# HIGHLY CONNECTED SUBGRAPHS WITH LARGE CHROMATIC NUMBER

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Abstract. For integers $k \geq 1$ and $m \geq 2$, let $g(k, m)$ be the least integer $n \geq 1$ such that every graph with chromatic number at least $n$ contains a $(k+1)$-connected subgraph with chromatic number at least $m$. Refining the recent result Girão and Narayanan that $g(k-1, k) \leq 7k+1$ for all $k \geq 2$, we prove that $g(k, m) \leq \max(m+2k-2, \lceil(3+\frac{1}{16})k\rceil)$ for all $k \geq 1$ and $m \geq 2$. This sharpens earlier results of Alon, Kleitman, Saks, Seymour, and Thomassen, of Chudnovsky, Penev, Scott, and Trotignon, and of Penev, Thomassé, and Trotignon.

Our result implies that $g(k, k+1) \leq \lceil(3+\frac{1}{16})k\rceil$ for all $k \geq 1$, making a step closer towards a conjecture of Thomassen from 1983 that $g(k, k+1) \leq 3k+1$, which was originally a result with a false proof and was the starting point of this research area.

## 1. Introduction

All graphs in this paper are finite and with no loops or parallel edges. Given a graph $G$ with vertex set $V(G)$, a *cutset* in $G$ is a (possibly empty) set $X \subseteq V(G)$ whose removal from $G$ results in a disconnected graph. For an integer $k \geq 1$, $G$ is said to be *$k$-connected* if it has more than $k$ vertices and has no cutset of cardinality less than $k$. A *stable set* of $G$ is a vertex set with pairwise nonadjacent vertices in $G$. The *chromatic number* of $G$, denoted by $\chi(G)$, is the least integer $m \geq 0$ such that $V(G)$ can be partitioned into $m$ stable sets.

The starting point of this paper is the following conjecture of Thomassen [13, Theorem 11] from 1983 which was originally a result with a false proof.

**Conjecture 1.1.** *For every integer $k \geq 1$, every graph with chromatic number more than $3k$ contains a $(k+1)$-connected graph with chromatic number more than $k$.*

For integers $k \geq 1$ and $m \geq 2$, let $g(k, m)$ be the smallest integer $n \geq 1$ such that every graph with chromatic number at least $n$ contains a $(k+1)$-connected subgraph with chromatic number at least $m$. Thus $g(k, 2)$ is simply the least integer $n \geq 1$ such that every graph with chromatic number at least $n$ contains a $(k+1)$-connected subgraph, $g(1, m) = \max(m, 3)$ for all $m \geq 2$, and Conjecture 1.1 says $g(k, k+1) \leq 3k+1$ for all $k \geq 1$. Here are the known estimates on $g(k, m)$ for $k, m \geq 2$.

- Alon, Kleitman, Saks, Seymour, and Thomassen [1], seeking for a remedy for the incorrect proof of Conjecture 1.1 in [13], initiated the study of $g(k, m)$ and proved that[1]

$$\max(m+k-1, 2k+1) \leq g(k, m) \leq \max(m+10k^2, 100k^3+1).$$

- Partly motivated by the study of *$\chi$-boundedness* (see [12] for a survey on this topic), Chudnovsky, Penev, Scott, and Trotignon [5] proved (among other things) that

$$g(k, m) \leq \max(m+2k^2, 2k^2+k+1)$$

and Penev, Thomassé, and Trotignon [11] proved that

$$g(k, m) \leq \max(m+2k-2, 2k^2+1).$$

- Motivated by recent progress on Hadwiger's conjecture [6], Girão and Narayanan [8] proved that $g(k-1, k) \leq 7k+1$; in fact their argument can be slightly modified to get $g(k-1, k) \leq 4k$.

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[1]The authors of [1] only stated that $g(k, m) \geq m+k-1$, but their lower bound construction in fact shows that $g(k, m) \geq \max(m+k-1, 2k+1)$.

A classical result of Mader [9] (see also [7, Theorem 1.4.3]) says that for every $k \geq 1$, every graph with average degree at least $4k$ contains a $(k+1)$-connected subgraph with more than $2k$ vertices. This leads to two natural questions:

- What is the smallest constant $C > 0$ such that every graph with average degree at least $Ck$ contains a $(k+1)$-connected subgraph with more than $2k$ vertices? Carmesin [4] recently showed that $C = 3 + \frac{1}{3}$ is the correct answer.
- What if we just ask for the $(k+1)$-connectivity without any demands on the number of vertices of the subgraph? Mader [10] conjectured that every graph with average degree at least $3k-1$ contains a $(k+1)$-connected subgraph, and the current record on this problem is held by Bernshteyn and Kostochka [2] who proved that $(3+\frac{1}{6})k$ suffices as long as the host graph has at least $\frac{5}{2}k$ vertices.

Our main result, which can be considered as an analogue of Carmesin's theorem for the chromatic number in some sense, improves all of the aforementioned upper bounds on $g(k,m)$. It shows that

$$g(k,m) \leq \max\left(m + 2k - 2, \left\lceil \left(3 + \frac{1}{16}\right) k \right\rceil\right)$$

for all $k \geq 1$ and $m \geq 2$, approaching the lower bound $g(k,m) \geq \max(m+k-1, 2k+1)$; and consequently, Conjecture 1.1 is true with 3 replaced by $3 + \frac{1}{16}$.

**Theorem 1.2.** *For every integer $k \geq 1$, every graph $G$ with $\chi(G) \geq (3+\frac{1}{16})k$ contains a $(k+1)$-connected subgraph with more than $\chi(G) - k$ vertices and chromatic number at least $\chi(G) - 2k + 2$.*

We suspect that the constant factor $3 + \frac{1}{16}$ in Theorem 1.2 can be replaced by 3, which would verify Conjecture 1.1.

## 2. Templates and inextensibility

The proof of Theorem 1.2 employs the "template-inextensibility" method first appeared in [5] (under the name "coloring constraints") and developed further in [8], with a number of modifications. Given sets $A \subseteq B$ and a map $f$ with domain $B$, let $f|_A$ be the restriction of $f$ to $A$, and let $f(A) := \{f(a) : a \in A\}$. For a graph $G$, let $|G|$ denote the number of vertices of $G$. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of $G$ with vertex set $S$ and edges whose endpoints are in $S$; a graph $H$ is an *induced subgraph* of $G$ if there is $S \subseteq V(G)$ with $H = G[S]$, and $H$ is a *proper* induced subgraph of $G$ if $|H| < |G|$.

For a finite set $\mathcal{C}$ of colors, a *proper $\mathcal{C}$-coloring* of $G$ is a function $f\colon V(G) \to \mathcal{C}$ satisfying $f(u) \neq f(v)$ for all $u, v$ adjacent vertices of $G$; thus $\chi(G)$ is the least integer $m \geq 0$ such that there is a proper $\mathcal{C}$-coloring of $G$ with $|\mathcal{C}| = m$. A *$\mathcal{C}$-template on* $G$ is a triple $T = (S, c, F)$ where

- $S$ is a subset of $V(G)$;
- $c\colon S \to \mathcal{C}$ is a proper $\mathcal{C}$-coloring of $G[S]$; and
- $F$ is a map from $V(G) \setminus S$ to the family of all subsets of $\mathcal{C}$.

It might be helpful to think of the vertices of $S$ as the *precolored* vertices, and each $v \in V(G) \setminus S$ as an *uncolored* vertex with a list of *forbidden colors* specified by $F(v)$.

For an integer $k \geq 1$, given a $\mathcal{C}$-template $T = (S, c, F)$ on $G$, define the *$k$-cost* of $T$ by

$$\operatorname{cost}_k(T) := k|S| + \sum_{v \in V(G) \setminus S} |F(v)|.$$

Every $A \subseteq V(G)$ naturally gives rise to the $\mathcal{C}$-template $T_A := (S \cap A, c|_{S \cap A}, F|_{A \setminus S})$ on $G[A]$. Note that $\operatorname{cost}_k$ is additive under disjoint unions, that is, $\operatorname{cost}_k(T_{A \cup B}) = \operatorname{cost}_k(T_A) + \operatorname{cost}_k(T_B)$ for all disjoint $A, B \subseteq V(G)$.

A proper $\mathcal{C}$-coloring $f$ of $G$ is said to *respect* $T$ if $f|_S = c$ and $f(v) \in \mathcal{C} \setminus F(v)$ for all $v \in V(G) \setminus S$. Say that $G$ is *$\mathcal{C}$-inextensible* if there exists a $\mathcal{C}$-template $T = (S, c, F)$ on $G$ such that

- $\operatorname{cost}_k(T) < 2k^2$;
- $|F(v)| \leq k$ for all $v \in V(G) \setminus S$; and
- there is no proper $\mathcal{C}$-coloring of $G$ respecting $T$.

In this case, say that $T$ *witnesses* the $\mathcal{C}$-inextensibility of $G$; note that $|S| < 2k$. Observe that if $\chi(G) > 0$, then there exists $\mathcal{C}$ so that $G$ is $\mathcal{C}$-inextensible: indeed, such a $\mathcal{C}$ can be chosen to be any set of $\chi(G) - 1$ colors, then the $\mathcal{C}$-inextensibility of $G$ is witnessed by the empty $\mathcal{C}$-template with no precolored vertices and no forbidden colors at each uncolored vertex. Say that $G$ is $\mathcal{C}$-*extensible* if it is not $\mathcal{C}$-inextensible.

