THE AVERAGE CUT-RANK OF GRAPHS

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Abstract. The cut-rank of a set X of vertices in a graph G is defined as the rank of the $X \times (V(G) \setminus X)$ matrix over the binary field whose (i, j)-entry is 1 if the vertex i in X is adjacent to the vertex j in $V(G) \setminus X$ and 0 otherwise. We introduce the graph parameter called the *average cut-rank* of a graph, defined as the expected value of the cut-rank of a random set of vertices. We show that this parameter does not increase when taking vertex-minors of graphs and a class of graphs has bounded average cut-rank if and only if it has bounded neighborhood diversity. This allows us to deduce that for each real α, the list of induced-subgraph-minimal graphs having average cut-rank larger than (or at least) α is finite. We further refine this by providing an upper bound on the size of obstruction and a lower bound on the number of obstructions for average cut-rank at most (or smaller than) α for each real $\alpha \geq 0$. Finally, we describe explicitly all graphs of average cut-rank at most 3/2 and determine up to 3/2 all possible values that can be realized as the average cut-rank of some graph.

1. Introduction

The *cut-rank function* of a graph G is a function $\rho_G : 2^{V(G)} \to \mathbb{Z}$ that maps every subset X of $V(G)$ to the rank of an $X \times (V(G) \setminus X)$ matrix over the binary field whose rows are indexed by X and columns are indexed by $V(G) \setminus X$ such that the (i, j)-entry is 1 if and only if the vertex i in X is adjacent to the vertex j in $V(G) \setminus X$. Roughly speaking, $\rho_G(X)$ is small if the set of all edges between X and $V(G) \setminus X$ form a simple structure to be described, though it could be dense. The rank-width of a graph, introduced by Oum and Seymour [14], uses the cut-rank function in its definition.

One of the most important properties of the cut-rank function is that it is preserved under the operation called local complementation. The *local complementation* at a vertex v of a graph G is an operation to obtain a new graph $G * v$ from G by complementing in the neighborhood of v. In other words, for all pairs x, y of neighbors of v, we delete xy if x, y are adjacent and add an edge xy otherwise to obtain $G * v$. A graph H is a *vertex-minor* of a graph G if H is an induced subgraph of a graph that can be obtained from G by some sequence of local complementations. Since local complementation preserves the cut-rank function [12], if H is a vertex-minor of G and $X \subseteq V(H)$, then $\rho_H(X) \leq \rho_G(X)$. It follows that the class of graphs of rank-width at most k is closed under taking vertex-minors [12].

It turns out that some of the theory developed for graph minors by Robertson and Seymour (see a survey of Lovász [11]) can be generalized for vertex-minors. For instance, Oum [13] showed that graphs of bounded rank-width are well-quasi-ordered under the vertex-minor relation and conjectured that graphs are well-quasi-ordered under the vertex-minor relation. If true, every class of graphs closed under taking vertex-minors would be characterized by a finite list of forbidden vertex-minors. This is why for each k, the class of graphs of rank-width at most k is characterized by finitely many forbidden vertex-minors [12]. Now there are many interesting problems regarding vertex-minors of graphs and yet we have only a few graph parameters that do not increase by taking

Date: July 14, 2021.

Nguyen was supported by KAIST Undergraduate Research Participation (URP) program. Oum was supported by the Institute for Basic Science (IBS-R029-C1).

vertex-minors. We need more examples to develop the theory of graph structure with respect to vertex-minors.

We aim to introduce one such graph parameter, called the average cut-rank. The *average cut-rank* of a graph G, denoted by $\mathbb{E}\rho(G)$, is the expectation of $\rho_G(X)$ for a uniformly chosen random subset X of $V(G)$. We will show that if H is a vertex-minor of G, then $\mathbb{E}\rho(H) \leq \mathbb{E}\rho(G)$.

Initially rank-width was introduced to study clique-width of a graph, introduced by Courcelle and Olariu [3]. Oum and Seymour [14] showed that

$$\mathrm{rw}(G) \leq \mathrm{cw}(G) \leq 2^{\mathrm{rw}(G)+1} - 1,$$

where $\mathrm{rw}(G)$, $\mathrm{cw}(G)$ denotes the rank-width, the clique-width of G, respectively. For each k, the class of graphs of clique-width at most k is closed under taking induced subgraphs but not under vertex-minors. Thus, rank-width is 'tied' to clique-width and yet it behaves better with vertex-minors than clique-width.

The average cut-rank also has such tied parameters. First let us describe a few graph parameters.

- The *neighborhood diversity* of a graph G, denoted by $\mathrm{nd}(G)$, is the number of equivalence classes on $V(G)$ where two vertices x, y are equivalent if and only if $x = y$ or x, y are twins in G. This was introduced by Lampis [10] in 2012 and an equivalent concept appeared earlier in Ding and Kotlov [5].
- The *maximum cut-rank* of a graph G is $\max \rho(G) := \max_{S\subseteq V(G)} \rho_G(S)$.
- For a field $\mathbb{F}$, the *minimum rank* of an n-vertex graph G, denoted by $\mathrm{mr}(\mathbb{F}, G)$ is the minimum rank of an $n \times n$ symmetric matrix $A = (a_{ij})$ over $\mathbb{F}$ such that for all $i \neq j$, $a_{ij} \neq 0$ if and only if the i-th vertex is adjacent to the j-th vertex. Note that any element of $\mathbb{F}$ is allowed in the diagonal entry of A. For more about the minimum rank of a graph, readers are referred to a survey by Fallat and Hogben [6]. We write $\mathbb{F}_q$ to denote the finite field with q elements.
- The *clique delta-cover number* of a graph G, denoted by $\mathrm{cd}(G)$, is the minimum integer t such that there exist t complete graphs $G_1, G_2, \ldots, G_t$ with the property that $E(G) = E(G_1)\triangle E(G_2)\triangle \cdots \triangle E(G_t)$, where $\triangle$ denotes the symmetric difference operation.

We prove that all these parameters are tied to each other, when $\mathbb{F}$ is a finite field as follows.

Theorem 1.1. *Let G be a graph with at least one edge. Then*
(i) $\mathbb{E}\rho(G) < \max \rho(G) \leq \mathrm{mr}(\mathbb{F}_2, G) \leq \mathrm{nd}(G) < 2^{2\max \rho(G)+2} \leq 2^{8\mathbb{E}\rho(G)+2}$,
(ii) $\mathbb{E}\rho(G) < \mathrm{cd}(G) \leq \frac{3}{2}\,\mathrm{mr}(\mathbb{F}_2, G) \leq \frac{3}{2}\,\mathrm{nd}(G) \leq \frac{3}{2}2^{\mathrm{cd}(G)}$, *and*
(iii) $\mathrm{nd}(G) \leq |\mathbb{F}|^{\mathrm{mr}(\mathbb{F},G)} \leq |\mathbb{F}|^{\mathrm{nd}(G)}$ *for every finite field* $\mathbb{F}$.

Ding and Kotlov [5, Lemma 2.3] showed that graphs of bounded neighborhood diversity are well-quasi-ordered under the induced subgraph relation. Independently, Ganian, Hliněný, Nešetřil, Obdržálek, and Ossona de Mendez [7] showed that a class of graphs has bounded neighborhood diversity if and only if it has *shrub-depth* 1 and proved that every class of graphs of bounded shrub-depth is well-quasi-ordered under the induced subgraph relation. Therefore we deduce the following corollary.

Corollary 1.2. *Every class of graphs of bounded average cut-rank is well-quasi-ordered under the induced subgraph relation.*

Note that for a vertex v of G, $\mathbb{E}\rho(G-v) \leq \mathbb{E}\rho(G) \leq \mathbb{E}\rho(G-v)+1$. Together with this easy fact, Corollary 1.2 implies that for each real α, there are only finitely many induced-subgraph-minimal graphs of average cut-rank at least α up to isomorphism, because those graphs have average cut-rank at most $\alpha + 1$.

We not only prove that there are finitely many of those graphs, but also provide an explicit upper bound on the number of vertices in each of them. Let us write log to denote $\log_2$, the binary logarithm. For every real x, let $\lfloor x \rfloor$ be the greatest integer not exceeding x and $\{x\} := x - \lfloor x \rfloor$ be the fractional part of x. For $\varepsilon \in [0,1)$, we define a sequence $\{x_n(\varepsilon)\}_{n\geq 0}$ by

$$x_0(\varepsilon) = \max(\lfloor 2 - \log(1-\varepsilon)\rfloor, 5),$$
$$x_n(\varepsilon) = 2^{8n+10}\lfloor x_{n-1}(\varepsilon) - \log(1 - \{2^{x_{n-1}(\varepsilon)}\varepsilon/2\}) + 1\rfloor.$$

It is not hard to see that $x_n(\varepsilon) \geq 2^{\Omega(n^2)}$ where the constant factor in the exponential term depends on ε. Now we are ready to present our second theorem.

Theorem 1.3. *Let $\alpha \geq 0$ and G be a graph with no isolated vertices. If $\mathbb{E}\rho(G) \geq \alpha$ and $\mathbb{E}\rho(G-v) \leq \alpha$ for all vertices v of G, then $|G| < x_{\lfloor\alpha\rfloor}(\{\alpha\})$.*

Theorem 1.3 implies that induced-subgraph-minimal graphs of average cut-rank at least α have bounded number of vertices for each α. Our third theorem shows that the number of such graphs cannot be too small. Indeed we prove a stronger statement in terms of vertex-minors. Our third theorem says that if we have a set $\mathcal{S}$ of graphs characterizing average cut-rank at most α in terms of forbidding graphs in $\mathcal{S}$ as a vertex-minor, then $|\mathcal{S}|$ cannot be too small. We remark that $\mathcal{S}$ does not need to contain all vertex-minor-minimal graphs having average cut-rank more than α, because if two graphs are locally equivalent (which we define in Section 4), then $\mathcal{S}$ does not need to have both of them.

Theorem 1.4. *There is some universal constant $c > 0$ so that the following holds. For every $\varepsilon \in [0,1)$ and $n \geq 0$, let $\mathcal{S}$ be a set of graphs such that the average cut-rank of a graph G is at most (or less than) $\varepsilon + n$ if and only if no graph in $\mathcal{S}$ is isomorphic to a vertex-minor of G. Then $\mathcal{S}$ contains at least $2^{cn\log(n+1)}$ graphs.*

Our final theorem characterizes graphs of average cut-rank at most $3/2$ completely and determines all possible reals up to $3/2$ that can be realized as the average cut-rank of some graph. For two graphs G and H, let $G + H$ be the *disjoint union* of G and H, and for an integer m, let mG be the disjoint union of m copies of G. For every $k \geq 0$, let E_k be $K_{1,k+1}$ with one edge subdivided.

Theorem 1.5. *Let G be a graph with no isolated vertices. Then G has average cut-rank at most $3/2$ if and only if it is isomorphic to a vertex-minor of one of P_5, $3K_2$, $2P_3$, $K_{1,k+1}$, $K_2 + K_{1,k+1}$, and E_k for $k \geq 0$. In addition, the set of all possible values for average cut-rank of graphs in the interval $[0, 3/2]$ is*

$$\left\{1 - \frac{1}{2^k} : k \geq 0\right\} \cup \left\{\frac{3}{2} - \frac{1}{2^{k+1}} : k \geq 0\right\} \cup \left\{\frac{3}{2} - \frac{3}{2^{k+2}} : k \geq 0\right\} \cup \left\{\frac{3}{2}\right\}.$$

This paper is organized as follows. In Section 2 we recall basic definitions and results. In Section 3 we discuss an equivalence relation involving cut-rank functions. We introduce and prove basic tools on the average cut-rank in Section 4. Sections 5, 6, 7, and 8 present the proofs of Theorems 1.1, 1.3, 1.4, and 1.5 respectively.

2. Preliminaries

2.1. **Basic notions on graphs.** For all positive integers k, let P_k be the path on k vertices, C_k be the cycle on k vertices, K_k be the complete graph on k vertices, and $K_{m,k}$ be the complete bipartite graph on m vertices one side and k vertices the other side. For the star $K_{1,k}$, we call the vertex at the singleton side the *central vertex*. (If $k = 1$ then we fix one vertex to be called central.)

For a graph G, denote $V(G), E(G), A(G)$, respectively, for its vertex set, edge set, and adjacency matrix. For disjoint sets $S, T \subseteq V(G)$, let $N_G(S,T)$ be the set of vertices in T adjacent to at least one member in S. For $v \in V(G)$, let $N_G(v,S) := N_G(\{v\}, S)$ and let $N_G(v) := N_G(v, V(G))$ be the set of all *neighbors* of v in G. Let $d_G(v) := |N_G(v)|$ be the *degree* of v in G. A vertex is *isolated* if it has degree zero, and a *leaf* if it has degree one.

Let $G[S]$ be the subgraph of G induced on the vertex set S; in this case we say $G[S]$ is an *induced subgraph* of G, and set $G - S := G[V(G) \setminus S]$ as well as $G - v := G - \{v\}$. For any two disjoint subsets X, Y of G, denote by $G[X, Y]$ the induced bipartite subgraph of G with bipartition (X, Y) consisting of edges having one end in X and the other in Y. For simplicity, set $|G| := |V(G)|$, and we sometimes write A_G instead of $A(G)$.

