NOTES ON TWO GRAPH THEORY PROBLEMS

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1. Problem 1: Computational topology of complex graphs

In Summer 2017, I began researching computational topology in Grenoble through the generous MIT-France Program, under Profs. Francis Lazarus and Arnaud de Mesmay. One fundamental open problem is to determine the "smallest" surface embedding of a complex network, with "size" measured by the topological genus of the surface. Practical motivation (for this basic research) includes a recent "growing interest in addressing combinatorial optimization problems using algorithms that exploit embeddings on surfaces to achieve provably higher-quality output or provably faster running times" [1]. In this vein, we seek new algorithms and theoretical results for the embeddability of graphs on surfaces. The plane $\mathbb{R}^2$ and sphere S^2 are well-understood, but other surfaces, such as the torus T^2 (surface of a donut) and projective plane $\mathbb{RP}^2$, are not.

From a theoretical angle, we seek to find useful characterizations of graphs embeddable in surfaces. An embedding is a drawing of a graph G in a surface S, such that every two edges intersect 0 times ("combinatorial intersection number" 0"). Two other fruitful notions are $\mathbb{Z}/2$-embeddings, where every two (non-adjacent) edges intersect an even number of times ("$\mathbb{Z}/2$-intersection number" 0); and $\mathbb{Z}$-embeddings ("signed intersection number" 0). A priori, on any surface S, embeddability implies $\mathbb{Z}$-embeddability, which in turn implies $\mathbb{Z}/2$-embeddability. A striking classical result of Hanani and Tutte says that if $S \in \{\mathbb{R}^2, S^2\}$, then the reverse implications also hold: $\mathbb{Z}/2$-embeddability guarantees $\mathbb{Z}$-embeddability and normal embeddability! The result is also known when $S = \mathbb{RP}^2$, but in August 2017, R. Fulek and J. Kynčl showed that the Hanani–Tutte theorem is false when S is the surface of a donut with four or more holes [2]. However, we suspect that there are fruitful notions other than $\mathbb{Z}/2$-embeddability and $\mathbb{Z}$-embeddability, for which a modified version of the Hanani–Tutte theorem should still be true.

From an algorithmic angle, we care mostly about the efficiency of theoretical characterizations. For example, $\mathbb{Z}/2$-embeddability is easy to test when $S \in \{\mathbb{R}^2, S^2\}$, so the Hanani–Tutte theorem gives rise to a known efficient algorithm for testing graph embeddability on S. However, even though the Hanani–Tutte theorem is true for $S = \mathbb{RP}^2$, there is no known efficient algorithm for testing $\mathbb{Z}/2$-embeddability in that case.

For both angles, we suspect that a deeper understanding may come from an area of algebraic topology known as *obstruction theory*, which has recently started to find its way into combinatorial (as opposed to topological) problems. Obstruction theory provides a motivating framework for $\mathbb{Z}$-embeddability and Hanani–Tutte when $S = \mathbb{R}^2$, as well as a natural context for the known efficient algorithms on S. When $S = \mathbb{RP}^2$ (in the realm of the unknown), we are cautiously optimistic about a possible quaternion group algorithm

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Based on a research statement from December 16, 2017. I have not worked on either of the problems here in a while, but thought these notes might be worth sharing anyways (e.g. for amusement).

for testing $\mathbb{Z}/2$-embeddability. But serious technical difficulties remain, because while the fundamental group $\pi_1(\mathbb{R}^2 \times \mathbb{R}^2 \setminus \Delta, *) = \mathbb{Z}$ (arising through obstruction theory, where Δ is the diagonal) is Abelian, the group $\pi_1(\mathbb{RP}^2 \times \mathbb{RP}^2 \setminus \Delta, *) = Q_8$ is non-Abelian (outside the reach of classical obstruction theory).

2. Problem 2: Ramsey theory of complex graphs

The *transitive tournament on N vertices* is the directed graph on vertices $1, 2, \ldots, N$ with an edge $i \to j$ whenever $i < j$. A classical result of Erdős and Szekeres states that any 2-coloring of the edges of the N-vertex transitive tournament has a *monochromatic directed path* with $\lceil N^{1/2} \rceil$ vertices. As Motzkin would say, "complete disorder is impossible". This is the philosophy of Ramsey theory, a fundamental part of extremal combinatorics, with broad applications to computer science and other parts of math and science.

