Conditional approaches to sums of cubes

Victor Wang

Princeton University
Advised by Peter Sarnak

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The story of 33

Via computer, Booker obtained (at "five past nine in the morning on the 27th of February 2019")

$$(8866128975287528)^3 + (-8778405442862239)^3 + (-2736111468807040)^3 = 33.$$

Remark

See the Youtube video "33 and all that" for a nice talk by Booker (with T-shirt and mug links) on the discovery of this and related results. (And for some drama involving an old version of Browning's website.)

Exercise

Try Google Calculator, then Wolfram Alpha.

Main qualitative results (roughly)

Theorem (W., 2021)

Assume certain standard conjectures on

- *L-functions (e.g. GRH and the Ratios Conjectures), and*
- *"unlikely" divisors (the Square-free Sieve Conjecture).*

Then the following hold:

1. 100% *of* admissible[a] *integers are sums of three cubes.*
2. *A positive fraction are sums of three* nonnegative *cubes.*

[a] i.e. locally represented; i.e. $\not\equiv \pm 4 \bmod 9$

Remark (Based on Cassels–Guy, 1966)

In the analog of (1) for the ternary cubic $5x^3 + 12y^3 + 9z^3$, "almost all" cannot be replaced with "all".

Homogeneously expanding point counts

Definition

Given $P \in \mathbb{Z}[\boldsymbol{x}]$ in $s \geq 2$ variables, let

$$N_{P,\Omega}(X) := \#\{\boldsymbol{x} \in \mathbb{Z}^s \cap X\Omega : P(\boldsymbol{x}) = 0\}$$

for each nice[a] region $\Omega \subset \mathbb{R}^s$ and scalar $X > 0$.

[a] say compact, semi-algebraic

Example

- If $\Omega = [-1, 1]^s$, then $N_{P,\Omega}(X)$ is the number of integral solutions $\boldsymbol{x} \in [-X, X]^s$ to $P(\boldsymbol{x}) = 0$.
- Say $P = y_1^3 + y_2^3 + y_3^3 - b \in \mathbb{Z}[\boldsymbol{y}]$. If $\Omega = [0, 1]^3$ and $X \geq b^{1/3}$, then $N_{P,\Omega}(X) = r_3(b)$.

The usual randomness heuristic in ℓ^1 (intro)

Fix $P_0 := x_1^3 + \cdots + x_s^3$ ($s \geq 2$) and $\Omega \subset \mathbb{R}^s$ (nice, non-null). For $X > 0$, the map

$$\mathbb{Z}^s \cap X\Omega \xrightarrow{P_0} \mathbb{Z}$$

has images $P_0(\boldsymbol{x}) \ll X^3$ and fibers

$$\underbrace{\{P_0(\boldsymbol{x}) = b\}}_{\text{size } N_{P_0(\boldsymbol{x})-b,\Omega}(X)} \to b.$$

Observation

As $X \to \infty$, the total space $\mathbb{Z}^s \cap X\Omega$ has $\asymp X^s$ points $\boldsymbol{x}$, each with $P_0(\boldsymbol{x}) \ll X^3$. So $N_{P_0(\boldsymbol{x})-b,\Omega}(X)$ is

- 0 if $|b| \gg X^3$ is sufficiently large, and
- $\asymp X^{s-3}$ *on average* (in ℓ^1) over $b \ll X^3$.

The usual randomness heuristic in ℓ^1 (cont'd)

Fix $P_0 := x_1^3 + \cdots + x_s^3$ ($s \geq 3$) and $\Omega \subset \mathbb{R}^s$ (nice, non-null).

Observation

As $X \to \infty$, the point count $N_{P_0(\boldsymbol{x})-b,\Omega}(X)$ is

- 0 if $|b| \gg X^3$ is sufficiently large, and
- $\asymp X^{s-3}$ *on average* (in ℓ^1) over $b \ll X^3$.

Remark

This is a *real* observation. More precise *real* considerations, alongside *p-adic* analogs, lead to the *Hardy–Littlewood prediction*, a "randomness heuristic" roughly of the form

$$N_{P_0(\boldsymbol{x})-b,\Omega}(X) \approx c^{\text{fin}}_{\text{HL},P_0}(b) \cdot c^{\infty}_{\text{HL},P_0,\Omega}(b/X^3) \cdot X^{s-3}$$

($X \to \infty$; b/X^3 fixed).

