

LINEAR-SIZED MINORS WITH GIVEN EDGE DENSITY

TUNG H. NGUYEN

ABSTRACT. It is proved that for every $\varepsilon > 0$, there exists $K > 0$ such that for every integer $t \geq 2$, every graph with chromatic number at least Kt contains a minor with t vertices and edge density at least $1 - \varepsilon$. Indeed, building on recent work of Delcourt and Postle on linear Hadwiger’s conjecture, for $\varepsilon \in (0, \frac{1}{2})$ we can take $K = C \log \log \frac{1}{\varepsilon}$ where $C > 0$ is a universal constant, which extends their recent $O(t \log \log t)$ bound on the chromatic number of graphs with no K_t minor.

1. INTRODUCTION

All graphs in this paper are finite and with no loops or parallel edges. Given a graph G , a *minor* of G is a graph obtained from a subgraph of G by a sequence of edge contractions; and the *chromatic number* of G , denoted by $\chi(G)$, is the least integer $n \geq 0$ such that the vertices of G can be coloured by n colours so that no two adjacent vertices get the same colour.

For integer $t \geq 2$, let K_t denote the complete graph on t vertices, and let $h(t)$ be the least positive integer such that every graph with chromatic number at least $h(t)$ contains a K_t minor. Hadwiger’s well-known conjecture [8] from 1943 states that $h(t) \leq t$ for all $t \geq 2$. This is a vast generalization of the four-colour theorem and is wide open for $t \geq 7$. (See the survey [22] for more details.) Linear Hadwiger’s conjecture [20], one of its weakenings, is still open and asks for the existence of a universal constant $C > 0$ such that $h(t) \leq Ct$.

Since every graph with chromatic number at least $n \geq 1$ has a subgraph with minimum degree at least $n - 1$, Mader [13] initiated the approach via the average degree and proved that for some constant $C > 0$, every graph with average degree at least $Ct \log t$ contains a K_t minor¹. Kostochka [11, 12] and Thomason [25] independently improved this to $Ct\sqrt{\log t}$, which is optimal up to the constant factor C by random graph constructions. Thus $h(t) \leq O(t\sqrt{\log t})$, which remained the record for nearly four decades until Norin, Postle, and Song [17] showed that $h(t) \leq C_\alpha t (\log t)^{1/4+\alpha}$ for every $\alpha > 0$ and $t \geq 2$, where $C_\alpha > 0$ is a constant depending on α . Following subsequent improvements, Delcourt and Postle [4] very recently achieved the best known bound $h(t) \leq O(t \log \log t)$.

Another natural relaxation for Hadwiger’s conjecture seeks to optimize the edge density of a minor on t vertices of every graph with chromatic number at least t . By a theorem of Mader [13], such a graph contains a minor with at most t vertices and minimum degree at least $\frac{1}{2}t$, and hence contains a minor on t vertices and with at least $\frac{1}{4}\binom{t}{2}$ edges. Norin and Seymour [18] recently showed that for all sufficiently large integers n , every graph on n vertices and with stability number two has a minor with $\lceil \frac{1}{2}n \rceil$ vertices and fewer than $\frac{1}{76}$ of all possible edges missing².

The main purpose of this paper is to provide a similar relaxation for linear Hadwiger’s conjecture. For $\varepsilon > 0$ and an integer $t \geq 2$, a graph is (ε, t) -dense if it has t vertices and at least $(1 - \varepsilon)\binom{t}{2}$ edges. Thus every (ε, t) -dense graph is a copy of K_t if $\varepsilon \leq t^{-2}$; and by averaging, every (ε, n) -dense graph contains an (ε, t) -dense subgraph for every $n \geq t \geq 2$. Here is our main result.

Theorem 1.1. *There is an integer $C > 0$ such that for every $\varepsilon \in (0, \frac{1}{2})$ and every integer $t \geq 2$, every graph with chromatic number at least $Ct \log \log \frac{1}{\varepsilon}$ contains an (ε, t) -dense minor.*

Theorem 1.1 has two implications: first, given any number strictly smaller than 1, one can find in a graph a minor whose edge density is at least that number and whose size is linear in terms of the chromatic number of the host graph; second, by letting $\varepsilon = t^{-2}$ for t large, one can recover the $O(t \log \log t)$ bound of Delcourt and Postle. In fact, our proof of Theorem 1.1 is a modification of their argument, illustrating that the “redundant” log factor can be interpreted as a quantity depending on the edge density of the

Date: June 20, 2022; revised April 7, 2024.