In what follows, when there is no danger of ambiguity, we drop the prefix $\mathcal{C}$ from the notions of proper $\mathcal{C}$-colorings, $\mathcal{C}$-templates, and $\mathcal{C}$-inextensibility. We also drop the prefix $k$ from the notion of $k$-costs, and drop the subscript $k$ from the notation $\operatorname{cost}_k$.

## 3. Connectivity

We wish to show that, given a set $\mathcal{C}$ of colors, if $|\mathcal{C}|$ is sufficiently large, then every inextensible graph contains a $(k+1)$-connected subgraph. Here is an example showing that $|\mathcal{C}| \geq 3k-1$ is necessary: if $|\mathcal{C}| = 3k-2$, let $G$ be a star[2] with $2k$ vertices, and consider a template on $G$ where the leaves are precolored by different colors and the center has $k-1$ forbidden colors which are not used for the leaves; then this template witnesses the inextensibility of $G$ while $G$ certainly has no 2-connected subgraphs. It turns out that $|\mathcal{C}| \geq 3k-1$ is also sufficient. To see this, let us say that a template $T = (S, c, F)$ on an inextensible graph $G$ is *good* if $T$ witnesses the inextensibility of $G$ and $|F(v)| \leq k-1$ for all $v \in V(G) \setminus S$, that is, each uncolored vertex has fewer than $k$ forbidden colors. The following lemma says that every inextensible graph has a good template as long as $|\mathcal{C}| \geq 3k-1$.

**Lemma 3.1.** *If $|\mathcal{C}| \geq 3k-1$ and $G$ is inextensible, then there is a good template on $G$.*

**Proof.** Since $G$ is inextensible, there exists a template $T = (S, c, F)$ witnessing its inextensibility; choose $T$ with $|S|$ maximal. Suppose there is $v \in V(G) \setminus S$ with $|F(v)| = k$; let $S' := S \cup \{v\}$. We have that

$$2k^2 - k|S| > \operatorname{cost}(T) - k|S| \geq |F(v)| = k,$$

and so $|S| < 2k-1$. Because $|\mathcal{C}| \geq 3k-1$, it follows that (recall that $c(S) = \{c(u) : u \in S\}$)

$$|\mathcal{C} \setminus (c(S) \cup F(v))| \geq |\mathcal{C}| - |S| - |F(v)| > (3k-1) - (2k-1) - k = 0,$$

so $\mathcal{C} \setminus (c(S) \cup F(v)) \neq \emptyset$. Let $T' := (S', c', F|_{V(G)\setminus S'})$ be a template on $G$ with $c'$ satisfying $c'|_S = c$ and $c'(v) \in \mathcal{C} \setminus (c(S) \cup F(v))$. Then

$$\operatorname{cost}(T') = \operatorname{cost}(T) + k - |F(v)| = \operatorname{cost}(T) < 2k^2$$

so $T'$ would witness the inextensibility of $G$ with $|S'| > |S|$, contradicting the maximality of $|S|$. Therefore $|F(v)| \leq k-1$ for all $v \in V(G) \setminus S$, that is, $T$ is good on $G$. This proves Lemma 3.1. ■

A graph is said to be *minimally inextensible* if it is inextensible while its proper induced subgraphs are extensible. It is immediate that every inextensible graph contains a minimally inextensible induced subgraph; and as the following lemma shows, every minimally inextensible graph is $(k+1)$-connected as long as $|\mathcal{C}| \geq 3k-1$. This constitutes the connectivity part of Theorem 1.2.

**Lemma 3.2.** *If $|\mathcal{C}| \geq 3k-1$, then every minimally inextensible graph has more than $|\mathcal{C}| - k + 1$ vertices and is $(k+1)$-connected.*

**Proof.** Let $G$ be a minimally inextensible graph. By Lemma 3.2, there is a good template $T = (S, c, F)$ on $G$.

**Claim 3.1.** *Every vertex in $S$ has more than $k$ neighbors in $V(G) \setminus S$.*

*Proof.* Let $u \in S$, and let $M$ be the set of neighbors of $u$ in $V(G) \setminus S$. Let $T' := (S \setminus \{u\}, c|_{S\setminus\{u\}}, F')$ be the template on $G \setminus u$ with $F'$ defined by

- $F'(v) := F(v)$ for all $v \in V(G) \setminus (S \cup M)$; and
- $F'(v) := F(v) \cup \{c(u)\}$ for all $v \in M$.

[2]A *star* is a complete bipartite graph with one part having only one vertex (called the *center*); and the vertices in the other part are called the *leaves*. If both parts have size one each then the center can be either one of the two vertices.

As $|F(v)| \le k-1$ for all $v \in V(G) \setminus S$, we have that $|F'(v)| \le k$ for all $v \in V(G) \setminus S$. Observe that

$$\begin{aligned}\mathrm{cost}(T') &\le k|S \setminus \{u\}| + \sum_{v \in V(G)\setminus(S\cup M)} |F(v)| + \sum_{v\in M}(|F(v)|+1)\\ &= k|S| - k + \sum_{v\in V(G)\setminus S}|F(v)| + |M| = \mathrm{cost}(T) + |M| - k.\end{aligned}$$

If $|M| \le k$, then $\mathrm{cost}(T') \le \mathrm{cost}(T) < 2k^2$; so the extensibility of $G \setminus u$ would give a proper coloring of $G \setminus u$ respecting $T'$ and so a proper coloring of $G$ respecting $T$, a contradiction. This shows that $|M| > k$, as required. This proves Claim 3.1. □

**Claim 3.2.** *Every vertex in $V(G) \setminus S$ has more than $|\mathcal{C}| - k$ neighbors in $G$.*

*Proof.* Let $v \in V(G) \setminus S$, and let $N$ be the set of neighbors of $v$ in $G$. The extensibility of $G \setminus v$ yields a proper coloring $c'$ of $G \setminus v$ respecting $T_{V(G)\setminus\{v\}}$. If $|N| \le |\mathcal{C}| - k$, then by the goodness of $T$

$$|\mathcal{C} \setminus (c'(N) \cup F(v))| \ge |\mathcal{C}| - |N| - |F(v)| > |\mathcal{C}| - (|\mathcal{C}| - k) - k = 0$$

so there would be a proper coloring of $G$ respecting $T$, a contradiction. This proves Claim 3.2. □

Now, Claims 3.1 and 3.2 together imply that $|G| > |\mathcal{C}| - k + 1 \ge 2k \ge k+1$. Next, suppose that $G$ is not $(k+1)$-connected, then there would be a cutset $X$ of $G$ with $|X| \le k$ and with disjoint nonempty sets of vertices $A, B$ with $A \cup B = V(G) \setminus X$ and no edges between them. Since cost is additive under disjoint unions, we have $\mathrm{cost}(T_A) + \mathrm{cost}(T_B) \le \mathrm{cost}(T) < 2k^2$; and so we may assume $\mathrm{cost}(T_B) < k^2$. Let $D := A \cup X \cup S$; then $|D| < |G|$ since $B \not\subseteq S$ by Claim 3.1. Thus $G[D]$ is extensible, and so has a proper coloring $c'$ respecting $T_D$. Let $T' := (S', c'', F|_{B\setminus S})$ be the template on $G[B \cup X]$ with $S' := X \cup (B \cap S)$ and $c''$ defined by $c''|_X := c'|_X$ and $c''|_{B\cap S} := c|_{B\cap S}$; note that $c''$ is a proper coloring of $G[S']$. Since

$$\mathrm{cost}(T') = k|X \cup (B \cap S)| + \sum_{v \in B\setminus S} |F(v)| = k|X| + \mathrm{cost}(T_B) < k^2 + k^2 = 2k^2,$$

and since $G[B \cup X]$ is extensible, it has a proper coloring $c'''$ respecting $T'$. As $c'|_X = c''|_X = c'''|_X$, gluing $c'$ and $c'''$ would give a proper coloring of $G$ respecting $T$, a contradiction. This proves Lemma 3.2. ■

## 4. Chromatic number

This section deals with the chromatic part of Theorem 1.2. We aim to prove that if $|\mathcal{C}|$ is sufficiently large then every inextensible graph has chromatic number as large as desired. To do so, we prove that if an inextensible graph $G$ has small chromatic number, then we can find a proper coloring of $G$ respecting a good template on $G$ (this is similar to the approach in the proof of Lemma 3.2). For an integer $n \ge 0$, let $[n]$ be $\{1, 2, \dots, n\}$ if $n \ge 1$ and $\emptyset$ if $n = 0$. Here is the main result of this section.

**Lemma 4.1.** *If $|\mathcal{C}| \ge (3 + \frac{1}{16})k - 1$, then every inextensible graph $G$ satisfies $\chi(G) \ge |\mathcal{C}| - 2k + 3$.*

It is worth noting that the bound $|\mathcal{C}| - 2k + 3$ is optimal in Lemma 4.1, since it is false if we ask for $|\mathcal{C}| - 2k + 4$ under the weaker condition $|\mathcal{C}| \ge 2k - 1$: indeed, let $m := |\mathcal{C}| - 2k + 4 \ge 3$, let $S$ be a stable set of cardinality $2k - 1$, let $K$ be a complete graph on $m - 2$ vertices, and let $H_{k,m}$ be the graph obtained by joining every vertex in $S$ to every vertex in $K$; then $\chi(H_{k,m}) = m - 1$, but the template on $H_{k,m}$ with the vertices in $S$ precolored differently and no forbidden colors at the vertices in $K$ witnesses the inextensibility of $H_{k,m}$. However, we do not know whether Lemma 4.1 still holds when $|\mathcal{C}| \ge 3k - 1$, which was the condition on $|\mathcal{C}|$ needed to guarantee the $(k+1)$-connectivity of minimally inextensible graphs back in Section 3.

