Conditional approaches to sums of cubes

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Sec 0: Intro

Example (3-var cubics soluble/$\mathbb{Z}$)

1. Covid: $(x+y+z)^3 = 100x + 10y + z$
   Pf: $\exists$ Zoomers (512)
2. Ghosh–Sarnak '17: $x^2 + y^2 + z^2 - xyz = b$ for 100% of admissible (locally rep'd) ints b
3. Let $g := x^3 + y^3 + z^3$
   Booker '19: $g = 33$
   Wooley '95+: $g = b$ for $\gg A^{0.917}$ ints $b \leq A$ $(A \to \infty)$
   Hooley '86+: " " for $\gg_\epsilon A^{1-\epsilon}$ ints, under Hypo HW ($\approx$ modularity + GRH for Hasse–Weil L-fn's)

1. Rel to (2)–(4), (1) is less interesting (not log Calabi-Yau?)
2. Pf: Variance analysis + counts for $4 + 2\epsilon$ vars
3. Pf: Use $16 + 16 + 16$ digits (large!)
   See "33 and all that"; algo is based on min (not max).
4. Pfs: 2nd mom't method + bounds for 6 vars
   (The precise H–W L-fn's: later.)

Theorem (W.)

Roughly: Assume standard NT conj's on L-fn's (e.g. Hypo HW + "RMT") & "unlikely" divisors ("$p^2 \mid \Delta(\boldsymbol{c})$")
Then 100% (resp. $> 0\%$) of admiss. ints b are sums of 3 cubes (resp. 3 cubes > 0)

Remark (Re: 100% Hasse)

For $5x^3 + 12y^3 + 9z^3$, $\exists$ Hasse failures (Cassels–Guy '66 $+ \epsilon$)

1. $> 0\%$, i.e. $\geq \delta\%$
2. Results: "$+$ flavor"
   Hypo's: "$\times$ flavor"

Thm pf hint.

$$d = 3, m = 6 \implies m - d = \frac{m}{2} + \cancel{O(\epsilon)} = \frac{d}{4}(m - \underline{\underline{2}}) + \cancel{O(4\epsilon)}$$
$$= 3 + \cancel{O(5\epsilon)}$$

$+$ Stats 101 & 102. □

Stats 101: Zero/Level sets (Counting basics)

For $P = x_1^3 + \cdots + x_s^3$ $(s = 3, 6)$, $K \subset \mathbb{R}^s$ nice (cpt, semi-alg), $X \to \infty$, let $N_{P-b,K}(X) := \#\{\boldsymbol{x} \in \mathbb{Z}^s \cap XK : P = b\}$ $(b \in \mathbb{Z})$

Example

$K = [-1, 1]^s \implies XK = [-X, X]^s$,

$$\mathbb{Z}^s \cap XK \xrightarrow{P} \mathbb{Z}$$

$$\boldsymbol{x} \mapsto P \ll X^3.$$

So $N_{P-b,K}(X)$ is $\asymp X^{s-3}$ *on avg* (in ℓ^1) over $b \ll X^3$.

HL ("randomness") prediction: $N_{P-b,K}(X) \approx\approx X^{s-3} \prod_{v \leq \infty} \sigma_v$

1. The 100% result needs weirder K,
   e.g. $K_\lambda := \{\boldsymbol{v} \in [-\lambda, \lambda]^3 : |g(\boldsymbol{v})| \leq 3\}$; $\lambda \to \infty$.
2. As $X \to \infty$, distribute $\asymp X^s$ pts $\boldsymbol{x}$ over values $P(\boldsymbol{x}) \ll X^3$.
3. Here and elsewhere, $\approx\approx$ means "roughly approximately" or "(roughly) looks like". It means I may be lying a bit. It is not meant to have a precise definition.
4. HL prediction (with dependencies spelled out slightly more precisely): $N_{P-b,K}(X) \approx\approx c^{\mathrm{fin}}(b) \cdot c_K^{\infty}(b/X^3) \cdot X^{s-3}$ (b/X^3 fixed)

Stats 102: Doubling (Rags to riches)

Let $g := y_1^3 + y_2^3 + y_3^3$. From $\mathbb{Z}^3 \xrightarrow{g} \mathbb{Z}$, get (the 2nd moment map, or "fiber-wise square")

$$\mathbb{Z} \leftarrow \mathbb{Z}^3 \times_g \mathbb{Z}^3 = \{(\boldsymbol{y}, \boldsymbol{z}) \in (\mathbb{Z}^3)^2 : g(\boldsymbol{y}) = g(\boldsymbol{z})\}.$$

Here $g(\boldsymbol{y}) = g(\boldsymbol{z}) \iff F(\boldsymbol{y}, -\boldsymbol{z}) = 0$ ($F := x_1^3 + \cdots + x_6^3$).