From stingy ternary to rich senary, via ℓ^2 (intro)

Let $P_0 := y_1^3 + y_2^3 + y_3^3 \in \mathbb{Z}[\boldsymbol{y}]$. Let $F := x_1^3 + \cdots + x_6^3 \in \mathbb{Z}[\boldsymbol{x}]$. If $\boldsymbol{x} = (\boldsymbol{y}, -\boldsymbol{z})$, then

$$P_0(\boldsymbol{y}) = P_0(\boldsymbol{z}) \iff F(\boldsymbol{x}) = 0.$$

Observation (Second moment method)

Let $\Omega := [-1, 1]^6$. If $N_{F,\Omega}(X) \ll X^3 = X^{6-3}$ $(X \to \infty)$, then $\{b \in \mathbb{Z} : r_3(b) > 0\}$ has positive lower density in $\mathbb{Z}$.

One can go further by *variance* analysis. For *critical* problems like $P_0(\boldsymbol{y}) = b$, cf. Ghosh–Sarnak (2017).[1]

[1] They study the Markoff-type equations $x^2 + y^2 + z^2 - xyz = b$.

From stingy ternary to rich senary, via ℓ^2 (cont'd)

Let $P_0 := y_1^3 + y_2^3 + y_3^3 \in \mathbb{Z}[\boldsymbol{y}]$. Let $F := x_1^3 + \cdots + x_6^3 \in \mathbb{Z}[\boldsymbol{x}]$.

Theorem (Cf. S. Diaconu, 2019)

Say for all *nice* $\Omega \subset \mathbb{R}^6$, *Hooley's conjecture*[a] *holds on* Ω. *Then* 100% *of admissible* $b \in \mathbb{Z}$ *are sums of three cubes.*

[a] of the form $N_{F,\Omega}(X) \approx c_{\mathrm{HLH},F,\Omega} \cdot X^3$

Remark (Cf. S. Diaconu, 2019)

To capture 100% of b's, we *must* let Ω deform.[a]

[a] For each *fixed* Ω, there is a "stingy" arithmetic progression of b's.

The senary state of art

Let $F := x_1^3 + \cdots + x_6^3 \in \mathbb{Z}[\boldsymbol{x}]$ and $\Omega := [-1,1]^6$. Then

$$N_{F,\Omega}(X) = \int_{\mathbb{R}/\mathbb{Z}} d\theta \, |T(\theta)|^6,$$

where $T(\theta) := \sum_{|x| \le X} e(\theta x^3)$. What is known here?

- Hua (1938) proved $N_{F,\Omega}(X) \ll X^{7/2+\epsilon}$, by Cauchy between $\int_{\mathbb{R}/\mathbb{Z}} d\theta \, |T(\theta)|^4 \ll X^{2+\epsilon}$ (structure) and $\int_{\mathbb{R}/\mathbb{Z}} d\theta \, |T(\theta)|^8 \ll X^{5+\epsilon}$ (randomness).
- By isolating a new source of randomness,[2] Vaughan (1986, 2020) gave a more robust proof of Hua's bound, ultimately leading to $N_{F,\Omega}(X) \ll X^{7/2} (\log X)^{\epsilon - 5/2}$ $(X \to \infty)$.
- Under standard conjectures on L-functions (e.g. GRH), Hooley (1986, 1997) proved $N_{F,\Omega}(X) \ll X^{3+\epsilon}$.

[2] "typical divisors" of integers $x \ll X$

Senary randomness and structure

Let $F := x_1^3 + \cdots + x_6^3$. For $\Omega \subset \mathbb{R}^6$ nice, Hardy–Littlewood predicts $\approx c_{\mathrm{HL},F,\Omega} \cdot X^{6-3}$ solutions $\boldsymbol{x} \in X\Omega$ to $F(\boldsymbol{x}) = 0$ "arising randomly" (as $X \to \infty$). But $F(\boldsymbol{x}) = 0$ also has $\asymp X^{6/2}$ "special structured" solutions $\boldsymbol{x} \ll X$.

Definition

Call $\boldsymbol{x} \in \mathbb{Z}^6$ *trivial* (or *diagonal-type*) if there exists $\pi \in S_6$ such that $x_{\pi(1)} + x_{\pi(2)} = x_{\pi(3)} + x_{\pi(4)} = x_{\pi(5)} + x_{\pi(6)} = 0$.

