Conditional approaches to sums of cubes

Victor Wang

Princeton University
Advised by Peter Sarnak

PANTHA, September 2021
Last modified: September 20, 2021

The story of 33

Via computer, Booker obtained (at "five past nine in the morning on the 27th of February 2019")

$$(8866128975287528)^3 + (-8778405442862239)^3 + (-2736111468807040)^3 = 33.$$

Remark

See the Youtube video "33 and all that" for a nice talk by Booker (with T-shirt and mug links) on the discovery of this and related results. (And for some drama involving an old version of Browning's website.)

Exercise

Try Google Calculator, then Wolfram Alpha.

Main qualitative results (roughly)

Theorem (W., 2021)

Assume certain standard NT conjectures on

- *L-functions (Langlands-type conjectures, GRH, the Ratios Conjectures, and an effective Krasner-type lemma), and*
- *"unlikely" divisors (the Square-free Sieve Conjecture).*

Then the following hold:

1. *Integral diagonal cubic equations in* 6 *variables satisfy the Hasse principle.*
2. *Almost all (i.e. asymptotically* 100% *of) integers* $t \not\equiv \pm 4 \bmod 9$ *are sums of three integer cubes.*
3. *A positive fraction of integers are sums of three* nonnegative *integer cubes.*

1. In particular, we are proving results of an “additive” flavor under hypotheses of a “multiplicative” flavor.
2. Conditionally on BSD-type hypotheses (namely finiteness of Tate–Shafarevich groups of certain elliptic curves over quadratic fields), Swinnerton-Dyer (2001) showed that the Hasse principle holds for all integral diagonal cubic equations in 5 variables (and for most in 4 variables); the same methods would presumably work in the case of 6 variables too.
   Our work has the advantage that it passes through a *quantitative* statement, proving not just (1) but also (2)–(3).
   Our methods might also allow one (with some more work) to prove weak approximation in (1), and perhaps (with a lot more work, and a bit of luck) also to prove results for non-diagonal smooth cubic forms. Furthermore, our hypotheses could plausibly be more tractable than BSD-type hypotheses; to test this, one could try working in the function field setting.

Main qualitative results (roughly; cont'd)

Theorem (W., 2021)

Assume certain standard NT conjectures on L-functions and "unlikely" divisors. Then the following hold:

1. *Integral diagonal cubic equations in* 6 *variables satisfy the Hasse principle.*
2. *Almost all (i.e. asymptotically* 100% *of) integers* $t \not\equiv \pm 4 \bmod 9$ *are sums of three integer cubes.*
3. *A positive fraction of integers are sums of three* nonnegative *integer cubes.*

Remark (Based on Cassels–Guy, 1966)

In the analog of (2) for the ternary cubic $5x^3 + 12y^3 + 9z^3$, "almost all" cannot be replaced with "all".

Homogeneously expanding point counts

Definition

Given a polynomial $P \in \mathbb{Z}[\boldsymbol{x}]$ in $s \geq 2$ variables, let

$$N_{P,\Omega}(X) := \#\{\boldsymbol{x} \in \mathbb{Z}^s \cap X\Omega : P(\boldsymbol{x}) = 0\}$$

for each nice region $\Omega \subset \mathbb{R}^s$ (e.g. a finite union of boxes) and scalar $X > 0$.

Example

- If $\Omega = [-1,1]^s$, then $N_{P,\Omega}(X)$ is the number of integral solutions $\boldsymbol{x} \in [-X,X]^s$ to $P(\boldsymbol{x}) = 0$.
- Say $P = y_1^3 + y_2^3 + y_3^3 - t \in \mathbb{Z}[\boldsymbol{y}]$. If $\Omega = [0,1]^3$ and $X \geq t^{1/3}$, then $N_{P,\Omega}(X)$ is $r_3(t)$, the number of ways to write t as a sum of three *nonnegative* integer cubes.

1. This slide defines point counts in homogeneously expanding regions.
2. One should restrict to nice regions, e.g. bounded sets well-approximated above and below (i.e. "well-inscribed" and "well-exhausted") by *finite unions of boxes*.
3. Equivalently, we could define $N_{P,\Omega}(X)$ as $\sum_{\boldsymbol{x}\in\mathbb{Z}^s} 1_{P(\boldsymbol{x})=0} \cdot 1_{\boldsymbol{x}\in X\Omega}$, where 1_E denotes the *indicator function* (or rather *indicator value*) of an event E.
4. More generally, one could look at *weighted* point counts $N_{P,w}(X)$ (for nice weights w), but this would slightly obfuscate the exposition.

