A FURTHER EXTENSION OF RÖDL'S THEOREM

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Abstract. Fix $\varepsilon > 0$ and a nonnull graph H. A well-known theorem of Rödl from the 80s says that every graph G with no induced copy of H contains a linear-sized ε-*restricted* set $S \subseteq V(G)$, which means S induces a subgraph with maximum degree at most $\varepsilon|S|$ in G or its complement. There are two extensions of this result:

- quantitatively, Nikiforov (and later Fox and Sudakov) relaxed the condition "no induced copy of H" into "at most $\kappa|G|^{|H|}$ induced copies of H for some $\kappa > 0$;" and
- qualitatively, Chudnovsky, Scott, Seymour, and Spirkl recently showed that there is $N > 0$ such that G is (N, ε)-*restricted*, which means $V(G)$ has a partition into at most N subsets that are ε-restricted.

A natural common generalization of these two asserts that every graph G with at most $\kappa|G|^{|H|}$ induced copies of H is (N, ε)-restricted for some $\kappa, N > 0$. This is unfortunately false, but we prove κ and N still exist so that for every $d \geq 0$, every graph with at most $\kappa d^{|H|}$ induced copies of H can be made (N, ε)-restricted by removing at most d vertices. This unifies the two aforementioned theorems, and is optimal up to κ and N for every value of d.

1. Introduction

Graphs in this paper are finite and simple. For a graph G with vertex set $V(G)$ and edge set $E(G)$, let $|G| := |V(G)|$, and let $\overline{G}$ denote its complement. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S, and let $G \setminus S := G[V(G) \setminus S]$. For a nonnull graph H, a *copy* of H in G is a graph isomorphism from H to $G[S]$ for some $S \subseteq V(G)$. Let $\mathrm{ind}_H(G)$ be the number of copies of H in G; and say that G is H-*free* if $\mathrm{ind}_H(G) = 0$. Given $\varepsilon > 0$, a subset $S \subseteq V(G)$ is ε-*restricted* in G if one of $G[S], \overline{G}[S]$ has maximum degree at most $\varepsilon|S|$. The following well-known theorem of Rödl [12] from 1986 has become a standard tool in the investigation of the Erdős–Hajnal conjecture[1] [6, 7] (see [3] for a survey).

Theorem 1.1. *For every $\varepsilon > 0$ and every graph H, there exists $\delta = \delta(H, \varepsilon) > 0$ such that every H-free graph G contains an ε-restricted set of size at least $\delta|G|$.*

Since its inception, Theorem 1.1 has found many extensions. Among these is the following quantitative improvement first proved by Nikiforov [11], which is useful in several situations (for example see [5, 8]).

Theorem 1.2. *For every $\varepsilon > 0$ and every graph H, there exist $\delta = \delta_{1.2}(H, \varepsilon) > 0$ and $\kappa = \kappa_{1.2}(H, \varepsilon) > 0$ such that for every graph G with $\mathrm{ind}_H(G) \leq \kappa|G|^{|H|}$, G has an ε-restricted set with at least $\delta|G|$ vertices.*

Rödl's original proof of Theorem 1.1 and Nikiforov's proof of Theorem 1.2 both employ the regularity lemma, and so give bounds on δ^{-1} and κ^{-1} which are towers of twos of height polynomial in ε^{-1} with constants depending on H. Fox and Sudakov [9] offered an alternative proof of Theorem 1.2 showing that both δ and κ can be chosen as $2^{-c \log^2(\varepsilon^{-1})}$ for some constant $c > 0$ depending on H, and conjectured that δ can in fact be taken to be a polynomial of ε in Theorem 1.1, which would imply the Erdős–Hajnal conjecture itself (see [8] for current progress on this topic).

Recently, Chudnovsky, Scott, Seymour, and Spirkl [4] provided a qualitative refinement of Theorem 1.1, which says that the vertex set of every H-free graph can even be partitioned into a bounded number of ε-restricted subsets. Formally, for $\varepsilon, N > 0$, a graph G is (N, ε)-*restricted* if there is a partition of $V(G)$ into at most N subsets that are ε-restricted in G; thus G is (N, ε)-restricted if and only if $\overline{G}$ is. Here is the main theorem of [4].

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[1] The very last sentence of [6] was actually the first time Erdős and Hajnal formally stated their famous conjecture.

Theorem 1.3. *For every $\varepsilon > 0$ and every graph H, every H-free graph is (N, ε)-restricted for some $N = N(H, \varepsilon) > 0$.*

The *edge density* of a graph G is the quotient $|E(G)|/\binom{|G|}{2}$. For $\varepsilon > 0$, a subset $S \subseteq V(G)$ is *weakly ε-restricted* in G if one of $G[S], \overline{G}[S]$ has edge density at most ε. Thus if S is $\frac{1}{2}\varepsilon$-restricted in G then it is weakly ε-restricted; and if S is weakly $\frac{1}{4}\varepsilon$-restricted in G then it has an ε-restricted subset of size $\lceil \frac{1}{2}|S| \rceil$. Hence the strength of Theorems 1.1 and 1.2 remain unaffected if "ε-restricted" is replaced by "weakly ε-restricted." As discussed in [4], however, Theorem 1.3 becomes significantly weaker if "(N, ε)-restricted" is replaced by "*weakly (N, ε)-restricted*," which means $V(G)$ has a partition into at most N subsets that are weakly ε-restricted in G. Indeed, repeated applications of Theorem 1.2 yield the following.

Theorem 1.4. *For every $\varepsilon > 0$ and every graph H, there exist $\kappa = \kappa(H, \varepsilon) > 0$ and $N = N(H, \varepsilon) > 0$ such that every graph G with $\operatorname{ind}_H(G) \le \kappa|G|^{|H|}$ is weakly (N, ε)-restricted.*

(As shown in [9, 11], with more care one can even take the corresponding weakly ε-restricted sets to have size differences at most 1 in this result.) It thus would be natural (and quite tempting) to conjecture the following, which would have unified Theorems 1.2 and 1.3 and strengthened Theorem 1.4 considerably.

Conjecture 1.5 (false)**.** *For every $\varepsilon > 0$ and every graph H, there exist $N = N(H, \varepsilon) > 0$ and $\kappa = \kappa(H, \varepsilon) > 0$ such that every graph G with $\operatorname{ind}_H(G) \le \kappa|G|^{|H|}$ is (N, ε)-restricted.*

Unfortunately, the following proposition[2] refutes this conjecture in a strong sense.

Proposition 1.6. *Let $N \ge 1$. Then for all integers m, n with $n \ge m \ge 20N^2$, every $\varepsilon \in (0, \frac{1}{18})$, and every graph H with $h := |H| \ge 2$, there is a graph on n vertices which has at most hmn^{h-1} copies of H and is not (N, ε)-restricted. In particular, for every $\kappa > 0$ and every integer $n \ge 20\kappa^{-1}hN^2$, there is a graph on n vertices which has at most κn^h copies of H and is not (N, ε)-restricted.*

Proof. In what follows, $\Delta(G)$ denotes the maximum degree of a graph G. By taking complements if necessary, we may assume H is connected, and so H has at least one edge as $h \ge 2$.

Let F be a random graph on $m \ge 20N^2$ vertices where each edge appears independently with probability $\frac{1}{2}$. For every $T \subseteq V(F)$ with $|T| \ge \frac{1}{N}m$, since $6\varepsilon < \frac{1}{3}$, Hoeffding's inequality [10] implies that T is weakly 6ε-restricted in F with probablity at most $2\exp(-\frac{1}{72}\binom{|T|}{2}) \le 2\exp(-\frac{1}{300N^2}m^2)$; and so, since $2^m \cdot 2\exp(-\frac{1}{300N^2}m^2) < 1$ (as $m \ge 20N^2$), there is a choice of F with no weakly 6ε-restricted set of size at least $\frac{1}{N}m$. Consequently F has no 3ε-restricted subset of size at least $\frac{1}{N}m$.