Theorem (Hooley, 1986')

There exists $\Omega \subset \mathbb{R}^6$ nice, and $\delta > 0$, such that (as $X \to \infty$)

$$N_{F,\Omega}(X) \geq \delta X^3 + \max\left(c_{\mathrm{HL},F,\Omega} \cdot X^3, \#\{\text{trivial } \boldsymbol{x} \in \mathbb{Z}^6 \cap X\Omega\}\right).$$

Hooley's conjecture (interpreted for general Ω)

(Let $F := x_1^3 + \cdots + x_6^3$.)

Definition

Call $\boldsymbol{x} \in \mathbb{Z}^6$ *trivial* (or *diagonal-type*) if there exists $\pi \in S_6$ such that $x_{\pi(1)} + x_{\pi(2)} = x_{\pi(3)} + x_{\pi(4)} = x_{\pi(5)} + x_{\pi(6)} = 0$.

If $\boldsymbol{x} \in \mathbb{Z}^6$ is trivial, then $F(\boldsymbol{x}) = 0$.

Conjecture (Hooley, 1986', interpreted generally)

For any nice $\Omega \subset \mathbb{R}^6$, we have (as $X \to \infty$)

$$N_{F,\Omega}(X) = c_{\mathrm{HL},F,\Omega} \cdot X^3 + \#\{\text{trivial } \boldsymbol{x} \in \mathbb{Z}^6 \cap X\Omega\} + o(X^3).$$

(Under this conjecture, Diaconu's methods show that 100% of admissible $b \in \mathbb{Z}$ are sums of three cubes.)

The delta method (intro)

Let $F := x_1^3 + \cdots + x_6^3$ and $\Omega := [-1, 1]^6$.

- Under Hypothesis HW (practically "modularity, plus GRH") for certain Hasse–Weil L-functions, Hooley (1986, 1997) proved $N_{F,\Omega}(X) \ll X^{3+\epsilon}$.[3]
- Hooley used an "upper-bound precursor" to the delta method. The latter has the advantage that it is an equality rather than an inequality.[4]

Remark

In a nutshell, the delta method relates NT of a "$+$" flavor to NT of a "$\times$" flavor. It is a modern version of the "completed averaging" method of H. Kloosterman (1926).

[3] Actually, a large-sieve hypothesis a la Bombieri–Vinogradov would suffice (W., 2021).

[4] See DFI (1993) and Heath-Brown (1996, 1998).

The delta method (cont'd)

Let $F := x_1^3 + \cdots + x_6^3$. Fix a nice $\Omega \subset \mathbb{R}^6$. In the circle method for $N_{F,\Omega}(X)$, one encounters sums resembling

$$\sum_{q \le X^{3/2}} \frac{1}{qX^{3/2}} \sum_{a \bmod q}{}' \sum_{\boldsymbol{x} \in \mathbb{Z}^6} w(\boldsymbol{x}/X) \cdot e_q(aF(\boldsymbol{x})),$$

where "$w \approx 1_\Omega$" ($w \in C_c^\infty$), and "$\prime$" means $a \perp q$. (Here $e_q(t) := e^{2\pi i t/q}$.)

Remark

In this setting, H. Kloosterman (1926) would suggest

1. averaging over a (q fixed), and
2. "completing" incomplete sums over $\boldsymbol{x}$,

in either order.

The delta method (cont'd^2)

Let $F := x_1^3 + \cdots + x_6^3$. Fix a nice $\Omega \subset \mathbb{R}^6$. In the circle method for $N_{F,\Omega}(X)$, one encounters sums resembling

$$\sum_{q \le X^{3/2}} \frac{1}{qX^{3/2}} \sum_{a \bmod q}{}' \sum_{\boldsymbol{x} \in \mathbb{Z}^6} w(\boldsymbol{x}/X) \cdot e_q(aF(\boldsymbol{x})).$$

Remark

H. Kloosterman (1926) would rewrite the sum above as

$$\sum_{q \le X^{3/2}} \frac{1}{qX^{3/2}} \sum_{\boldsymbol{c} \in \mathbb{Z}^6} S_{\boldsymbol{c}}(q) \cdot (X/q)^6 \hat{w}(X\boldsymbol{c}/q),$$

where

$$S_{\boldsymbol{c}}(q) := \sum_{a \bmod q}{}' \sum_{\boldsymbol{x} \bmod q} e_q(aF(\boldsymbol{x}) + \boldsymbol{c} \cdot \boldsymbol{x}).$$