A randomness heuristic (intro)

Fix $s \geq 2$ and let $P_0 := x_1^3 + \cdots + x_s^3$. Fix a nice $\Omega \subset \mathbb{R}^s$.

Question

As $X \to \infty$, how are the $\asymp X^s$ points $\boldsymbol{x} \in \mathbb{Z}^s \cap X\Omega$ distributed among the fibers $\{P_0 = t\}$ ($t \in \mathbb{Z}$) of $P_0 \colon \mathbb{Z}^s \to \mathbb{Z}$?

Observation

The fiber $\{P_0(\boldsymbol{x}) = t\}$ ($\boldsymbol{x} \in \mathbb{Z}^s \cap X\Omega$) has $N_{P_0(\boldsymbol{x})-t,\Omega}(X)$ points. Therefore, $N_{P_0(\boldsymbol{x})-t,\Omega}(X)$ is

- 0 if $|t| \gg X^3$ is sufficiently large, and
- $\asymp X^{s-3}$ *on average* (in ℓ^1) over $t \ll X^3$.

1. We have suppressed some Ω-dependencies.

A randomness heuristic (cont'd)

Fix $s \geq 3$ and let $P_0 := x_1^3 + \cdots + x_s^3$. Fix a nice $\Omega \subset \mathbb{R}^s$.

Observation

For $t \in \mathbb{Z}$, the point count $N_{P_0(\boldsymbol{x})-t,\Omega}(X)$ is

- 0 if $|t| \gg X^3$ is sufficiently large, and
- $\asymp X^{s-3}$ *on average* (in ℓ^1) over $t \ll X^3$.

Remark

This is a *real* observation. More precise *real* considerations, alongside *p-adic* analogs, lead to the *Hardy–Littlewood prediction*, a "randomness heuristic" roughly of the form

$$N_{P_0(\boldsymbol{x})-t,\Omega}(X) \sim c^{\mathrm{fin}}_{\mathrm{HL},P_0}(t) \cdot c^{\infty}_{\mathrm{HL},P_0,\Omega}(t/X^3) \cdot X^{s-3} \quad (X \to \infty).$$

1. When $s \leq 4$, the Hardy–Littlewood prediction sometimes takes a subtler form.

A randomness heuristic (cont'd^2)

Fix $s \geq 3$ and let $P_0 := x_1^3 + \cdots + x_s^3$. Fix a nice $\Omega \subset \mathbb{R}^s$.

Remark

Let $t := 0$, or hold t/X^3 constant. Then for fibers $P_0(\boldsymbol{x}) = t$, the Hardy–Littlewood *prediction* roughly takes the form

$$N_{P_0(\boldsymbol{x})-t,\Omega}(X) \sim c^{\mathrm{fin}}_{\mathrm{HL},P_0}(t) \cdot c^{\infty}_{\mathrm{HL},P_0,\Omega}(t/X^3) \cdot X^{s-3} \quad (X \to \infty),$$

where $c^{\mathrm{fin}}_{\mathrm{HL},P_0}(t)$ is a product of p-adic densities.

Remark

When $s \leq 4$, the Hardy–Littlewood *prediction* sometimes takes a more complicated form (and the *full truth* even more so!). But when $s \geq 5$, the word "roughly" can be removed (in the context of the *prediction*).

1. At least *typically* when $s \geq 3$, the Hardy–Littlewood prediction might still more or less make sense (provided $c^{\mathrm{fin}}_{\mathrm{HL},P_0}(t)$ is sufficiently well-behaved).

A critical issue

Consider the *critical case* $s = 3$; let $P_0 := y_1^3 + y_2^3 + y_3^3 \in \mathbb{Z}[\boldsymbol{y}]$.

Observation (S. Diaconu, 2019)

Fix a bounded region $\Omega \subset \mathbb{R}^3$. For each $t \in \mathbb{Z}$, let $X := t^{1/3}$. Then there exists an arithmetic progression $a + d\mathbb{Z}$ ($a, d \in \mathbb{Z}$; $d \neq 0$) on which $P_0(\boldsymbol{y}) = t$ is locally solvable for all t's, yet $N_{P_0(\boldsymbol{y})-t,\Omega}(X) = 0$ for $\geq 99\%$ of t's.

Proof idea.

Arrange for "$c^{\mathrm{fin}}_{\mathrm{HL},P_0}(t)$" to be small over $t \equiv a \bmod d$. □

Remark

$P_0(\boldsymbol{y}) = t$ is locally unsolvable if and only if $t \equiv \pm 4 \bmod 9$.