Now, fix such an F; and for every $n \ge m$, let G be a graph obtained from F by adding $n - m$ isolated vertices and making each of them adjacent to every vertex in $V(F)$. Since H has at least one edge, every copy of H in G has at least one image vertex in $V(F)$, and so

$$\operatorname{ind}_H(G) \le \sum_{i=1}^{h} \binom{h}{i} m^i (n-m)^{h-i} = n^h - (n-m)^h = m \sum_{i=1}^{h} n^{i-1}(n-m)^{h-i} \le hmn^{h-1}.$$

It thus remains to show that G is not (N, ε)-restricted. Suppose not; and let $A_1 \cup \cdots \cup A_k$ be a partition of $V(G)$ for some $k \le N$ such that A_i is ε-restricted for all $i \in \{1, 2, \ldots, k\}$. Then $\bigcup_{i=1}^{k}(A_i \cup V(F))$ is a partition of $V(F)$, and so we may assume $T := A_1 \cup V(F)$ has size at least $\frac{1}{N}m$. Thus T is not 3ε-restricted in F; hence $S := A_1 \setminus V(F)$ is nonempty. It follows that

$$\Delta(G[A_1]) = |S| + \Delta(F[T]) > \varepsilon|S| + 3\varepsilon|T| > \varepsilon(|S| + |T|) = \varepsilon|A_1|,$$
$$\Delta(\overline{G}[A_1]) = \max(|S| - 1, \Delta(\overline{F}[T])) \ge \max(|S| - 1, 3\varepsilon|T|) > \varepsilon(|S| + |T|) = \varepsilon|A_1|.$$

Therefore A_1 is not ε-restricted in G, a contradiction. This proves Proposition 1.6. ∎

The graphs constructed in Proposition 1.6 suggest that an "exceptional" set of vertices should necessarily be removed in order for the remaining vertices to admit a partition into a bounded number of ε-restricted pieces. Our main theorem shows that this is also sufficient.

[2] We remark that Alex Scott (personal communication) independently discovered similar counterexamples.

Theorem 1.7. *For every $\varepsilon > 0$ and every graph H, there exist $\kappa = \kappa_{1.7}(H, \varepsilon) > 0$ and $N = N_{1.7}(H, \varepsilon) > 0$ such that for every $d \geq 0$ and every graph G with $\operatorname{ind}_H(G) \leq \kappa d^{|H|}$, there is $S \subseteq V(G)$ with $|S| \leq d$ such that $G \setminus S$ is (N, ε)-restricted; equivalently, G can be made (N, ε)-restricted by removing at most $C \cdot \operatorname{ind}_H(G)^{1/|H|}$ vertices where $C = \kappa^{-1/|H|}$.*

We would like to make three remarks. First, Theorem 1.3 is a special case of Theorem 1.7 with $d = 0$; and taking $d = \varepsilon|G|$ in Theorem 1.7 yields Theorem 1.2. Thus Theorem 1.7 can be viewed as a remedy for the false Conjecture 1.5; and the counterexamples in Proposition 1.6 (with suitable choices of m, n depending on d and more isolated vertices added) show that Theorem 1.7 is optimal up to κ and N for *any* given value of d.

Second, Theorem 1.7 is related to the induced removal lemma [1, 13] which also implies Theorem 1.2. Here, we are dealing with the property of being (N, ε)-restricted which is weaker than H-freeness (by Theorem 1.3) and not closed under the induced subgraph relation. But the trade-off is worth considering: removing only a handful of vertices instead of adding/deleting edges; and working well for *all* graphs, including those with subquadratic number of edges and only few copies of H.

Third, our proof of Theorem 1.7 generalizes the proof of Theorem 1.3 given in [4], demonstrating that the argument there can be extended to graphs with a bounded number of copies of H (at the cost of removing a small number of vertices). The resulting bounds on $\kappa_{1.7}^{-1}(H, \varepsilon)$ and $N_{1.7}(H, \varepsilon)$, as a result, are better than what the regularity lemma could provide (but still huge functions, namely towers of twos of height depending solely on $|H|$ with ε^{-1} on top).

In what follows, for an integer $k \geq 0$, let $[k]$ denote $\{1, 2, \ldots, k\}$ if $k \geq 1$ and $\emptyset$ if $k = 0$. The vertex set of H will always be $\{v_1, \ldots, v_h\}$ for some $h \geq 1$; and we drop the subscript H from the notation ind_H.

2. A slight digression

This section provides a short and self-contained proof of Theorem 1.2 without using the regularity lemma, which will be used frequently in the proof of Theorem 1.7. The presentation here mostly follows [9].

For $\varepsilon > 0$, a graph G, and disjoint subsets A, B of $V(G)$, B is *ε-sparse* to A in G if every vertex in B is adjacent to fewer than $\varepsilon|A|$ vertices of A in G, and *ε-dense* to A in G if it is ε-sparse to A in $\overline{G}$. Say that B is *ε-tight* to A if it is either ε-sparse or ε-dense to A. The following lemma implicitly appears in [9, Lemma 4.1], which in turn generalizes an old result of Erdős and Hajnal [7, Theorem 1.5]. This result will also be useful later on.

Lemma 2.1. *Let H be a graph, and let $\varepsilon_1, \ldots, \varepsilon_{h-1}, \delta_1, \ldots, \delta_{h-1} \in (0, 1)$. Let G be a graph, and let $D_1, \ldots, D_h$ be disjoint nonempty subsets of $V(G)$ such that for all indices i, j with $1 \leq i < j \leq h$, there do not exist $A \subseteq D_i$ and $B \subseteq D_j$ with $|A| \geq \prod_{t=j}^{h-1} \varepsilon_t \cdot |D_i|$ and $|B| \geq \frac{\delta_{j-1}}{j-1} \prod_{t=j}^{h-1} \varepsilon_t \cdot |D_j|$ satisfying B is ε_j-sparse to A if $v_i v_j \in E(H)$ and ε_j-dense to A if $v_i v_j \notin E(H)$. Then there are at least $\prod_{t=1}^{h-1} (1 - \delta_t) \varepsilon_t^t \cdot \prod_{i=1}^{h} |D_i|$ copies φ of H in G with $\varphi(v_i) \in D_i$ for all $i \in [h]$.*

Proof. Induction on $h \geq 1$. We may assume that $h \geq 2$. For $i \in [h-1]$, let P_i be the set of vertices in D_h with fewer than $\varepsilon_{h-1}|D_i|$ neighbors in D_i if $v_i v_h \in E(H)$ and the set of vertices in D_h with fewer than $\varepsilon_{h-1}|D_i|$ nonneighbors in D_i if $v_i v_h \notin E(H)$. By the hypothesis, $|P_i| \leq \frac{\delta_{h-1}}{h-1}|D_h|$ for all $i \in [h-1]$. Let $D_h' := D_h \setminus (\bigcup_{i \in [h-1]} P_i)$; then $|D_h'| \geq (1 - \delta_{h-1})|D_h|$.