Let the *complement* of G, denoted by $\overline{G}$, be the graph with vertex set $V(G)$ and edge set $\{uv : u \neq v, uv \notin E(G)\}$.

Two distinct vertices x, y of G are called *twins* if $N_G(x) \setminus \{x, y\} = N_G(y) \setminus \{x, y\}$. If, in addition, they are adjacent then we call them *true twins*, otherwise we call them *false twins*.

In G, a subset $S \subseteq V(G)$ is a *clique* if every two vertices in S are adjacent, and an *independent set* every two vertices in S are nonadjacent.

For two disjoint subsets $A, B \subseteq V(G)$, A is *complete* to B if every vertex in A is adjacent to all vertices of B and A is *anticomplete* to B if every vertex in A is nonadjacent to all vertices of B.

For two sets A and B, let $A \triangle B := (A \setminus B) \cup (B \setminus A)$. For two graphs G_1 and G_2, let the *symmetric difference* of G_1 and G_2, denoted by $G_1 \triangle G_2$, be the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \triangle E(G_2)$. When $E_1 \cap E_2 = \emptyset$ and $G = G_1 \triangle G_2$ we say G admits a *decomposition* into G_1 and G_2.

For a subset S of $V(G)$, *identifying* S is the operation of replacing all vertices in S by a new vertex and joining it to every vertex in $N_G(S, V(G) \setminus S)$. For an equivalence relation $\equiv$ on $V(G)$, the *quotient graph* of G induced by $\equiv$ is the graph obtained from G by identifying each equivalence class C of $(V(G), \equiv)$ to a vertex denoted by C.

For two graphs G_1 and G_2, we say G_1 is *isomorphic* to G_2, if there is a bijection $\varphi : V(G_1) \to V(G_2)$ satisfying for $u, v \in V(G)$, $\varphi(u)\varphi(v)$ is an edge of G_2 if and only if uv is an edge of G_1.

2.2. **Local complementations and vertex-minors.** For a graph G and its vertex v, let $G * v$ be the graph obtained from G by switching all adjacencies between neighbors of v. To be precise, two vertices x and y are adjacent in $G * v$ if and only if in G, either

(1) they are adjacent and at least one of them is non-adjacent to v, or
(2) they are nonadjacent but both are adjacent to v.

Indeed, $V(G * v) = V(G)$ and $G * v * v$ is G itself for every $v \in V(G)$. We call such an operation the *local complementation* at v. We say two graphs are *locally equivalent* if one can be obtained from the other by a series of local complementations.

We say that a graph H is a *vertex-minor* of G if it can be obtained from G by a series of local complementations and vertex deletions. A simple observation points out that given such a series, we may rearrange the operations so that all the local complementations are executed before the vertex deletions without changing the output graph. Thus, if H is a vertex-minor of G, then H is actually an induced subgraph of a graph locally equivalent to G.

For an edge uv of G, the *pivot* of G on uv is an operation to obtain a graph, denoted by $G \wedge uv$, from G by three local complementations, $G * u * v * u$. This is well defined because $G * u * v * u = G * v * u * v$ whenever u, v are adjacent, see [12, Proposition 2.1].

2.3. **Cut-rank.** For a matrix $M := (m_{ij} : i \in R, j \in C)$, let $\mathrm{rank}(M)$ be its rank. If $X \subseteq R$ and $Y \subseteq C$, denote by $M[X, Y]$ the submatrix of M obtained by taking the rows indexed by X and the columns indexed by Y so that $M[X, Y] = (m_{ij} : i \in X, j \in Y)$.

For a graph G and two disjoint subsets X and Y, let us write $\rho^*_G(X, Y) = \mathrm{rank}(A_G[X, Y])$ where A_G is considered as a matrix over the binary field. The *cut-rank function* of a graph G is a function $\rho_G : 2^{V(G)} \to \mathbb{Z}$ such that $\rho_G(S) := \rho^*_G(S, V(G) \setminus S)$. This implies immediately that ρ_G is symmetric, that is, $\rho_G(S) = \rho_G(V(G) \setminus S)$ for all $S \subseteq V(G)$.

In this paper we need the following property of cut-rank functions, which shows that local complementations preserve the cut-rank function of a graph G.

Proposition 2.1 (Oum [12, Proposition 2.6])**.** *For a graph G and $v \in V(G)$, we have $\rho_G(S) = \rho_{G*v}(S)$ for all $S \subseteq V(G)$.*

2.4. **Well-quasi-ordering and forbidden lists.** Given a set $\mathcal{X}$ and a relation $\leq$ on $\mathcal{X}$, $(\mathcal{X}, \leq)$ is a *quasi-order* if

(i) for every $x \in \mathcal{X}$ we have $x \leq x$;
(ii) for any $x, y, z \in \mathcal{X}$, if $x \leq y$ and $y \leq z$ then $x \leq y \leq z$.

We say two elements x, y of $\mathcal{X}$ are *comparable* if $x \leq y$ or $y \leq x$.

We say $\leq$ is a *well-quasi-ordering* on $\mathcal{X}$, or $\mathcal{X}$ is *well-quasi-ordered* under $\leq$, or $(\mathcal{X}, \leq)$ is a *well-quasi-order*, if for every infinite sequence $\{x_n\}_{n\geq 0}$ of elements of $\mathcal{X}$, there are indices $i < j$ satisfying $x_i \leq x_j$.

An *antichain* is a subset of $\mathcal{X}$ having no two distinct comparable elements. A subclass S of $\mathcal{X}$ is *closed* by $\leq$ if $y \in S$ and $x \leq y$ imply $x \in S$. An antichain $\mathcal{C}$ is called a *forbidden list for S by* $\leq$ if for all $x \in \mathcal{X}$, x belongs to S if and only if there is no $y \in \mathcal{C}$ satisfying $y \leq x$. When $\mathcal{X}$ is a class of graphs, $\mathcal{X}$ is *hereditary* if $\mathcal{X}$ is closed under induced subgraphs; that is, if $G \in \mathcal{X}$ and H is isomorphic to an induced subgraph of G then $H \in \mathcal{X}$.

3. An equivalence relation involving cut-rank functions

An *attached star* in a graph G is an induced subgraph isomorphic to a star whose noncentral vertices are leaves in G. In other words, an attached star in G is an induced subgraph, say $G[S]$, isomorphic to a star such that the set of noncentral vertices is anticomplete to $V(G) \setminus S$. The *size* of an attached star is the number of its vertices.

In G, let $\equiv_G$ be a binary relation on $V(G)$ such that for $x, y \in V(G)$, $x \equiv_G y$ if $\rho_G(\{x\}) = \rho_G(\{y\}) \geq \rho_G(\{x, y\})$. It is easy to see that $x \equiv_G y$ if and only if one of the following holds:

(i) x and y are twins in G, or
(ii) one of them is a leaf in G whose unique neighbor is the other.

Furthermore, $\equiv_G$ is in fact an equivalence relation on $V(G)$, as shown by the following.

Proposition 3.1. *The relation $\equiv_G$ is an equivalence relation on $V(G)$. Moreover, each equivalence class of $(V(G), \equiv_G)$ is one of the following types in G: the vertex set of an attached star, a clique of true twins, and an independent set of false twins.*

Proof. By definition, it is obvious that for $x, y \in V(G)$, $x \equiv_G x$, and if $x \equiv_G y$, then $y \equiv_G x$. Thus $\equiv_G$ is reflexive and symmetric. To prove that $\equiv_G$ is an equivalence relation on $V(G)$, it remains to show that $x \equiv_G y$ and $y \equiv_G z$ imply $x \equiv_G z$. We may assume that x, y, z are distinct. We have three cases to consider.

(1) $d_G(y) = 0$. We have x, y, z are isolated in G and $x \equiv_G z$.
(2) $d_G(y) = 1$. If $N_G(y) \not\subseteq \{x, z\}$ then trivially x, y, z are leaves in G with a unique common neighbor, so $x \equiv_G z$. If $N_G(y) \subseteq \{x, z\}$, we may assume that $xy \in E(G)$ and $yz \notin E(G)$, then y, z are twins in G which implies that z is a leaf in G whose unique neighbor is x, and thus $x \equiv_G z$.
(3) $d_G(y) \geq 2$. Now, if x is a leaf in G, then y is the unique neighbor of x in G and so x is non-adjacent to z, which implies z is a leaf whose unique neighbor is y because $y \equiv_G z$ and therefore $x \equiv_G z$. By symmetry, if z is a leaf in G, then $x \equiv_G z$. If neither x nor z is a leaf in G, then $\{x, y\}$ and $\{y, z\}$ are two pairs of twins in G, thus x and z are also twins in G, and so $x \equiv_G z$.

Now let C be an equivalence class in $(V(G), \equiv_G)$. If there is a vertex x in C which is a leaf in G then its unique neighbor, say y, must also be in C; so every vertex in $C \setminus \{x, y\}$, being a twin of x, is also a leaf in G whose unique neighbor is y, which implies that $G[C]$ is an attached star in G. On the other hand, if C contains no leaves in G, then necessarily they are pairwise twins.

It is well known that a set of pairwise twins is either a clique of true twins or an independent set of false twins in G. So, we conclude that C is the vertex set of an attached star, a clique of true twins, or an independent set of false twins. $\square$

In addition, an immediate consequence of Proposition 2.1 is that local complementations preserve every equivalence class in $(V(G), \equiv_G)$.

Corollary 3.2. *For every $x, y, v \in V(G)$, $x \equiv_G y$ if and only if $x \equiv_{G*v} y$. In other words, the equivalence classes of $(V(G), \equiv_G)$ remain unchanged in $(V(G), \equiv_{G*v})$.*

4. Average cut-rank

The *average cut-rank* of G is defined as

$$\mathbb{E}\rho(G) := \frac{1}{2^{|G|}} \sum_{S \subseteq V(G)} \rho_G(S).$$

In other words, $\mathbb{E}\rho(G)$ is the expected value of $\rho_G(S)$ where S is chosen uniformly at random among all subsets of $V(G)$. Note that due to the symmetry of ρ_G, $\mathbb{E}\rho(G)$ is a rational number whose denominator in closed form is a positive integer dividing $2^{|G|-1}$.

One of the reasons to study the average cut-rank is that it does not increase when taking vertex-minors. The following theorem not only shows this but also shows that the average cut-rank strictly decreases whenever we take a vertex-minor except for some trivial cases.

Theorem 4.1. *If H is a vertex-minor of a graph G, then*

$$\mathbb{E}\rho(H) \leq \mathbb{E}\rho(G).$$

In addition, if $V(G) \setminus V(H)$ has at least one non-isolated vertex, then

$$\mathbb{E}\rho(H) \leq \mathbb{E}\rho(G) - 2^{-|H|}.$$

Proof. Since H is a vertex-minor of G, H is an induced subgraph of some graph G' which is locally equivalent to G. By Proposition 2.1, $\mathbb{E}\rho(G) = \mathbb{E}\rho(G')$. Because an isolated vertex remains isolated after each local complementation, we may assume that H is an induced subgraph of G. Let S be a subset of $V(G)$ chosen uniformly at random. So $S \cap V(H)$ is a random subset of $V(H)$. We have

$$\rho_G(S) = \operatorname{rank}(A_G[S, V(G) \setminus S]) \geq \operatorname{rank}(A_H[S \cap V(H), V(H) \setminus S) = \rho_H(S \cap V(H)),$$

so $\mathbb{E}\rho(G) = \mathbb{E}_S \rho_G(S) \geq \mathbb{E}_T \rho_H(T) = \mathbb{E}\rho(H)$.

Suppose that $V(G) \setminus V(H)$ has at least one non-isolated vertex. If there is some vertex in $V(G) \setminus V(H)$, say v, having at least one neighbor in $V(H)$, then any subset S of $V(G) \setminus V(H)$ containing v or any subset S of $V(G)$ containing $V(H)$ but not v satisfies $\rho_G(S) \geq 1$ while $\rho_H(S \cap V(H)) = 0$.

If no vertex in $V(G) \setminus V(H)$ has a neighbor in $V(H)$, then $G \setminus V(H)$ has at least one edge, say uv for $u, v \in V(G) \setminus V(H)$. Then any subset S of $V(G)$ such that S contains only one of u, v and $S \cap V(H)$ is $\emptyset$ or $V(H)$ satisfies $\rho_G(S) \geq 1$ and $\rho_G(S \setminus U) = 0$.

In both cases, we have

$$\mathbb{E}\rho(G) \geq \mathbb{E}\rho(H) + \frac{2 \cdot 2^{|G|-|H|-1}}{2^{|G|}} = \mathbb{E}\rho(H) + 2^{-|H|}. \qquad \square$$

As an example, we compute the average cut-rank of complete graphs and complete bipartite graphs. We omit its easy proof.

Lemma 4.2. *For integers $m, k \geq 1$,*

$$\mathbb{E}\rho(K_k) = 1 - 2^{1-k}, \quad \mathbb{E}\rho(K_{m,k}) = \frac{(2^m - 1)(2^k - 1)}{2^{m+k-1}}.$$

In particular $\mathbb{E}\rho(K_{1,k}) = 1 - 2^{-k} = \mathbb{E}\rho(K_{k+1})$.

The following result shows that $1 - 2^{1-|G|}$ is in fact the smallest possible average cut-rank of any graph G with no isolated vertices. The equality holds if G is a complete graph or a star, by Lemma 4.2.