Observation

Let $K = [-1, 1]^6$. If $N_{F,K}(X) \ll X^3$ ($X \to \infty$), then $> 0\%$ of $\mathbb{Z}$ lies in $g(\mathbb{Z}^3_{>0})$.

Proof.

C–S ineq (2nd mom't method) □

Hooley '86a: HL misses triv. sol's
(e.g. $x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = 0$). But:

Conjecture (HLH)

For any nice $K \subset \mathbb{R}^6$,

$$N_{F,K}(X) = c_{\mathrm{HL},F,K} \cdot X^3 + \#\{\text{triv. } \boldsymbol{x} \in \mathbb{Z}^6 \cap XK\} + o(X^3)$$

($X \to \infty$).

Theorem (S. Diaconu '19 + ϵ)

Say, $\forall$ nice $K \subset \mathbb{R}^6$, HLH holds. Then 100% *Hasse holds.*

Proof.

Variance analysis (for log C–Y's) (cf. Ghosh–Sarnak '17) □

1. Must allow gen'l K! $\forall$ fixed K, $\exists$ "stingy" AP of b's.

Sec 3: What's known?

Hua '38: $N_{F,K}(X) \ll X^{7/2+\epsilon}$ (by Cauchy b/w structure and randomness).
Vaughan '86+: " " $\ll X^{7/2}(\log X)^{\epsilon-5/2}$ (by new source of randomness).
Hooley '86+: " " $\ll X^{3+\epsilon}$, under Hypo HW.

Remark

A large-sieve hypo[a] would suffice (W.).
(It's open! But)
$\exists$ uncond. apps to $x^2+y^3+z^3$ (W., via Brüdern '91 + Duke–Kowalski '00 + Wiles et al).

[a] a la Bombieri–Vinogradov

Proposition (δ-method: Kloosterman '26, Duke–Friedlander–Iwaniec '93, Heath-Brown '96)

$$N_{F,K}(X) \approx\approx \mathbb{E}_{\boldsymbol{c} \ll X^{1/2}} \mathbb{E}_{n \leq X^{3/2}} [n^{-1} S_{\boldsymbol{c}}(n)] =: \star$$

(Hooley '86: $\ll$), where

$$S_{\boldsymbol{c}}(n) := \sum_{a \bmod n}' \sum_{\boldsymbol{x} \bmod n} e_n(aF(\boldsymbol{x}) + \boldsymbol{c} \cdot \boldsymbol{x})$$

($e_n(t) := e^{2\pi i t/n}$) (Don't worry about the "$'$"; it means $a \perp n$)

"Pf".

$$N_{F,K}(X) \approx\approx \sum_{n \le X^{3/2}} \frac{1}{nX^{3/2}} \sum_{a \bmod n}{}' \sum_{\boldsymbol{x} \ll X} e_n(aF(\boldsymbol{x})) \quad (\circ\text{-method})$$

$$\approx\approx \sum_{n \le X^{3/2}} \frac{1}{nX^{3/2}} \mathbb{E}_{\boldsymbol{c} \ll n/X}[S_{\boldsymbol{c}}(n)] \quad (\text{"complexity" } n/X)$$

$$\approx\approx \star$$

□

(In gen'l, $\sum'_{a \bmod n} \sum_{\boldsymbol{x} \ll X} e_n(aF(\boldsymbol{x}))$ is "incomplete" mod n,[1] but still a wt'd avg. of the complete sums $S_{\boldsymbol{c}}(n)$, by Poisson (Nyquist–Shannon))
(Re: sampling complexity, give analogy to movies where car goes too fast, and wheels look like they're going backwards)

[1] such "sparsity" is a large part of the difficulty of analytic NT

Let $\widetilde{S}_{\boldsymbol{c}}(n) := n^{-7/2} S_{\boldsymbol{c}}(n)$
(Related to) $\mathcal{V}_{\boldsymbol{c}} := \{[\boldsymbol{x}] \in \mathbb{P}^5 : F(\boldsymbol{x}) = \boldsymbol{c} \cdot \boldsymbol{x} = 0\}$
Fact: $\exists$ disc poly $\Delta \in \mathbb{Z}[\boldsymbol{c}]$ measuring singularities of $\mathcal{V}_{\boldsymbol{c}}$

Lemma (Hooley)

If $\Delta(\boldsymbol{c}) \neq 0$, then $\widetilde{S}_{\boldsymbol{c}}(n)$ look (to 1st order) like the coeffs $\mu_{\boldsymbol{c}}(n)$ of $1/L(s, V_{\boldsymbol{c}})$ ($V_{\boldsymbol{c}} := (\mathcal{V}_{\boldsymbol{c}})_{\mathbb{Q}}$).

(Keys: F homog; $V_{\boldsymbol{c}} \cong$ odd-dim hypersurface; LTF.)