A critical fix

Consider the *critical case* $s = 3$; let $P_0 := y_1^3 + y_2^3 + y_3^3 \in \mathbb{Z}[\boldsymbol{y}]$.

Observation (S. Diaconu, 2019)

Fix a bounded region $\Omega \subset \mathbb{R}^3$. For each $t \in \mathbb{Z}$, let $X := t^{1/3}$. Then there exists an arithmetic progression $a + d\mathbb{Z}$ ($a, d \in \mathbb{Z}$; $d \neq 0$) on which $P_0(\boldsymbol{y}) = t$ "fails the $X\Omega$-restricted Hasse principle" for $\geq 99\%$ of t's.

But if we repeatedly *enlarge* Ω, the problem goes away: "every few new digits", we expect new solutions $\boldsymbol{y} \in \mathbb{Z}^3$ to $P_0(\boldsymbol{y}) = t$, for most if not all $t \not\equiv \pm 4 \bmod 9$.[1]

[1] This is consistent with the folklore conjecture that perhaps all integers $t \not\equiv \pm 4 \bmod 9$ are sums of three cubes. (See e.g. Heath-Brown, 1992.)

From stingy ternary to rich senary (intro)

- Even with larger regions $\Omega \subset \mathbb{R}^3$ tailored to "producing" small sums of three cubes (e.g.

$$\Omega_\lambda := \{\boldsymbol{v} \in [-\lambda, \lambda]^3 : |v_1^3 + v_2^3 + v_3^3| \le 3\}$$

  as $\lambda \to \infty$), the expected solution sets are still fairly sparse (e.g. only larger by a factor of $\asymp \log \lambda$).
- But for all Ω, we can do better *statistically*, using the *second moment method* (classical) or a *variance analysis* (cf. Ghosh–Sarnak, 2017, for $x^2 + y^2 + z^2 - xyz = t$).
- The key players are the "first-moment map"

$$P_0 \colon \mathbb{Z}^3 \to \mathbb{Z},\ \boldsymbol{y} \mapsto P_0(\boldsymbol{y})$$

  and the "second-moment map"

$$\{(\boldsymbol{y}, \boldsymbol{z}) \in (\mathbb{Z}^3)^2 : P_0(\boldsymbol{y}) = P_0(\boldsymbol{z})\} \to \mathbb{Z},\ (\boldsymbol{y}, \boldsymbol{z}) \mapsto P_0(\boldsymbol{y}).$$

1. The idea of analyzing variances is also classical (being the basis of most results on "exceptional sets in Waring's problem"). We have singled out Ghosh–Sarnak (2017) because the Markoff-type equations $x^2 + y^2 + z^2 - xyz = t$ are *critical* (just like $x^3 + y^3 + z^3 = t$).

From stingy ternary to rich senary (cont'd)

Let $F := x_1^3 + \cdots + x_6^3 \in \mathbb{Z}[\boldsymbol{x}]$. If $\boldsymbol{x} = (\boldsymbol{y}, -\boldsymbol{z})$, then

$$P_0(\boldsymbol{y}) = P_0(\boldsymbol{z}) \iff F(\boldsymbol{x}) = 0.$$

Observation (Second moment method)

Let $\Omega := [-1,1]^6$. If $N_{F,\Omega}(X) \ll X^3 = X^{6-3}$ $(X \to \infty)$, then the set $\{t \in \mathbb{Z} : r_3(t) \neq 0\}$ has positive lower density in $\mathbb{Z}$.

Theorem (Based on Diaconu, 2019)

Suppose that for all nice regions $\Omega \subset \mathbb{R}^6$, Hooley's conjecture[a] *(interpreted on Ω) holds. Then almost all $t \not\equiv \pm 4 \bmod 9$ are sums of three cubes.*

[a] of the form $N_{F,\Omega}(X) \sim c_{\mathrm{HLH},F,\Omega} \cdot X^3$

The senary state of art

Let $F := x_1^3 + \cdots + x_6^3 \in \mathbb{Z}[\boldsymbol{x}]$ and $\Omega := [-1,1]^6$. Then

$$N_{F,\Omega}(X) = \int_{\mathbb{R}/\mathbb{Z}} d\theta \, |T(\theta)|^6,$$

where $T(\theta) := \sum_{|x| \le X} e(\theta x^3)$. What is known here?