For the rest of this section we make use of the following setup. Let $|\mathcal{C}| \ge 3k-1$, let $G$ be an inextensible graph, let $\chi := \chi(G)$, and let $S_1 \cup \dots \cup S_\chi$ be a partition of $V(G)$ into $\chi$ stable sets. Let $T = (S, c, F)$ be a good template of $G$ given by Lemma 3.1, that is, $|F(v)| < k$ for all $v \in V(G) \setminus S$. For every $P \subseteq V(G)$, let $w(P) := \sum_{v\in P} |F(v)|$ be the *weight* of $P$; note that $w$ is additive under disjoint unions. For every

$i \in [\chi]$, let $P_i := S_i \setminus S$ and $p_i := \lfloor w(P_i)/k \rfloor$; we may assume $P_i \neq \emptyset$, since we can add an isolated vertex to $P_i$ if $P_i = \emptyset$. Let $p := p_1 + \cdots + p_\chi$ and $t := 2k - |S|$, then

$$kt = 2k^2 - k|S| > \mathrm{cost}(T) - k|S| \geq w(P_1) + \cdots + w(P_\chi). \tag{1}$$

Thus, since $w(P_i) \geq kp_i$ for all $i \in [\chi]$, we obtain $kt > k(p_1 + \cdots + p_\chi) = kp$ which yields

$$p \leq t - 1. \tag{2}$$

4.1. $4k-1$ **with a loss of** $2k-1$ **suffices.** To give a better explanation of the main argument, we give a quick proof of the bound $g(k,m) \leq \max(m + 2k - 1, 4k - 1)$ for all $k \geq 1$ and $m \geq 2$, whose chromatic part follows from the following weaker version of Lemma 4.1.

**Lemma 4.2.** *If* $|\mathcal{C}| \geq 4k - 2$ *then* $\chi \geq |\mathcal{C}| - 2k + 2$.

The proof of this lemma resembles the argument in [8], utilizing the following simple fact.

**Lemma 4.3.** *Let* $k, q \geq 1$ *be integers, and let* $Q$ *be a set of integers (repeated members allowed) with* $0 \leq a \leq k$ *for all* $a \in Q$ *and* $qk \leq \sum_{a \in A} a < (q+1)k$. *Then there exists a partition* $Q = Q_1 \cup \cdots \cup Q_q$ *such that* $\sum_{a \in Q_i} a < 2k$ *for all* $i \in [q]$.

**Proof.** The lemma is trivial for $q = 1$. Let $q > 1$ and assume that it folds for $q - 1$. Let $R$ be a maximal subset of $Q$ with $\sum_{a \in R} a \geq (q-1)k$, then $\sum_{a \in R} a < qk$ as $0 \leq a \leq k$ for all $a \in Q$; so by induction there is a partition $R = Q_1 \cup \cdots \cup Q_{q-1}$ with $\sum_{a \in Q_i} a < 2k$ for all $i \in [q-1]$. Then let $Q_q := Q \setminus R$ and note that $\sum_{a \in Q_q} a < (q+1)k - (q-1)k = 2k$. This proves Lemma 4.3. ∎

**Proof of Lemma 4.2.** Let $I_0 := \{i \in [\chi] : p_i = 0\}$, let $I_1 := \{i \in [\chi] : p_i \geq 1\}$, and let $P := \bigcup_{i \in I_1} P_i$; note that $p = \sum_{i \in I_1} p_i$. By Lemma 4.3, for each $i \in I_1$ there is a partition $P_i = P_{i1} \cup \cdots \cup P_{ip_i}$ such that $w(P_{ij}) < 2k$ for all $j \in [p_i]$. For every $i \in I_1$ and $j \in [p_i]$, let $L_{ij} := \mathcal{C} \setminus (c(S) \cup \bigcup_{v \in P_{ij}} F(v))$, then since $p \leq t - 1 = 2k - |S| - 1$ by (2) and since $|\mathcal{C}| \geq 4k - 2$, we see that

$$|L_{ij}| \geq |\mathcal{C}| - |S| - w(P_{ij}) \geq |\mathcal{C}| - (2k - p - 1) - (2k - 1) = |\mathcal{C}| - (4k - 2) + p \geq p.$$

Hence we can assign $p = \sum_{i \in I_1} p_i$ different colors to the stable sets in $\bigcup_{i \in I_1} \{P_{ij} : j \in [p_i]\}$, obtaining a proper coloring $c'$ of $G[S \cup P]$ respecting $T_{S \cup P}$ with $|c'(P)| \leq p$. Put $S' := S \cup P$ and $t' := t - p \geq 1$; then

$$|c'(S')| \leq |S| + p = 2k - t + p = 2k - t'.$$

Now, suppose for a contradiction that $\chi \leq |\mathcal{C}| - 2k + 1$, and let $L_i := \mathcal{C} \setminus (c'(S') \cup \bigcup_{v \in P_i} F(v))$ for all $i \in I_0$. Let $I' := \{i \in I_0 : w(P_i) \geq t'\}$. Note that for every $i \in I' \setminus I_0$, we have that

$$|L_i| \geq |\mathcal{C}| - |c'(S')| - w(P_i) \geq |\mathcal{C}| - (2k - t') - (t' - 1) = |\mathcal{C}| - 2k + 1 \geq \chi \geq |I_0|.$$

Thus, if we can assign $|I'|$ different colors to the stable sets in $\{P_i : i \in I'\}$ such that each $P_i$ gets a color in $L_i$, then we can assign $|I_0|$ different colors to the stable sets in $\{P_i : i \in I_0\}$ such that each $P_i$ gets a color in $L_i$. This would give a proper coloring of $G$ respecting $T$, a contradiction.

We now assign colors to $\{P_i : i \in I'\}$. We may assume that $I' \neq \emptyset$. Let $x_i := w(P_i)$ for all $i \in I'$, and assume that $I' = [n]$ and $x_1 \geq \ldots \geq x_n$ for some $n \geq 1$. Since $p = \sum_{i \in I_1} p_i$, by (1) we see that for every $i \in [n]$,

$$kt > \sum_{i \in I_0} x_i + \sum_{i \in I_1} w(P_i) \geq ix_i + k \sum_{i \in I_1} p_i = ix_i + kp$$

so $i < kt'/x_i$. It follows that, since $x_i = w(P_i) \leq k$,

$$x_i + i < x_i + \frac{kt'}{x_i} = k + t' - \frac{(k - x_i)(x_i - t')}{x_i} \leq k + t'$$

so $x_i + i \leq k + t' - 1$, which yields (note that $|\mathcal{C}| \geq 4k - 2 \geq 3k - 1$)

$$|L_i| - i \geq |\mathcal{C}| - |c'(S')| - x_i - i \geq |\mathcal{C}| - (2k - t') - (k + t' - 1) = |\mathcal{C}| - (3k - 1) \geq 0.$$

Thus, $|L_i| \geq i$ for all $i \in [n]$, and so we can greedily assign $n = |I'|$ different colors to the stable sets $P_1, \ldots, P_n$ such that each $P_i$ gets a color in $L_i$. This proves Lemma 4.2. ∎

We can now give a proof that $g(k,m) \leq \max(m+2k-1, 4k-1)$ for all $k \geq 1$ and $m \geq 2$.

**Proposition 4.1.** *For every integer $k \geq 1$, every graph $G$ with $\chi(G) \geq 4k-1$ contains a $(k+1)$-connected subgraph with more than $\chi(G) - k$ vertices and chromatic number at least $\chi(G) - 2k + 1$.*

**Proof.** Let $\mathcal{C}$ be a set of $\chi(G) - 1$ colors, then $G$ is $\mathcal{C}$-inextensible; so it has a minimally $\mathcal{C}$-inextensible subgraph $H$. Then $H$ is $(k+1)$-connected and has more than $\chi(G) - k$ vertices by Lemma 3.2, and satisfies $\chi(H) \geq |\mathcal{C}| - 2k + 2 = \chi(G) - 2k + 1$ by Lemma 4.2. This proves Proposition 4.1. ■

4.2. **Reduction step.** The proof of Lemma 4.2 consists of two steps: coloring the stable sets $P_i$ with $p_i \geq 1$ by $p$ colors, then coloring those $P_i$ with $p_i = 0$. As shown in the proof, the second step can be done smoothly under the condition $|\mathcal{C}| \geq 3k-1$ as long as the first step has been carried out, which is possible when $|\mathcal{C}| \geq 4k-2$ thanks to Lemma 4.3. In order to go below $4k-2$ significantly, it might be tempting to improve the constant factor 2 in Lemma 4.3 to a (much) smaller constant independent of $q \geq 1$. This, however, is not possible even if one asks for a partition of $Q$ into $q+r$ sets each with sum of elements less than $(2-\varepsilon)k$, for any given integer $r \geq 0$ and any given small $\varepsilon > 0$.[3] Another potential shortcoming of the proof of Lemma 4.2 is that the goodness of the template $T$ was never really used, since the calculations involving every $|F(v)|$ only require them to be at most $k$. On the other hand, given the goodness of $T$, every partition of $P_i$ into stable sets of weight less than $k$ must have at least $p_i + 1$ components as $w(P_i) \geq p_i k$. In fact, we shall prove that under the condition $|\mathcal{C}| \geq 3k-1$, it is possible to get around Lemma 4.3 and reduce Lemma 4.1 essentially to the case when each $P_i$ has a partition into exactly $p_i + 1$ sets of weight less than $k$. More precisely, for each $i \in [\chi]$, there is a nonnegative integer $q_i \leq p_i$ and a subset of $P_i$ of weight at least $q_i k$ whose vertices can be colored by at most $q_i$ colors and whose complement in $P_i$ can be partitioned into $p_i - q_i + 1$ stable sets each of weight less than $k$. This is the content of the following lemma.