Now, for each $u \in D_h'$ and $i \in [h-1]$, let D_i^u be the set neighbors of u in D_i if $v_i v_h \in E(H)$ and the set of nonneighbors of u in D_i if $v_i v_h \notin E(H)$; then $|D_i^u| \geq \varepsilon_{h-1}|D_i|$ for all $i \in [h-1]$. Thus for all indices i, j with $1 \leq i < j \leq h-1$, there do not exist $A \subseteq D_i^u$ and $B \subseteq D_j^u$ with $|A| \geq \prod_{t=j}^{h-2} \varepsilon_t \cdot |D_i^u|$ and $|B| \geq \frac{\delta_{j-1}}{j-1} \prod_{t=j}^{h-2} \varepsilon_t \cdot |D_j^u|$ such that B is ε_j-sparse to A if $v_i v_j \in E(H)$ and ε_j-dense to A if $v_i v_j \notin E(H)$. So by induction, there are at least $\prod_{t=1}^{h-2} (1 - \delta_t) \varepsilon_t^t \cdot \prod_{i=1}^{h-1} |D_i^u|$ copies φ_u of $H \setminus v_h$ in $G \setminus D_h$ with $\varphi_u(v_i) \in D_i^u$ for all $i \in [h-1]$. Summing up over all $u \in D_h'$, we deduce that there are at least

$$\sum_{u \in D_h'} \left(\prod_{t=1}^{h-2} (1 - \delta_t) \varepsilon_t^t \cdot \prod_{i=1}^{h-1} |D_i^u| \right) \geq |D_h'| \left(\prod_{t=1}^{h-2} (1 - \delta_t) \varepsilon_t^t \right) \left(\varepsilon_{h-1}^{h-1} \prod_{i=1}^{h-1} |D_i| \right) \geq \prod_{t=1}^{h-1} (1 - \delta_t) \varepsilon_t^t \cdot \prod_{i=1}^{h} |D_i|$$

copies φ of H in G such that $\varphi(v_i) \in D_i$ for all $i \in [h]$. This proves Lemma 2.1. ■

Corollary 2.2. *Let $\varepsilon \in (0,1)$, let H be a graph, and let $\kappa = \kappa_{2.2}(H,\varepsilon) := (4h)^{-h}\varepsilon^{\binom{h}{2}}$. Then every G with $\operatorname{ind}(G) \le \kappa|G|^h$ contains disjoint $A, B \subseteq V(G)$ with $|A|, |B| \ge (2h)^{-2}\varepsilon^{h-1}|G|$ such that B is ε-tight to A.*

Proof. We may assume $|G| \ge h$. Let $D_1, \dots, D_h$ be disjoint subsets of $V(G)$ each of size $\lfloor \frac{1}{h}|G| \rfloor$; then $|D_t| \ge \frac{1}{2h}|G|$ for all $t \in [h]$. It suffices to apply Lemma 2.1 with $\varepsilon_t = \varepsilon$ and $\delta_t = \frac{1}{2}$ for all $t \in [h]$. ■

For $\varepsilon_1, \varepsilon_2, \kappa > 0$ and a graph H, let $\beta(H,\kappa,\varepsilon_1,\varepsilon_2)$ be the largest constant β with $0 < \beta \le 1$ such that every graph G with $\operatorname{ind}(G) \le \kappa|G|^h$ has an induced subgraph with at least $\beta|G|$ vertices and edge density at most ε_1 or at least $1-\varepsilon_2$; then $\beta(H,\kappa,\varepsilon_1,\varepsilon_2)$ is decreasing in κ and $\beta(H,\kappa,\varepsilon_1,\varepsilon_2) = 1$ for all $\kappa > 0$ whenever $\varepsilon_1 + \varepsilon_2 \ge 1$ (and so whenever $\varepsilon_1\varepsilon_2 \ge 1$). We need the following lemma.

Lemma 2.3. *Let $\varepsilon_1, \varepsilon_2 > 0$, let H be a graph, and let $\eta := \eta_{2.3}(H,\varepsilon_1,\varepsilon_2) := \frac{1}{2}(2h)^{-2}(\frac{1}{4}\varepsilon)^{h-1}$ where $\varepsilon = \min(\varepsilon_1,\varepsilon_2)$. Then for every κ with $0 < \kappa \le \kappa_{2.2}(H,\frac{1}{4}\varepsilon)$, we have*

$$\beta(H,\kappa,\varepsilon_1,\varepsilon_2) \ge \eta \cdot \min(\beta(H,\eta^{-h}\kappa,\tfrac{3}{2}\varepsilon_1,\varepsilon_2), \beta(H,\eta^{-h}\kappa,\varepsilon_1,\tfrac{3}{2}\varepsilon_2)).$$

Proof. Let $\beta_1 := \beta(H,\eta^{-h}\kappa,\frac{3}{2}\varepsilon_1,\varepsilon_2)$, $\beta_2 := \beta(H,\eta^{-h}\kappa,\varepsilon_1,\frac{3}{2}\varepsilon_2)$, and $\beta_0 := \eta \cdot \min(\beta_1,\beta_2)$. Let G be a graph with $\operatorname{ind}(G) \le \kappa|G|^h$; we need to show there exists $S \subseteq V(G)$ with $|S| \ge \beta_0|G|$ such that $G[S]$ has edge density at most ε_1 or at least $1-\varepsilon_2$. By Corollary 2.2, G has disjoint subsets $A, B \subseteq V(G)$ with $|A|, |B| \ge 2\eta|G|$ such that B is $\frac{1}{4}\varepsilon$-tight to A; and we may assume B is $\frac{1}{4}\varepsilon$-sparse to A.

Because $\operatorname{ind}(G[B]) \le \kappa|G|^h \le \eta^{-h}\kappa|B|^h$, by the definition of β and by averaging, there exists $B_1 \subseteq B$ with $|B_1| = \lceil \beta_1\eta|G| \rceil \ge \beta_0|G|$ such that $G[B_1]$ has edge density at most $\frac{3}{2}\varepsilon_1$ or at least $1-\varepsilon_2$. If the latter holds then we are done, so we may assume the former holds.

Let A_0 be the set of vertices in A each with at most $\frac{1}{2}\varepsilon|B_1|$ neighbors in B_1. Since G has fewer than $\frac{1}{4}\varepsilon|A||B_1|$ edges between A and B_1, we have $|A_0| \ge \frac{1}{2}|A| \ge \eta|G|$. Thus $\operatorname{ind}(G[A_0]) \le \eta^{-h}\kappa|A_0|^h$, and so by the definition of β and by averaging, there exists $A_1 \subseteq A_0$ with $|A_1| = \lceil \beta_1\eta|G| \rceil \ge \beta_0|G|$ such that $G[A_1]$ has edge density at most $\frac{3}{2}\varepsilon_1$ or at least $1-\varepsilon_2$. Again, we may assume the former holds.

Now, let $S := A_1 \cup B_1$; then $|S| = 2|A_1| = 2|B_1| \ge 2\beta_0|G|$. Since $G[A_1], G[B_1]$ each have edge density at most $\frac{3}{2}\varepsilon_1$ and G has at most $\frac{1}{2}\varepsilon|A_1||B_1|$ edges between A_1 and B_1, we deduce that

$$\begin{aligned}|E(G[S])| &\le |E(G[A_1])| + |E(G[B_1])| + \frac{1}{2}\varepsilon|A_1||B_1| \\ &\le \frac{3}{2}\varepsilon_1\binom{|A_1|}{2} + \frac{3}{2}\varepsilon_1\binom{|B_1|}{2} + \frac{1}{2}\varepsilon_1|A_1||B_1| = 3\varepsilon_1\binom{|A_1|}{2} + \frac{1}{2}\varepsilon_1|A_1|^2 \le \varepsilon_1\binom{2|A_1|}{2} = \varepsilon_1\binom{|S|}{2}.\end{aligned}$$

Therefore S has the desired property. This proves Lemma 2.3. ■

We are now give a proof of Theorem 1.2 in the following equivalent form, which leads to the dependence of $\delta_{1.2}(H,\varepsilon)$ and $\kappa_{1.2}(H,\varepsilon)$ on ε and h as mentioned in the Introduction.