To prove Erdős–Szekeres, say with colors red and blue, one defines a canonical Record map as follows: assign to vertex i the *pair* (R_i, B_i), where R_i (resp. B_i) denotes the vertex-length of the longest red (resp. blue) path ending at i. These N pairs must all be distinct, so by pigeonhole, the longest monochromatic path has length at least $N^{1/2}$. The argument immediately generalizes to show that any k-colored tournament has a monochromatic path of length at least $N^{1/k}$. Recently, Loh [5] introduced the simplest unsolved variant of these foundational Ramsey-theoretic questions.

Question 2.1 (Tournaments formulation)**.** *Must every 3-coloring of the edges of the N-vertex transitive tournament contain a 1-color-*avoiding *directed path (i.e. path using only two of three colors) with at least $N^{2/3}$ vertices?*

I have always enjoyed sharing and discussing this natural question and its history. Yet at the same time, it seems deep, important, and rich in connections. For instance, it is completely equivalent to the more *geometric* Question 2.3 below.

Definition 2.2. Call a set of triples $S \subseteq \mathbb{R}^3$ *ordered* if the triples can be listed as $L_1, \ldots, L_{|S|}$ such that for every $i < j$, the coordinate-wise difference $L_j - L_i$ has at least two strictly positive coordinates.

Question 2.3 (Triples formulation)**.** *Must an ordered set $S \subseteq [n]^3$ satisfy $|S| \le n^{3/2}$?*

It turns out that there is a "trivial" bound $|S| \le n^2$. Loh [5] improved the bound to $|S| \le n^2/e^{\log^*(n)}$ using a powerful multipurpose tool (the triangle removal lemma), where $\log^*$ is the iterated logarithm. However, the method cannot give a power bound $|S| \le n^\alpha$ with $\alpha < 2$, due to the Behrend construction from additive combinatorics. Gowers and Long [3] later proved such a power bound inductively, but their method apparently cannot achieve the conjectured $\alpha = 3/2$ exponent.

To prove the *reduction* from Question 2.1 to 2.3, Loh [5] used a canonical Record map analogous to that from the proof of Erdős–Szekeres. Loh also used a *non-canonical* Color map to *reduce* Question 2.3 to 2.1. Jonathan Tidor, Ben Yang, and I instead used a *canonical* Color map to make a *stronger reduction*, ultimately showing in [7] that Questions 2.1 and 2.3 are both equivalent to an easier question about a more structured class of tournaments, which we called *geometric* RGBK-*tournaments* (capturing both the combinatorics and the geometry of the situation).

In other words, the maps Color and Record yield extra transitivity structure "for free". This, together with an inductive Cauchy–Schwarz argument, resolves our question in some special cases (including some addressed independently by Wagner [8]), but perhaps more importantly suggests directions for future innovation:

- I can try finding new transformations as useful as Color and Record.
- The Cauchy–Schwarz tournaments satisfy a *global-to-local* principle, so that *any counterexample* must contain a *more extreme counterexample inside*. New progress may reveal interesting new global-to-local arguments.

Thinking about the equivalent Question 2.3 has led to even more possible directions:

- We established a rigorous connection, [7, Theorem 4.8], from a geometric problem of independent interest [6]. So progress on [6] would lead to progress on Question 2.3.
- We raised a natural and stronger L^2-question on slice-counts [7, Question 4.12], and suggested several plausible approaches. I suggested one approach, [7, Section 4.4.2], via the foundational "sum of squares of dimensions" theorem from representation theory. If successful, this would differ from typical applications of representation theory to extremal combinatorics based on character theory.
- Ben noted in [7, Observation 4.16] that some of the tight examples for Question 2.3 are also tight examples for the joints problem from incidence geometry. The joints problem originated from computer science, but has a surprisingly clean proof using elementary algebraic geometry [4]. However, joints have simple algebraic structure, while the triples in Question 2.3 do not. Thus any geometric progress would likely reveal new phenomena and provide an interesting comparison with the joints problem.

References

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