**Lemma 4.4.** *If $|\mathcal{C}| \geq 3k-1$, then there exist integers $q_1, \ldots, q_\chi$ with $0 \leq q_i \leq p_i$ for all $i \in [\chi]$ and subsets $P'_1, \ldots, P'_\chi$ of $P_1, \ldots, P_\chi$ respectively, such that*

- *for every $i \in [\chi]$, $w(P'_i) \geq q_i k$, and there is a partition $P_i \setminus P'_i = \bigcup_{j \in [t_i+1]} P_{ij}$ with $w(P_{ij}) < k$ for all $j \in [t_i + 1]$ where $t_i := p_i - q_i$; and*
- *for $P^1 := \bigcup_{i \in [\chi]} P'_i$, there is a proper coloring $c_1$ of $G[S \cup P^1]$ respecting $T_{S \cup P^1}$ with $|c_1(P'_i)| \leq q_i$ for all $i \in [\chi]$, in particular $|c_1(P^1)| \leq q_1 + \cdots + q_\chi$.*

To prove Lemma 4.4, we need several notions. Given a subset $P$ of $V(G)$, a sequence $\mathcal{Q} = (Q_1, \ldots, Q_n)$ of nonempty disjoint subsets of $P$ is called *fit* if $0 \leq w(Q_j) < k$ for all $j \in [n]$ and $P = Q_1 \cup \cdots \cup Q_n$. Then every sequence $(v_1, \ldots, v_{|P|})$ of vertices in $P$ is a fit sequence of $P$ since $w(\{v_j\}) = |F(v_j)| < k$ for all $j \in [n]$ by the goodness of $T$; also, note that permuting the components of $\mathcal{Q}$ preserves the fitness of the resulting sequence. A fit sequence $\mathcal{Q} = (Q_1, \ldots, Q_n)$ is said to be *increasing* if $w(Q_1) \leq \ldots \leq w(Q_n)$; observe that a fit sequence of $\mathcal{Q}$ can be made increasing by sorting its components, and so $P_i$ always has an increasing fit sequence.

Given a fit sequence $\mathcal{Q} = (Q_1, \ldots, Q_n)$ of $P$, let $w_j(\mathcal{Q}) := w(Q_1 \cup \cdots \cup Q_j)$ for all $j \in [n]$; then $w_n(\mathcal{Q}) = w(P)$. For $j \in [n]$, say that $Q_j$ is a *jump* of $\mathcal{Q}$ if $\lfloor w_j(\mathcal{Q})/k \rfloor = \lfloor w_{j-1}(\mathcal{Q})/k \rfloor + 1$, that is, there is an integer $q \geq 1$ such that

$$(q-1)k \leq w_{j-1}(\mathcal{Q}) < qk \leq w_j(\mathcal{Q}) < (q+1)k$$

and say that $Q_j$ is a *non-jump* of $\mathcal{Q}$ if $\lfloor w_j(\mathcal{Q})/k \rfloor = \lfloor w_{j-1}(\mathcal{Q})/k \rfloor$, that is, there is an integer $q \geq 0$ with

$$qk \leq w_{j-1}(\mathcal{Q}) \leq w_j(\mathcal{Q}) < (q+1)k,$$

and in this case say that $q$ is a *landmark* of $\mathcal{Q}$. Observe that, as $0 \leq w(Q_j) < k$ for all $j \in [n]$,

- $Q_1$ is always a non-jump, in particular 0 is always a landmark of $\mathcal{Q}$;
- each $Q_j$ is either a jump or a non-jump;
- there are $\lfloor w(P)/k \rfloor$ jumps in total; and

[3] To see this, let $\varepsilon > 0$ be sufficiently small, let $k$ be much larger than $1/\varepsilon$, let $q$ be such that $r/\varepsilon < (q+r)/2 < (r+1)/\varepsilon$, and let $Q$ be some set of $q+r+1$ integers at least $(1-\varepsilon/2)k$ and smaller than $k$.

- every landmark of $\mathcal{Q}$ is at most $\lfloor w(P)/k\rfloor$.

A fit sequence of $P$ is called *critical* if it has no two consecutive non-jumps. By choosing an increasing fit sequence of $P$ then combining two consecutive non-jumps (if exist) and sorting the new fit sequence repeatedly, we see that $P$ always has an increasing critical sequence. Also, given a critical sequence $\mathcal{Q} = (Q_1, \dots, Q_n)$ of $P$ with landmarks $q_1, \dots, q_\ell$ satisfying $0 = q_1 < \dots < q_\ell \le \lfloor w(P)/k\rfloor$, it is not hard to see that

- $\ell = n - \lfloor w(P)/k\rfloor$;
- for every $r \in [\ell-1]$, there are exactly $q_{r+1} - q_r$ jumps of $\mathcal{Q}$ between $q_r k$ and $(q_{r+1}+1)k$, in particular $q_r k \le w_{q_r+r}(\mathcal{Q}) < (q_r + 1)k$ for all $r \in [\ell]$; and
- there are exactly $\lfloor w(P)/k\rfloor - q_\ell$ jumps of $\mathcal{Q}$ after $q_\ell k$.

Given the above definitions, the proof of Lemma 4.4 relies on an iterative procedure which recolors and swaps stable sets within fit sequences of $P_1, \dots, P_\chi$, as follows.

**Proof of Lemma 4.4.** It suffices to iterate the following claim for $i = 1, 2, \dots, \chi$ in turn.

**Claim 4.1.** *Let $i \in [\chi]$, and assume that there exist integers $q_1, \dots, q_{i-1}$ with $0 \le q_h \le p_h$ for all $h \in [i-1]$ and subsets $P'_1, \dots, P'_{i-1}$ of $P_1, \dots, P_{i-1}$ respectively, such that*

- *for every $h \in [i-1]$, $w(P'_h) \ge q_h k$, and there is a partition $P_h \setminus P'_h = \bigcup_{j\in[t_h+1]} P_{hj}$ such that $w(P_{hj}) < k$ for all $j \in [t_h + 1]$ where $t_h := p_h - q_h$; and*
- *for $P' := \bigcup_{h\in[i-1]} P'_h$, there is a proper coloring $c'$ of $G[S \cup P']$ respecting $T_{S\cup P'}$ with $|c'(P'_h)| \le q_h$ for all $h \in [i-1]$, in particular $|c'(P')| \le q_1 + \dots + q_{i-1}$.*

*Then there exists an integer $q_i$ with $0 \le q_i \le p_i$ and a subset $P'_i$ of $P_i$ such that for $t_i := p_i - q_i$,*

- *$w(P'_i) \ge q_i k$, and there is a partition $P_i \setminus P'_i = \bigcup_{j\in[t_i+1]} P_{ij}$ with $w(P_{ij}) < k$ for all $j \in [t_i + 1]$; and*
- *we can color the vertices of $P'_i$ by at most $q_i$ colors such that every $v \in P'_i$ gets a color not in $c'(S \cup P') \cup F(v)$.*

*Proof.* Let $\mathcal{Q} = (Q_1, \dots, Q_n)$ be an increasing critical sequence of $P_i$. Let $q^i_1, \dots, q^i_\ell$ with $0 = q^i_1 < \dots < q^i_\ell \le \lfloor w(P_i)/k\rfloor = p_i$ be the landmarks of $\mathcal{Q}$, and let $n_r := q^i_r + r$ for all $r \in [\ell]$; then

$$q^i_r \le w_{n_r-1}(\mathcal{Q}) \le w_{n_r}(\mathcal{Q}) < (q^i_r + 1)k$$

by the criticality of $\mathcal{Q}$. Let $\mathcal{Q}^1 := \mathcal{Q}$; and for $r \in [\ell - 1]$, assume that there is a fit sequence $\mathcal{Q}^r = (Q^r_1, \dots, Q^r_n)$ of $P_i$ such that

- $(Q^r_1, \dots, Q^r_{n_r})$ is a permutation of $(Q_1, \dots, Q_{n_r})$;
- $Q^r_j = Q_j$ for all $j \in [n] \setminus [n_r]$; and
- we have assigned at most $q^i_r$ colors to the stable sets $Q^r_1, \dots, Q^r_{n_r-1}$ such that each $v \in Q^r_1 \cup \dots \cup Q^r_{n_r-1}$ gets a color not in $c'(S \cup P') \cup F(v)$.