Proposition 4.3. *A graph G without isolated vertices has average cut-rank at least $1-2^{1-|G|}$. The equality holds if and only if G is a star or a complete graph.*

Proof. If G is connected, then for every nonempty proper subset S of $V(G)$, $G[S, V(G) \setminus S]$ has at least one edge, hence $\rho_G(S) \geq 1$. Because there are $2^{|G|} - 2$ subsets S of this type, we obtain

$$\mathbb{E}\rho(G) = \frac{1}{2^{|G|}} \sum_{S \subseteq V(G)} \rho_G(S) \geq \frac{2^{|G|}-2}{2^{|G|}} = 1 - 2^{1-|G|}.$$

If G is disconnected, then since G has no isolated vertices, G contains an induced subgraph isomorphic to $2K_2$. It follows that, by Theorem 4.1,

$$\mathbb{E}\rho(G) \geq \mathbb{E}\rho(2K_2) = 1 > 1 - 2^{1-|G|}.$$

Now we consider the equality case. The preceding argument shows that if $\mathbb{E}\rho(G) = 1 - 2^{1-|G|}$, then G is necessarily connected and $\rho_G(S) = 1$ for all nonempty proper subsets S of $V(G)$. In particular, it follows that for all $x, y \in V(G)$, we have $\rho_G(\{x\}) = \rho_G(\{y\}) = \rho_G(\{x, y\}) = 1$, or equivalently $x \equiv_G y$. Therefore, $(V(G), \equiv_G)$ has only one equivalence class, so Proposition 3.1 implies that G is a star, a complete graph, or an edgeless graph. Because G is connected, G is thus a star or a complete graph. Lemma 4.2 then completes the proof. $\square$

Theorem 4.1 provides a lower bound on $\mathbb{E}\rho(G) - \mathbb{E}\rho(H)$ when H is a vertex-minor of G. The next proposition gives an upper bound on this difference.

Proposition 4.4. *Let G_1 and G_2 be graphs and $G = G_1 \triangle G_2$. Then*

$$\mathbb{E}\rho(G) \leq \mathbb{E}\rho(G_1) + \mathbb{E}\rho(G_2),$$

and the equality holds if (but not necessarily only if) $V(G_1) \cap V(G_2) = \emptyset$, i.e. G is the disjoint union of G_1 and G_2. In particular, for every vertex $v \in V(G)$,

$$\mathbb{E}\rho(G - v) \geq \mathbb{E}\rho(G) - 1 + 2^{-d_G(v)} \geq \mathbb{E}\rho(G) - 1 + 2^{1-|G|}.$$

Proof. For $i = 1, 2$, let H_i be the graph with vertex set $V(G)$ and edge set $E(G_i)$. Then $G = H_1 \triangle H_2$ and by Theorem 4.1, $\mathbb{E}\rho(H_i) = \mathbb{E}\rho(G_i)$ for $i = 1, 2$. Choose a subset S of $V(G)$ uniformly at random and set $T := V(G) \setminus S$. Then since $G = H_1 \triangle H_2$, we have

$$\begin{aligned}\rho_G(S) = \operatorname{rank}(A_G[S,T]) &= \operatorname{rank}(A_{H_1}[S,T] + A_{H_2}[S,T]) \\ &\leq \operatorname{rank}(A_{H_1}[S,T]) + \operatorname{rank}(A_{H_2}[S,T]) = \rho_{H_1}(S) + \rho_{H_2}(S).\end{aligned}$$

This implies immediately that

$$\mathbb{E}\rho(G) \leq \mathbb{E}\rho(H_1) + \mathbb{E}\rho(H_2) = \mathbb{E}\rho(G_1) + \mathbb{E}\rho(G_2).$$

If $V(G_1) \cap V(G_2) = \emptyset$, for $i = 1, 2$ let S_i be a random subset of $V(G_i)$ and let $T_i := V(G_i) \setminus S_i$. Then $S_1 \cup S_2$ is a random subset of $V(G)$ and $T_1 \cup T_2 = V(G) \setminus (S_1 \cup S_2)$. It is easy to see that

$$\rho_{G_1}(S_1) + \rho_{G_2}(S_2) = \operatorname{rank}(A_{G_1}[S_1, T_1]) + \operatorname{rank}(A_{G_2}[S_2, T_2]) = \operatorname{rank}(A_G[S_1 \cup S_2, T_1 \cup T_2]).$$

As a result, we deduce $\mathbb{E}\rho(G) = \mathbb{E}\rho(G_1) + \mathbb{E}\rho(G_2)$.

Now for any vertex $v \in V(G)$, let $G_1 := G - v$ and $G_2 := G[\{v\}, N_G(v)]$, which is isomorphic to $K_{1,d_G(v)}$. By Lemma 4.2 we obtain

$$\mathbb{E}\rho(G) \leq \mathbb{E}\rho(G - v) + \mathbb{E}\rho(K_{1,d_G(v)}) = \mathbb{E}\rho(G - v) + 1 - 2^{-d_G(v)} \leq \mathbb{E}\rho(G - v) + 1 - 2^{1-|G|}. \quad \square$$

It can be seen that the lower bound from Theorem 4.1 and the upper bound from Proposition 4.4 are pretty far apart, because they apply for all graphs in general. In many cases, we need the upper and lower bounds on $\mathbb{E}\rho(G) - \mathbb{E}\rho(H)$ to be close enough, when H is a vertex-minor of G. The next propositions provide a tighter upper bound compared to Proposition 4.4 and a tighter lower bound compared to Theorem 4.1, when we have distinctive structures involving false twins, in particular the attached stars.

Proposition 4.5. *Let G be a graph in which $u_1, \ldots, u_k$ are pairwise false twins, where $k \geq 1$. Let $d := d_G(u_1)$. Then*

$$\mathbb{E}\rho(G - u_1) \geq \mathbb{E}\rho(G) - \frac{2^d - 1}{2^{k+d-1}}.$$

In particular if $d_G(u_1) = 1$ then

$$\mathbb{E}\rho(G - u_1) \geq \mathbb{E}\rho(G) - 2^{-k}.$$

Proof. Let $n := |G|$, $H := G - u_1$, $V := V(G)$, and

$$\mathcal{F} := \{S \subseteq V : \{u_1\} \subsetneq S \cap \{u_1, \ldots, u_k\} \text{ or } \{u_1\} \cup N_G(u_1) \subseteq S\}.$$

Observe that for $S \in \mathcal{F}$, we have $\rho_G(S) = \rho_H(S \setminus \{u_1\})$, because if $\{u_1\} \subsetneq S \cap \{u_1, \ldots, u_k\}$ then there is some $j \in \{2, \ldots, k\}$ such that $u_j \in S$ and so the row vectors corresponding to u_1 and u_j in $A_G[S, V \setminus S]$ are the same, and if $u_1 \cup N_G(u_1) \subseteq S$ then the row vector corresponding to u_1 in $A_G[S, V \setminus S]$ is zero. Obviously, $\rho_G(S) \leq \rho_H(S \setminus \{u_1\}) + 1$ for all $S \subseteq V$. Therefore, because ρ_G is symmetric and there are exactly $2^{n-d-k}(2^d - 1)$ subsets S of V such that $u_1 \in S \notin \mathcal{F}$, we have

$$\begin{aligned}
\mathbb{E}\rho(G) &= \mathbb{E}(\rho_G(S) \mid u_1 \in S \subseteq V) \\
&= \mathbb{E}(\rho_G(S) \mid u_1 \in S \in \mathcal{F}) \cdot \mathbb{P}(S \in \mathcal{F} \mid u_1 \in S) + \mathbb{E}(\rho_G(S) \mid u_1 \in S \notin \mathcal{F}) \cdot \mathbb{P}(S \notin \mathcal{F} \mid u_1 \in S) \\
&\leq \mathbb{E}(\rho_H(S \setminus \{u_1\}) \mid u_1 \in S \in \mathcal{F}) \cdot \mathbb{P}(S \in \mathcal{F} \mid u_1 \in S) \\
&\quad + \mathbb{E}(\rho_H(S \setminus \{u_1\}) + 1 \mid u_1 \in S \notin \mathcal{F}) \cdot \mathbb{P}(S \notin \mathcal{F} \mid u_1 \in S) \\
&\leq \mathbb{E}(\rho_H(S \setminus \{u_1\}) \mid u_1 \in S \in \mathcal{F}) \cdot \mathbb{P}(S \in \mathcal{F} \mid u_1 \in S) \\
&\quad + \mathbb{E}(\rho_H(S \setminus \{u_1\}) \mid u_1 \in S \notin \mathcal{F}) \cdot \mathbb{P}(S \notin \mathcal{F} \mid u_1 \in S) + \mathbb{P}(S \notin \mathcal{F} \mid u_1 \in S) \\
&= \mathbb{E}(\rho_H(S \setminus \{u_1\}) \mid u_1 \in S \subseteq V) + \mathbb{P}(S \notin \mathcal{F} \mid u_1 \in S) \\
&= \mathbb{E}(\rho_H(T) \mid T \subseteq V(H)) + \frac{2^{n-d-k}(2^d - 1)}{2^{n-1}} \\
&= \mathbb{E}\rho(H) + \frac{2^d - 1}{2^{k+d-1}},
\end{aligned}$$

which completes the proof of the proposition. ☐

Proposition 4.6. *Let G be a graph on $n \geq 1$ vertices, T be the vertex set of an attached star in G, and $H := G - T$. Then*

$$\mathbb{E}\rho(G) - 1 < \mathbb{E}\rho(H) \leq \mathbb{E}\rho(G) - 1 + 2^{1-|T|}.$$

Proof. Let $V := V(G)$ and let $T = \{u_1, \ldots, u_k, v\}$ where v is the central vertex and $u_1, \ldots, u_k$ are leaves of $G[T]$. The left hand side inequality is trivial by Proposition 4.4 and Lemma 4.2, because

$$\mathbb{E}\rho(G) \leq \mathbb{E}\rho(H) + \mathbb{E}\rho(F) = \mathbb{E}\rho(H) + 1 - 2^{-d_G(v)} < \mathbb{E}\rho(H) + 1,$$

where F is the connected subgraph of G consisting of all edges incident with v.

We move on to the right hand side inequality. Observe that for every $S \subseteq V$, we have $\rho_G(S) \geq \rho_H(S \setminus T)$. If furthermore $v \in S$ and $T \not\subseteq S$, then in $A_G[S, V \setminus S]$, the row vectors corresponding to $T \cap S \setminus \{v\}$ are all zero vectors, and for every $u_j \in T \setminus S$, the column vector corresponding to u_j has only one 1 as its common entry with the row vector corresponding to v. It follows that in $A_G[S, V \setminus S]$, this row vector is linearly independent to the other row vectors, and in $A_G[S \setminus \{v\}, V \setminus S]$, the column vectors corresponding to $T \setminus S$ are all zero vectors. Hence $\rho_G(S) = \rho_H(S \setminus T) + 1$. Therefore, due to the symmetry of ρ_G,

$$\begin{aligned}
\mathbb{E}\rho(G) &= \mathbb{E}(\rho_G(S) \mid v \in S \subseteq V) \\
&= \mathbb{E}(\rho_G(S) \mid v \in T \subseteq S) \cdot \mathbb{P}(T \subseteq S \mid v \in S) + \mathbb{E}(\rho_G(S) \mid v \in S, T \not\subseteq S) \cdot \mathbb{P}(T \not\subseteq S \mid v \in S) \\
&\geq \mathbb{E}(\rho_H(S \setminus T) \mid v \in T \subseteq S) \cdot \mathbb{P}(T \subseteq S \mid v \in S) \\
&\quad + \mathbb{E}(\rho_H(S \setminus T) + 1 \mid v \in S, T \not\subseteq S) \cdot \mathbb{P}(T \not\subseteq S \mid v \in S) \\
&\geq \mathbb{E}(\rho_H(S \setminus T) \mid v \in T \subseteq S) \cdot \mathbb{P}(T \subseteq S \mid v \in S) \\
&\quad + \mathbb{E}(\rho_H(S \setminus \{v\}) \mid v \in S, T \not\subseteq S) \cdot \mathbb{P}(T \not\subseteq S \mid v \in S) + \mathbb{P}(T \not\subseteq S \mid v \in S) \\
&= \mathbb{E}(\rho_H(S \setminus T) \mid v \in S \subseteq V) + \mathbb{P}(T \not\subseteq S \mid v \in S) \\
&= \mathbb{E}(\rho_H(U) \mid U \subseteq V(H)) + \frac{2^{n-k-1}(2^k - 1)}{2^{n-1}} \\
&= \mathbb{E}\rho(H) + 1 - 2^{1-|T|}.
\end{aligned}$$

This completes the proof. □

5. Characterization of classes of graphs of bounded average cut-rank

In this section, we will prove Theorem 1.1, which characterizes classes of graphs of bounded average cut-rank and relates them to existing concepts. We will also discuss some corollaries on well-quasi-ordering.

We start with some definitions solely used in this section. In a graph G, two vertices x, y are called *twin-equivalent* if either $x = y$ or they are twins. It is easy to verify that the relation "twin-equivalent" is an equivalence relation on $V(G)$. Thus the vertex set of G can be partitioned into *twin classes*. The *neighborhood diversity* of G, first defined by Lampis [10], is the number of twin classes in G.

Here is a fundamental property on the rank and the number of distinct rows of a 0-1 matrix.