- Hua (1938) proved $N_{F,\Omega}(X) \ll X^{7/2+\epsilon}$, by Cauchy between $\int_{\mathbb{R}/\mathbb{Z}} d\theta \, |T(\theta)|^4 \ll X^{2+\epsilon}$ (divisor bound) and $\int_{\mathbb{R}/\mathbb{Z}} d\theta \, |T(\theta)|^8 \ll X^{5+\epsilon}$ (Cauchy, then divisor bound).
- Via clever Cauchy, and other ideas, Vaughan (1986, 2020) gave a more robust proof of Hua's bound, ultimately leading to $N_{F,\Omega}(X) \ll X^{7/2}(\log X)^{\epsilon-5/2}$ $(X \to \infty)$.
- Under Langlands-type hypotheses and GRH (for certain Hasse–Weil L-functions), Hooley (1986, 1997) proved $N_{F,\Omega}(X) \ll X^{3+\epsilon}$.

The senary failure of randomness (intro)

Let $F := x_1^3 + \cdots + x_6^3$. Recall that for a nice region $\Omega \subset \mathbb{R}^6$, Hardy–Littlewood predicts $\sim c_{\mathrm{HL},F,\Omega} \cdot X^{6-3}$ solutions to $F(\boldsymbol{x}) = 0$ "arising randomly". But this is not the full truth!

Proposition (Randomness failure)

If $\Omega := [-1,1]^6$, then $N_{F,\Omega}(X) - c_{\mathrm{HL},F,\Omega} \cdot X^3 \gg X^3$ $(X \to \infty)$.

Proof sketch.

Recall that if $\Omega := [-1,1]^6$, then $N_{F,\Omega}(X) = \int_{\mathbb{R}/\mathbb{Z}} d\theta\, |T(\theta)|^6$. Now choose sensible major and minor arcs $\mathfrak{M}, \mathfrak{m}$. Then $\int_{\mathfrak{M}} d\theta\, |T(\theta)|^6 \sim c_{\mathrm{HL},F,\Omega} \cdot X^{6-3}$. But $\int_{\mathfrak{m}} d\theta\, |T(\theta)|^2 \asymp X$, so Hölder implies $\int_{\mathfrak{m}} d\theta\, |T(\theta)|^6 \gg X^3$. $\square$

The senary failure of randomness (cont'd)

(Let $F := x_1^3 + \cdots + x_6^3$.) One can show more.

Definition

Say $\boldsymbol{x} \in \mathbb{Z}^6$ is of *diagonal type* if there exists $\pi \in S_6$ such that $x_{\pi(1)} + x_{\pi(2)} = x_{\pi(3)} + x_{\pi(4)} = x_{\pi(5)} + x_{\pi(6)} = 0$.

If $\boldsymbol{x} \in \mathbb{Z}^6$ is of diagonal type, then $F(\boldsymbol{x}) = 0$.

Theorem (Hooley, 1986')

There exists a nice region $\Omega \subset \mathbb{R}^6$, and a real number $\delta > 0$, such that $N_{F,\Omega}(X)$ is (for all sufficiently large $X \gg 1$)

$$\geq \delta X^3 + \max\left(c_{\mathrm{HL},F,\Omega} \cdot X^3, \#\{\text{diagonal-type } \boldsymbol{x} \in \mathbb{Z}^6 \cap X\Omega\}\right).$$

Hooley's conjecture (interpreted for general Ω)

(Let $F := x_1^3 + \cdots + x_6^3$.)

Definition

Say $\boldsymbol{x} \in \mathbb{Z}^6$ is of *diagonal type* if there exists $\pi \in S_6$ such that $x_{\pi(1)} + x_{\pi(2)} = x_{\pi(3)} + x_{\pi(4)} = x_{\pi(5)} + x_{\pi(6)} = 0$.

If $\boldsymbol{x} \in \mathbb{Z}^6$ is of diagonal type, then $F(\boldsymbol{x}) = 0$.

Conjecture (Hooley, 1986', interpreted generally)

For any nice region $\Omega \subset \mathbb{R}^6$, we have (as $X \to \infty$)

$$N_{F,\Omega}(X) \sim c_{\mathrm{HL},F,\Omega} \cdot X^3 + \#\{\text{diagonal-type } \boldsymbol{x} \in \mathbb{Z}^6 \cap X\Omega\}.$$

(Under this conjecture, Diaconu's methods show that almost all $t \not\equiv \pm 4 \bmod 9$ are sums of three cubes.)

The delta method (intro)

Let $F := x_1^3 + \cdots + x_6^3$ and $\Omega := [-1, 1]^6$.

- Under Langlands-type hypotheses and GRH (for certain Hasse–Weil L-functions), Hooley (1986, 1997) proved $N_{F,\Omega}(X) \ll X^{3+\epsilon}$.
- Using the delta method, Heath-Brown (1996, 1998) gave a slightly more systematic proof of Hooley's conditional bound $N_{F,\Omega}(X) \ll X^{3+\epsilon}$ (under the same hypotheses).