Let $L := \mathcal{C} \setminus c'(S \cup P')$. Because $q_h \le p_h$ for all $h \in [i-1]$ and $p \le t - 1$ by (2), we see that

$$|c'(P')| \le q_1 + \dots + q_{i-1} \le p_1 + \dots + p_{i-1} \le p \le t - 1.$$

Thus, as $|\mathcal{C}| \ge 3k - 1$, we obtain

$$|L| \ge |\mathcal{C}| - |c'(S \cup P')| \ge |\mathcal{C}| - |S| - |c'(P')| \ge |\mathcal{C}| - (2k - t) - (t - 1) = |\mathcal{C}| - (2k - 1) \ge k.$$

For $j \in \{n_r, n_r + 1, \dots, n_{r+1}\}$, let $L_j := L \setminus (\bigcup_{v\in Q^r_j} F(v))$; note that $w(Q^r_{n_r}) \le w(Q_{n_r})$ since $\mathcal{Q}$ is increasing, and $w(Q^r_j) = w(Q_j)$ for all $j \in [n_{r+1}] \setminus [n_r]$, in particular $|L_j| \ge |L| - w(Q^r_{n_r}) > k - k = 0$ and so $L_j$ is nonempty for all $j \in \{n_r, n_r + 1, \dots, n_{r+1}\}$. Since $q^i_r$ and $q^i_{r+1}$ are landmarks of $\mathcal{Q}$, we see that

$$\sum_{j=n_r}^{n_{r+1}} w(Q^r_j) \le \sum_{j=n_r}^{n_{r+1}} w(Q_j) = w_{n_{r+1}}(\mathcal{Q}) - w_{n_r-1}(\mathcal{Q}) < (q^i_{r+1} + 1)k - q^i_r k = (n_{r+1} - n_r)k.$$

Therefore, as $|L| \geq k$, we deduce that

$$\sum_{j=n_r}^{n_{r+1}} |L_j| \geq \sum_{j=n_r}^{n_{r+1}} (|L| - w(Q_j^r)) > (n_{r+1} - n_r + 1)|L| - (n_{r+1} - n_r)k = (n_{r+1} - n_r)(|L| - k) + |L| \geq |L|.$$

It follows that $L_{n_r}, \dots, L_{n_{r+1}}$ are not pairwise disjoint, thus there is a color common to two among them; and so, since they are all nonempty, we can assign at most $n_{r+1} - n_r = q_{r+1}^i - q_r^i + 1$ colors to the stable sets in $\{Q_j^r : j \in \{n_r, \dots, n_{r+1}\}\}$ such that each $Q_j^r$ gets a color in $L_j$. Then

- if in fact at most $q_{r+1}^i - q_r^i$ colors are needed, then obviously we have assigned at most $q_{r+1}^i - q_r^i$ colors to the stable sets $Q_{n_r}^r, \dots, Q_{n_{r+1}-1}^r$ such that $Q_j^r$ gets a color in $L_j$ for all $j \in \{n_r, \dots, n_{r+1} - 1\}$; we then uncolor $Q_{n_{r+1}}^r$ and let $\mathcal{Q}^{r+1} := \mathcal{Q}^r$;
- if $q_{r+1}^i - q_r^i + 1$ colors are needed and $Q_{n_{r+1}}^r$ gets a color not already used for $Q_{n_r}^r, \dots, Q_{n_{r+1}-1}^r$, then again we have assigned at most $q_{r+1}^i - q_r^i$ colors to the stable sets $Q_{n_r}^r, \dots, Q_{n_{r+1}-1}^r$ such that $Q_j^r$ gets a color in $L_j$ for all $j \in \{n_r, \dots, n_{r+1} - 1\}$; we then uncolor $Q_{n_{r+1}}^r$ and let $\mathcal{Q}^{r+1} := \mathcal{Q}^r$; and
- if $q_{r+1}^i - q_r^i + 1$ colors are needed and we have to color $Q_{n_{r+1}}^r$ and $Q_j^r$ with $j \in \{n_r, \dots, n_{r+1} - 1\}$ by the same color, then pick $j' \in \{n_r, \dots, n_{r+1} - 1\} \setminus \{j\}$ (this is possible as $n_{r+1} - n_r = q_{r+1}^i - q_r^i + 1 > 1$), uncolor $Q_{j'}^r$, and swap $Q_{j'}^r$ and $Q_{n_{r+1}}^r$ in $\mathcal{Q}^r$ to obtain a new fit sequence $\mathcal{Q}^{r+1}$ of $P_i$.

In this way, we have shown that there exists a fit sequence $\mathcal{Q}^{r+1} = (Q_1^{r+1}, \dots, Q_n^{r+1})$ of $P_i$ such that

- $(Q_1^{r+1}, \dots, Q_{n_{r+1}}^{r+1})$ is a permutation of $(Q_1, \dots, Q_{n_{r+1}})$;
- $Q_j^{r+1} = Q_j$ for all $j \in [n] \setminus [n_{r+1}]$; and
- we have assigned at most $q_r^i + (q_{r+1}^i - q_r^i) = q_{r+1}^i$ colors to the stable sets $Q_1^{r+1}, \dots, Q_{n_{r+1}-1}^{r+1}$ such that each $v \in Q_1^{r+1} \cup \dots \cup Q_{n_{r+1}-1}^{r+1}$ gets a color not in $c'(S \cup P') \cup F(v)$.

Iterating this procedure for $r = 1, 2, \dots, \ell - 1$ in turn, we have shown that there is a fit sequence $\mathcal{Q}^\ell = (Q_1^\ell, \dots, Q_n^\ell)$ of $P_i$ such that

- $(Q_1^\ell, \dots, Q_{n_\ell}^\ell)$ is a permutation of $(Q_1, \dots, Q_{n_\ell})$;
- $Q_j^\ell = Q_j$ for all $j \in [n] \setminus [n_\ell]$; and
- we can assign at most $q_\ell^i$ colors to the stable sets $Q_1^\ell, \dots, Q_{n_\ell - 1}^\ell$ such that each $v \in Q_1^\ell \cup \dots \cup Q_{n_\ell - 1}^\ell$ gets a color not in $c'(S \cup P') \cup F(v)$.

Now, let $q_i := q_\ell^i$, $t_i := p_i - q_i = n - n_\ell$, $P_i' := \bigcup_{j \in [n_\ell - 1]} Q_j^\ell$, and $P_{ij} := Q_{j+n_\ell-1}^\ell$ for all $j \in [t_i + 1]$; then

- $w(Q_{n_\ell}^\ell) \leq w(Q_{n_\ell})$ since $\mathcal{Q}$ is increasing, and $w_{n_\ell}(\mathcal{Q}^\ell) = w_{n_\ell}(\mathcal{Q})$ since $(Q_1^\ell, \dots, Q_{n_\ell}^\ell)$ is a permutation of $(Q_1, \dots, Q_{n_\ell})$, therefore
$$w(P_i') = w_{n_\ell - 1}(\mathcal{Q}^\ell) = w_{n_\ell}(\mathcal{Q}^\ell) - w(Q_{n_\ell}^\ell) \geq w_{n_\ell}(\mathcal{Q}) - w(Q_{n_\ell}) = w_{n_\ell - 1}(\mathcal{Q}) \geq q_\ell^i k = q_i k;$$
- because there are exactly $\lfloor w(P_i)/k \rfloor - q_i = t_i$ jumps of $\mathcal{Q}$ after $q_i k$, $\bigcup_{j \in [t_i+1]} P_{ij}$ is a partition of $P_i \setminus P_i'$ with $w(P_{ij}) = w(Q_{j+n_\ell-1}^\ell) = w(Q_{j+n_\ell-1}) < k$ for all $j \in [t_i + 1]$; and
- we can color the vertices of $P_i'$ by at most $q_\ell^i = q_i$ colors such that every $v \in P_i'$ gets a color not in $c'(S \cup P') \cup F(v)$. This proves Claim 4.1. □

The proof of Lemma 4.4 is complete. ■

4.3. **Finishing the proof.** We now come to the rest of the proof of Lemma 4.1. As explained in the previous subsection, Lemma 4.4 allows us to have in mind that each $P_i$ has a partition into $p_i + 1$ stable sets of weight less than $k$; but for technical reason (there might be some $i$ with $w(P_i \setminus P_i') < t_i k$) it might be wise to present the full argument without assuming that. We shall go through several coloring steps. To achieve the bound $\chi \geq |\mathcal{C}| - 2k + 3$ which is optimal as we have seen, we observe that each vertex in $P_i \setminus P_i'$ (which is yet to be colored) need not get a color not already used for $(S \cap S_i) \cup P_i'$. Then, careful calculations with the supposition $\chi \leq |\mathcal{C}| - 2k + 2$ (for a contradiction) will enable us to basically work with the supposition $\chi \leq |\mathcal{C}| - 2k + 1$ as in the second step in the proof of Lemma 4.2 (with more twists).

We obtain the constant factor $3 + \frac{1}{16}$ from the estimate in the last coloring step. Let us proceed with the details.