Theorem 2.4. *For every $\varepsilon > 0$ and every graph H, there exist $\delta = \delta(H,\varepsilon) > 0$ and $\kappa = \kappa(H,\varepsilon) > 0$ such that every graph G with $\operatorname{ind}(G) \le \kappa|G|^h$ contains a weakly ε-restricted set of size at least $\delta|G|$.*

Proof. Let $s := \lceil \log_{\frac{3}{2}}(\varepsilon^{-2}) \rceil$, $\eta := \eta_{2.3}(H,\varepsilon,\varepsilon)$, $\delta := \eta^s$, and $\kappa := \eta^{sh} \cdot \kappa_{2.2}(H,\frac{1}{4}\varepsilon)$. Note that $\eta_{2.3}(H,\cdot,\cdot)$ is decreasing in each of the last two components. Thus, since $\beta(H,\cdot,\cdot,\cdot)$ is decreasing in the second component and equals 1 whenever the last two components have product at least 1, applying Lemma 2.3 for s times yields $\beta(H,\kappa,\varepsilon,\varepsilon) \ge \eta^s = \delta$. This proves Theorem 2.4. ■

3. Key lemma

This section introduces and proves our key lemma, the following.

Lemma 3.1. *For all $\varepsilon, \eta, \theta \in (0, \frac{1}{2})$ and every graph H, there are $\kappa = \kappa_{3.1}(H,\varepsilon,\eta,\theta) > 0$ and $N = N_{3.1}(H, \varepsilon,\eta,\theta) > 0$ with the following property. For every $d \ge 0$ and every graph G with $\operatorname{ind}(G) \le \kappa d^h$, there is $S \subseteq V(G)$ with $|S| \le d$ such that $V(G) \setminus S$ can be partitioned into nonempty sets*

$$A_1, \dots, A_m;\ B_1, \dots, B_m;\ C_1, \dots, C_n$$

where $m \le \binom{h}{2}$ *and* $n \le N$*, such that*

- $A_1, \dots, A_m, C_1, \dots, C_n$ *are* ε*-restricted in* G*; and*
- *for every* $i \in [m]$*,* $|B_i| \le \eta |A_i|$ *and* B_i *is* θ*-tight to* A_i*.*

This contains [4, Theorem 1.5] as a special case with $d = 0$, and already gives Theorem 1.2 with $\varepsilon = \eta = \theta$ and $d = \varepsilon|G|$. We shall employ the same approach as in [4, Section 2], and recommend reading the detailed sketch there first. Here we explain the modifications.

We recall some definitions. For $c, \varepsilon > 0$ and a graph G, a pair (A, B) of disjoint nonempty subsets of $V(G)$ is (c, ε)-*full* in G if for every $A_1 \subseteq A$ and $B_1 \subseteq B$ with $|A_1| \ge c|A|$ and $|B_1| \ge c|B|$, G has at least $\varepsilon|A_1||B_1|$ edges between A_1, B_1; and (A, B) is (c, ε)-*empty* in G if it is (c, ε)-full in $\overline{G}$. Thus for every $c' > c$ and every $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \ge c'|A|$ and $|B'| \ge c'|B|$, (A', B') is $(c/c', \varepsilon)$-full if (A, B) is (c, ε)-full and is $(c/c', \varepsilon)$-empty if (A, B) is (c, ε)-empty. A collection $\{D_1, \dots, D_h\}$ of disjoint nonempty subsets of $V(G)$ is a (c, ε)-*blowup* of H if for all distinct $i, j \in [h]$, (D_i, D_j) is (c, ε)-full if $v_i v_j \in E(H)$ and is (c, ε)-empty if $v_i v_j \notin E(H)$.

In proving Lemma 3.1, we shall be concerned with partitions of $V(G)$ into "rows" of subsets and pairs of subsets as follows:

- *first row*: pairs $(A_1, B_1), \dots, (A_m, B_m)$ for some $m \ge 0$ such that for all $i \in [m]$, A_i is ε-restricted, B_i is very tight to A_i and has size smaller than a tiny fraction of A_i (B_i might be empty);
- *second row*: ε-restricted nomempty sets $C_1, \dots, C_n$ for some $n \ge 0$;
- *third row*: ε'-restricted nonempty sets $D_1, \dots, D_t$ for some t with $0 \le t \le h$, such that $\{D_1, \dots, D_t\}$ is a (c, ξ)-blowup of $H[\{v_1, \dots, v_t\}]$ for some appropriately chosen $c, \varepsilon', \xi > 0$; and
- *fourth row*: the set L of "leftover" vertices such that whenever $t > 0$, L has size smaller than a tiny fraction of each D_i.

Such a partition certainly exists, with $m = n = t = 0$ and $L = V(G)$. Starting from $t = 0$ with this partition, we shall attempt to increase t one by one for at most h steps. Let S be the set of vertices in L with the "correct adjacencies" to the collection $\{D_1, \dots, D_t\}$, that is, those having at least a small fraction of neighbors in D_i if $v_{t+1} v_i \in E(H)$ and at least a small fraction of nonneighbors in D_i if $v_{t+1} v_i \notin E(H)$. Then $L \setminus S$ can be partitioned into (possibly empty) sets $L_1, \dots, L_t$ such that L_i is (very) tight to D_i for every $i \in [t]$. As the notation suggests, if $|S| \le d$ then we stop the iteration and rearrange the sets $A_1, \dots, A_m,\ B_1, \dots, B_m,\ C_1, \dots, C_n,\ D_1, \dots, D_t,\ L_1, \dots, L_t$ to form a partition of $V(G) \setminus S$ with the desired property (this is not hard, and the bounds on m and n will come up later).

So let us assume $|S| > d$. We can then apply Theorem 1.2 to find an ε'-restricted subset S_0 of S. Keeping in mind that $\{D_1, \dots, D_t, S_0\}$ now form a "partial" blowup of $H[\{v_1, \dots, v_t, v_{t+1}\}]$, we iteratively construct a nested sequence $S_0 \supseteq S_1 \supseteq \ldots \supseteq S_t$ and subsets $P_1 \subseteq D_1, \dots, P_t \subseteq D_t$ such that each pair (S_i, P_i) is reasonably full (if $v_{t+1} v_i \in E(H)$) or reasonably empty (if $v_{t+1} v_i \notin E(H)$); then the collection $\{P_1, \dots, P_t, S_t\}$ will be a sufficiently good blowup of $H[\{v_1, \dots, v_t, v_{t+1}\}]$ while $P_1, \dots, P_t, S_t$ are still ε-restricted (for suitable c, ε', ξ). To execute this process, we need the following useful lemma which allows us to extract decent fullness/emptiness from moderate denseness/sparseness; this is a straightforward consequence of the regularity lemma, but recently Scott, Seymour, and Spirkl [14, Lemma 2.2] gave a short proof giving much better bounds.

Lemma 3.2. *For every* $c, \varepsilon, \tau \in (0, 1)$ *with* $\varepsilon < \tau \le 8/9$*, there is* $\gamma = \gamma_{3.2}(c, \varepsilon, \tau) \in (0, \frac{1}{3})$ *for which the following holds. Let* G *be a graph with* $A, B \subseteq V(G)$ *disjoint and nonempty such that* G *has at least* $\tau|A||B|$ *edges between* A *and* B*. Then there exist* $A' \subseteq A$ *and* $B \subseteq B'$ *with* $|A'| \ge \gamma|A|$ *and* $|B'| \ge \gamma|B|$ *such that* (A', B') *is* (c, ε)*-full.*

Observe that L is nonempty since S is, which implies each D_i is quite large, and so we can take each P_i to have size at least a (small) fraction of D_i yet at most half of D_i simultaneously. Then each L_i is still quite tight to and tiny compared to $D_i \setminus P_i$; and we can move each pair $(D_i \setminus P_i, L_i)$ to the first row.