The delta method (cont'd^3)

Let $F := x_1^3 + \cdots + x_6^3$. Fix a nice $\Omega \subset \mathbb{R}^6$. In the delta method for $N_{F,\Omega}(X)$, one encounters sums resembling

$$\sum_{q \leq X^{3/2}} \frac{1}{qX^{3/2}} \sum_{\boldsymbol{c} \in \mathbb{Z}^6} S_{\boldsymbol{c}}(q) \cdot (X/q)^6 \hat{w}(X\boldsymbol{c}/q),$$

where

$$S_{\boldsymbol{c}}(q) := \sideset{}{'}\sum_{a \bmod q} \sum_{\boldsymbol{x} \bmod q} e_q(aF(\boldsymbol{x}) + \boldsymbol{c} \cdot \boldsymbol{x}).$$

Observation (Partly classical; used in Hooley, 1986)

F is *homogeneous*, so $S_{\boldsymbol{c}}(mn) = S_{\boldsymbol{c}}(m) S_{\boldsymbol{c}}(n)$ if $(m,n) = 1$. Also, if $p \nmid \boldsymbol{c}$, then $p^{-7/2} S_{\boldsymbol{c}}(p) \approx \widetilde{E}_{\boldsymbol{c}}(p)$, where $\widetilde{E}_{\boldsymbol{c}}(p)$ measures the "bias modulo p" of the cubic 3-fold $\mathcal{V}_{\boldsymbol{c}} : F(\boldsymbol{x}) = \boldsymbol{c} \cdot \boldsymbol{x} = 0$. Here if $p \nmid \Delta(\boldsymbol{c})$, then $|\widetilde{E}_{\boldsymbol{c}}(p)| \leq 10$ (Weil conjectures).

The delta method (cont'd^4)

Let $F := x_1^3 + \cdots + x_6^3$. Fix a nice $\Omega \subset \mathbb{R}^6$. In the delta method for $N_{F,\Omega}(X)$, one encounters sums resembling

$$\sum_{q \le X^{3/2}} \frac{1}{qX^{3/2}} \sum_{\boldsymbol{c} \in \mathbb{Z}^6} S_{\boldsymbol{c}}(q) \cdot (X/q)^6 \hat{w}(X\boldsymbol{c}/q). \tag{1}$$

If $\Delta(\boldsymbol{c}) \neq 0$, then the normalized sums $\widetilde{S}_{\boldsymbol{c}}(q) := q^{-7/2} S_{\boldsymbol{c}}(q)$ look (to first order) like the coefficients $\mu_{\boldsymbol{c}}(q)$ of the *reciprocal* Hasse–Weil L-function $1/L(s, V_{\boldsymbol{c}})$ associated to the cubic 3-fold $V_{\boldsymbol{c}} := V(F, \boldsymbol{c} \cdot \boldsymbol{x})/\mathbb{Q}$ (Hooley, 1986).

Exercise (Cf. Hooley, 1986)

Assuming that $\Delta(\boldsymbol{c}) \neq 0$ for all $\boldsymbol{c}$, that $\widetilde{S}_{\boldsymbol{c}}(q) = \mu_{\boldsymbol{c}}(q)$ for all $\boldsymbol{c}, q$, and that $\sum_{n \le N} \mu_{\boldsymbol{c}}(n) \ll_\epsilon \|\boldsymbol{c}\|^\epsilon N^{1/2+\epsilon}$ for all $\boldsymbol{c}, N$ ($N \geq 1$), show that the sum (1) above is $\ll_\epsilon X^{3+\epsilon}$.

Main quantitative results (roughly)

Theorem (W., 2021)

Assume certain standard conjectures on

- $L(s, V_{\boldsymbol{c}})$ *and some "second-order" relatives ("Hypothesis HW2", the Ratios Conjectures, and an effective Krasner-type lemma), and*
- *"unlikely" divisors (the Square-free Sieve Conjecture for the discriminant polynomial* $\Delta \in \mathbb{Z}[\boldsymbol{c}]$*).*

Then for 6*-variable diagonal cubic forms* $F \in \mathbb{Z}[\boldsymbol{x}]$*, and for nice* $\Omega \subset \mathbb{R}^6$ *with* $\Omega \cap \operatorname{hess} F = \emptyset$*,*[a] *we have* $N_{F,\Omega}(X) \ll_{F,\Omega} X^3$*, and in fact Hooley's conjecture*[b] *holds for* F, Ω*.*