Remark

In a nutshell, the delta method relates NT of a "$+$" flavor to NT of a "$\times$" flavor. It is a modern version of the "completed averaging" method of H. Kloosterman (1926).

1. Many of the keys ideas in Hooley (1986, 1997) and Heath-Brown (1996, 1998) coincide. The differences between the two approaches are all technical, though Heath-Brown's work has the advantage that it is based on an equality (the delta method) rather than an inequality (as in Hooley's "upper-bound precursor" to the delta method).

The delta method (cont'd)

Let $F := x_1^3 + \cdots + x_6^3$. Fix a nice $\Omega \subset \mathbb{R}^6$. In the circle method for $N_{F,\Omega}(X)$, one encounters sums resembling

$$\sum_{q \le X^{3/2}} \frac{1}{qX^{3/2}} \sum_{a \bmod q}' \sum_{\boldsymbol{x} \in \mathbb{Z}^6} w(\boldsymbol{x}/X) \cdot e_q(aF(\boldsymbol{x})),$$

where w is a smooth weight "approximating" Ω, and $a \bmod q$ is restricted to residues coprime to q. (Here $e_q(t) := e^{2\pi i t/q}$.)

Remark

In this setting, H. Kloosterman (1926) would suggest

1. averaging over fractions a/q with q fixed, and
2. using Poisson summation (over each fixed $\boldsymbol{x} \bmod q$) to "complete" incomplete exponential sums over $\boldsymbol{x}$,

in either order.

The delta method (cont'd^2)

Let $F := x_1^3 + \cdots + x_6^3$. Fix a nice $\Omega \subset \mathbb{R}^6$. In the circle method for $N_{F,\Omega}(X)$, one encounters sums resembling

$$\sum_{q \le X^{3/2}} \frac{1}{qX^{3/2}} \sum_{a \bmod q}{}' \sum_{\boldsymbol{x} \in \mathbb{Z}^6} w(\boldsymbol{x}/X) \cdot e_q(aF(\boldsymbol{x})).$$

Remark

H. Kloosterman (1926) would rewrite the sum above as

$$\sum_{q \le X^{3/2}} \frac{1}{qX^{3/2}} \sum_{\boldsymbol{c} \in \mathbb{Z}^6} S_{\boldsymbol{c}}(q) \cdot (X/q)^6 \hat{w}(X\boldsymbol{c}/q),$$

where

$$S_{\boldsymbol{c}}(q) := \sum_{a \bmod q}{}' \sum_{\boldsymbol{x} \bmod q} e_q(aF(\boldsymbol{x}) + \boldsymbol{c} \cdot \boldsymbol{x}).$$

The delta method (cont'd^3)

Let $F := x_1^3 + \cdots + x_6^3$. Fix a nice $\Omega \subset \mathbb{R}^6$. In the delta method for $N_{F,\Omega}(X)$, one encounters sums resembling

$$\sum_{q \le X^{3/2}} \frac{1}{qX^{3/2}} \sum_{\boldsymbol{c} \in \mathbb{Z}^6} S_{\boldsymbol{c}}(q) \cdot (X/q)^6 \hat{w}(X\boldsymbol{c}/q),$$

where

$$S_{\boldsymbol{c}}(q) := \sideset{}{'}\sum_{a \bmod q} \sum_{\boldsymbol{x} \bmod q} e_q(aF(\boldsymbol{x}) + \boldsymbol{c} \cdot \boldsymbol{x}).$$

Observation (Partly classical; used in Hooley, 1986)

F is *homogeneous*, so $S_{\boldsymbol{c}}(mn) = S_{\boldsymbol{c}}(m) S_{\boldsymbol{c}}(n)$ if $(m, n) = 1$. Also, if $p \nmid \boldsymbol{c}$, then $p^{-7/2} S_{\boldsymbol{c}}(p) \approx \widetilde{E}_{\boldsymbol{c}}(p)$, where $\widetilde{E}_{\boldsymbol{c}}(p)$ measures the "bias modulo p" of the cubic 3-fold $\mathcal{V}_{\boldsymbol{c}} : F(\boldsymbol{x}) = \boldsymbol{c} \cdot \boldsymbol{x} = 0$. Here if $p \nmid \Delta(\boldsymbol{c})$, then $|\widetilde{E}_{\boldsymbol{c}}(p)| \le 10$ (Weil conjectures).