**Proof of Lemma 4.1.** Picking up where we left off, we let $P'_1, \dots, P'_\chi$, $P^1$, $q_1, \dots, q_\chi$, $t_1, \dots, t_\chi$, and $c_1$ be given by Lemma 4.4. Let $S^1 := S \cup P^1$, and let $T^1 := (S^1, c_1, F|_{V(G)\setminus S^1})$ be a template on $G$. Let $q := q_1 + \dots + q_\chi$, then for every $i \in [\chi]$,

$$|c_1(S^1 \setminus S_i)| \le |c(S \setminus S_i)| + \sum_{h\in[\chi]\setminus\{i\}} |c_1(P'_i)| \le |S \setminus S_i| + \sum_{h\in[\chi]\setminus\{i\}} q_h = |S| - |S \cap S_i| + q - q_i. \tag{3}$$

For every $i \in [\chi]$, let $X_i := P_i \setminus P'_i = \bigcup_{j\in[t_i+1]} P_{ij}$ and $x_i := w(X_i) - t_i k$; then $x_i < k$, and since $w(P'_i) \ge q_i k$, we have

$$w(P_i) - x_i = w(P_i) - w(X_i) + t_i k = w(P'_i) + t_i k \ge (q_i + t_i)k = p_i k.$$

Let $t' := t - p$, then $t' \ge 1$ by (2), and

$$kt > \mathrm{cost}(T) - k|S| = \sum_{i\in[\chi]} w(P_i) \ge \sum_{i\in[\chi]} (p_i k + x_i) = pk + \sum_{i\in[\chi]} x_i$$

by (1), which implies

$$k(t-p) = kt' > \sum_{i\in[\chi]} x_i. \tag{4}$$

Now, let $I_0 := \{i \in [\chi] : t_i = 0\}$; by adding isolated vertices we may assume $X_i \neq \emptyset$ for all $i \in I_0$. Let $I_2$ be a maximal subset of $[\chi]$ such that we can properly color $\bigcup_{i\in I_2} X_i$ by $\sum_{i\in I_2} t_i$ colors so that for every $i \in I_2$, each $v \in X_i$ gets a color not in $c_1(S^1 \setminus S_i) \cup F(v)$. Then $I_0, I_2$ are disjoint. Let $I_1 := [\chi] \setminus (I_0 \cup I_2)$, and let $s_1 := \sum_{i\in I_1} t_i$ and $s_2 := \sum_{i\in I_2} t_i$. Let $P^2 := \bigcup_{i\in I_2} X_i$, and let $c_2$ be a proper coloring of $G[S^1 \cup P^2]$ repsecting $T^1_{S^1\cup P^2}$ with $|c_2(P^2)| \le s_2$ whose existence is guaranteed by the definition of $I_2$. Observe that

$$s_1 + s_2 = \sum_{i\in I_1} t_i + \sum_{i\in I_2} t_i = \sum_{i\in[\chi]} t_i = \sum_{i\in[\chi]} (p_i - q_i) = p - q.$$

Let $S^2 := S^1 \cup P^2$, and let $T^2 := (S^2, c_2, F|_{V(G)\setminus S^2})$ be a template on $G$. Let $I := I_0 \cup I_1$, then $I \cup I_2$ is a partition of $[\chi]$. For every $i \in I$, as $|c_1(S^1 \setminus S_i)| \le |S| - |S \cap S_i| + q - q_i$ by (3) and $s_1 + s_2 = p - q$,

$$\begin{aligned} |c_2(S^2 \setminus S_i)| \le |c_1(S^1 \setminus S_i)| + |c_2(P^2)| &\le |S| - |S \cap S_i| + q - q_i + s_2 \\ &= (2k - t' - p) - |S \cap S_i| + q - q_i + (p - q - s_1) \\ &= 2k - t' - s_1 - |S \cap S_i| - q_i. \end{aligned} \tag{5}$$

**Claim 4.2.** *$x_i \ge t_i(|\mathcal{C}| - 3k + t' + s_1 + |S \cap S_i| + q_i)$ for all $i \in I$.*

*Proof.* Suppose that there is $i \in I$ such that $x_i < t_i(|\mathcal{C}| - 3k + t' + s_1 + |S \cap S_i| + q_i)$; then $i \in I_1$. Let $L := \mathcal{C} \setminus c_2(S^2 \setminus S_i)$. Since $|c_2(S^2 \setminus S_i)| \le 2k - t' - s_1 - |S \cap S_i| - q_i$ by (5), we see that

$$\begin{aligned} |L| &\ge |\mathcal{C}| - |c_2(S^2 \setminus S_i)| \\ &\ge |\mathcal{C}| - (2k - t' - s_1 - |S \cap S_i| - q_i) = |\mathcal{C}| - 2k + t' + s_1 + |S \cap S_i| + q_i, \end{aligned}$$

therefore $|L| \ge |\mathcal{C}| - 2k + 1$ since $t' \ge 1$. For every $j \in [t_i + 1]$, let $L_j := L \setminus (\bigcup_{v\in P_{ij}} F(v))$, then

$$|L_j| \ge |L| - w(P_{ij}) \ge |\mathcal{C}| - 2k + 1 - (k - 1) = |\mathcal{C}| - (3k - 2) > 0.$$

As $p_i = q_i + t_i$ and by our supposition on $x_i$, we have that

$$\begin{aligned} \sum_{j\in[t_i+1]} |L_j| \ge (t_i + 1)|L| - \sum_{j\in[t_i+1]} w(P_{ij}) &\ge |L| + t_i(|\mathcal{C}| - 2k + t' + s_1 + |S \cap S_i| + q_i) - (x_i + t_i k) \\ &= |L| + t_i(|\mathcal{C}| - 3k + t' + s_1 + |S \cap S_i| + q_i) - x_i > |L| \end{aligned}$$

so $L_1, \dots, L_{t_i+1}$ are nonempty and not pairwise disjoint. Therefore we can assign at most $t_i$ colors to the stable sets in $\{P_{ij} : j \in [t_i + 1]\}$ so that each $P_{ij}$ gets a color in $L_j$, and so $I_2 \cup \{i\}$ contradicts the maximality of $I_2$. This proves Claim 4.2. □

For each $i \in I$, assume $w(P_{i1}) = \min_{j\in[t_i+1]} w(P_{ij}) =: y_i$, and let $Y_i := P_{i1}$; then $y_i = w(Y_i)$. Let $P^3 := \bigcup_{i\in I_1}(X_i \setminus Y_i)$; we next extend $c_2$ to $P^3$ by at most $s_1$ colors, as follows.

**Claim 4.3.** *There is a proper coloring $c_3$ of $G[S^2 \cup P^3]$ respecting $T^2_{S^2\cup P^3}$ and satisfying $|c_3(X_i \setminus Y_i)| \le t_i$ for all $i \in I_1$.*

*Proof.* Note that for every $i \in I_1$, $P^3 \cap S_i = X_i \setminus Y_i$ is the disjoint union of $t_i$ stable sets of the form $P_{ij}$ where $j \in [t_i + 1] \setminus \{1\}$. For $i \in I_1$ and $j \in [t_i + 1] \setminus \{1\}$, let $L_{ij} := \mathcal{C} \setminus (c_2(S^2 \setminus S_i) \cup \bigcup_{v\in P_{ij}} F(v))$, then as $|c_2(S^2 \setminus S_i)| \le 2k - t' - s_1$ by (5), as $p_i = q_i + t_i$, and as $|\mathcal{C}| \ge 3k - 1$, we have

$$
\begin{aligned}
|L_{ij}| &\ge |\mathcal{C}| - |c_2(S^2 \setminus S_i)| - w(P_{ij})\\
&\ge |\mathcal{C}| - (2k - t' - s_1) - (k - 1) = |\mathcal{C}| - (3k - 1) + t' + s_1 \ge s_1.
\end{aligned}
$$

Hence we can assign $\sum_{t\in I_1} t_i$ different colors to the stable sets in $\bigcup_{i\in I_1}\{P_{ij} : j \in [t_i + 1] \setminus \{1\}\}$ such that each $P_{ij}$ gets a color in $L_{ij}$. This proves Claim 4.3. □

Let $S^3 := S^2 \cup P^3$, and let $T^3 := (S^3, c_3, F|_{V(G)\setminus S^3})$ be the template on $G$ with $c_3$ given by Claim 4.3. For $i \in I$, since $|c_3(P^3 \setminus S_i)| \le \sum_{h\in I_1\setminus\{i\}}|c_3(X_h \setminus Y_h)| \le \sum_{h\in I_1\setminus\{i\}} t_h = s_1 - t_i$, since $|c_2(S^2 \setminus S_i)| \le 2k - t - |S \cap S_i| - q_i - s_1$ by (5), and since $p_i = q_i + t_i$, we have that

$$
\begin{aligned}
|c_3(S^3 \setminus S_i)| &\le |c_2(S^2 \setminus S_i)| + |c_3(P^3 \setminus S_i)|\\
&\le (2k - t' - |S \cap S_i| - q_i - s_1) + (s_1 - t_i) = 2k - t' - |S \cap S_i| - p_i.
\end{aligned}
\tag{6}
$$

So far our arguments only concern $|\mathcal{C}|$ instead of its relation to $\chi$. Now, assume for a contradiction that $\chi \le |\mathcal{C}| - 2k + 2$. Let $L_i := \mathcal{C} \setminus (c_3(S^3 \setminus S_i) \cup \bigcup_{v\in Y_i} F(v))$ for all $i \in I$. As promised, the following claim will enable us to basically work with the supposition $\chi \le |\mathcal{C}| - 2k + 1$.