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[a] This is general enough for our main qualitative needs.
[b] of the form $N_{F,\Omega}(X) = (c_{\mathrm{HLH},F,\Omega} + o(1)) \cdot X^3$

"Glossary" (for our hypotheses)

1. "Second-order" relatives: $L(s, V_{\boldsymbol{c}}, \bigwedge^2)$ and $L(s, V(F))$.
2. Hypothesis HW2: Need modularity, and need $1/L(s)$ to be holomorphic on the region $\Re(s) > 1/2$. And other technical things (e.g. basic expected properties of conductors and γ-factors).
3. Ratios Conjectures: Give predictions of Random Matrix Theory (RMT) type for the mean values of $1/L(s, V_{\boldsymbol{c}})$ and $1/L(s_1, V_{\boldsymbol{c}})L(s_2, V_{\boldsymbol{c}})$ over natural families of $\boldsymbol{c}$'s.
4. Effective Krasner-type lemma: Need control on $\boldsymbol{c}$-variation of the local factors $L_p(s, V_{\boldsymbol{c}})$ (easy if e.g. $p \nmid \Delta(\boldsymbol{c})$).
5. Square-free Sieve Conjecture: Need

$$\Pr\left[\boldsymbol{c} \in [-Z, Z]^6 : \exists \text{ sq-full } q \in [Q, 2Q] \text{ with } q \mid \Delta(\boldsymbol{c})\right]$$

to be $\ll Q^{-\delta}$, uniformly over $Z \geq 1$ and $Q \leq Z^6$.

Proof ideas and themes (overview)

In the delta method for $N_{F,\Omega}(X)$, one roughly encounters

$$\sum_{\boldsymbol{c}\in\mathbb{Z}^6} \sum_{q\leq X^{3/2}} \frac{1}{qX^{3/2}} \cdot q^{7/2}\widetilde{S}_{\boldsymbol{c}}(q) \cdot (X/q)^6 \hat{w}(X\boldsymbol{c}/q),$$

for various $w \in C_c^\infty(\mathbb{R}^6)$ with $w(0) = 0$.

1. Roughly speaking, the conjectured main term $c_{\mathrm{HLH},F,\Omega} \cdot X^3$ comes, *unconditionally*, from the locus $\Delta(\boldsymbol{c}) = 0$.
2. Meanwhile, *conditionally*, the remaining sum (over $\Delta(\boldsymbol{c}) \neq 0$) is roughly (at least in key ranges) like

$$\sum_{\text{finite set}} (\text{typically bounded}) \times (\text{RMT-type sum}).$$

GRH would only prove these "RMT-type sums" to be $O_\epsilon(X^{3+\epsilon})$. But *conditionally*, each sum is $0 + O(X^{3-\delta})$, in part because $w(0) = 0$ and $w \in C_c^\infty$.

Proof ideas and themes (extracting main terms)

The main term $c_{\mathrm{HLH},F,\Omega} \cdot X^3$ comes from the locus $\Delta(\boldsymbol{c}) = 0$.

1. The "randomness prediction" $c_{\mathrm{HL},F,\Omega} \cdot X^3$ comes from $\boldsymbol{c} = 0$.
2. On the other hand, $\#\{\text{trivial } \boldsymbol{x} \in \mathbb{Z}^6 \cap X\Omega\}$ comes from $\boldsymbol{c} \neq 0$ with $\Delta(\boldsymbol{c}) = 0$.

Remark

For random $\boldsymbol{c}$, one expects $\widetilde{S}_{\boldsymbol{c}}(n) \ll_\epsilon n^\epsilon$ (most n). But for some special $\boldsymbol{c}$, we have $|\widetilde{S}_{\boldsymbol{c}}(n)| \gg_\epsilon n^{1/2-\epsilon}$ (many/all n).

Proof ideas and themes (sums to primes)

Let $\mathcal{V}_{\boldsymbol{c}} \subseteq \mathbb{P}^5$ be cut out by $x_1^3 + \cdots + x_6^3 = \boldsymbol{c} \cdot \boldsymbol{x} = 0$.