Now, we want to use Theorem 1.2 to pull out as many ε-restricted sets as possible from $S \setminus S_t$ (assuming this is nonempty) so that the resulting new "leftover" set L' still has size smaller than a tiny fraction of S_t and of each P_i; then we can move those new restricted sets to the third row. A potential issue here is that Theorem 1.2 may not be applicable if $S \setminus S_t$ is not large enough while most of the copies of H in

G are "concentrated" on $G[S \setminus S_t]$. This can be avoided, conveniently, by making sure that $|S_0|$ is not too large compared to $|S|$ right in the first place (if $|S| \geq 2$), which will be done by the following simple corollary of Theorem 1.2 itself (we believe this is well-known, but still include a proof for completeness).

Corollary 3.3. *For every $\varepsilon > 0$ and every graph H, there exist $\delta = \delta_{3.3}(H, \varepsilon) \in (0, \frac{1}{4})$ and $\kappa = \kappa_{3.3}(H, \varepsilon) > 0$ such that for every graph G with $\operatorname{ind}(G) \leq \kappa|G|^h$, G has an ε-restricted set T with $|T| = \lceil \delta|G| \rceil$; in particular $|T| = 1$ if $|G| = 1$ and $|G \setminus T| \geq \frac{1}{2}|G|$ if $|G| \geq 2$.*

Proof. Let $\delta := \frac{1}{4} \cdot \delta_{1.2}(H, \frac{1}{8}\varepsilon)$ and $\kappa := \kappa_{1.2}(H, \frac{1}{8}\varepsilon)$. By Theorem 1.2, G has an $\frac{1}{8}\varepsilon$-restricted set U with $|U| \geq 2\delta|G|$; in particular U is weakly $\frac{1}{4}\varepsilon$-restricted. By averaging, there is a weakly $\frac{1}{4}\varepsilon$-restricted subset U' of U such that $|U'| = \lceil 2\delta|G| \rceil$, and so there is $T \subseteq U'$ with $|T| = \lceil \frac{1}{2}|U'| \rceil = \lceil \frac{1}{2}\lceil 2\delta|G| \rceil \rceil = \lceil \delta|G| \rceil$ such that T is ε-restricted in G. In particular, if $|G| \leq 4$ then $|T| = 1$; and if $|G| > 4$ then $|G \setminus T| > |G| - 1 - \delta|G| \geq \frac{3}{4}|G| - \frac{1}{4}|G| = \frac{1}{2}|G|$. This proves Corollary 3.3. ∎

For $\delta, \eta \in (0, 1)$, let $\phi(\delta, \eta)$ be the least integer $p \geq 1$ with $(1 - \delta)^p \leq \eta$; then $\phi(\delta, \eta) \leq \delta^{-1} \log \eta$. The next corollary of Theorem 1.2 formalizes the process of repeatedly pulling out ε-restricted sets from $S \setminus S_t$.

Corollary 3.4. *For every $\varepsilon, \eta \in (0, 1)$, for every graph H, and for $\delta := \delta_{1.2}(H, \varepsilon) > 0$, there exists $\kappa = \kappa_{3.4}(H, \varepsilon, \eta) > 0$ such that for every graph G with $\operatorname{ind}(G) \leq \kappa|G|^h$, there is $T \subseteq V(G)$ with $|T| \leq \eta|G|$ such that $G \setminus T$ is $(\phi(\delta, \eta), \varepsilon)$-restricted.*

Proof. Let $\kappa := \eta^h \cdot \kappa_{1.2}(H, \varepsilon)$. We may assume $|G| \geq 1$. Let $U_0 := V(G)$; and for $i \geq 0$, as long as U_i is defined and $|U_i| > \eta|G|$, let $U_{i+1} \subseteq U_i$ such that $U_i \setminus U_{i+1}$ is ε-restricted and $|U_i \setminus U_{i+1}| \geq \delta|U_i|$, which is possible by Theorem 1.2 since

$$\operatorname{ind}(G[U_i]) \leq \kappa|G|^h = \kappa_{1.2}(H, \varepsilon) \cdot (\eta|G|)^h < \kappa_{1.2}(H, \varepsilon) \cdot |U_i|^h.$$

This produces a chain of sets $V(G) = U_0 \supseteq U_1 \supseteq \ldots \supseteq U_n$ for some $p \geq 1$ such that $|U_{i+1}| \leq (1 - \delta)|U_i| \leq (1 - \delta)^{i+1}|G|$ and $|U_i| > \eta|G|$ for all $i \in \{0, 1, \ldots, p - 1\}$. In particular $\eta|G| < |U_{n-1}| \leq (1 - \delta)^{n-1}|G|$; thus $p - 1 < \phi(\delta, \eta)$ and so $p \leq \phi(\delta, \eta)$. Let $T := U_p$; then $\bigcup_{i=1}^{p}(U_i \setminus U_{i-1})$ is a partition of $V(G) \setminus T$ into p subsets which are ε-restricted in G. This proves Corollary 3.4. ∎

Now assume we have reached $t = h$ and obtained a decent blowup $\{D_1, \ldots, D_h\}$ of H. Observe that to be able to reach $t = h$ means the "exceptional" set S in each step always had size more than d; so it is not hard to see that each $|D_i|$ is still more than a (tiny) fraction of d. It thus suffices to apply the following, which is direct corollary of Lemma 2.1 and is an analogue of the induced counting lemma [2, Lemma 3.2].

Corollary 3.5. *Let $\varepsilon \in (0, \frac{1}{2})$, let H be a graph, and let G be a graph with an $(\varepsilon^h, \varepsilon)$-blowup $\{D_1, \ldots, D_h\}$ of H. Then there are at least $(1 - \varepsilon)^{h-1}\varepsilon^{\binom{h}{2}}|D_1| \cdots |D_h|$ copies φ of H in G with $\varphi(v_i) \in D_i$ for all $i \in [h]$.*

Proof. This follows from Lemma 2.1 with $\varepsilon_t := \varepsilon$ and $\delta_t := t \cdot \varepsilon^t$ for all $t \in [h - 1]$; note that $\delta_t \leq t2^{-t+1}\varepsilon \leq \varepsilon$ since $\varepsilon \in (0, \frac{1}{2})$. ∎

We are now ready to prove Lemma 3.1.

Proof of Lemma 3.1. Let $\xi := \frac{1}{4}\theta$ and $\varepsilon_h := \min(\varepsilon, \xi^h)$. Let $\Gamma_{t,t} = \lambda_{t,t} := 1$; and for $t = h - 1, h - 2, \ldots, 0$ in turn, do the following:

- for $i = t - 1, t - 2, \ldots, 0$ in turn, let $\Gamma_{t,i} := \lambda_{t,i+1}\Gamma_{t,i+1}$ and $\lambda_{t,i} := \gamma_{3.2}(\frac{1}{3}\varepsilon_{t+1}\Gamma_{t,i+1}, 2\xi, \xi)$; and
- let $\varepsilon_t := \varepsilon_{t+1}\lambda_{t,0}$.