Exercise (Cf. Hooley, "$\underline{\underline{2}}\times$-Kloosterman")

"Assume" $\forall \boldsymbol{c}, n, N$: $\Delta(\boldsymbol{c}) \neq 0$, $\widetilde{S}_{\boldsymbol{c}}(n) = \mu_{\boldsymbol{c}}(n)$,
$\sum_{n \leq N} \mu_{\boldsymbol{c}}(n) \ll \|\boldsymbol{c}\|^{\epsilon} N^{1/2+\epsilon}$. Then $\star \ll X^{3+\epsilon}$.

1. Here F is homog (& a is summed), so $S_{\boldsymbol{c}}(n)$ is multiplicative. Locally: If $p \nmid \boldsymbol{c}$, then $\widetilde{S}_{\boldsymbol{c}}(p) = \widetilde{E}_{\boldsymbol{c}}(p) + O(p^{-1/2})$, where $\widetilde{E}_{\boldsymbol{c}}(p) := p^{-3/2}[\#\mathcal{V}_{\boldsymbol{c}}(\mathbb{F}_p) - \#\mathbb{P}^3(\mathbb{F}_p)]$.
2. So in a nutshell, δ-method relates NT of a "$+$" flavor to NT of a "$\times$" flavor.
3. The modern definition of $L(s, V_{\boldsymbol{c}})$ (see Taylor, 2004) is a bit technical, and is based on the Galois representation $H^3(V_{\boldsymbol{c}} \times \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$ for a choice of auxiliary prime ℓ. (The choice of ℓ should not matter; for our specific representations, this is probably known unconditionally.)
4. In the earlier "cryptic pf. outline", the $\underline{\underline{2}}$ in $\frac{d}{4}(m - \underline{\underline{2}})$ comes from averaging over a, n ("*double* Kloosterman method"). The $dm/4$ corresponds to a heuristic of square-root cancellation over $\boldsymbol{x}$ mod n.

Sec 4: What's new?

Theorem (W.)

Assume standard NT conj's on

- $L(s, V_{\boldsymbol{c}}), L(s, V_{\boldsymbol{c}}, \bigwedge^2), L(s, V(F))$ *(Hypo HW2 + Ratios Conj's + Krasner*[a]*), and*
- *"unlikely" divisors ("*$p^2 \mid \Delta(\boldsymbol{c})$*").*

Then for any nice $K \subset \mathbb{R}^6$ *w/* $K \cap \operatorname{hess} F = \emptyset$,[b] *(we have)* $N_{F,K}(X) \ll X^3$*, & in fact HLH Conj. holds. (Actual hypo's for former are cleaner than those for latter.)*

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[a] "effective version of Kisin's thesis (*Local constancy in p-adic families of Galois representations*)"

[b] This could probably be removed with enough work, but is mild enough for our main qualitative needs.

1. If pressed for time after stating the thm, just write “Pf ingredients incl. (R2’) & partial results toward a dichotomy/$\mathbb{F}_p$” and after that briefly state dichotomy conj and (R2’).

Glossary for hypo's

1. HW2 (skip? similar in spirit to Hooley's Hypo HW): Need modularity, $1/L(s)$ to be holom. on $\Re(s) > 1/2$, & other technical things (e.g. basic expected properties of conductors and γ-factors).
2. Ratios (cover): Give predictions of Random Matrix Theory (RMT) type for mean values of $1/L(s, V_{\boldsymbol{c}})$ and $1/L(s_1, V_{\boldsymbol{c}})L(s_2, V_{\boldsymbol{c}})$ over (natural) fam's of $\boldsymbol{c}$'s.
3. Krasner (cover? since haven't said anything about it? skip is fine too): Need $L_p(s, V_{\boldsymbol{c}})$ to only depend on $\boldsymbol{c}$ mod $p\Delta(\boldsymbol{c})^{1000}$ (cf. Kisin's thesis).
4. SFSC (skip? already sketched intuition): Need (for $Z \geq 1$, $P \leq Z^3$)

$$\Pr\left[\boldsymbol{c} \in [-Z, Z]^6 : \exists\, p \in [P, 2P] \text{ with } p^2 \mid \Delta(\boldsymbol{c})\right] \ll P^{-\delta}.$$

Fairy-tale proof sketch

Recall (the toy sum) $\star := \mathbb{E}_{\boldsymbol{c} \ll X^{1/2}} \mathbb{E}_{n \le X^{3/2}} [n^{-1} S_{\boldsymbol{c}}(n)]$. There are (maybe) 5 sources of ϵ in Hooley/Heath-Brown, incl. (what I'll call) II, IIIG, IIIBp.