Side Conjecture (Randomness vs. structure over $\mathbb{F}_p$)

If $p \geq 100$ and $\boldsymbol{c} \in \mathbb{F}_p^6$ with $|\#\mathcal{V}_{\boldsymbol{c}}(\mathbb{F}_p) - \#\mathbb{P}^3(\mathbb{F}_p)| \geq 10^{10} p^{3/2}$, then $\mathcal{V}_{\boldsymbol{c}} \bmod p$ contains a plane $P \subseteq \mathcal{V} \bmod p$, i.e. there exists $\pi \in S_6$ with $c_{\pi(1)}^3 - c_{\pi(2)}^3 = c_{\pi(3)}^3 - c_{\pi(4)}^3 = c_{\pi(5)}^3 - c_{\pi(6)}^3 = 0$.

Remark (R. Kloosterman)

A characteristic 0 analog of a stronger version of the conjecture (in the nodal case) holds (with a Hodge-theoretic proof).

We prove *partial* results towards the conjecture, by combining work of Katz (1991) or Skorobogatov (1992) on the one hand with work of Lindner (2020) on the other. We then *apply* these partial results with the aid of the Square-free Sieve Conjecture.

Proof ideas and themes (sums to prime powers)

We prove new boundedness and vanishing criteria for sums of the form $\widetilde{S}_{\boldsymbol{c}}(p^{\geq 2})$. Again, we *apply* these results in conjunction with the Square-free Sieve Conjecture.

Proof ideas and themes (integrals)

I lied a bit. In the delta method, the archimedean (integral) factors are more complicated than $(X/q)^6 \hat{w}(X\boldsymbol{c}/q)$.

Remark

In reality, we prove new oscillatory integral estimates (using a more precise stationary phase analysis than that of Hooley or Heath-Brown), somewhat parallel to our work on $\widetilde{S}_{\boldsymbol{c}}(p^{\geq 2})$, to establish some decay for small moduli q (which if not handled would lose a critical X^{ϵ}).

Proof ideas and themes (typical boundedness)

Remark (Worst- vs. average-case behavior)

Let q denote a prime power $\ll X^{3/2}$. For *all* $\boldsymbol{c} \in \mathbb{Z}^6$ with $\Delta(\boldsymbol{c}) \neq 0$, Hooley (1986) roughly gives $|\widetilde{S}_{\boldsymbol{c}}(q)| \ll$

- $q^{\geq 1/2+\delta}$ for cube-full $q \mid \Delta(\boldsymbol{c}), (c_1 \cdots c_6)^\infty$ (very few q),
- $q^{1/2}$ for the remaining $q \mid \Delta(\boldsymbol{c})$ (a few q), and
- 10 for the rest (most q).

For *conjecturally typical* $\boldsymbol{c} \in \mathbb{Z}^6$,[a] W. (2021) roughly gives

- $|\widetilde{S}_{\boldsymbol{c}}(q)| \ll 10^{10}$ for all q.

In general, we "interpolate" between this and Hooley (1986).

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[a] e.g. $\boldsymbol{c}$ with $\Delta(\boldsymbol{c})$ "nearly" square-free

Proof ideas and themes (RMT-type predictions)

Remark (Ratios Conjectures)

Conrey–Farmer–Zirnbauer (2008) give a heuristic recipe for predicting all the main terms (up to a power-saving error term) for mean values of L-function ratios (e.g. $L, 1/L, L/L$) over natural families of L-functions.

- The recipe for $1/L$'s is fairly simple; "randomness" pervades. One essentially replaces "incomplete" local averages with their "complete" analogs.
- The recipe for L's is more complicated, and involves "duality". While we do not need this directly, it supports our overall belief in the validity of the recipe.

More on RMT-type predictions

How does $\boldsymbol{c} \mapsto L(s, V_{\boldsymbol{c}})$ behave on average? RMT predictions originated for L-zeros "in the bulk" from Montgomery–Dyson, and "near 1/2" from Katz–Sarnak. CFKRS (2005) developed *full main term* predictions for L-powers, and CFZ (2008) for L-ratios; e.g. for some $\delta > 0$, one expects the following:

Conjecture (R1, roughly)

Over $\Delta(\boldsymbol{c}) \neq 0$, we have (for $X \geq 1$; $s = \sigma + it$; $\sigma > 1/2$)

$$\mathbb{E}'_{\boldsymbol{c} \ll X^{1/2}} \left[\frac{1}{L(s, V_{\boldsymbol{c}})} - \underbrace{\zeta(2s) L(s + 1/2, V)}_{\text{polar factors}} A_F(s) \right] \ll_{\sigma,t} X^{-\delta}.$$

Here $A_F(s) \ll 1$ for $\Re(s) \geq 1/2 - \delta$.