Lemma 5.1. *Any* 0-1 *matrix* M *has at most* $2^{\mathrm{rank}(M)}$ *distinct rows.*

Proof. Let $r = \mathrm{rank}(M)$. Then M has a non-singular $r \times r$ submatrix, whose columns are indexed by I. Note that $|I| = r$ and each row vector is completely determined by the 0-1 values on the entries in I and therefore M has at most $2^{|I|}$ distinct rows. □

The authors would like to thank Alex Scott (personal communication) for suggesting the proof of the following lemma.

Lemma 5.2. *For every graph* G, $\max \rho(G) \leq 4\mathbb{E}\rho(G)$.

Proof. Let k be the maximum cut-rank of G. Then there are two disjoint subsets A, B of $V(G)$ satisfying $|A| = |B| = k$ and $\rho^*_G(A, B) = k$. Let $H := G[A \cup B]$. Since $\mathbb{E}\rho(G) \geq \mathbb{E}\rho(H)$, it suffices to show that $\mathbb{E}\rho(H) \geq k/4$.

Let S be a subset of $V(H) = A \cup B$ chosen uniformly at random. Then $S \cap A$ is a random subset of A, and $B \setminus S$ is a random subset of B. Since

$$\rho_H(S) = \rho^*_H(S, (A \cup B) \setminus S) \geq \rho^*_H(S \cap A, B \setminus S),$$

we have

$$\mathbb{E}\rho(H) = \mathbb{E}_S \rho_G(S) \geq \mathbb{E}_S \rho^*_H(S \cap A, B \setminus S) = \mathbb{E}_{X,Y} \rho^*_H(X, Y),$$

where the last expression indicates the expected value of $\rho^*_H(X, Y)$ with X, Y selected from 2^A, 2^B, respectively.

Fix $X \subseteq A$ and let $Y \subseteq B$ be random. Because $\rho^*_H(A,B) = \rho^*_G(A,B) = k$ and $|A| = |B| = k$, $\rho^*_H(X,B) = |X|$, so there is a subset Z of B satisfying $|Z| = |X|$ and $A_H[X,Z]$ has full rank. Because $Y \subseteq B$ is random, $Y \cap Z$ is a random subset of Z, which implies that

$$\mathbb{E}_Y \rho^*_H(X,Y) \geq \mathbb{E}_Y \rho^*_H(X, Y\cap Z) = \mathbb{E}_Y |Y \cap Z| = \frac{|X|}{2}.$$

Therefore

$$\mathbb{E}_{X,Y} \rho^*_G(X,Y) = \mathbb{E}_X \mathbb{E}_Y \rho^*_G(X,Y) \geq \mathbb{E}_X \frac{|X|}{2} = \frac{k}{4}.$$

Thus $\mathbb{E}\rho(H) \geq k/4$ and the conclusion follows. $\square$

The following lemma shows that a hereditary class of graphs is of bounded maximum cut-rank (or average cut-rank) if and only if it is of bounded neighborhood diversity. This result is also essential to the proof of Theorem 1.3.

Lemma 5.3. *For every graph G,* $\mathrm{nd}(G) < 2^{2\max\rho(G)+2}$.

Proof (Adapted from the proof of Lemma 4.5 in [4]*).* Let A be a maximal subset of $V(G)$ without any pair of twins in G. We construct a complete graph H on the vertex set A and label every edge uv of H as follows: uv is labeled by w for some $w \in V(G) \setminus \{u,v\}$ adjacent to only one among u and v in G. This labeling exists because of the definition of A. Let $m := |A|$, and let S be a random subset of $A = V(H)$ where each vertex is included independently at random with probability $p := 1/\sqrt{m}$. For every edge $e \in E(H)$, let X_e be the indicator random variable for the event that the ends and the label of e in H are in S, and put $X := \sum_{e\in E(H)} X_e$. Then for all $e \in E(H)$, $\mathbb{E}[X_e] = p^3$ if the label of e is in A and $\mathbb{E}[X_e] = 0$ otherwise. By linearity of expectation

$$\mathbb{E}[|S| - X] = \mathbb{E}[|S|] - \mathbb{E}[X] \geq pm - p^3\binom{m}{2} > pm - \frac{p^3m^2}{2} = \frac{\sqrt{m}}{2}.$$

Thus, there is a subset S of A such that $|S| - X > \sqrt{m}/2$; that is, there are fewer than $|S| - \sqrt{m}/2$ edges in H having ends and labels in S. Then, by deleting one end for each such edge, we get a subset T of S satisfying $|T| > \sqrt{m}/2$ and for every distinct $u,v \in T$, in H the label of uv does not belong to T. This means that for every distinct $u, v \in T$, in G there is a vertex w outside T which is adjacent to only one of u and v, which implies that $A_G[T, V(G)\setminus T]$ has more than $\sqrt{m}/2$ distinct rows. Hence, by Lemma 5.1,

$$2^{\max\rho(G)} \geq 2^{\rho_G(T)} > \frac{\sqrt{m}}{2},$$

which implies $|A| = m < 2^{2\max\rho(G)+2}$. As every vertex in $V(G)\setminus A$ is a twin of some vertex in A (A is maximal), we conclude that $V(G)$ can be partitioned into less than $2^{2\max\rho(G)+2}$ twin classes. $\square$

Now we are ready to prove Theorem 1.1.

Theorem 1.1. *Let G be a graph with at least one edge. Then*

(i) $\mathbb{E}\rho(G) < \max\rho(G) \leq \mathrm{mr}(\mathbb{F}_2, G) \leq \mathrm{nd}(G) < 2^{2\max\rho(G)+2} \leq 2^{8\mathbb{E}\rho(G)+2}$,
(ii) $\mathbb{E}\rho(G) < \mathrm{cd}(G) \leq \frac{3}{2}\mathrm{mr}(\mathbb{F}_2, G) \leq \frac{3}{2}\mathrm{nd}(G) \leq \frac{3}{2}2^{\mathrm{cd}(G)}$, *and*
(iii) $\mathrm{nd}(G) \leq |\mathbb{F}|^{\mathrm{mr}(\mathbb{F},G)} \leq |\mathbb{F}|^{\mathrm{nd}(G)}$ *for every finite field* $\mathbb{F}$.

Proof. As G has at least one edge, $\mathbb{E}\rho(G) < \max\rho(G)$ trivially. Since $\rho_G(X) \leq \mathrm{mr}(\mathbb{F}_2, G)$ for all $X \subseteq V(G)$ trivially, $\max\rho(G) \leq \mathrm{mr}(\mathbb{F}_2, G)$.

To prove $\mathrm{mr}(\mathbb{F}, G) \leq \mathrm{nd}(G)$ for any field $\mathbb{F}$, let us assume that $k = \mathrm{nd}(G)$ and so G has exactly k twin classes. Starting from the adjacency matrix of G, we change the diagonal entry of a vertex v to 1 if v belongs to a twin class that is a clique of G. The resulting matrix has k distinct rows and so its rank is at most k. This proves that $\mathrm{mr}(\mathbb{F}, G) \leq k$.

Since every matrix of rank k over $\mathbb{F}$ has at most $|\mathbb{F}|^k$ distinct rows, we have $\mathrm{nd}(G) \le |\mathbb{F}|^{\mathrm{mr}(\mathbb{F},G)}$. This was shown by Ding and Kotlov [5, Corollary 2.2].

Lemmas 5.2 and 5.3 show that $\mathrm{nd}(G) < 2^{2\max\rho(G)+2} \le 2^{8\mathbb{E}\rho(G)+2}$.

Let $t = \mathrm{cd}(G)$. Then there are complete graphs $G_1, \dots, G_t$ such that $G = G_1 \triangle \cdots \triangle G_t$. As $E(G) \ne \emptyset$, $t \ge 1$. By Proposition 4.4 and Lemma 4.2, we see that $\mathbb{E}\rho(G) \le \sum_{i=1}^t \mathbb{E}\rho(G_i) < \mathrm{cd}(G)$. Also, $V(G)$ can be partitioned into 2^t subsets, each of them is a set of pairwise twins, based on the inclusion of $V(G_1)$, $V(G_2)$, …, $V(G_t)$. This leads to an inequality that $\mathrm{nd}(G) \le 2^{\mathrm{cd}(G)}$.

Now, it remains to prove that $\mathrm{cd}(G) \le \frac{3}{2}\,\mathrm{mr}(\mathbb{F}_2, G)$. Let A be a symmetric matrix over $\mathbb{F}_2$ of rank m realizing $\mathrm{mr}(\mathbb{F}_2, G)$. It is known that every symmetric matrix of rank m can be written as a sum of $m - 2s$ rank-1 symmetric matrices and s rank-2 symmetric matrices, see Godsil and Royle [9, Lemma 8.9.3]. As the field is binary, we can also deduce easily that in the outcome, the rank-2 symmetric matrices have zero diagonals, by using the proof of [9, Lemma 8.10.1]. Rank-1 symmetric matrices over $\mathbb{F}_2$ are of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

where 1 represents an all-1 matrix, 0 represents an all-0 matrix, and the diagonal entries represent square matrices. Thus, every rank-1 symmetric matrix over $\mathbb{F}_2$ is the adjacency matrix of one complete graph with some isolated vertices, while changing a few diagonal entries to 1. Rank-2 symmetric matrices over $\mathbb{F}_2$ with zero diagonals are of the form

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

and so every rank-2 symmetric matrix over $\mathbb{F}_2$ with zero diagonals can be written as the sum of three rank-1 symmetric matrices as follows.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Thus, A can be written as a sum of at most $3m/2$ rank-1 symmetric matrices over $\mathbb{F}_2$. This proves that $\mathrm{cd}(G) \le \frac{3}{2}\,\mathrm{mr}(\mathbb{F}_2, G)$. □

Corollary 1.2 yields the following corollary.

Corollary 5.4. *Let $\mathcal{C}$ be a hereditary class of graphs. If graphs in $\mathcal{C}$ have bounded average cut-rank, then there exists a finite list of graphs H_1, H_2, …, H_k such that a graph G is in $\mathcal{C}$ if and only if G has no induced subgraph isomorphic to H_i for every $i = 1, \dots, k$.*

Proof. Let α be a real such that every graph in $\mathcal{C}$ has average cut-rank at most α. If H is an induced-subgraph-minimal graph not in $\mathcal{C}$, then H has average cut-rank at most $\alpha + 1$ by Proposition 4.4. By Corollary 1.2, there are only finitely many induced-subgraph-minimal graphs not in $\mathcal{C}$, because they form an antichain. □

Let $S_{\mathbb{E}\rho}$ be the set of all reals α such that there exists a graph with average cut-rank α. By the definition of average cut-rank, this set is trivially a subset of $\{p/2^q : p \in \mathbb{N} \cup \{0\}, q \in \mathbb{N}\}$. By the previous corollary, we deduce the following topological property of $S_{\mathbb{E}\rho}$.

Proposition 5.5. *For any $\alpha \ge 0$ there is some $\delta_\alpha > 0$ such that every graph has average cut-rank outside $(\alpha, \alpha + \delta_\alpha)$. This implies that $S_{\mathbb{E}\rho}$ is not dense in any interval, hence is nowhere dense in $[0, \infty)$.*

Proof. By Corollary 5.4, there exists a finite list $\{G_1, \dots, G_m\}$ of forbidden induced subgraphs for the class of graphs of average cut-rank at most α. Because $\alpha < \mathbb{E}\rho(G_j) =: r_j$ for all $j = 1, \dots, m$, we have $\alpha < \min\{r_1, \dots, r_m\} =: q_\alpha$. Hence there is no graph having average cut-rank lying inside (α, q_α). The conclusion thus follows for $\delta_\alpha = q_\alpha - \alpha$. □

6. Upper bound on the size of induced subgraph obstructions

Ding and Kotlov [5] proved that each forbidden induced subgraph for the class of graphs of minimum rank over a finite field $\mathbb{F}$ at most k has at most $(|\mathbb{F}|^k/2+1)^2$ vertices.

We can find an upper bound on the size of each forbidden induced subgraph for the class of graphs of maximum cut-rank at most k as follows.

Theorem 6.1. *If* $\max \rho(G) > k$ *and* $\max \rho(G - v) \le k$ *for all vertices* v *of* G*, then* $|G| = 2k+2$*.*

Proof. If $\max \rho(G) > k$, then there exists a pair (X, Y) of disjoint sets of vertices such that $|X| = |Y| = k+1$ and the rank of $A(G)[X, Y] = k+1$. If $|G| > 2k+2$, then there is a vertex $v \notin X \cup Y$ and therefore $\max \rho(G - v) \ge \operatorname{rank} A(G)[X, Y] = k+1$, contradicting the assumption. Trivially, if $|G| < 2k+2$, then $\max \rho(G) \le k$. □

Now we will find such an upper bound for the class of graphs of average cut-rank at most α, thus proving Theorem 1.3. For convenience, we recall the sequence $\{x_n(\varepsilon)\}_{n \ge 0}$ defined in Section 1, as follows.

$$\begin{aligned} x_0(\varepsilon) &= \max(\lfloor 2 - \log(1-\varepsilon) \rfloor, 5), \\ x_n(\varepsilon) &= 2^{8n+10} \lfloor x_{n-1}(\varepsilon) - \log(1 - \{2^{x_{n-1}(\varepsilon)}\varepsilon/2\}) + 1 \rfloor \quad \text{for all integers } n \ge 1. \end{aligned}$$

Theorem 1.3. *Let* $\alpha \ge 0$ *and* G *be a graph with no isolated vertices. If* $\mathbb{E}\rho(G) \ge \alpha$ *and* $\mathbb{E}\rho(G - v) \le \alpha$ *for all vertices* v *of* G*, then* $|G| < x_{\lfloor \alpha \rfloor}(\{\alpha\})$*.*

Proof. Let $\alpha = \varepsilon + n$ where $\varepsilon = \{\alpha\} \in [0, 1)$ and $n = \lfloor \alpha \rfloor \in \mathbb{N} \cup \{0\}$. We fix ε and proceed by induction on $n \ge 0$. For convenience, set $x_n := x_n(\varepsilon)$ for all $n \ge 0$.