**Claim 4.4.** *There exists $J \subseteq I$ with $I_1 \subseteq J$, $|J| \le |\mathcal{C}| - 2k + 1$, and $|L_i| \ge |I|$ for all $i \in I \setminus J$.*

*Proof.* If $|I| \le |\mathcal{C}| - 2k + 1$ then we can just take $J$ to be $I$; so we may assume $|I| \ge |\mathcal{C}| - 2k + 2$, then $|I| = \chi = |\mathcal{C}| - 2k + 2$, $I = [\chi]$, and $I_2 = \emptyset$. It suffices to show that there exists $i \in I_0$ with $|L_i| \ge |I|$; then we can take $J := I \setminus \{i\}$. Suppose not; then $|L_i| < |I| = |\mathcal{C}| - 2k + 2$ for all $i \in I_0$, so since $x_i = y_i$ for $i \in I_0$ and $|c_3(S^3 \setminus S_i)| \le 2k - t' - |S \cap S_i| - p_i$ by (6),

$$
\begin{aligned}
x_i = y_i \ge |\mathcal{C}| - |c_3(S^3 \setminus S_i)| - |L_i| &> |\mathcal{C}| - (2k - t' - |S \cap S_i| - p_i) - (|\mathcal{C}| - 2k + 2)\\
&= t' - 2 + |S \cap S_i| + p_i,
\end{aligned}
$$

hence $x_i \ge t' - 1 + |S \cap S_i| + p_i$ for all $i \in I_0$. For each $i \in I_1$, since $|\mathcal{C}| \ge 3k - 1$, since $s_1 + q_i \ge t_i + q_i = p_i$, and since $t_i \ge 1$, by Claim 4.2 we have that

$$x_i \ge t_i(|\mathcal{C}| - 3k + t' + s_1 + |S \cap S_i| + q_i) \ge t' - 1 + |S \cap S_i| + p_i.$$

Hence, since $|I| = \chi = |\mathcal{C}| - 2k + 2 \ge k + 1$ and $\sum_{i\in I}|S \cap S_i| = |S| = 2k - t' - p$, we obtain

$$
\begin{aligned}
\sum_{i\in I} x_i &\ge \sum_{i\in I}(t' - 1 + |S \cap S_i| + p_i)\\
&= |I|(t' - 1) + |S| + p \ge (k + 1)(t' - 1) + (2k - t' - p) + p = kt' + k - 1 \ge kt',
\end{aligned}
$$

contradicting (4). This proves Claim 4.4. □

Let $J$ be the subset of $I$ obtained from Claim 4.4, and let $I' := \{i \in J : x_i \ge t'\}$; then $I_1 \subseteq I'$ as $x_i \ge t_i(|\mathcal{C}| - 3k + t' + s_1) \ge t'$ for all $i \in I_1$ by Claim 4.2 since $s_1 \ge t_i \ge 1$ and $|\mathcal{C}| \ge 3k - 1$. The following claim is sufficient to complete the proof of Lemma 4.1.

**Claim 4.5.** *It is possible to assign $|I'|$ different colors to the stable sets $\{Y_i : i \in I'\}$ so that each $Y_i$ gets a color in $L_i$.*

To see how Claim 4.5 leads to a contradiction and finishes the proof of Lemma 4.1, observe that for every $i \in J \setminus I' \subseteq I_0$, since $y_i = x_i \le t' - 1$ and $|c_3(S^3 \setminus S_i)| \le 2k - t'$ by (6),

$$|L_i| \ge |\mathcal{C}| - |c_3(S^3 \setminus S_i)| - y_i \ge |\mathcal{C}| - (2k - t') - (t' - 1) = |\mathcal{C}| - 2k + 1 \ge |J|,$$

so we can assign $|J|$ different colors to the stable sets in $\{Y_i : i \in J\}$ so that each $Y_i$ gets a color in $L_i$. Thus, since $|L_i| \ge |I|$ for all $i \in I \setminus J$, we can assign $|I|$ different colors to the stable sets in $\{Y_i : i \in I\}$ so that each $Y_i$ gets a color in $L_i$. In this way we would obtain a proper coloring of $G$ respecting $T$, a contradiction.

Hence, the rest of the proof will be devoted to Claim 4.5. We may assume that $I' \neq \emptyset$. Put $d := |\mathcal{C}| - 3k + 1 \ge \frac{1}{16}k$ and $d' := d + s_1 - 1$; then Claim 4.2 yields $x_i \ge t_i(d + t' + s_1 - 1) = t_i(t' + d')$ for all $i \in I_1$. For every $i \in I'$, as $|c_3(S^3 \setminus S_i)| \le 2k - t'$ by (6), we see that

$$|L_i| - i \ge |\mathcal{C}| - |c_3(S^3 \setminus S_i)| - y_i - i \ge |\mathcal{C}| - (2k - t') - (y_i + i) = d + k + t' - (y_i + i) - 1. \tag{7}$$

**Claim 4.6.** *We may assume that $t' \ge d' + 1$.*

*Proof.* Suppose that $t' \le d'$, then $x_i \ge t_i(t'+d') \ge t_i(t_i+1)t' \ge (t_i+1)t'$ for all $i \in I'$. Assume that $I' = [n]$ for some $n \ge 1$, and that $x_1 \ge \ldots \ge x_n$. For every $i \in [n]$, since $t_i k + x_i = \sum_{j \in [t_i+1]} w(P_{ij}) \ge (t_i + 1)y_i$, and since $i < kt'/x_i$ which follows from $kt' > x_1 + \cdots + x_n \ge ix_i$ by (2), we have that

$$y_i + i < \frac{t_i k + x_i}{t_i + 1} + \frac{kt'}{x_i} = k + t' + \frac{(k - x_i)((t_i + 1)t' - x_i)}{x_i(t_i + 1)} \le k + t'$$

and so $y_i + i \le k + t' - 1$; thus (7) yields

$$|L_i| - i \ge d + k + t' - (k + t' - 1) - 1 = d \ge 0.$$

It follows that we can greedily assign $n = |I'|$ different colors to the stable sets $Y_1, \ldots, Y_n$ so that each $Y_i$ gets a color in $L_i$, proving Claim 4.5. Thus we may assume that $t' \ge d' + 1$, as claimed. □

Now, put $s := s_1 - 1$; then $d' = d + s$ and $t' \ge d' + 1 \ge s + 1 \ge |I_1|$ by Claim 4.6.

**Claim 4.7.** *We may assume that $s > d + s(t' + d')/k$.*

*Proof.* Suppose that $s \le d + s(t' + d')/k$. For every $i \in I_1$, by (7) and since $t' \ge |I_1|$, we have that

$$|L_i| \ge d + k + t' - y_i - 1 \ge d + k + t' - (k - 1) - 1 = d + t' \ge |I_1|,$$

so we can assign $|I_1|$ different colors to the stable sets in $\{Y_i : i \in I_1\}$ so that each $Y_i$ gets a color in $L_i$.

Now, assume that $I' \setminus I_1 = [n] \subseteq I_0$ for some $n \ge 0$, and that $x_1 \ge \ldots \ge x_n$. If $n = 0$ then Claim 4.5 is proved, so we may assume that $n \ge 1$. Since $x_i \ge t_i(t' + d')$ for all $i \in I_1$, (4) implies that

$$kt' > \sum_{i \in I_1} t_i(t' + d') + \sum_{i \in [n]} x_i = s_1(t' + d') + \sum_{i \in [n]} x_i.$$

Thus, for every $i \in [n]$, $i < (kt' - s_1(t' + d'))/x_i$, and so $y_i + i = x_i + i < x_i + \frac{kt' - s_1(t'+d')}{x_i} = \phi(x_i)$ where $\phi\colon [t', \infty) \to \mathbb{R}$ is defined by $\phi(x) := x + \frac{kt' - s_1(t'+d')}{x}$. Since $\phi$ is convex and $t' \le x_i \le k - 1$, we have that $\phi(x_i) \le \max(\phi(t'), \phi(k-1))$. Observe that $\phi(t') = k + t' - s_1 - s_1 d'/t' < k + t' - s_1$, and that

$$\begin{aligned}\phi(k - 1) &= k - 1 + t' - \frac{(s_1 - 1)t' + s_1 d'}{k - 1} \\ &< k - 1 + t' - \frac{s}{k}(t' + d') \le k - 1 + t' - s + d = k + t' - s_1 + d\end{aligned}$$

where the last inequality follows from our supposition. Thus for every $i \in [n]$, $\phi(x_i) < k + t' - s_1 + d$ which yields $y_i + i \le k + t' - s_1 + d - 1$, and so (7) implies that

$$|L_i| - i \ge d + k + t' - (y_i + i) - 1 \ge d + k + t' - (k + t' - s_1 + d - 1) - 1 = s_1 \ge |I_1|.$$

Hence $|L_i| \ge i + |I_1|$ for all $i \in [n]$, and so we can greedily assign $n = |I' \setminus I_1|$ colors to the stable sets $Y_1, \ldots, Y_n$ such that each $Y_i$ gets a color in $L_i$ not already used for any stable set in $\{Y_{i'} : i' \in I_1\}$. Since this proves Claim 4.5, we may thus assume that $s > d + s(t' + d')/k$, as claimed. □

The last part of the proof reduces to proving that

$$\frac{(k-(t'+d'))(t'-d')}{t'+d'} \le 2d. \tag{8}$$

To see how (8) concludes the proof of Claim 4.5 and thus the proof of Lemma 4.1, assume that $I' = [n]$ for some $n \ge 1$ and $x_1 \ge \ldots \ge x_n$. Then as in the proof of Claim 4.6, we have that for every $i \in [n]$,

$$y_i + i < k + t' + \frac{(k-x_i)((t_i+1)t' - x_i)}{x_i(t_i+1)}.$$

If $x_i \ge (t_i+1)t'$ then $y_i + i < k + t'$. If $x_i \le (t_i+1)t' - 1$, then $i \in I_1$ which yields $t_i \ge 1$; thus since $k - 1 \ge x_i \ge t_i(t'+d') \ge t' + d'$ and $t' \ge d' + 1$ by Claim 4.6, by (8) we would obtain

$$\frac{(k-x_i)((t_i+1)t'-x_i)}{x_i(t_i+1)} \le \frac{(k-x_i)(t'-d')}{x_i(t_i+1)} \le \frac{(k-(t'+d'))(t'-d')}{2(t'+d')} \le d$$

so $y_i + i < k + t' + d$. We would have thus shown that $y_i + i \le k + t' + d - 1$ for all $i \in [n]$, so (7) implies

$$|L_i| - i \ge d + k + t' - (k + t' + d - 1) - 1 = 0,$$

hence we could greedily assign $n = |I'|$ different colors to the stable sets $Y_1, \ldots, Y_n$ such that each $Y_i$ gets a color in $L_i$, proving Claim 4.5.