Partially supported by AFOSR grants A9550-10-1-0187 and FA9550-22-1-0234, and NSF grant DMS-2154169.

¹In this paper \log denotes the natural logarithm.

²By a theorem of Plummer, Stiebitz, and Toft [19], Hadwiger’s conjecture for graphs with stability number two is equivalent to the statement that every graph with n vertices and stability number two has a $K_{\lceil \frac{1}{2}n \rceil}$ minor.

desired minors instead of the number of vertices. Along the way, we shall see in the next sections that this phenomenon, in fact, also holds for several known results in the area. For example, we shall prove that in order to colour graphs with no (ε, t) -dense minors, it is enough to colour those with not too many vertices; this is similar to [4, Theorem 1.6] but with $1/\varepsilon$ inside all of the log terms.

Theorem 1.2. *There is an integer $C_{1.2} > 0$ such that the following holds. Let $\varepsilon \in (0, \frac{1}{2})$, and let $t \geq (2 \log \frac{1}{\varepsilon})^2$ be an integer. For every graph G , let³*

$$f_{1.2}(G, \varepsilon, t) := 1 + \max_{F \subseteq G} \left\{ \frac{\chi(F)}{a} : a \geq \frac{t}{(2 \log \frac{1}{\varepsilon})^{1/2}}, |F| \leq C_{1.2} a \left(\log \frac{1}{\varepsilon} \right)^4, F \text{ is } (\varepsilon, a)\text{-dense-minor-free} \right\}.$$

If $\chi(G) \geq C_{1.2} t \cdot f_{1.2}(G, \varepsilon, t)$, then G contains an (ε, t) -dense minor.

Let us see how Theorem 1.2 yields Theorem 1.1. We need a result first proved by Woodall [27, Theorem 4], which also follows from an elegant argument of Duchet and Meyniel [6] (see [22, Theorem 4.2]).

Theorem 1.3. *For every integer $t \geq 2$, every graph G with no K_t minor contains an induced subgraph with chromatic number less than t and at least $\frac{1}{2}|G|$ vertices.*

Here is an immediate corollary of Theorem 1.3.

Corollary 1.4. *For every integer $t \geq 2$ and every K_t -minor-free graph G , $\chi(G) \leq t(2 + \log_2(\frac{1}{t}|G|))$.*

Proof. Let $q := \frac{1}{t}|G|$. Theorem 1.3 gives an integer $s \geq 0$ and a chain of sets $V(G) = V_0 \supset V_1 \supset \dots \supset V_s = \emptyset$ with $|V_i| \leq \frac{1}{2}|V_{i-1}|$ and $\chi(G[V_{i-1} \setminus V_i]) < t$ for all $i \in [s]$; in particular $|V_i| \leq 2^{-i}|G| = 2^{-i}tq$ for all $i \in [s]$. Let $r \geq 0$ be maximal with $|V_r| \geq t$; then $|V_{r+1}| < t$ and $2^{-r}tq \geq |V_r| \geq t$ which imply $2^r \leq q$ and so $r \leq \log_2 q$. Therefore

$$\chi(G) \leq \sum_{i=1}^{r+1} \chi(G[V_{i-1} \setminus V_i]) + \chi(G[V_{r+1}]) \leq (r+1)t + t \leq t(2 + \log_2 q) = t(2 + \log_2(\frac{1}{t}|G|)). \quad \blacksquare$$

The proof of Theorem 1.1 now follows shortly.

Proof of Theorem 1.1, assuming Theorem 1.2. Choose C such that $C \geq 7C_{1.2}^2$ and every graph with average degree at least $\frac{1}{4}Ct \log_2 t$ has a K_t minor. Let $\Gamma := \log(1/\varepsilon) \geq 2$, and let G be a graph with $\chi(G) \geq Ct \log_2 \Gamma$. If $t \leq 4\Gamma^2$, then $t \leq \Gamma^4$, so $\chi(G) \geq \frac{1}{4}Ct \log_2 t$ and G has a K_t minor. If $t \geq 4\Gamma^2$, then for every integer $a \geq (2\Gamma)^{-1/2}t$ and every (ε, a) -dense-minor-free subgraph F of G with $|F| \leq C_{1.2}\Gamma^4 a$, Corollary 1.4 applied to F and a gives $\chi(F) \leq a(2 + \log_2(\frac{1}{a}|F|)) \leq a(2 + \log_2(C_{1.2}\Gamma^4)) \leq 6C_{1.2} \log_2 \Gamma$. Thus $f_{1.2}(G, \varepsilon, t) \leq 1 + 6C_{1.2} \log_2 \Gamma \leq 7C_{1.2} \log_2 \Gamma$, and so $\chi(G) \geq Ct \log_2 \Gamma \geq C_{1.2} t \cdot f_{1.2}(G, \varepsilon, t)$. Thus by Theorem 1.2, G contains an (ε, t) -dense minor. This proves Theorem 1.1. \blacksquare