First let us assume that $n = 0$. If there is a vertex $v \in V(G)$ such that $G - v$ has no isolated vertices, then by Proposition 4.3, $\varepsilon \ge \mathbb{E}\rho(G - v) \ge 1 - 2^{1-(|G|-1)}$, which implies that $|G| \le \lfloor 2 - \log(1-\varepsilon) \rfloor \le x_0$. Thus we may assume that the deletion of every vertex of G yields a graph with some isolated vertex. It follows that $E(G)$ is a perfect matching and therefore $\varepsilon \ge \mathbb{E}\rho(G-v) = \mathbb{E}\rho(\frac{|G|-1}{2} K_2) = \frac{|G|-1}{4}$. This implies that $|G| \le 4\varepsilon + 1 < 5 \le x_0$.

Now we may assume that $n > 0$. Suppose for the sake of contradiction that $|G| \ge x_n$. Observe that by Theorem 1.1, for any vertex v, $\operatorname{nd}(G) \le 2 \operatorname{nd}(G - v) + 1 < 2 \cdot 2^{8(n+\varepsilon)+2} + 1 \le x_n \le |G|$ and therefore there is a vertex v having a twin. Then $\operatorname{nd}(G) = \operatorname{nd}(G - v) < 2^{8(n+\varepsilon)+2} < 2^{8n+10}$ and therefore G has a twin class C with $|C| > |G|/2^{8n+10}$.

Note that $|C| > x_{n-1} \ge x_0 \ge 5$. Let x, z be distinct vertices in C.

- If C is a clique of true twins in G, then $(G*x)[C]$ is an attached star in $G*x$. Let $G' := G*x$.
- If C is an independent set of false twins in G, then since the vertices in C are nonisolated in G, there is some $y \in N_G(C, V(G) \setminus C)$. Then C is a clique of true twins in $G * y$ and $(G * y * x)[C]$ is an attached star in $G * y * x$. Let $G' := G * y * x$.

Let $S := C \setminus \{z\}$, $H := G - z$, and $H' := G' - z$. Then in both cases, H' is locally equivalent to H, $H'[S]$ is an attached star in H', and $G' - C = H' - S$. By Proposition 4.6 and Theorem 4.1 we deduce that

$$\mathbb{E}\rho(H) = \mathbb{E}\rho(H') \ge \mathbb{E}\rho(H' - S) = \mathbb{E}\rho(G' - C) > \mathbb{E}\rho(G') - 1 = \mathbb{E}\rho(G) - 1 \ge \varepsilon + n - 1,$$

thus $H' - S$ contains some induced-subgraph-minimal graph of average cut-rank larger than $\varepsilon + n - 1$, say F, as an induced subgraph. Note that F has no isolated vertices because deleting isolated vertices does not change the average cut-rank. By the induction hypothesis, F has less than x_{n-1}

vertices. Then, $\mathbb{E}\rho(F)$ is a rational number larger than $\varepsilon+n-1$ whose denominator divides $2^{x_{n-1}-1}$, so by Theorem 4.1 we see that

$$\mathbb{E}\rho(H'-S) \geq \mathbb{E}\rho(F) \geq \varepsilon + n - 1 + \frac{1-\{2^{x_{n-1}}\varepsilon/2\}}{2^{x_{n-1}}/2}.$$

By Theorem 4.1 and Proposition 4.6, we thus obtain

$$\begin{aligned}\varepsilon + n \geq \mathbb{E}\rho(H) = \mathbb{E}\rho(H') &\geq \mathbb{E}\rho(H'-S) + 1 - 2^{1-|S|} \\ &\geq \varepsilon + n - 1 + \frac{1-\{2^{x_{n-1}}\varepsilon/2\}}{2^{x_{n-1}}/2} + 1 - 2^{1-|S|}.\end{aligned}$$

Thus, we deduce that

$$1-|S| \geq -x_{n-1} + 1 + \log(1-\{2^{x_{n-1}}\varepsilon/2\})$$

and so $|S| \leq \lfloor x_{n-1} - \log(1-\{2^{x_{n-1}}\varepsilon/2\})\rfloor$ and $|C| \leq \lfloor x_{n-1} - \log(1-\{2^{x_{n-1}}\varepsilon/2\}) + 1\rfloor$. This is a contradiction because $|C| > |G|/2^{8n+10} \geq x_n/2^{8n+10}$. □

7. Average cut-rank and forbidden vertex-minors

7.1. **Forbidden vertex-minors.** By Corollary 5.4, we can observe the following.

> Let $\mathcal{C}$ be a class of graphs closed under taking vertex-minors. If $\mathcal{C}$ has bounded average cut-rank, then there exists a finite list of graphs G_1, G_2, …, G_m such that a graph G is in $\mathcal{C}$ if and only if G has no vertex-minor isomorphic to G_j for every $j = 1, \ldots, m$.

A minimal such list is called a *list of forbidden vertex-minors* for $\mathcal{C}$. A list of forbidden vertex-minors is not unique, as one can replace a graph in the list with any locally equivalent graph.

But essentially the list is determined up to some equivalence relation. For two classes $\mathcal{S}_1$ and $\mathcal{S}_2$ of graphs, we say that $\mathcal{S}_1$ is *locally equivalent* to $\mathcal{S}_2$, denoted by $\mathcal{S}_1 \simeq \mathcal{S}_2$, if for every $G \in \mathcal{S}_1$ there is some $H \in \mathcal{S}_2$ isomorphic to a graph locally equivalent to G and for every $H \in \mathcal{S}_2$ there is some $G \in \mathcal{S}_1$ isomorphic to a graph locally equivalent to H. Then we can easily verify that the relation $\simeq$ is an equivalence relation and for every class of graphs closed under taking vertex-minors, the list of forbidden vertex-minors for $\mathcal{C}$ is determined up to local equivalence. As the list is an antichain with respect to the vertex-minor relation, every list of forbidden vertex-minors for $\mathcal{C}$ has the same size.

Let $\mathcal{L}_{\leq\alpha}$ be the class of all graphs H satisfying $\mathbb{E}\rho(H) > \alpha$ and any proper vertex-minor of H has average cut-rank at most α, and let $\mathcal{L}_{<\alpha}$ be the class of all graphs H satisfying $\mathbb{E}\rho(H) \geq \alpha$ and any proper vertex-minor of H has average cut-rank smaller than α. Then by Proposition 4.4, every graph in $\mathcal{L}_{\leq\alpha}$ or $\mathcal{L}_{<\alpha}$ has average cut-rank smaller than $\alpha+1$. By Corollary 1.2, both $\mathcal{L}_{\leq\alpha}$ and $\mathcal{L}_{<\alpha}$ are finite. We can also easily deduce that

> a graph has average cut-rank larger than (or at least) α if and only if it contains a vertex-minor in $\mathcal{L}_{\leq\alpha}$ (or $\mathcal{L}_{<\alpha}$, respectively).

Therefore for every $\alpha \geq 0$, $\mathcal{L}_{\leq\alpha}$ is locally equivalent to every list of forbidden vertex-minors for the class of graphs of average cut-rank at most α. Similarly, $\mathcal{L}_{<\alpha}$ is locally equivalent to every list of forbidden vertex-minors for the class of graphs of average cut-rank smaller than α.

7.2. **Lower bound on the number of vertex-minor obstructions.** Recall that for every $\alpha \geq 0$, every list of forbidden vertex-minors for the class of graphs of average cut-rank at most α is finite and has the same size; the same happens for the lists of forbidden vertex-minors for the class of graphs of average cut-rank smaller than α. We shall show that, there is some universal constant $c > 0$ such that for any $\varepsilon \in [0,1)$ and nonnegative integer n, every list of forbidden vertex-minors for the class of graphs of average cut-rank at most (or smaller than) $\varepsilon + n$ contains at least $2^{cn\log(n+1)}$ graphs. To do so, we construct a set of at least $2^{cn\log(n+1)}$ vertex-minor-minimal graphs of average cut-rank larger than $\varepsilon + n$, such that no two of them are locally equivalent to each other. Then, we can obtain from this set another set of at least $2^{cn\log(n+1)}$ vertex-minor-minimal graphs of average cut-rank at least $\varepsilon + n$ such that no two of them are locally equivalent to each other. Let us start with several notions to make our arguments clearer.

For a graph G, let $\pi(G)$ denote the quotient graph of G induced by $\equiv_G$. It is not difficult to see that a graph F without isolated vertices is a forest if and only if $\pi(F)$ is a forest and every equivalence class of $(V(F), \equiv_F)$ induces an attached star in F. In this case, let $R(F)$ be the set of central vertices in the equivalence classes of $(V(F), \equiv_F)$. Then it is not difficult to check that $F[R(F)]$ is isomorphic to $\pi(F)$. We regard $\pi(F)$ as a weighted graph by assigning each vertex C of $\pi(F)$ the weight $|C|$.

For two forests F_1 and F_2 without isolated vertices, we shall write $\pi(F_1) \cong \pi(F_2)$ if there is an isomorphism keeping weights from $\pi(F_1)$ to $\pi(F_2)$. From the definitions we can deduce the following easily.

Lemma 7.1. *Two forests F_1 and F_2 without isolated vertices are isomorphic if and only if $\pi(F_1) \cong \pi(F_2)$.*

The following is another useful characterization of isomorphic forests.

Lemma 7.2 (Bouchet [2, Corollary 5.4])**.** *For two forests F_1 and F_2, F_1 is isomorphic to F_2 if and only if F_1 is isomorphic to a graph locally equivalent to F_2.*

For two graphs G and H, H is called an *elementary* vertex-minor of G if H is a vertex-minor of G and $|H| = |G| - 1$. The following theorem of Bouchet [1] characterizes elementary vertex-minors of a graph up to local equivalence. Geelen and Oum [8] provided a direct proof.

Proposition 7.3 (Bouchet [1, Corollary 9.2])**.** *Let v be a vertex of a graph G. If H is a vertex-minor of G with $V(H) = V(G) \setminus \{v\}$, then H is locally equivalent to one of $G - v, (G * v) - v$, and $(G \wedge uv) - v$ for any u adjacent to v in G.*

For a graph G, a vertex $v \in V(G)$, and an integer $k \geq 0$, we denote by $G +_v K_{1,k}$ the graph obtained from the disjoint union of G and $K_{1,k}$ by adding an edge between v and the central vertex of $K_{1,k}$. The following lemma is crucial for our construction.

Lemma 7.4. *Let $G \in \mathcal{L}_{\leq \varepsilon+n}$ and $d \geq 1$ be the size of the largest attached star in G. Then there exists a unique positive integer $q_1 = q_1(G)$ such that $G + K_{1,q_1} \in \mathcal{L}_{\leq \varepsilon+n+1}$ and $q_1 \geq d$. Furthermore, for each $v \in V(G)$, there exists a unique positive integer $q_2 = q_2(G, v) \in \{q_1 - 1, q_1\}$ such that $G +_v K_{1,q_2} \in \mathcal{L}_{\leq \varepsilon+n+1}$.*

Proof. First, we prove that

$$\mathbb{E}\rho(G) \leq \varepsilon + n + 2^{1-d} \leq \varepsilon + n + 1. \tag{1}$$

Indeed, if $d = 1$ then for any $u \in V(G)$ we have, by Proposition 4.6,

$$\mathbb{E}\rho(G) < \mathbb{E}\rho(G - u) + 1 \leq \varepsilon + n + 1 = \varepsilon + n + 2^{1-d}.$$

If $d > 1$, then let u be a leaf in an attached star of size d in G. By Proposition 4.5 and the fact that $G \in \mathcal{L}_{\leq \varepsilon+n}$, we have

$$\mathbb{E}\rho(G) \leq \mathbb{E}\rho(G - u) + 2^{1-d} \leq \varepsilon + n + 2^{1-d} \leq \varepsilon + n + 1,$$

and (1) is proved. Hence, because $\mathbb{E}\rho(G) > \varepsilon + n$, by Lemma 4.2 and Proposition 4.4, there is some $q_1 \geq 1$ such that for all $k \geq q_1$

$$\mathbb{E}\rho(G + K_{1,k}) = \mathbb{E}\rho(G) + 1 - 2^{-k} > \varepsilon + n + 1, \tag{2}$$

and for all $1 \leq k < q_1$

$$\mathbb{E}\rho(G + K_{1,k}) = \mathbb{E}\rho(G) + 1 - 2^{-k} \leq \mathbb{E}\rho(G) + 1 - 2^{1-q_1} \leq \varepsilon + n + 1. \tag{3}$$

Thus, since (by (1) and (3))

$$\varepsilon + n + 2^{1-d} \geq \mathbb{E}\rho(G) > \varepsilon + n + 1 - (1 - 2^{-q_1}) = \varepsilon + n + 2^{-q_1},$$

we obtain $q_1 \geq d$. We show that $G + K_{1,q_1} \in \mathcal{L}_{\leq \varepsilon+n+1}$. Indeed, if H is a proper vertex-minor of $G + K_{1,q_1}$, then H is the disjoint union of H_1 and H_2 where H_1 is a vertex-minor of G and H_2 is a vertex-minor of K_{1,q_1} such that at least one of these two containments is proper. If H_1 is a proper vertex-minor of G, then since $G \in \mathcal{L}_{\leq \varepsilon+n}$,

$$\mathbb{E}\rho(H) = \mathbb{E}\rho(H_1) + \mathbb{E}\rho(H_2) \leq \varepsilon + n + 1 - 2^{-q_1} < \varepsilon + n + 1,$$

and if H_2 is a proper vertex-minor of K_{1,q_1}, then

$$\mathbb{E}\rho(H) = \mathbb{E}\rho(H_1) + \mathbb{E}\rho(H_2) \leq \mathbb{E}\rho(G) + 1 - 2^{1-q_1} \leq \varepsilon + n + 1.$$

Thus $G + K_{1,q_1} \in \mathcal{L}_{\leq \varepsilon+n+1}$. This proves the first claim.