Now, to prove (8), observe that

$$\begin{aligned}\frac{(k-(t'+d'))(t'-d')}{t'+d'} &= \frac{(k-(t'+d'))((t'+d')-2d')}{t'+d'} \\ &= k + 2d' - \frac{2kd'}{t'+d'} - (t'+d') \le k + 2d' - 2\sqrt{2kd'} = (\sqrt{k} - \sqrt{2d'})^2\end{aligned}$$

where the inequality follows from the arithmetic mean–geometric mean inequality. If $d' \ge 4d$, then since $d \ge \frac{1}{16}k$ we would have $d' \ge \frac{1}{4}k$ so $(\sqrt{k} - \sqrt{2d'})^2 \le (1 - \frac{1}{\sqrt{2}})^2 k < 2d$ and (8) is proved. So we may assume $d' < 4d$ which yields $s < 3d$ and so $\frac{4}{3}s < d'$ (recall that $d' = d + s$).

By Claim 4.7, we have that $s > d$ and $t' + d' < (1 - d/s)k$. We claim that $(1 - d/s)k < \sqrt{2kd'}$. Indeed, this is equivalent to $d/s + \sqrt{2d'/k} > 1$, which is true since

$$\frac{d}{s} + \sqrt{\frac{2d'}{k}} = \frac{d}{s} + \sqrt{\frac{d'}{2k}} + \sqrt{\frac{d'}{2k}} \ge 3\sqrt[3]{\frac{dd'}{2sk}} > 1$$

where the penultimate inequality follows from the arithmetic mean–geometric mean inequality, and the last inequality is true because $d \ge \frac{1}{16}k$ and $d' > \frac{4}{3}s$.

Consequently, $t' + d' < (1 - d/s)k < \sqrt{2kd'}$. Thus, because the function $x \mapsto 2kd'/x + x$ is strictly decreasing over $(0, \sqrt{2kd'})$, we deduce that

$$\frac{2kd'}{t'+d'} + t' + d' > \frac{2sd'}{s-d} + \left(1 - \frac{d}{s}\right)k = k + 2d' + \frac{2dd'}{s-d} - \frac{kd}{s} = k + 2d' - 2d + \frac{4sd}{s-d} - \frac{kd}{s},$$

and so

$$\frac{(k-(t'+d'))(t'-d')}{t'+d'} = k + 2d' - \frac{2kd'}{t'+d'} - (t'+d') < 2d - \frac{4ds}{s-d} + \frac{kd}{s} = 2d - \frac{d(4s^2 - k(s-d))}{s(s-d)}.$$

To complete the proof, it is enough to show that $4s^2 - k(s-d) \ge 0$ which is equivalent to $4(s - \frac{1}{8}k)^2 + k(d - \frac{1}{16}k) \ge 0$. This proves (8) and Lemma 4.1. ■

We are now ready to finish the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Let $|\mathcal{C}|$ be a set of $\chi(G) - 1$ colors, then $G$ is $\mathcal{C}$-inextensible; so there is a minimally $\mathcal{C}$-inextensible subgraph $H$ of $G$. Then $H$ is $(k+1)$-connected and has more than $\chi(G) - k$ vertices by Lemma 3.2, and satisfies $\chi(H) \ge |\mathcal{C}| - 2k + 3 = \chi(G) - 2k + 2$ by Lemma 4.1. This proves Theorem 1.2. ■

## 5. Concluding remarks

There are some final points we would like to make. First, as remarked in the Introduction, we conjecture that the constant $\frac{1}{16}$ in Theorem 1.2 can be removed, which would yield $g(k,m) \le \max(m+2k-2, 3k)$ for all $k \ge 1$ and $m \ge 2$; a possible way to approach this conjecture is to prove Lemma 4.1 when $|\mathcal{C}| \ge 3k-1$. Note that one can simplify the proof of Lemma 4.1 to get $\chi(G) \ge |\mathcal{C}| - 3k+4$ for every inextensible graph $G$ whenever $|\mathcal{C}| \ge 3k-1$, which together with Lemma 3.2 implies that $g(k,m) \le m+3k-3$ for all $k \ge 1$ and $m \ge 3$.

Second, there is an analogue of $g(k,m)$ for list colorings as well (see [7, Section 5.4] for preliminaries). For integers $k \ge 1$ and $m \ge 2$, let $g_\ell(k,m)$ be the least integer $n \ge 1$ such that every graph with choosability at least $n$ contains a $(k+1)$-connected subgraph with choosability at least $m$. It was proved in [8] that $g_\ell(k-1,k) \le 4k$ for all $k \ge 2$; by modifying the argument outlined there and adapting the notions of templates and inextensibility in Section 2 to the list coloring setting, one could show the following which yields $g_\ell(k,m) \le m+3k-2$ for all $k \ge 1$ and $m \ge 2$.

**Proposition 5.1.** *For every integer $k \ge 1$, every graph $G$ with $\chi_\ell(G) \ge 3k$ contains a $(k+1)$-connected subgraph with more than $\chi_\ell(G) - k$ vertices and choosability at least $\chi_\ell(G) - 3k + 2$.*

Note that the lower bound construction in [1] also shows that $g_\ell(k,m) \ge \max(m+k-1, 2k+1)$ for all $k \ge 1$ and $m \ge 2$. It would be interesting to narrow the gap between the lower and upper bounds on $g_\ell(k,m)$, for instance to decide whether $g_\ell(k,m) \le \max(m+2k, Ck)$ for some universal constant $C > 0$, similar to Theorem 1.2.

Finally, given the bounds $\max(m+k-1, 2k+1) \le g(k,m) \le \max(m+2k-2, \lceil (3+\frac{1}{16})k \rceil)$ for all $k \ge 1$ and $m \ge 2$, it would be nice to prove that $g(k,m) \le \max(m+(1+\varepsilon)k, Ck)$ for some universal constants $\varepsilon \in (0,1)$ and $C > 0$. New methods would be needed to accomplish this. Indeed, a loss of $2k-2$ on the chromatic number seems to be the best that the "template-inextensibility" method could produce, because of the optimality of the bound $|\mathcal{C}| - 2k + 3$ in Lemma 4.1 as discussed in Section 4. When $k = 2$, the above bounds only yield $g(2,m) \in \{m+1, m+2\}$ for all $m \ge 5$, but Alex Scott and Paul Seymour (personal communication) recently showed that $g(2,m) = m+1$ for all $m \ge 4$ by proving the following equivalent statement which was conjectured by the author.

**Theorem 5.2.** *For every integer $m \ge 4$ and for every graph $G$ with chromatic number at least $m$, if there are nonadjacent vertices $u, v$ in $G$ for which every optimal coloring of $G$ assigns different colors to $u, v$, then $G$ contains a 3-connected subgraph with chromatic number at least $m$.*

(Here an *optimal coloring* of $G$ is a proper coloring of $G$ using $\chi(G)$ colors.) To see how Theorem 5.2 is equivalent to the statement that $g(2,m) = m+1$ for $m \ge 4$, first assume that Theorem 5.2 holds. Let $G$ be a graph with $\chi(G) = m+1$; we may assume that every subgraph of $G$ with fewer edges than $G$ has chromatic number at most $m$. Suppose that $G$ has no 3-connected subgraphs; then $G$ would have a cutset $S = \{u, v\}$ with $u, v$ nonadjacent in $G$ and two nonempty vertex sets $A, B$ with $A \cup B = V(G) \setminus S$ such that there are no edges between $A$ and $B$. By [3, Theorem 14.9], there would be $H \in \{G[S \cup A], G[S \cup B]\}$ such that $\chi(H) = m$ and every optimal coloring of $H$ assigns to $u, v$ different colors; so $H$ would be a counterexample to Theorem 5.2, a contradiction. Hence $G$ is 3-connected and so $g(2,m) = m+1$ for all $m \ge 4$.

Now, assume that there exists a counterexample $G$ to Theorem 5.2 with two special vertices $u, v$. Let $G_1, G_2, G_3$ be disjoint copies of $G$ with $u_i, v_i$ being the respective copies of $u, v$ for $i = 1, 2, 3$, and let $H_{2,m}$ be the graph constructed in the discussion on Lemma 4.1 with stable set $S = \{x_1, x_2, x_3\}$; then for $i = 1, 2, 3$ identify $u_i$ with $x_i$ and $v_i$ with $x_{i+1}$ (here $x_1 = x_4$), and let $H$ be the resulting graph. If $\chi(H) = m$, then $\chi(G) = m$, and any optimal coloring of $H$ would color $u_i, v_i$ differently for each $i \in \{1, 2, 3\}$ which implies that $x_1, x_2, x_3$ would receive different colors; but this is a contradiction since each of them is adjacent to the vertices in $H_{2,m} \setminus S$ which is a complete graph on $m-2$ vertices. This shows that $\chi(H) = m+1$. Moreover, it is not hard to see that $H$ has no 3-connected subgraphs with chromatic number at least $m$, which yields $g(2,m) = m+2$.

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