The locus $\Delta(\boldsymbol{c}) = 0$ in $\star$ *unconditionally* produces the conj'd main term $c_{\mathrm{HLH}} \cdot X^3$ (cf. II). (Here $\boldsymbol{c} = 0$, n small, gives "random" part; $\Delta(\boldsymbol{c}) = 0$, n large, gives "structured" part. Key: $S_{\boldsymbol{c}}(n)$ is biased for special $\boldsymbol{c}$'s.)

The remaining sum (over $\Delta(\boldsymbol{c}) \neq 0$) is *conditionally*

$$\approx\approx \sum_{\text{finite set}} (\text{typically } O(1))^2 \times (\text{RMT-type sum}).$$

To prove "typical-$O(1)$" (*under* SFSC), re: IIIBp, need partial results towards a dichotomy conj. $/\mathbb{F}_p$; use "worst-case" results of Skorobogatov '92 (or Katz '91) and "average-case" results of Lindner '20 (or Debarre–Laface–Roulleau '17). (We *apply* these partial results with the aid of SFSC.)
Here each "RMT-type sum" is $0 + O(X^{3-\delta})$ (*under* Ratios), improving on GRH bound $O_\epsilon(X^{3+\epsilon})$ (cf. IIIG).
(Put everything together to finish.)

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[2]needs proof; loosely resembles Sarnak(–Xue) "density philosophy"

Dichotomy conjecture $/\mathbb{F}_p$

Side Conjecture

If $p \geq 100$ and $\boldsymbol{c} \in \mathbb{F}_p^6$ with $|\#\mathcal{V}_{\boldsymbol{c}}(\mathbb{F}_p) - \#\mathbb{P}^3(\mathbb{F}_p)| \geq 10^{10} p^{3/2}$, then $\mathcal{V}_{\boldsymbol{c}} \bmod p$ contains a plane $P \subseteq \{F = 0\} \bmod p$ (i.e. $c_1^3 - c_2^3 = c_3^3 - c_4^3 = c_5^3 - c_6^3 = 0$ or...).

Remark (R. Kloosterman)

A char. 0 analog of a stronger conj. (in the nodal case) holds (with a Hodge-theoretic proof).

(Lindner '20 proves partial results towards the "stronger conjecture".)

"RMT"

How does $\boldsymbol{c} \mapsto L(s, V_{\boldsymbol{c}})$ behave on average? RMT predictions originated for L-zeros "in the bulk" from Montgomery–Dyson, and "near 1/2" from Katz–Sarnak. CFKRS (2005) developed *full main term* predictions for L-powers, and CFZ (2008) for L-ratios; e.g. for some $\delta > 0$, one expects the following:

Conjecture (R1, roughly)

$$\mathbb{E}'_{\boldsymbol{c} \ll X^{1/2}} \left[\frac{1}{L(s, V_{\boldsymbol{c}})} - \underbrace{\zeta(2s) L(s + 1/2, V(F))}_{\text{polar factors}} A_F(s) \right] \ll_{\sigma,t} X^{-\delta}$$

(over $\Delta(\boldsymbol{c}) \neq 0$) (for $X \geq 1$; $s = \sigma + it$; $\sigma > 1/2$)
Here $A_F(s) \ll 1$ for $\Re(s) \geq 1/2 - \delta$.

1. The polar factors are related to the RMT symmetry type, which is conjecturally determined by knowing (enough about) Sato–Tate groups or analogous things (Universality Conjecture; see e.g. Sarnak–Shin–Templier '16).
   In our case, the latter can be computed to be symplectic, either by point-counting $/\mathbb{F}_q$ (using Lindner at one point!), or by "visiting monodromy.com" (in some sense; one can basically quote various works of Deligne, Katz, and Sarnak);
   cf. Deligne's interesting quote "I did not know at first how far I could go. The first case I could handle was a hypersurface of odd dimension in projective space. But that was a completely new case already, so then I had confidence that one could go all the way..." recorded in Milne's pRH.pdf or LEC.pdf.

We really care about *integrals* over s.

Conjecture (R2', roughly)

For certain holomorphic $f(s)$, e.g. e^{s^2}, we have

$$\mathbb{E}'_{\boldsymbol{c} \ll X^{1/2}} \left| \int_{(\sigma)} ds \, \frac{\zeta(2s)^{-1} L(s+1/2, V(F))^{-1}}{L(s, V_{\boldsymbol{c}})} \cdot f(s) N^s \right|^2 \ll_f N$$

($\sigma > 1/2$; $1 \ll N \ll X^{3/2}$).

- There are no $\log N$ or $\log X$ factors on the RHS! The numerator $\zeta(2s)^{-1} L(s+1/2, V(F))^{-1}$ serves as a *mollifier*, and $\int ds$ also helps.
- We use (R2') for $N_{F,K}(X) \ll X^3$, and a "slight adelic perturbation" of (R1) for HLH.