Now, define

$$\varepsilon' := \min_{t \in \{0,1,\ldots,h-1\}} \varepsilon_{t+1}\Gamma_{t,0}, \qquad \delta' := \delta_{3.3}(H, \varepsilon'),$$

$$\eta' := \frac{1}{2}\eta\delta' \cdot \min_{t \in \{0,1,\ldots,h-1\}} \Gamma_{t,0}, \qquad N := \binom{h}{2} + (h - 1) \cdot \phi(\delta', \eta').$$

Also, for $i = 1, 2, \ldots, h$ in turn, do the following:

- let $\Lambda_{i,i} := \delta'\Gamma_{i-1,0}$; and

- for $t = i, i+1, \dots, h-1$ in turn, let $\Lambda_{t+1,i} := \lambda_{t,i}\Lambda_{t,i}$.

Finally, put

$$\kappa := \min\left((1-\xi)^{h-1}\xi^{\binom{h}{2}}\Lambda_{h,1}\cdots\Lambda_{h,h},\ \kappa_{3.3}(H,\varepsilon'),\ 2^{-h}\cdot\kappa_{3.4}(H,\varepsilon,\eta')\right).$$

For integers $m, n, t \geq 0$ with $t \leq h$, an (m,n,t)-*partition* in G is a partition of $V(G)$ into (not necessarily nonempty) subsets

$$A_1, \dots, A_m;\ B_1, \dots, B_m;\ C_1, \dots, C_n;\ D_1, \dots, D_t;\ L$$

such that

- $m \leq \binom{t}{2}$ and $n \leq t\cdot\phi(\delta',\eta')$;
- $A_1, \dots, A_m, C_1, \dots, C_n$ are nonempty and ε-restricted;
- for every $i \in [m]$, $|B_i| \leq \eta|A_i|$ and B_i is θ-tight to A_i;
- $\{D_1, \dots, D_t\}$ is an (ε_t, ξ)-blowup of $H[\{v_1, \dots, v_t\}]$; and
- if $t > 0$, then $|D_i| > \max(\Lambda_{t,i}d, 2\eta^{-1}|L|)$ and D_i is ε_t-restricted for every $i \in [t]$.

For the readers' convenience, let us write such a partition as follows

$$\begin{gathered}(A_1, B_1), \dots, (A_m, B_m);\\ C_1, \dots, C_n;\\ D_1, \dots, D_t;\\ L.\end{gathered}$$

Observe that $V(G)$ itself is a $(0,0,0)$-partition in G. Thus, there is $t \in \{0, 1, \dots, h\}$ maximal such that there is an (m,n,t)-partition in G. If $t = h$, then $\{D_1, \dots, D_h\}$ would be a (ξ^{h-1}, ξ)-blowup of H; so by Corollary 3.5, G would contain at least

$$(1-\xi)^h\xi^{\binom{h}{2}}|D_1|\cdots|D_h| > (1-\xi)^h\xi^{\binom{h}{2}}\Lambda_{h,1}\cdots\Lambda_{h,h}d^h \geq \kappa d^h \geq \operatorname{ind}(G)$$

copies of H, a contradiction. Thus $t < h$.

Let S be the set of vertices u in L with the property that for every $i \in [t]$, u has at least $2\xi|D_i|$ neighbors in D_i if $v_{t+1}v_i \in E(H)$ and at least $2\xi|D_i|$ nonneighbors in D_i if $v_{t+1}v_i \notin E(H)$. Then there is a partition $L \setminus S = L_1 \cup \dots \cup L_t$ such that L_i is 2ξ-tight to D_i for all $i \in [t]$. We shall prove that S satisfies the lemma; and the following crucial claim is the key step.

Claim 3.6. *$|S| \leq d$.*

Proof. Suppose that $|S| > d$; then $|S| \geq 1$. Since $\delta' = \delta_{3.3}(H,\varepsilon')$ and

$$\operatorname{ind}(G[S]) \leq \kappa d^h < \kappa_{3.3}(H,\varepsilon')\cdot|S|^h$$

by the definitions of δ' and κ, Corollary 3.3 yields an ε'-restricted subset S_0 of S with $|S_0| = \lceil\delta'|G|\rceil$; in particular $|S_0| = 1$ if $|S| = 1$ and $|S \setminus S_0| \geq \frac{1}{2}|S| > \frac{1}{2}d$ if $|S| \geq 2$. We shall define a chain of sets $S_0 \supseteq S_1 \supseteq \ldots \supseteq S_t$ together with sets $P_1, \dots, P_t$ of vertices such that for all $i \in [t]$,

- $|S_i| \geq \lambda_{t,i}|S_{i-1}|$;
- $P_i \subseteq D_i$ and $\lambda_{t,i}|D_i| \leq |P_i| \leq \frac{1}{2}|D_i|$; and
- (S_i, P_i) is $(\varepsilon_{t+1}\Gamma_{t,i}, \xi)$-full if $v_iv_{t+1} \in E(H)$ and is $(\varepsilon_{t+1}\Gamma_{t,i}, \xi)$-empty if $v_iv_{t+1} \notin E(H)$.

To this end, assume that for $i \in [t]$, S_{i-1} and P_{i-1} have been defined. Assume $v_iv_{t+1} \in E(H)$ without loss of generality; then by the maximality of L_i, every vertex of $S_{i-1} \subseteq S$ has at least $2\xi|D_i|$ neighbors in D_i. Thus Lemma 3.2 and the definition of $\lambda_{t,i}$ yield $S_i \subseteq S_{i-1}$ and $D_i' \subseteq D_i$ with $|S_i| \geq \lambda_{t,i}|S_{i-1}|$ and $|D_i'| \geq \lambda_{t,i}|D_i|$ so that (S_i, D_i') is $(\frac{1}{3}\varepsilon_{t+1}\Gamma_{t,i}, \xi)$-full. Let $P_i \subseteq D_i'$ with $|P_i| = \min(|D_i'|, \lfloor\frac{1}{2}|D_i|\rfloor)$; then since $\lfloor\frac{1}{2}|D_i|\rfloor \geq \frac{1}{3}|D_i| \geq \lambda_{t,i}|D_i|$ (as $|D_i| > 2\eta^{-1}|S| > 1$), $\max(\lambda_{t,i}|D_i|, \frac{1}{3}|D_i'|) \leq |P_i| \leq \frac{1}{2}|D_i|$. In particular, (S_i, P_i) is $(\varepsilon_{t+1}\Gamma_{t,i}, \xi)$-full. This defines S_i and P_i in the case $v_iv_{t+1} \in E(H)$; and similar arguments with $(\frac{1}{3}\varepsilon_{t+1}\Gamma_{t,i}, \xi)$-full replaced by $(\frac{1}{3}\varepsilon_{t+1}\Gamma_{t,i}, \xi)$-empty also define S_i and P_i in the case $v_iv_{t+1} \notin E(H)$.

From the above construction, we see that $|S_t| = 1$ if $|S| = 1$ and $|S \setminus S_t| \geq |S \setminus S_0| > \frac{1}{2}d$ if $|S| \geq 2$. Let $L' := \emptyset$ if the former holds; and if the latter holds, then since

$$\operatorname{ind}(G[S \setminus S_t]) \leq \kappa d^h < \kappa_{3.4}(H,\varepsilon,\eta')\cdot|S \setminus S_t|^h$$

by the definition of κ, Corollary 3.4 yields $L' \subseteq S \setminus S_t$ with $|L'| \le \eta'|S \setminus S_t|$ such that $S \setminus (S_t \cup L')$ is $(\phi(\delta', \eta'), \varepsilon)$-restricted. Thus there is always a subset $L' \subseteq S \setminus S_t$ with $|L'| \le \eta'|S \setminus S_t|$ such that $S \setminus (S_t \cup L')$ has a partition into nonempty ε-restricted subsets $Q_1, \dots, Q_s$ for some $s \le \phi(\delta', \eta')$.