The delta method (cont'd^4)

Let $F := x_1^3 + \cdots + x_6^3$. Fix a nice $\Omega \subset \mathbb{R}^6$. In the delta method for $N_{F,\Omega}(X)$, one encounters sums resembling

$$\sum_{q \le X^{3/2}} \frac{1}{qX^{3/2}} \sum_{\boldsymbol{c} \in \mathbb{Z}^6} S_{\boldsymbol{c}}(q) \cdot (X/q)^6 \hat{w}(X\boldsymbol{c}/q). \qquad (1)$$

If $\Delta(\boldsymbol{c}) \neq 0$, then the normalized sums $\widetilde{S}_{\boldsymbol{c}}(q) := q^{-7/2} S_{\boldsymbol{c}}(q)$ look (to first order) like the coefficients $\mu_{\boldsymbol{c}}(q)$ of the *reciprocal* Hasse–Weil L-function $1/L(s, V_{\boldsymbol{c}})$ associated to the cubic 3-fold $V_{\boldsymbol{c}} := V(F, \boldsymbol{c} \cdot \boldsymbol{x})/\mathbb{Q}$ (Hooley, 1986).

Exercise (Cf. Hooley, 1986)

Assuming that $\Delta(\boldsymbol{c}) \neq 0$ for all $\boldsymbol{c}$, that $\widetilde{S}_{\boldsymbol{c}}(q) = \mu_{\boldsymbol{c}}(q)$ for all $\boldsymbol{c}, q$, and that $\sum_{n \le N} \mu_{\boldsymbol{c}}(n) \ll_\epsilon \|\boldsymbol{c}\|^\epsilon N^{1/2+\epsilon}$ for all $\boldsymbol{c}, N$ ($N \ge 1$), show that the sum (1) above is $\ll_\epsilon X^{3+\epsilon}$.

1. The modern definition of $L(s, V_{\boldsymbol{c}})$ (see Taylor, 2004) is a bit technical, and is based on the Galois representation $H^3(V_{\boldsymbol{c}} \times \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$ for a choice of auxiliary prime ℓ. (The choice of ℓ should not matter; for our specific representations, this is probably known unconditionally.)

Main quantitative results (roughly)

Theorem (W., 2021)

Assume certain standard NT conjectures on

- *the Hasse–Weil L-functions* $L(s, V_{\boldsymbol{c}})$ *and some "second-order" relatives (Langlands-type conjectures, GRH, the Ratios Conjectures, and an effective Krasner-type lemma), and*
- *"unlikely" divisors (the Square-free Sieve Conjecture for the discriminant polynomial* $\Delta \in \mathbb{Z}[\boldsymbol{c}]$*).*

Then for diagonal cubic forms $F \in \mathbb{Z}[\boldsymbol{x}]$ *in* 6 *variables, and for a large class*[a] *of regions* $\Omega \subset \mathbb{R}^6$*, Hooley's conjecture*[b] *(interpreted for* F, Ω*) holds.*

---

[a] large enough for our main qualitative needs

[b] of the form $N_{F,\Omega}(X) \sim c_{\mathrm{HLH},F,\Omega} \cdot X^3$

Some details on the hypotheses

1. "Second-order" relatives: $L(s, V_{\boldsymbol{c}}, \bigwedge^2)$ and $L(s, V(F))$.
2. Langlands-type conjectures and GRH: Need Selberg-type axioms (analytic continuation, Ramanujan bound, etc.) to hold, and need $1/L(s, V_{\boldsymbol{c}})$ to be holomorphic on the region $\Re(s) > 1/2$. And other technical things (e.g. basic expected properties of conductors and γ-factors).
3. Ratios Conjectures: Give predictions of Random Matrix Theory (RMT) type for the mean values of $1/L(s, V_{\boldsymbol{c}})$ and $1/L(s_1, V_{\boldsymbol{c}})L(s_2, V_{\boldsymbol{c}})$ over natural families of $\boldsymbol{c}$'s.
4. Effective Krasner-type lemma: Need limited $\boldsymbol{c}$-variation of the local factors $L_p(s, V_{\boldsymbol{c}})$ (easy if e.g. $p \nmid \Delta(\boldsymbol{c})$).
5. Square-free Sieve Conjecture: Need

$$\Pr\left[\boldsymbol{c} \in [-Z, Z]^6 : \exists \text{ sq-full } q \in [Q, 2Q] \text{ with } q \mid \Delta(\boldsymbol{c})\right]$$

to be $\ll Q^{-\delta}$, uniformly over $Z \geq 1$ and $Q \leq Z^6$.

