $\ll$), where

$$S_{\boldsymbol{c}}(n) := \sum_{a \bmod n}' \sum_{\boldsymbol{x} \bmod n} e_n(aF(\boldsymbol{x}) + \boldsymbol{c} \cdot \boldsymbol{x})$$

($e_n(t) := e^{2\pi i t/n}$) (Don't worry about the "$'$"; it means $a \perp n$)

“Pf”.

$$N_{F,K}(X) \approx\approx \sum_{n \le X^{3/2}} \frac{1}{nX^{3/2}} \sum_{a \bmod n}' \sum_{\boldsymbol{x} \ll X} e_n(aF(\boldsymbol{x})) \quad (\circ\text{-method})$$

$$\approx\approx \sum_{n \le X^{3/2}} \frac{1}{nX^{3/2}} \mathbb{E}_{\boldsymbol{c} \ll n/X}[S_{\boldsymbol{c}}(n)] \quad (\text{“complexity” } n/X)$$

$$\approx\approx \star$$

□

(In gen'l, $\sum'_{a \bmod n} \sum_{\boldsymbol{x} \ll X} e_n(aF(\boldsymbol{x}))$ is “incomplete” mod n,[1] but still a wt'd avg. of the complete sums $S_{\boldsymbol{c}}(n)$, by Poisson (Nyquist–Shannon))

(Re: sampling complexity, give analogy to movies where car goes too fast, and wheels look like they're going backwards)

[1] such “sparsity” is a large part of the difficulty of analytic NT

Let $\widetilde{S}_{\boldsymbol{c}}(n) := n^{-7/2} S_{\boldsymbol{c}}(n)$
(Related to) $\mathcal{V}_{\boldsymbol{c}} := \{[\boldsymbol{x}] \in \mathbb{P}^5 : F(\boldsymbol{x}) = \boldsymbol{c} \cdot \boldsymbol{x} = 0\}$
Fact: $\exists$ disc poly $\Delta \in \mathbb{Z}[\boldsymbol{c}]$ measuring singularities of $\mathcal{V}_{\boldsymbol{c}}$

Lemma (Hooley)

If $\Delta(\boldsymbol{c}) \neq 0$, then $\widetilde{S}_{\boldsymbol{c}}(n)$ look (to 1st order) like the coeffs $\mu_{\boldsymbol{c}}(n)$ of $1/L(s, V_{\boldsymbol{c}})$ ($V_{\boldsymbol{c}} := (\mathcal{V}_{\boldsymbol{c}})_{\mathbb{Q}}$).

(Keys: F homog; $V_{\boldsymbol{c}} \cong$ odd-dim hypersurface; LTF.)

Exercise (Cf. Hooley, "$\underline{\underline{2}}\times$-Kloosterman")

"Assume" $\forall \boldsymbol{c}, n, N$: $\Delta(\boldsymbol{c}) \neq 0$, $\widetilde{S}_{\boldsymbol{c}}(n) = \mu_{\boldsymbol{c}}(n)$, $\sum_{n \leq N} \mu_{\boldsymbol{c}}(n) \ll \|\boldsymbol{c}\|^{\epsilon} N^{1/2+\epsilon}$. Then $\star \ll X^{3+\epsilon}$.

Sec 4: What's new?

Theorem (W.)

Assume standard NT conj's on

- $L(s, V_{\boldsymbol{c}}), L(s, V_{\boldsymbol{c}}, \bigwedge^2), L(s, V(F))$ *(Hypo HW2 + Ratios Conj's + Krasner*[a]*), and*
- *"unlikely" divisors ("*$p^2 \mid \Delta(\boldsymbol{c})$*").*

Then for any nice $K \subset \mathbb{R}^6$ *w/* $K \cap \operatorname{hess} F = \emptyset$*,*[b] *(we have)* $N_{F,K}(X) \ll X^3$*, & in fact HLH Conj. holds. (Actual hypo's for former are cleaner than those for latter.)*

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[a] "effective version of Kisin's thesis (*Local constancy in p-adic families of Galois representations*)"

[b] This could probably be removed with enough work, but is mild enough for our main qualitative needs.

Glossary for hypo's

1. HW2 (skip? similar in spirit to Hooley's Hypo HW): Need modularity, $1/L(s)$ to be holom. on $\Re(s) > 1/2$, & other technical things (e.g. basic expected properties of conductors and γ-factors).
2. Ratios (cover): Give predictions of Random Matrix Theory (RMT) type for mean values of $1/L(s, V_{\boldsymbol{c}})$ and $1/L(s_1, V_{\boldsymbol{c}})L(s_2, V_{\boldsymbol{c}})$ over (natural) fam's of $\boldsymbol{c}$'s.
3. Krasner (cover? since haven't said anything about it? skip is fine too): Need $L_p(s, V_{\boldsymbol{c}})$ to only depend on $\boldsymbol{c} \bmod p\Delta(\boldsymbol{c})^{1000}$ (cf. Kisin's thesis).
4. SFSC (skip? already sketched intuition): Need (for $Z \geq 1$, $P \leq Z^3$)

$$\Pr\left[\boldsymbol{c} \in [-Z, Z]^6 : \exists\, p \in [P, 2P] \text{ with } p^2 \mid \Delta(\boldsymbol{c})\right] \ll P^{-\delta}.$$