Now, let $P_{t+1} := S_t$; we shall prove that the following partition of $V(G)$

$$(A_1, B_1), \dots, (A_m, B_m), (D_1 \setminus P_1, L_1), \dots, (D_t \setminus P_t, L_t);$$
$$C_1, \dots, C_n, Q_1, \dots, Q_s;$$
$$P_1, \dots, P_t, P_{t+1};$$
$$L'$$

is an $(m + t, n + s, t + 1)$-partition in G, which contradicts the maximality of t. To this end, observe the following.

- Since $m \le \binom{t}{2}$ and $n \le t \cdot \phi(\delta', \eta')$, we have $m + t \le \binom{t+1}{2}$ and $n + s \le (t + 1) \cdot \phi(\delta', \eta')$.
- $A_1, \dots, A_m, C_1, \dots, C_n, Q_1, \dots, Q_s$ are nonempty and ε-restricted by definition; and for every $i \in [t]$, $|D_i \setminus P_i| \ge \frac{1}{2}|D_i| > 0$ from the construction of P_i, in particular $D_i \setminus P_i$ is ε-restricted since D_i is ε_t-restricted and $2\varepsilon_t \le \varepsilon$.
- For every $i \in [m]$, $|B_i| \le \eta|A_i|$ and B_i is θ-tight to A_i by definition; and for every $i \in [t]$, $|L_i| \le |L| < \frac{1}{2}\eta|D_i| \le \eta|D_i \setminus P_i|$ and L_i is θ-tight to $D_i \setminus P_i$ since it is 2ξ-tight to D_i while $\xi = \frac{1}{4}\theta$.
- $\{P_1, \dots, P_t\}$ is an (ε_{t+1}, ξ)-blowup of $H[\{v_1, \dots, v_t\}]$ since $|P_i| \ge \lambda_{t,i}|D_i| \ge \varepsilon_t\varepsilon_{t+1}^{-1}|D_i|$ for all $i \in [t]$ and $\{D_1, \dots, D_t\}$ is an (ε_t, ξ)-blowup of $H[\{v_1, \dots, v_t\}]$. Also, from the above construction, for all $i \in \{0, 1, \dots, t\}$, we have $|P_{t+1}| = |S_t| \ge (\lambda_{t,t}\lambda_{t,t-1} \cdots \lambda_{t,i+1})|S_i| = \Gamma_{t,i}|S_i|$, in particular (P_i, P_{t+1}) is (ε_{t+1}, ξ)-full if $v_iv_{t+1} \in E(H)$ and is (ε_{t+1}, ξ)-empty if $v_iv_{t+1} \notin E(H)$. Thus $\{P_1, \dots, P_t, P_{t+1}\}$ is an (ε_{t+1}, ξ)-blowup of $H[\{v_1, \dots, v_t, v_{t+1}\}]$.
- For every $i \in [t]$, since $\Lambda_{t+1,i} = \lambda_{t,i}\Lambda_{t,i}$, $|L'| \le \eta'|S \setminus S_t| < \eta'|S| \le \eta'|L|$ as $|S_t| > 0$, and $\eta' \le \Gamma_{t,0} < \lambda_{t,i}$ by the definition of η', we have

$$|P_i| \ge \lambda_{t,i}|D_i| > \lambda_{t,i} \max(\Lambda_{t,i}d, 2\eta^{-1}|L|) \ge \max(\Lambda_{t+1,i}d, 2\eta^{-1}\lambda_{t,i}(\eta')^{-1}|L'|) \ge \max(\Lambda_{t+1,i}d, 2\eta^{-1}|L'|);$$

  and since $\Lambda_{t+1,t+1} = \delta'\Gamma_{t,0}$ and $\eta' \le \frac{1}{2}\eta\delta'\Gamma_{t,0}$ by definition, we deduce that

$$|P_{t+1}| = |S_t| \ge \Gamma_{t,0}|S_0| \ge \Gamma_{t,0}\delta'|S| > \max(\Lambda_{t+1,t+1}d, \Gamma_{t,0}\delta'(\eta')^{-1}|L'|) \ge \max(\Lambda_{t+1,t+1}d, 2\eta^{-1}|L'|).$$

  Also, for every $i \in [t]$, P_i is ε_{t+1}-restricted since D_i is ε_t-restricted; and $P_{t+1} = S_t$ is ε_{t+1}-restricted since S_0 is ε'-restricted and $\varepsilon' \le \varepsilon_{t+1}\Gamma_{t,0}$ by the definition of ε'.

This proves Claim 3.6. □

Now, recall that $V(G) \setminus S$ is partitioned into (possibly empty) subsets

$$A_1, \dots, A_m;\ B_1, \dots, B_m;\ C_1, \dots, C_n;\ D_1, \dots, D_t;\ L_1, \dots, L_t$$

such that

- $m \le \binom{t}{2}$ and $n \le t \cdot \phi(\delta', \eta')$;
- $A_1, \dots, A_m, C_1, \dots, C_n$ are nonempty and ε-restricted;
- for every $i \in [m]$, $|B_i| \le \eta|A_i|$ and B_i is θ-tight to A_i; and
- if $t > 0$, then for every $i \in [t]$, $|D_i| > 2\eta^{-1}|L_i| \ge \eta^{-1}|L_i|$, D_i is ε-restricted (since it is ε_t-restricted), and L_i is θ-tight to D_i.

By renumbering the above sets if necessary, we may assume there exist $q \in \{0, 1, \dots, m\}$ and $r \in \{0, 1, \dots, t\}$ such that $B_i \ne \emptyset$ for all $i \in [q]$ and $B_i = \emptyset$ for all $i \in [m] \setminus [q]$, and $L_i \ne \emptyset$ for all $i \in [r]$ and $L_i = \emptyset$ for all $i \in [t] \setminus [r]$. Then the following partition of $V(G)$

$$A_1, \dots, A_q, D_1, \dots, D_r;\ B_1, \dots, B_q, L_1, \dots, L_r;\ A_{q+1}, \dots, A_m, D_{r+1}, \dots, D_t, C_1, \dots, C_n$$

has the desired property, because $q + r \le m + t \le \binom{t}{2} + t = \binom{t+1}{2} \le \binom{h}{2}$, and

$$(m - q) + (t - r) + n \le \binom{t}{2} + t + t \cdot \phi(\delta', \eta') \le \binom{h}{2} + (h - 1) \cdot \phi(\delta', \eta') = N.$$

This proves Lemma 3.1. ■

We remark that the proof of Lemma 3.2 in [14] shows that $\gamma_{3.2}^{-1}(c,\varepsilon,\tau)$ can be chosen to be roughly $2^{((\tau-\varepsilon)c^2)^{-1}\log^2(c^{-1})}$ for sufficiently small c,ε,τ. This leads to bounds on $\kappa_{3.1}^{-1}(H,\varepsilon,\eta,\theta)$ and $N_{3.1}(H,\varepsilon,\eta,\theta)$ which are towers of twos of height depending on h with $(\varepsilon\eta\theta)^{-1}$ on top, which results in the tower-type dependence of $\kappa_{1.7}^{-1}(H,\varepsilon)$ and $N_{1.7}(H,\varepsilon)$ on ε^{-1} and h mentioned in the Introduction.