Proof ideas and themes (overview)

In the delta method for $N_{F,\Omega}(X)$, one roughly encounters

$$\sum_{\boldsymbol{c}\in\mathbb{Z}^6} \sum_{q\le X^{3/2}} \frac{1}{qX^{3/2}} \cdot q^{7/2}\widetilde{S}_{\boldsymbol{c}}(q) \cdot (X/q)^6 \hat{w}(X\boldsymbol{c}/q),$$

for various $w \in C_c^\infty(\mathbb{R}^6)$ with $w(0) = 0$.

1. Roughly speaking, the conjectured main term $c_{\mathrm{HLH},F,\Omega} \cdot X^3$ comes, *unconditionally*, from the locus $\Delta(\boldsymbol{c}) = 0$.
2. Meanwhile, *conditionally*, the remaining sum (over $\Delta(\boldsymbol{c}) \neq 0$) roughly decomposes, at least in key ranges, as a finite linear combination of products of the form

   $$\text{(typically bounded)} \times \text{(RMT-susceptible sums)}.$$

   GRH would only prove these "RMT-susceptible sums" to be $O_\epsilon(X^{3+\epsilon})$. But *conditionally*, each sum is $0 + O(X^{3-\delta})$, in part because $w(0) = 0$ and $w \in C^\infty$.

Proof ideas and themes (extracting main terms)

The main term $c_{\mathrm{HLH},F,\Omega} \cdot X^3$ comes from the locus $\Delta(\boldsymbol{c}) = 0$.

1. The "randomness prediction" $c_{\mathrm{HL},F,\Omega} \cdot X^3$ comes from $\boldsymbol{c} = 0$.
2. On the other hand, $\#\{\text{diagonal-type } \boldsymbol{x} \in \mathbb{Z}^6 \cap X\Omega\}$ comes from $\boldsymbol{c} \neq 0$ with $\Delta(\boldsymbol{c}) = 0$.

Remark

For random $\boldsymbol{c}, n$, one heuristically expects $\widetilde{S}_{\boldsymbol{c}}(n) \ll_\epsilon n^\epsilon$—which can be formalized in a way key to the locus $\Delta(\boldsymbol{c}) \neq 0$—but key to the locus $\Delta(\boldsymbol{c}) = 0$ is that for certain special $\boldsymbol{c}$'s, the truth can easily be a factor of $\gg_\epsilon n^{1/2-\epsilon}$ larger. (Thus whereas the locus $\Delta(\boldsymbol{c}) \neq 0$ centers around L-functions, the the locus $\Delta(\boldsymbol{c}) = 0$ centers around algebraic geometry.)

Proof ideas and themes (sums to primes)

Let $\mathcal{V}_{\boldsymbol{c}} \subseteq \mathbb{P}^5$ be cut out by $x_1^3 + \cdots + x_6^3 = \boldsymbol{c} \cdot \boldsymbol{x} = 0$.

Side Conjecture (Randomness vs. structure over $\mathbb{F}_p$)

If $p \geq 1000$ and $\boldsymbol{c} \in \mathbb{F}_p^6$ with $|\#\mathcal{V}_{\boldsymbol{c}}(\mathbb{F}_p) - \#\mathbb{P}^3(\mathbb{F}_p)| \geq 10^{10} p^{3/2}$, then $\mathcal{V}_{\boldsymbol{c}} \bmod p$ contains a plane $P \subseteq \mathcal{V} \bmod p$, i.e. there exists $\pi \in S_6$ with $c_{\pi(1)}^3 - c_{\pi(2)}^3 = c_{\pi(3)}^3 - c_{\pi(4)}^3 = c_{\pi(5)}^3 - c_{\pi(6)}^3 = 0$.

Remark (R. Kloosterman)

A characteristic 0 analog of a stronger version of the conjecture in the case $c_1 \cdots c_6 \neq 0$ holds (with a Hodge-theoretic proof).

We prove *partial* results towards the conjecture, by combining work of Katz (1991) or Skorobogatov (1992) on the one hand with work of Lindner (2020) on the other. We then *apply* these partial results with the aid of the Square-free Sieve Conjecture.

1. Lindner (2020) proves partial results towards the "stronger version of the conjecture".
2. One can think of Katz/Skorobogatov as providing general "worst-case" information, and Lindner as providing helpful "average-case" information.

Proof ideas and themes (sums to prime powers)

We prove new boundedness and vanishing criteria for sums of the form $\widetilde{S}_{\boldsymbol{c}}(p^{\geq 2})$. Again, we *apply* these results in conjunction with the Square-free Sieve Conjecture.

Proof ideas and themes (integrals)

I lied a bit. In the delta method, the archimedean (integral) factors are more complicated than $(X/q)^6 \hat{w}(X\boldsymbol{c}/q)$.