Now let v be a vertex of G. By Proposition 4.6 and the construction of q_1, for all $k \geq q_1$,

$$\mathbb{E}\rho(G +_v K_{1,k}) \geq \mathbb{E}\rho(G) + 1 - 2^{-k} > \varepsilon + n + 1,$$

and for all $1 \leq k < q_1 - 1$, by Proposition 4.4,

$$\mathbb{E}\rho(G +_v K_{1,k}) \leq \mathbb{E}\rho(G) + 1 - 2^{-1-k} \leq \mathbb{E}\rho(G) + 1 - 2^{1-q_1} \leq \varepsilon + n + 1.$$

Because $G +_v K_{1,q_1-1}$ is a proper induced subgraph of $G +_v K_{1,q_1}$ and the average cut-rank is strictly monotone with respect to the induced subgraph relation by Theorem 4.1, there is a unique $q_2 = q_2(G, v) \in \{q_1 - 1, q_1\}$ such that

$$\begin{aligned} \mathbb{E}\rho(G +_v K_{1,k}) &> \varepsilon + n + 1 \quad \text{for all } k \geq q_2, \\ \mathbb{E}\rho(G +_v K_{1,k}) &\leq \varepsilon + n + 1 \quad \text{for all } 1 \leq k < q_2. \end{aligned}$$

In the formation of $G' := G +_v K_{1,q_2}$, let x be the central vertex of K_{1,q_2} that is adjacent to v and $S := V(K_{1,q_2})$. We show that $G' \in \mathcal{L}_{\leq \varepsilon+n+1}$. Indeed, suppose for the contrary that H is an elementary vertex-minor of G' with $V(G') = V(H) \cup \{u\}$ such that $\mathbb{E}\rho(H) > \varepsilon + n + 1$. By Proposition 7.3, H is locally equivalent to one of $G' - u, (G' * u) - u$, and $(G' \wedge uw) - u$ for any w adjacent to u in G'. We may assume without loss of generality that H is one of these graphs. There are three cases to consider.

(1) If $H = G' - u$, then u belongs to one of $V(G) \setminus \{v\}$, $\{v\}$, $\{x\}$, and $S \setminus \{x\}$.
   (a) If $u \in V(G) \setminus \{v\}$ then $H = (G - u) +_v K_{1,q_2}$. Because $\mathbb{E}\rho(G - u) \leq \varepsilon + n$, we have, by Proposition 4.4,

   $$\varepsilon + n + 1 < \mathbb{E}\rho(H) \leq \mathbb{E}\rho(G - u) + \mathbb{E}\rho(K_{1,q_2+1}) < \varepsilon + n + 1,$$

   a contradiction.
   (b) If $u = v$ then $H = (G - v) + K_{1,q_2}$. Similarly we obtain a contradiction.
   (c) If $u = x$ then H is the disjoint union of G with q_2 isolated vertices, so H and G have the same average cut-rank which is smaller than $\varepsilon + n + 1$, a contradiction.
   (d) If $u \in S \setminus \{x\}$ then $H = G +_v K_{1,q_2-1}$ which has average cut-rank smaller than $\varepsilon + n + 1$ by the definition of q_2, a contradiction.

(2) If $H = (G' * u) - u$, then from the first case we may assume that u is not a leaf in G', hence $u \notin S \setminus \{x\}$. There are three subcases to consider.
   (a) If $u \in V(G) \setminus \{v\}$ then $H = ((G * u) - u) +_v K_{1,q_2}$, which leads to a contradiction.
   (b) If $u = v$ then $H - S$ is an elementary vertex-minor of G, $N_H(x) = (S \setminus \{x\}) \cup N_G(v)$, and $H[S]$ is an attached star of H of size $q_2 + 1$ with the central vertex x, so by Proposition 4.4,

   $$\varepsilon + n + 1 < \mathbb{E}\rho(H) \leq \mathbb{E}\rho(H - S) + \mathbb{E}\rho(K_{1,q_2+d_G(v)}) < \varepsilon + n + 1,$$

   a contradiction.

(c) If $u = x$ then $H * z$ is isomorphic to $G +_v K_{1,q_2-1}$ where z is a vertex in $S \setminus \{x\}$, thus has average cut-rank smaller than $\varepsilon + n + 1$, a contradiction.

(3) If $H = (G' \wedge uw) - u$, then we may assume that u is neither a leaf nor a neighbor of a leaf in G', because otherwise w either is the unique neighbor of u in G' or can be chosen to be a leaf adjacent to u, and so $H = (G' \wedge uw) - u$ is isomorphic to $G' - w$, returning to the first case. There are two subcases to consider.

(a) If $u \in V(G) \setminus \{v\}$ then we may assume that $w \neq v$ (if there is no other choice then u is a leaf in G'). Now $H - S$ is an elementary vertex-minor of G and $H = (H-S) +_v K_{1,q_2}$. We obtain a contradiction.

(b) If $u = v$ then because G has no isolated vertices, we may choose $w = x$. Then $H[(V(G) \setminus \{v\}) \cup \{x\}]$ is isomorphic to G via some isomorphism bringing x to v and fixing every vertex in $V(G) \setminus \{v\}$. Furthermore, in H, $S \setminus \{x\}$ is an independent set and complete to $N_G(v) \cup \{x\}$ as well as anticomplete to $V(G) \setminus (N_G(v) \cup \{x\})$. Thus, for some $z \in S \setminus \{x\}$ we have $H * x * z$ is isomorphic to $G +_v K_{1,q_2-1}$, which brings a contradiction.

Therefore $G +_v K_{1,q_2} \in \mathcal{L}_{\leq \varepsilon+n+1}$, completing the proof of the lemma. $\square$

Now we come to the construction. Let $\mathcal{F}_\varepsilon := \{K_{1,\lfloor 1-\log(1-\varepsilon) \rfloor}\}$ for all $\varepsilon \in [0,1)$, and for all integers $k \geq 0$,

$$\mathcal{F}_{\varepsilon+2k+1} := \{F + K_{1,q_1(F)} : F \in \mathcal{F}_{\varepsilon+2k}\},$$
$$\mathcal{F}_{\varepsilon+2k+2} := \{(F + K_{1,q_1(F)}) +_v K_{1,q_2(F+K_{1,q_1(F)},v)} : F \in \mathcal{F}_{\varepsilon+2k}, v \in R(F)\},$$

where $q_1(F)$ and $q_2(F,v)$ are defined as in Lemma 7.4. Note that no graphs in $\mathcal{F}_{\varepsilon+n}$ have isolated vertices.

Corollary 7.5. *$\mathcal{F}_{\varepsilon+n} \subseteq \mathcal{L}_{\leq \varepsilon+n}$ for all $n \geq 0$.*

Proof. By Lemma 4.2, $K_{1,\lfloor 1-\log(1-\varepsilon) \rfloor} \in \mathcal{L}_{\leq \varepsilon}$ for all $\varepsilon \in [0,1)$. The conclusion thus follows inductively by Lemma 7.4. $\square$

Here is another consequence of Lemma 7.4.

Corollary 7.6. *For all $F \in \mathcal{F}_{\varepsilon+2n}$ and $v \in R(F)$, $q_2(F + K_{1,q_1(F)}, v)$ is at least $q_1(F)$, hence at least the maximum weight in $\pi(F)$.*

Proof. Let $H := F + K_{1,q_1(F)}$. By Lemma 7.4, $q_1(F)$ is at least the maximum weight in $\pi(F)$, so $q_1(F)+1$ is the largest weight in $\pi(H)$, which implies that $q_1(H) \geq q_1(F)+1$. Also by Lemma 7.4, $q_2(H,v) \geq q_1(H) - 1$, and thus $q_2(H,v)$ is at least $q_1(F)$, hence at least the maximum weight in $\pi(F)$. $\square$

Now we account for the restriction $v \in R(F)$ in the definition of $\mathcal{F}_{\varepsilon+2n+2}$: Because $q_2(H,v)$ can possibly be equal to $q_1(F)$, to deduce Lemmas 7.7 and 7.8 we require that the copy of $K_{1,q_2(H,v)}$ attached to v lies in a component different from a copy of $K_{1,q_1(F)}$.

Lemma 7.7. *For every $F \in \mathcal{F}_{\varepsilon+n}$, $\pi(F)$ has exactly $n+1$ vertices, and in $\pi(F)$, no positive integer appears more than twice as a weight; if some weight appears twice then the corresponding vertices are in different components and one of them is the smallest weight in its component.*

Proof. We proceed by induction on n. When $n = 0$ the lemma is trivial. Assuming that the lemma is true for $n = 2k$, we shall show that it is also true for $n = 2k+1$ and $2k+2$. Let $F \in \mathcal{F}_{\varepsilon+2k}$ and consider $H := F + K_{1,q_1(F)} \in \mathcal{F}_{\varepsilon+2k+1}$. Set $S := V(K_{1,q_1(F)})$. By Lemma 7.4, $q_1(F)$ is at least the maximum weight in $\pi(F)$, so the conclusion holds for $\pi(H)$ because it also holds for $\pi(F)$, which is done by the induction hypothesis.

Now consider $G := H +_v K_{1,q_2(H,v)} \in \mathcal{F}_{\varepsilon+2k+2}$ for $v \in R(F)$. By Corollary 7.6, $q_2(H,v)$ is at least $q_1(F)$ as well as the maximum weight in $\pi(F)$. So, since $v \in R(F)$, the weights in $\pi(F)$ are preserved in $\pi(G)$, hence by the induction hypothesis the conclusion for $\pi(G-S)$ indeed holds. Thus, to verify the conclusion for $\pi(G)$, it is enough to check two (unique) copies of $K_{1,q_2(H,v)}$ and $K_{1,q_1(F)}$ in G. But this is easy, since if $q_2(H,v) > q_1(F)$ then we are done, and if $q_2(H,v) = q_1(F)$ then those two copies must be in different components because $v \in R(F)$. □

Lemma 7.8. *For every $\varepsilon \in [0,1)$ and $n \geq 0$, no two distinct forests in $\mathcal{F}_{\varepsilon+n}$ are isomorphic.*

Proof. When $n = 0$ the lemma holds trivially. Assume that the lemma holds for $n = 2k$, we show that it also holds for $n = 2k+1$ and $n = 2k+2$. Consider $H_j := F_j + K_{1,q_1(F_j)} \in \mathcal{F}_{\varepsilon+2k+1}$ where $F_j \in \mathcal{F}_{\varepsilon+2k}$ for $j = 1,2$ and suppose that H_1 and H_2 are isomorphic. Since $q_1(F_j)$ is at least the maximum weight in $\pi(H_j)$ and $|\pi(H_j)| = 2k+2$ for $j = 1,2$, necessarily $q_1(F_1) = q_1(F_2)$ and so F_1 must be isomorphic to F_2, implying H_1 and H_2 are isomorphic.

Now consider $G_j := H_j +_{v_j} K_{1,q_2(H_j,v_j)} \in \mathcal{F}_{\varepsilon+2k+2}$ for $v_j \in R(F_j)$ for $j = 1,2$. Assume that G_1 and G_2 are isomorphic, then by Lemma 7.1 $\pi(G_1) \cong \pi(G_2)$. For $j = 1,2$, let T_j be the component in G_j containing the attached star $K_{1,q_2(H_j,v_j)}$ so that T_j is not the component isomorphic to $K_{1,q_1(F_j)}$ in G_j, by construction. By Corollary 7.6, $q_2(H_j,v_j)$ is at least the maximum weight in $\pi(F_j)$, so by Lemma 7.7 $q_2(H_j,v_j)$ is at least the maximum weight in $\pi(T_j)$, for $j = 1,2$. Thus, necessarily $q_2(H_1,v_1) = q_2(H_2,v_2)$ and $\pi(T_1) \cong \pi(T_2)$, which leads to $\pi(G_1 - V(T_1)) \cong \pi(G_2 - V(T_2))$. Hence, by deleting the vertex with label $q_2(H_1,v_1) = q_2(H_2,v_2)$ in each $\pi(G_j)$, we obtain $\pi(H_1) \cong \pi(H_2)$, so by Lemma 7.1 there is an isomorphism φ from H_1 to H_2. Thus, because the labels in T_j are distinct for $j = 1,2$ by Lemma 7.7, we have $\varphi(v_1) = v_2$. Therefore G_1 and G_2 are isomorphic and the proof is completed. □

Combining Lemmas 7.7 and 7.8, we deduce the number of pairwise nonisomorphic graphs in $\mathcal{F}_{\varepsilon+n}$ for all $\varepsilon \in [0,1)$ and $n \geq 0$. We employ the standard notation $k!! = \prod_{1 \leq j \leq k, j \equiv k \pmod 2} j$ for $k = 1,2,\ldots$ and the convention $(-1)!! = 0!! = 1$.