4. Finishing the proof

With Lemma 3.1 in hand, it now suffices to make obvious changes to [4, Section 3] to complete the proof of Theorem 1.7. For $\varepsilon>0$, an integer $k\ge 0$, and a graph G, a (k,ε)-*path-partition* of G is a sequence $(W_0,W_1,\dots,W_k)$ of disjoint nonempty sets with union $V(G)$ such that for every $i\in\{0,1,\dots,k-1\}$,

- W_i is ε-restricted in G;
- $|W_i|\ge 12|W_k|$; and
- $W_{i+1}\cup\cdots\cup W_k$ is $\frac{1}{12}\varepsilon$-tight to W_i.

We need the following result [4, Theorem 3.3].

Lemma 4.1. *For all $\varepsilon\in(0,\frac13)$, every graph with a $(\lceil 4\varepsilon^{-1}\rceil,\frac14\varepsilon)$-path-partition is $(2400\varepsilon^{-2},\varepsilon)$-restricted.*

The following lemma allows us to "lengthen" a given path-partition of length less than $\lceil 4\varepsilon^{-1}\rceil$ by one, at the cost of removing a small number of vertices.

Lemma 4.2. *Let $\varepsilon\in(0,\frac13)$ and $K:=\lceil 4\varepsilon^{-1}\rceil$. Let H be a graph with $V(H)=[h]$ for some $h\ge 2$. Let*

$$\varepsilon':=h^{-2K}\varepsilon,\qquad \eta:=h^{-2},\qquad \theta:=\frac{1}{12}h^{-2K}\varepsilon,$$
$$\kappa=\kappa_{4.2}(H,\varepsilon):=h^{-2Kh}\cdot\kappa_{3.1}(H,\varepsilon',\eta,\theta),\qquad N=N_{4.2}(H,\varepsilon):=N_{3.1}(H,\varepsilon',\eta,\theta).$$

Let k be an integer with $0\le k\le K$. Let $d\ge 0$, and let G be a graph with $\operatorname{ind}(G)\le\kappa d^h$ such that G has a $(k,h^{2(k-K)}\varepsilon)$-path-partition $(W_0,W_1,\dots,W_k)$. Then there exists $S\subseteq V(G)$ with $|S|\le h^{-2k}d$ such that $G\setminus S$ is $(h^{2(K-k)}(2400\varepsilon^{-2}+N)-N,\varepsilon)$-restricted.

Proof. We proceed by backward induction on k. If $k=K$ then the conclusion follows by Lemma 4.1. We may assume that $k<K$ and that the lemma holds for $k+1$. Since

$$\operatorname{ind}(G[W_k])\le\kappa d^h\le\kappa_{3.1}(H,\varepsilon',\eta,\theta)\cdot(h^{-2(k+1)}d)^h,$$

by Lemma 3.1 applied to $G[W_k]$ with d replaced by $h^{-2(k+1)}d$, there is $T\subseteq W_k$ with $|T|\le h^{-2(k+1)}d$ such that $W_k\setminus T$ can be partitioned into nonempty sets

$$A_1,\dots,A_m,B_1,\dots,B_m,C_1,\dots,C_n$$

where $m\le\binom{h}{2}$ and $n\le N$, such that

- $A_1,\dots,A_m,C_1,\dots,C_n$ are ε'-restricted in G; and
- for every $i\in[m]$, $|B_i|\le\eta|A_i|$ and B_i is θ-tight to A_i.

If $m=0$ then G is $(k+N,\varepsilon)$-restricted and we are done (note that $h\ge 2$ and $k\le 4\varepsilon^{-2}$); thus we may assume $m\ge 1$. It follows that $|W_k|\ge|A_1|\ge\eta^{-1}|B_1|\ge h^2\ge 2m$, and so $|W_i|\ge 12|W_k|\ge 24m$ for all $i\in\{0,1,\dots,k-1\}$. Thus for each such i, W_i has a partition $W_i^1\cup\cdots\cup W_i^m$ with $|W_i^j|\ge\lfloor\frac1m|W_i|\rfloor\ge\frac{1}{2m}|W_i|\ge h^{-2}|W_i|$ for all $j\in[m]$. Let $U_j:=\bigcup_{i=0}^{k-1}W_i^j\cup(A_j\cup B_j)$ for every $j\in[m]$.

Claim 4.3. *For all $j\in[m]$, $(W_0^j,W_1^j,\dots,W_{k-1}^j,A_j,B_j)$ is a $(k+1,h^{2(k+1-K)}\varepsilon)$-path-partition of $G[U_j]$.*

Proof. It suffices to observe the following.

- A_j is $h^{2(k+1-K)}\varepsilon$-restricted since it is ε'-restricted and $\varepsilon'=h^{-2K}\varepsilon$; and also, for each $i\in\{0,1,\dots,k-1\}$, W_i^j is $h^{2(k+1-K)}\varepsilon$-restricted since W_i is $h^{2(k-K)}\varepsilon$-restricted and $|W_i^j|\ge h^{-2}|W_i|$.
- For every $i\in\{0,1,\dots,k-1\}$, since $12|W_k|\le|W_i|\le h^2|W_i^j|$, we have

$$12|B_j|\le 12\eta|A_j|\le\min(12h^{-2}|W_k|,|A_j|)\le\min(|W_i^j|,|A_j|).$$

- B_j is $\frac{1}{12}h^{2(k+1-K)}\varepsilon$-tight to A_j by the definition of θ; and also, for every $i \in \{0,1,\ldots,k-1\}$, $(W^j_{i+1} \cup \cdots \cup W^j_{k-1}) \cup (A_j \cup B_j)$ is $\frac{1}{12}h^{2(h+1-K)}$-tight to W^j_i since $W_{i+1} \cup \cdots \cup W_k$ is $\frac{1}{12}h^{2(k-K)}\varepsilon$-tight to W_i and $|W^j_i| \geq h^{-2}|W_i|$.

This proves Claim 4.3. □

By Claim 4.3 and induction, for each $j \in [m]$, there is $S_j \subseteq U_j$ with $|S_j| \leq h^{-2(k+1)}d$ such that $G[U_j \setminus S_j]$ is $(h^{2(K-k-1)}(2400\varepsilon^{-2}+N)-N,\varepsilon)$-restricted. Put $S := \bigcup_{j\in[m]} S_j \cup T$; then $|S| \leq (m+1)h^{-2(k+1)}d \leq h^{-2k}d$ as $h \geq 2$, and since

$$\begin{aligned} m \cdot (h^{2(K-k-1)}(2400\varepsilon^{-2}+N)-N)+n &\leq h^2 \cdot (h^{2(K-k-1)}(2400\varepsilon^{-2}+N)-N)+N \\ &\leq h^{2(K-k)}(2400\varepsilon^{-2}+N)-N, \end{aligned}$$

we see that $G \setminus S$ is $(h^{2(K-k)}(2400\varepsilon^{-2}+N)-N,\varepsilon)$-restricted. This proves Lemma 4.2. ■

We are now ready to finish the proof of Theorem 1.7, which we restate here for the reader's convenience.

Theorem 4.4. *For every $\varepsilon > 0$ and every graph H, there exist $\kappa = \kappa(H,\varepsilon) > 0$ and $N = N(H,\varepsilon) > 0$ such that for every $d \geq 0$ and every graph G with $\operatorname{ind}(G) \leq \kappa d^h$, there is $S \subseteq V(G)$ with $|S| \leq d$ such that $G \setminus S$ is (N,ε)-restricted.*

Proof. We may assume $h \geq 2$. Let $\kappa := \kappa_{4.2}(H,\varepsilon)$ and $N := h^{2K}(2400\varepsilon^{-2} + N_{4.2}(H,\varepsilon))$. Then Theorem 4.4 follows from Lemma 4.2 applied to the $(0, h^{-2K}\varepsilon)$-partition $V(G)$ of G. ■

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