Remark

In reality, we prove new oscillatory integral estimates (using a more precise stationary phase analysis than that of Hooley or Heath-Brown), somewhat parallel to our work on $\widetilde{S}_{\boldsymbol{c}}(p^{\geq 2})$, to establish some decay for small moduli q (which if not handled would lose a critical X^{ϵ}).

Proof ideas and themes (RMT-type predictions)

Remark (Ratios Conjectures)

Conrey–Farmer–Zirnbauer (2008) give a heuristic recipe for predicting all the main terms (up to a power-saving error term) for mean values of L-function ratios (e.g. $L, 1/L, L/L$) over natural families of L-functions.

- The recipe for $1/L$'s is fairly simple.[a] One essentially replaces "incomplete" local averages with their "complete" analogs.
- The recipe for L's is more complicated, and involves "duality" (specifically, the approximate functional equation for L). While we do not need this directly, it supports our overall belief in the validity of the recipe.

[a] This is morally related to "Möbius randomness" heuristics.

A cartoon of today's main players

1. Let $P_0(\boldsymbol{y}) := y_1^3 + y_2^3 + y_3^3$ first.
2. Let $F(\boldsymbol{x}) := x_1^3 + \cdots + x_6^3$ second.

$$\underbrace{\mathbb{A}^3 \xrightarrow{\boldsymbol{y} \mapsto P_0(\boldsymbol{y})} \mathbb{A}^1 \xleftarrow{P_0(\boldsymbol{y})} \{(\boldsymbol{y}, \boldsymbol{z}) \in (\mathbb{A}^3)^2 : P_0(\boldsymbol{y}) = P_0(\boldsymbol{z})\}}_{\text{Cf. Hardy–Littlewood (1925)}}$$

$$\{(\boldsymbol{y}, \boldsymbol{z}) \in (\mathbb{A}^3)^2 : P_0(\boldsymbol{y}) = P_0(\boldsymbol{z})\} \cong \{F(\boldsymbol{x}) = 0\} = C(\mathcal{V})$$

$$\underbrace{C(\mathcal{V}) \dashrightarrow \mathcal{V} \xleftarrow{[\boldsymbol{x}]} \{([\boldsymbol{x}], [\boldsymbol{c}]) \in \mathcal{V} \times (\mathbb{P}^5)^\vee : \boldsymbol{c} \cdot \boldsymbol{x} = 0\} \xrightarrow{[\boldsymbol{c}]} (\mathbb{P}^5)^\vee}_{\text{Cf. Kloosterman (1926), Heath-Brown (1983), Hooley (1986), \ldots}}$$

Analogs?

- $c^2 + b^4 + a^4 = t$ has some similarity to $c^3 + b^3 + a^3 = t$.
- Allowing *negative* integers, one might go significantly further with "exceptional sets" for *non-critical* problems, like $c^2 + b^3 + a^3 = t$ or $c^2 + b^2 + a^3 = t$, than for the critical $c^3 + b^3 + a^3 = t$. This could be interesting in view of lower bounds on such sets (from Brauer–Manin obstructions).

Deformations?

- Let $N_{(q)}(X) := \#\{\boldsymbol{x} \in \mathbb{Z}^6 \cap [-X, X]^6 : q \mid x_1^3 + \cdots + x_6^3\}$. It is routine to estimate $N_{(q)}(X)$ if $q \leq X^{1-\delta}$. The delta method gives a way to estimate $N_{(q)}(X)$ for $q > 6X^3$. What can be proven in between these extremes?
- (Based on a comment from Wooley.) Let $N^{(\gamma)}(X)$ be the number of integral solutions to

$$x_1^3 + x_2^3 + x_3^3 = y_1^3 + y_2^3 + y_3^3$$

with $x_1, y_1 \in [10X^\gamma, 20X^\gamma]$ and $x_2, y_2, x_3, y_3 \in [X, 2X]$. Then $N^{(3/2)}(X) \asymp X^{7/2}$ unconditionally, while $N^{(1)}(X) \ll X^{7/2}$ unconditionally and $N^{(1)}(X) \asymp X^3$ conditionally. What about for $\gamma \in (1, 3/2)$?

1. Regarding $N^{(\gamma)}(X)$, my understanding (based on conversation with Wooley after the talk; but it is possible that I have misunderstood or misremembered what he said) is that it is known unconditionally (due to work of Vaughan on "diminishing ranges") that $N^{(\gamma)}(X) \ll_\epsilon X^{\gamma+2+\epsilon}$ for $\gamma \geq 6/5$ (and proving this for $\gamma \geq 6/5 - \delta$ might have interesting applications).