Fairy-tale proof sketch

Recall (the toy sum) $\star := \mathbb{E}_{\boldsymbol{c} \ll X^{1/2}} \mathbb{E}_{n \leq X^{3/2}} [n^{-1} S_{\boldsymbol{c}}(n)]$. There are (maybe) 5 sources of ϵ in Hooley/Heath-Brown, incl. (what I'll call) II, IIIG, IIIBp.

The locus $\Delta(\boldsymbol{c}) = 0$ in $\star$ *unconditionally* produces the conj'd main term $c_{\mathrm{HLH}} \cdot X^3$ (cf. II). (Here $\boldsymbol{c} = 0$, n small, gives "random" part; $\Delta(\boldsymbol{c}) = 0$, n large, gives "structured" part. Key: $S_{\boldsymbol{c}}(n)$ is biased for special $\boldsymbol{c}$'s.)

The remaining sum (over $\Delta(\boldsymbol{c}) \neq 0$) is *conditionally*

$$\approx\approx \sum_{\text{finite set}} (\text{typically } O(1))^2 \times (\text{RMT-type sum}).$$

To prove "typical-$O(1)$" (*under* SFSC), re: IIIBp, need partial results towards a dichotomy conj. $/\mathbb{F}_p$; use "worst-case" results of Skorobogatov '92 (or Katz '91) and "average-case" results of Lindner '20 (or Debarre–Laface–Roulleau '17). (We *apply* these partial results with the aid of SFSC.)
Here each "RMT-type sum" is $0 + O(X^{3-\delta})$ (*under* Ratios), improving on GRH bound $O_\epsilon(X^{3+\epsilon})$ (cf. IIIG).
(Put everything together to finish.)

[2] needs proof; loosely resembles Sarnak(–Xue) "density philosophy"

Dichotomy conjecture $/\mathbb{F}_p$

Side Conjecture

If $p \geq 100$ *and* $\boldsymbol{c} \in \mathbb{F}_p^6$ *with* $|\#\mathcal{V}_{\boldsymbol{c}}(\mathbb{F}_p) - \#\mathbb{P}^3(\mathbb{F}_p)| \geq 10^{10} p^{3/2}$, *then* $\mathcal{V}_{\boldsymbol{c}}$ mod p *contains a plane* $P \subseteq \{F = 0\}$ mod p *(i.e.* $c_1^3 - c_2^3 = c_3^3 - c_4^3 = c_5^3 - c_6^3 = 0$ *or...).*

Remark (R. Kloosterman)

A char. 0 analog of a stronger conj. (in the nodal case) holds (with a Hodge-theoretic proof).

(Lindner '20 proves partial results towards the "stronger conjecture".)

“RMT”

How does $\boldsymbol{c} \mapsto L(s, V_{\boldsymbol{c}})$ behave on average? RMT predictions originated for L-zeros “in the bulk” from Montgomery–Dyson, and “near 1/2” from Katz–Sarnak. CFKRS (2005) developed *full main term* predictions for L-powers, and CFZ (2008) for L-ratios; e.g. for some $\delta > 0$, one expects the following:

Conjecture (R1, roughly)

$$\mathbb{E}'_{\boldsymbol{c} \ll X^{1/2}} \left[\frac{1}{L(s, V_{\boldsymbol{c}})} - \underbrace{\zeta(2s) L(s + 1/2, V(F))}_{\text{polar factors}} A_F(s) \right] \ll_{\sigma,t} X^{-\delta}$$

(over $\Delta(\boldsymbol{c}) \neq 0$) (for $X \geq 1$; $s = \sigma + it$; $\sigma > 1/2$)
Here $A_F(s) \ll 1$ for $\Re(s) \geq 1/2 - \delta$.

We really care about *integrals* over s.

Conjecture (R2', roughly)

For certain holomorphic $f(s)$, e.g. e^{s^2}, we have

$$\mathbb{E}'_{\mathbf{c} \ll X^{1/2}} \left| \int_{(\sigma)} ds \, \frac{\zeta(2s)^{-1} L(s+1/2, V(F))^{-1}}{L(s, V_{\mathbf{c}})} \cdot f(s) N^s \right|^2 \ll_f N$$

($\sigma > 1/2$; $1 \ll N \ll X^{3/2}$).

- There are no $\log N$ or $\log X$ factors on the RHS! The numerator $\zeta(2s)^{-1} L(s+1/2, V(F))^{-1}$ serves as a *mollifier*, and $\int ds$ also helps.
- We use (R2') for $N_{F,K}(X) \ll X^3$, and a "slight adelic perturbation" of (R1) for HLH.