A sample corollary of the Ratios Conjectures

We really care about *integrals* over s.

Conjecture (R2', roughly)

For certain holomorphic $f(s)$, e.g. e^{s^2}, we have

$$\mathbb{E}'_{\boldsymbol{c} \ll X^{1/2}} \left| \int_{(\sigma)} ds \, \frac{\zeta(2s)^{-1} L(s+1/2, V)^{-1}}{L(s, V_{\boldsymbol{c}})} \cdot f(s) N^s \right|^2 \ll_f N$$

($\sigma > 1/2$; $1 \ll N \ll X^{3/2}$).

- There are no $\log N$ or $\log X$ factors on the RHS! The numerator $\zeta(2s)^{-1} L(s+1/2, V)^{-1}$ serves as a *mollifier*, and $\int ds$ also helps.
- We use (R2') for $N_{F,\Omega}(X) \ll X^3$, and a "slight adelic perturbation" of (R1) for HLH.

A cartoon of today's main players

1. Let $P_0(\boldsymbol{y}) := y_1^3 + y_2^3 + y_3^3$ first.
2. Let $F(\boldsymbol{x}) := x_1^3 + \cdots + x_6^3$ second.

$$\underbrace{\mathbb{A}^3 \xrightarrow{P_0} \mathbb{A}^1 \xleftarrow{P_0} \mathbb{A}^3 \times_{P_0} \mathbb{A}^3 \cong \{(\boldsymbol{y}, \boldsymbol{z}) \in (\mathbb{A}^3)^2 : P_0(\boldsymbol{y}) = P_0(\boldsymbol{z})\}}_{\text{Cf. Hardy–Littlewood (1925)}}$$

$$\{(\boldsymbol{y}, \boldsymbol{z}) \in (\mathbb{A}^3)^2 : P_0(\boldsymbol{y}) = P_0(\boldsymbol{z})\} \cong \{F(\boldsymbol{x}) = 0\} = C(\mathcal{V})$$

$$\underbrace{C(\mathcal{V}) \dashrightarrow \mathcal{V} \xleftarrow{[\boldsymbol{x}]} \{([\boldsymbol{x}], [\boldsymbol{c}]) \in \mathcal{V} \times (\mathbb{P}^5)^\vee : \boldsymbol{c} \cdot \boldsymbol{x} = 0\} \xrightarrow{[\boldsymbol{c}]} (\mathbb{P}^5)^\vee}_{\text{Cf. Kloosterman (1926), Heath-Brown (1983), Hooley (1986), ...}}$$

Analogs?

- $c^2 + b^4 + a^4 = t$ has some similarity to $c^3 + b^3 + a^3 = t$.
- Allowing *negative* integers, one might go significantly further with "exceptional sets" for *non-critical* problems, like $c^2 + b^3 + a^3 = t$ or $c^2 + b^2 + a^3 = t$, than for the critical $c^3 + b^3 + a^3 = t$. This could be interesting, especially in view of lower bounds on such sets (from Brauer–Manin obstructions).

Deformations?

- Let $N_{(q)}(X) := \#\{\boldsymbol{x} \in \mathbb{Z}^6 \cap [-X, X]^6 : q \mid x_1^3 + \cdots + x_6^3\}$. It is routine to estimate $N_{(q)}(X)$ if $q \leq X^{1-\delta}$. The delta method gives a way to estimate $N_{(q)}(X)$ for $q > 6X^3$. What can be proven in between these extremes?
- (Based on a comment from Wooley.) Let $N^{(\gamma)}(X)$ be the number of integral solutions to

$$x_1^3 + x_2^3 + x_3^3 = y_1^3 + y_2^3 + y_3^3$$

  with $x_1, y_1 \in [10X^\gamma, 20X^\gamma]$ and $x_2, y_2, x_3, y_3 \in [X, 2X]$. Then $N^{(3/2)}(X) \asymp X^{7/2}$ unconditionally, while $N^{(1)}(X) \ll X^{7/2}$ unconditionally and $N^{(1)}(X) \asymp X^3$ conditionally. What about for $\gamma \in (1, 3/2)$?