Corollary 7.9. *For every $\varepsilon \in [0,1)$ and $k \geq 0$, the number of pairwise nonisomorphic graphs in $\mathcal{F}_{\varepsilon+2k}$ and $\mathcal{F}_{\varepsilon+2k+1}$ is*

$$|\mathcal{F}_{\varepsilon+2k}| = |\mathcal{F}_{\varepsilon+2k+1}| = (2k-1)!!.$$

The next lemma describes properties of $\mathcal{L}_{\leq\alpha}$ and $\mathcal{L}_{<\alpha}$ to be used later.

Lemma 7.10. *Let $\alpha > 0$. Then the following statements hold.*

- *$\mathcal{L}_{<\alpha} \setminus \mathcal{L}_{\leq\alpha}$ is the class of all graphs without isolated vertices of average cut-rank exactly α.*
- *If $G \in \mathcal{L}_{\leq\alpha} \setminus \mathcal{L}_{<\alpha}$, then G has a proper vertex-minor H of average cut-rank exactly α in $\mathcal{L}_{<\alpha}$ such that $|G| - |H| \leq 2$. If the equality holds then H can be chosen so that G is isomorphic to $H + K_2$.*

Proof. Let $G \in \mathcal{L}_{<\alpha} \setminus \mathcal{L}_{\leq\alpha}$. Then G has no isolated vertices and $\mathbb{E}\rho(G) \geq \alpha$, so if $\mathbb{E}\rho(G) > \alpha$, G must have a proper vertex-minor, say H, in $\mathcal{L}_{\leq\alpha}$, but then $\mathbb{E}\rho(H) > \alpha$ so $G \notin \mathcal{L}_{<\alpha}$ by definition, a contradiction. On the other hand, by Theorem 4.1, if a graph G with no isolated vertices has average cut-rank α then $G \in \mathcal{L}_{<\alpha} \setminus \mathcal{L}_{\leq\alpha}$.

Now let $G \in \mathcal{L}_{\leq\alpha} \setminus \mathcal{L}_{<\alpha}$. Then G has a proper vertex-minor of average cut-rank at least α, say H, which also must have average cut-rank at most α. Thus $\mathbb{E}\rho(H) = \alpha$, and we may assume that $H \in \mathcal{L}_{<\alpha}$ by deleting isolated vertices. Since H is a proper vertex-minor of G, there is some $G' \in \mathcal{L}_{\leq\alpha}$ locally equivalent to G so that H is a proper induced subgraph of G'. Let $V(G) \setminus V(H) = \{v_0,\ldots,v_k\}$ where $k \geq 0$ and $H' := G' - v_0$. We may assume that $k \geq 1$, because otherwise the lemma holds. Because H' is a proper vertex-minor of G and contains $H \in \mathcal{L}_{<\alpha}$ as an induced subgraph, we have $\mathbb{E}\rho(H') = \mathbb{E}\rho(H) = \alpha$. Then by Theorem 4.1, $\{v_1,\ldots,v_k\} = V(H') \setminus V(H)$

consists of isolated vertices in H'. Hence, since $G' \in \mathcal{L}_{\leq\alpha}$ has no isolated vertices, $G'[\{v_0, \dots, v_k\}]$ is an attached star in G' of size $k+1$ where v_0 is the central vertex.

If v_0 is isolated in $G' - v_1$, then $k = 1$ and $G'[\{v_0, v_1\}]$ is a component of size 2 in G' and G' is isomorphic to $H + K_2$. Then G is isomorphic to $H'' + K_2$ where H'' is locally equivalent to H.

If v_0 is not isolated in $G' - v_1$, then $G' - v_1$ has no isolated vertices and contains H as a proper induced subgraph. This implies, again by Theorem 4.1, that $\alpha = \mathbb{E}\rho(H) < \mathbb{E}\rho(G' - v_1)$, contradicting the minimality of G. □

We remark that if α is a positive integer, both $\mathcal{L}_{\leq\alpha} \setminus \mathcal{L}_{<\alpha}$ and $\mathcal{L}_{<\alpha} \setminus \mathcal{L}_{\leq\alpha}$ are nonempty. For instance, $(2\alpha - 1)K_2 + K_{1,2}$ belongs to $\mathcal{L}_{\leq\alpha} \setminus \mathcal{L}_{<\alpha}$ and $2\alpha K_2$ belongs to $\mathcal{L}_{<\alpha} \setminus \mathcal{L}_{\leq\alpha}$.

To finish the proof of Theorem 1.4 we need one more lemma.

Lemma 7.11. *If $\varepsilon > 0$ or $n \geq 1$, then every forest F in $\mathcal{F}_{\varepsilon+n} \setminus \mathcal{L}_{<\varepsilon+n}$ has a leaf, say v, whose deletion yields a forest, say H, in $\mathcal{L}_{<\varepsilon+n}$ of average cut-rank exactly $\varepsilon + n$. Moreover, if v belongs to an equivalence class of size 2 of $(V(F), \equiv_F)$ and its unique neighbor has degree 2 in F then $|\pi(H)| = n$; otherwise $|\pi(H)| = n + 1$.*

Proof. Let $F \in \mathcal{F}_{\varepsilon+n} \setminus \mathcal{L}_{<\varepsilon+n}$. By Corollary 7.5 and Lemma 7.10, F has a proper vertex-minor, say H', of average cut-rank $\varepsilon+n$ such that $H' \in \mathcal{L}_{<\varepsilon+n}$ and $|F|-|H'| \leq 2$. Moreover, if $|F|-|H'| = 2$ then H' can be chosen so that F is isomorphic to $H' + K_2$, so F has a component of size 2. In this case, by the construction of $\mathcal{F}_{\varepsilon+n}$, Lemma 7.4, and Corollary 7.6 we deduce that $n \leq 1$. If $n = 0$ then F is isomorphic to K_2, so H' is empty, but this is absurd since $\varepsilon > 0$ by hypothesis; if $n = 1$ then H' is isomorphic to $K_{1,q}$ for some $q \geq 1$, a contradiction since $1 \leq 1+\varepsilon = \mathbb{E}\rho(H') = 1-2^{-q} < 1$. Thus, H' is an elementary vertex-minor of F.

Let $\{x\} = V(F) \setminus V(H')$, then by Proposition 7.3 we may assume without loss of generality that H' is one of $F - x$, $(F * x) - x$, and $(F \wedge xy) - x$ for any y adjacent to x in F.

(1) If $H' = F - x$ then since every equivalence class of $(V(F), \equiv_F)$ has at least two vertices (the construction of $\mathcal{F}_\varepsilon$, Lemma 7.4, and Corollary 7.6) and H' has no isolated vertices, x is necessarily a leaf in F, so we let $v = x$.
(2) If $H' = (F * x) - x$ then we may assume that $d_F(x) \geq 2$, so if y is a leaf adjacent to x then $F - y$ is isomorphic to $H' * y \in \mathcal{L}_{<\varepsilon+n}$, and we let $v = y$.
(3) If $H' = (F \wedge xy) - x$ then if furthermore x is a leaf in F then y is the unique neighbor of x in F, hence isolated in H', a contradiction. So, $d_F(x) \geq 2$, and since y can be chosen to be any neighbor of x in F, we may assume that y is a leaf adjacent to x. Then $F - y$ is isomorphic to $H' \in \mathcal{L}_{<\varepsilon+n}$ and we let $v = y$.

So, we have chosen v. Let $H := F - v$ and u be the unique neighbor of v in F. In all cases, H is locally equivalent to H' and therefore $H \in \mathcal{L}_{<\varepsilon+n}$. The first part of the lemma is proved.

We come to the second part of the lemma. If v belongs to an equivalence class of size 2 of $(V(F), \equiv_F)$ and $d_F(u) = 2$ then the neighbor of u other than v in F, say w, has degree at least two in F. Let C be the equivalence class of $(V(F), \equiv_F)$ containing w, then $C \cup \{u\}$ is an equivalence class of $(V(H), \equiv_H)$. It follows that $|\pi(H)| = |\pi(F)| - 1 = n$ by Lemma 7.7.

In the other cases, it is easy to check that $|\pi(H)| = |\pi(F)| = n + 1$. This completes the proof of the lemma. □

We are now ready to prove Theorem 1.4.

Theorem 1.4. *There is some universal constant $c > 0$ so that the following holds. For every $\varepsilon \in [0, 1)$ and $n \geq 0$, let $\mathcal{S}$ be a set of graphs such that the average cut-rank of a graph G is at most (or less than) $\varepsilon + n$ if and only if no graph in $\mathcal{S}$ is isomorphic to a vertex-minor of G. Then $\mathcal{S}$ contains at least $2^{cn\log(n+1)}$ graphs.*

Proof. Choose $c > 0$ to be some constant (independent of ε and n) such that

$$\frac{(2\lfloor n/2 \rfloor - 1)!!}{n+1} \geq 2^{cn\log(n+1)} \quad \text{for all } n \in \mathbb{N}.$$

First consider the case that $\mathcal{S}$ is a list of forbidden vertex-minors for the class of graphs of average cut-rank at most $\varepsilon + n$. Then $\mathcal{S}$ is locally equivalent to $\mathcal{L}_{\leq \varepsilon+n}$. By Corollary 7.5, $\mathcal{F}_{\varepsilon+n} \subseteq \mathcal{L}_{\leq \varepsilon+n}$, and by Lemmas 7.2 and 7.8, no two distinct forests F_1 and F_2 in $\mathcal{F}_{\varepsilon+n}$ are locally equivalent up to isomorphisms. Therefore, for every forest F in $\mathcal{F}_{\varepsilon+n}$, there is some member in $\mathcal{S}$ which is isomorphic to a graph locally equivalent to F and these members are pairwise not locally equivalent to each other. By Corollary 7.9,

$$|\mathcal{S}| \geq |\mathcal{F}_{\varepsilon+n}| = \left(2\left\lfloor \frac{n}{2} \right\rfloor - 1\right)!! \geq 2^{cn\log(n+1)}. \tag{4}$$

Now consider the case that $\mathcal{S}$ is a list of forbidden vertex-minors for the class of graphs of average cut-rank smaller than $\varepsilon + n$. Then $\mathcal{S}$ is locally equivalent to $\mathcal{L}_{<\varepsilon+n}$. We may assume that $\varepsilon + n > 0$. Let $\{F_1, \ldots, F_m\} = \mathcal{F}_{\varepsilon+n} \setminus \mathcal{L}_{<\varepsilon+n}$.

For every $j = 1, \ldots, m$, by Lemma 7.11, F_j has a leaf, whose deletion yields a forest in $\mathcal{L}_{<\varepsilon+n}$, say H_j, of average cut-rank exactly $\varepsilon + n$. Moreover, $|\pi(H_j)|$ is either n or $n+1$ depending on the condition written in the statement of Lemma 7.11.

Claim. *For every $j \in \{1, \ldots, m\}$, there are, up to isomorphism, at most $n+1$ forests F such that there is some leaf in F whose deletion yields H_j.*

Proof. There are two cases to consider.

(1) $|\pi(H_j)| = n$. The only way to obtain F from H_j is to add a new vertex to H_j and join it to some leaf in H_j (to create a new equivalence class of size 2). Because $(V(H_j), \equiv_{H_j})$ has n equivalence classes, each of which induces an attached star in H_j, there are at most n forests F satisfying the claim.

(2) $|\pi(H_j)| = n + 1$. The only way to obtain F from H_j is to add a new vertex to H_j and join it to the central vertex of some equivalence class of $(V(H_j), \equiv_{H_j})$. Because there are $n+1$ such equivalence classes, there are thus at most $n+1$ forests F satisfying the claim.

Hence there are at most $n+1$ desired forests F, completing the proof of the claim. ■

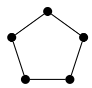
Let $\mathcal{G}$ be a graph on the vertex set $\{1, \ldots, m\}$ such that for distinct $j, k \in \{1, \ldots, m\}$, $jk \in E(\mathcal{G})$ if H_j is isomorphic to a graph locally equivalent to H_k. For $j \in \{1, \ldots, m\}$, by Lemma 7.2, $k \in N_{\mathcal{G}}(j)$ if and only if H_j is isomorphic to H_k, implying that there is some forest F'_k isomorphic to F_k such that H_j can be obtained by deleting some leaf of F'_k. Because the set $\{F_1, \ldots, F_m\}$ consists of pairwise nonisomorphic forests, by Lemma 7.8, so does the set $\{F'_k : k \in N_{\mathcal{G}}(j)\} \cup \{F_j\}$. It follows by the claim that $d_{\mathcal{G}}(j) \leq n$ for all $j = 1, \ldots, m$.

Let S be a maximal independent set in $\mathcal{G}$. Then every vertex outside of S is adjacent in $\mathcal{G}$ to some vertex in S whose degree is at most n. Hence $m = |\mathcal{G}| \leq |S| + n|S|$, or equivalently $|S| \geq \frac{m}{n+1}$.

Let $\mathcal{T}$ be the disjoint union of $\mathcal{F}_{\varepsilon+n} \cap \mathcal{L}_{<\varepsilon+n}$ and $\{H_j : j \in S\}$. Since S is an independent set in $\mathcal{G}$, for every distinct $j, k \in S$ we have H_j is not isomorphic to a graph locally equivalent to H_k. This implies, from our construction, that $\mathcal{T} \subseteq \mathcal{L}_{<\varepsilon+n}$ and no two distinct graphs in $\mathcal{T}$ are locally equivalent to each other up to isomorphisms. Furthermore, no two distinct forests in $\mathcal{F}_{\varepsilon+n} \cap \mathcal{L}_{<\varepsilon+n}$ are locally equivalent to each other up to isomorphisms. Therefore, by (4),

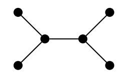
$$|\mathcal{S}| \geq |\mathcal{T}| = |\mathcal{F}_{\varepsilon+n} \cap \mathcal{L}_{<\varepsilon+n}| + |S| \geq |\mathcal{F}_{\varepsilon+n}| - m + \frac{m}{n+1} \geq \frac{|\mathcal{F}_{\varepsilon+n}|}{n+1} = \frac{(2\lfloor n/2 \rfloor - 1)!!}{n+1} \geq 2^{cn\log(n+1)},$$

and the theorem is completely proved. □

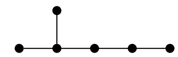


(A) C_5, $\mathbb{E}\rho(C_5) = \frac{25}{16}$.

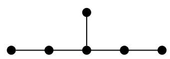
(B) P_6, $\mathbb{E}\rho(P_6) = \frac{57}{32}$.



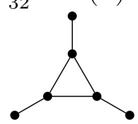
(C) $P_{4,1}$, $\mathbb{E}\rho(P_{4,1}) = \frac{49}{32}$.



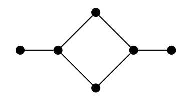
(D) $P_{5,1}$, $\mathbb{E}\rho(P_{5,1}) = \frac{7}{4}$.



(E) $P_{5,2}$, $\mathbb{E}\rho(P_{5,2}) = \frac{51}{32}$.



(F) $C_{3,1}$, $\mathbb{E}\rho(C_{3,1}) = \frac{57}{32}$.



(G) $C_{4,1}$, $\mathbb{E}\rho(C_{4,1}) = \frac{27}{16}$.

FIGURE 1. Small graphs and their average cut-rank.

8. Graphs of average cut-rank at most 3/2

We now aim to prove Theorem 1.5. Our plan is to bound the number of connected components and investigate the maximum induced path of every graph locally equivalent to a fixed graph. This approach not only characterizes graphs of average cut-rank at most 3/2 but also reveals $\mathcal{L}_{<1}, \mathcal{L}_{\leq 1}, \mathcal{L}_{<3/2}, \mathcal{L}_{\leq 3/2}$ up to local equivalence. For every graph G, we denote by $p(G)$ the maximum length of a path graph which is a vertex-minor of G.

Recall that for every $k \geq 0$, E_k is the graph $K_{1,k+1}$ with one edge subdivided. The following lemma computes $\mathbb{E}\rho(E_k)$, which explains why $3/2 - 3/2^{k+2}$ appears in Theorem 1.5. We omit its easy proof.

Lemma 8.1. *For all $k \geq 0$, we have $\mathbb{E}\rho(E_k) = \frac{3}{2} - \frac{3}{2^{k+2}}$.*

8.1. Graphs of average cut-rank at most 1.

We need the following lemma. We leave its easy proof to the readers.

Lemma 8.2. *If G is a connected graph having no path of length three as a vertex-minor then G is isomorphic to a star or a complete graph.*

Lemma 8.3. *If a graph with no isolated vertices has average cut-rank at most 1, then it is isomorphic to a graph locally equivalent to one of $2K_2$ and $K_{1,k}$ for $k \geq 1$. Moreover $\mathcal{L}_{<1}$ is locally equivalent to $\{2K_2, K_4\}$ and $\mathcal{L}_{\leq 1}$ is locally equivalent to $\{3K_2, K_2 + P_3, P_4\}$.*

Proof. It follows easily from Lemma 8.2 and the following observations: $\mathbb{E}\rho(2K_2) = 1$, $\mathbb{E}\rho(3K_2) = 3/2$, $\mathbb{E}\rho(K_2 + P_3) = 7/4$, and $\mathbb{E}\rho(P_4) = 9/8$. $\square$

8.2. Graphs of average cut-rank at most 3/2.

Let us start with several technical results whose proofs are left to the interested readers.

Lemma 8.4. *Let P be an induced path of length 3 in a graph G and v be a vertex of G outside P such that v has at least 2 neighbors in P. Then*

- *If v is adjacent to both ends of P, G contains a cycle of length 5 as a vertex-minor.*
- *Otherwise, G contains a path of length 4 as a vertex-minor.*

Lemma 8.5. *Every graph without isolated vertices on 5 vertices is isomorphic to a graph locally equivalent to to one of $K_2 + P_3$, $K_{1,4}$, P_5, E_2, and C_5.*

Lemma 8.6. *Let G be a graph on at most 5 vertices. If $\mathbb{E}\rho(G) \geq 3/2$, then G is isomorphic to a graph locally equivalent to C_5.*

Graphs C_5, P_6, $P_{4,1}$, $P_{5,1}$, $P_{5,2}$, $C_{3,1}$, and $C_{4,1}$ with their average cut-rank are listed in Figure 1. We deduce the following easily.

Corollary 8.7. *The graphs C_5, P_6, $P_{4,1}$, $P_{5,1}$, $P_{5,2}$, $C_{3,1}$, and $C_{4,1}$ belong to $\mathcal{L}_{<3/2} \cup \mathcal{L}_{\leq 3/2}$.*

The following lemma is a major step toward the proof of Theorem 1.5.

Lemma 8.8. *If a graph without isolated vertices has average cut-rank at most 3/2, then it is isomorphic to a graph locally equivalent to one of P_5, $3K_2$, $2P_3$, $K_{1,k+1}$, $K_2 + K_{1,k+1}$, and E_k for $k \geq 0$. Moreover*

$$\mathcal{L}_{<3/2} \simeq \{3K_2, K_2 + P_3, 2P_3, C_5, P_6, P_{4,1}, P_{5,1}, P_{5,2}, C_{3,1}, C_{4,1}\},$$
$$\mathcal{L}_{\leq 3/2} \simeq \{4K_2, 2K_2 + P_3, K_2 + P_4, P_3 + K_{1,3}, P_3 + P_4, C_5, P_6, P_{4,1}, P_{5,1}, P_{5,2}, C_{3,1}, C_{4,1}\}.$$

Proof. Let G be a graph such that either $\mathbb{E}\rho(G) \leq 3/2$ or $G \in \mathcal{L}_{<3/2} \cup \mathcal{L}_{\leq 3/2}$. By Lemmas 8.5 and 8.6, we may assume that $|G| > 5$. It is easy to check that $C_5, P_6, 4K_2, 2K_2 + P_3 \in \mathcal{L}_{\leq 3/2}$ and so we may assume that G has no vertex-minor isomorphic to C_5, P_6, $4K_2$, $2K_2 + P_3$. Thus G has at most 3 components.

If G has exactly 3 components, then G has an induced subgraph isomorphic to $3K_2$ which has average cut-rank 3/2, and if furthermore G has at least 7 vertices then it has a vertex-minor isomorphic to $2K_2 + P_3$ whose average cut-rank is 7/4. Hence if $\mathbb{E}\rho(G) \leq 3/2$ then G is isomorphic to $3K_2$, if $G \in \mathcal{L}_{<3/2}$ then G is isomorphic to $3K_2$, and if $G \in \mathcal{L}_{\leq 3/2}$ then G is isomorphic to a graph locally equivalent to $2K_2 + P_3$.

When G has exactly 2 components, if every component of G has at least 3 vertices then G has a vertex-minor isomorphic to $2P_3$ whose average cut-rank is 3/2, and if furthermore G has at least 7 vertices then G has a vertex-minor isomorphic to $P_3 + K_{1,3}$ or $P_3 + P_4$ whose average cut-rank is 13/8 or 15/8, respectively; if one component of G has only 2 vertices then $G = K_2 + H$ for some graph H with $\mathbb{E}\rho(H) = \mathbb{E}\rho(G) - 1/2$. By applying Lemma 8.3 to H, we deduce that if $\mathbb{E}\rho(G) \leq 3/2$ then G is isomorphic to a graph locally equivalent to $K_2 + K_{1,k}$ for some $k \geq 1$ or $2P_3$, if $G \in \mathcal{L}_{<3/2}$ then $2P_3$, and if $G \in \mathcal{L}_{\leq 3/2}$ then G is isomorphic to a graph locally equivalent to $P_3 + K_{1,3}$ or $P_3 + P_4$.

Now we assume that G is connected. By Lemma 8.3, we may assume that G has average cut-rank larger than 1, so by Lemma 8.2, $p(G) \geq 3$. By applying local complementaions if necessary, we may assume that G has an induced path of length $p(G)$.

If $p(G) \geq 4$, then let $P = abcde$ be an induced path of length 4. Then there is a vertex v outside P adjacent to some vertex of P. If v is adjacent to a, then it is easy to check that G has a vertex-minor isomorphic to C_5 or P_6, contradicting our assumption. Thus v is nonadjacent to a and by symmetry, nonadjacent to e. By considering all possible $N(v) \cap \{b, c, d\}$, we deduce that G has a vertex-minor isomorphic to $P_{5,1}$, $P_{5,2}$, $C_{4,1}$, or $C_{3,1}$. Hence, if $p(G) = 4$, then $\mathbb{E}\rho(G) > 3/2$ and in addition if $G \in \mathcal{L}_{<3/2} \cup \mathcal{L}_{\leq 3/2}$ then G is isomorphic to a graph locally equivalent to one of $P_{5,1}$, $P_{5,2}$, $C_{4,1}$, and $C_{3,1}$, by Corollary 8.7.

If $p(G) = 3$ then let $P = abcd$ be an induced path of length 3. Then by Lemma 8.5, $S := N_G(P) \setminus V(P) \neq \emptyset$. Pick $v \in S$. If $\{a, d\} \subseteq N_G(v)$ then G is isomorphic to a graph locally equivalent to C_5, contradicting our assumption. Thus we may assume that v is nonadjacent to d. If v is adjacent to a, then we may apply local complementations to find a vertex-minor isomorphic to P_5, contradicting the assumption that $p(G) = 3$. Thus, v is nonadjacent to a. By the same argument, we deduce that v is adjacent to exactly one of b and c. Hence, each vertex in S should be adjacent to only one of b, c in P. If all the vertices in S are pairwise nonadjacent and adjacent to the same among b, c, then G is isomorphic to E_k for some $k \geq 1$ and thus $\mathbb{E}\rho(G) < 3/2$. Otherwise, there are two vertices in S, say u, v, being adjacent to each other or adjacent to different vertices in $\{b, c\}$. In the case u, v are adjacent, if they are adjacent to the same among b, c then $P_{5,1}$ is isomorphic to an induced subgraph of $G * u$ which contradicts $p(G) = 3$, otherwise $G \wedge uv$ has an induced path of length 5, contradicting the assumption; in the case u, v are adjacent to different vertices in $\{b, c\}$, we only have to check when $uv \notin E(G)$, then G is isomorphic to a graph locally equivalent to $P_{4,1}$. Thus, if $p(G) = 3$ and $\mathbb{E}\rho(G) \leq 3/2$ then G is isomorphic to a graph locally equivalent to E_k for some $k \geq 0$, and if $p(G) = 4$ and $G \in \mathcal{L}_{<3/2} \cup \mathcal{L}_{\leq 3/2}$ then G is isomorphic to a graph locally equivalent to $P_{4,1}$ by Corollary 8.7. □

Theorem 1.5. *Let G be a graph with no isolated vertices. Then G has average cut-rank at most $3/2$ if and only if it is isomorphic to a vertex-minor of one of P_5, $3K_2$, $2P_3$, $K_{1,k+1}$, $K_2 + K_{1,k+1}$, and E_k for $k \geq 0$. In addition, the set of all possible values for average cut-rank of graphs in the interval $[0, 3/2]$ is*

$$\left\{1 - \frac{1}{2^k} : k \geq 0\right\} \cup \left\{\frac{3}{2} - \frac{1}{2^{k+1}} : k \geq 0\right\} \cup \left\{\frac{3}{2} - \frac{3}{2^{k+2}} : k \geq 0\right\} \cup \left\{\frac{3}{2}\right\}.$$

Proof. It suffices to combine Proposition 4.4, and Lemmas 8.1, 4.2, 8.8, and the fact that $\mathbb{E}\rho(P_5) = 23/16 = 3/2 - 1/2^4$. □

Acknowledgments

The authors would like to thank the anonymous reviewers for helpful comments.

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