The rest of this paper is devoted to proving Theorem 1.2. We make no serious attempt to optimize absolute constants throughout the paper.

Notation. For an integer $n \geq 0$, let $[n]$ denote $\{1, 2, \dots, n\}$ if $n \geq 1$ and \emptyset if $n = 0$. For a graph G with vertex set $V(G)$ and $E(G)$, let $|G| := |V(G)|$ and $e(G) := |E(G)|$. For $v \in V(G)$, let $N_G(v)$ be the set of neighbours of v in G , and let $d_G(v) := |N_G(v)|$. The *average degree* of G is $\frac{1}{|G|} \sum_{v \in V(G)} d_G(v) = \frac{2e(G)}{|G|}$, and its *edge density* is $e(G)/\binom{|G|}{2}$. Let $\delta(G)$ and $\Delta(G)$, respectively, denote the *minimum degree* and *maximum degree* of G . Let \overline{G} denote the graph with vertex set $V(G)$ and edge set $\{uv : u, v \in V(G), uv \notin E(G)\}$; then $\Delta(\overline{G}) = |G| - 1 - \delta(G)$ is the maximum number of nonneighbours of a vertex of G .

Two disjoint subsets $A, B \subseteq V(G)$ are *anticomplete* in G if G has no edge between them. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S ; and let $G \setminus S := G[V(G) \setminus S]$. A subset $X \subseteq V(G)$ is a *cutset* of G if there is a partition $A \cup B = V(G) \setminus X$ with A, B nonempty and anticomplete. For an integer $k \geq 0$, G is *k-connected* if $|G| > k$ and G has no cutset of size less than k ; and G is *connected* if $|G| = 1$ or it is 1-connected. The *connectivity* of G , denoted by $\kappa(G)$, is the minimum size of a cutset of G .

For an integer $t \geq 1$ and a collection $\mathcal{M} = \{M_1, \dots, M_t\}$ of subgraphs of G , let $\bigcup_{i=1}^t M_i$ be the subgraph of G on vertex set $V(\mathcal{M}) := \bigcup_{i=1}^t V(M_i)$ and edge set $\bigcup_{i=1}^t E(M_i)$. Given a graph H on vertex set $[t]$, \mathcal{M} is an *H model* in G if M_1, \dots, M_t are vertex-disjoint and for all distinct $i, j \in [t]$, G has an edge between $V(M_i)$ and $V(M_j)$ whenever $ij \in E(H)$; and each M_i is called a *fragment* of \mathcal{M} . Thus G has an H minor if and only if it has an H model. For a subset $S \subseteq V(\mathcal{M})$ of size t , \mathcal{M} is *rooted at S* if $|S \cap V(M_i)| = 1$ for all $i \in [t]$.

³Here $F \subseteq G$ means F is a subgraph of G .

2. PROOF SKETCH

This section summarizes the proof of Theorem 1.2, which follows the same route as that of [4, Theorem 1.6]. For integer $m \geq 0$, a graph G is m -chromatic-separable if there are vertex-disjoint subgraphs G_1, G_2 of G with $\chi(G_1), \chi(G_2) \geq \chi(G) - m$; and G is m -chromatic-inseparable if there do not exist such G_1, G_2 . Our proof of Theorem 1.2 treats the chromatic-inseparable case separately via the following theorem.

Theorem 2.1. *There is an integer $C_{2.1} > 0$ such that the following holds. Let $\varepsilon \in (0, \frac{1}{2})$, and let $t \geq (\log \frac{1}{\varepsilon})^{1/2}$ be an integer. For every graph G , let*

$$g_{2.1}(G, \varepsilon, t) := 1 + \max_{F \subseteq G} \left\{ \frac{\chi(F)}{t} : |F| \leq C_{2.1}t \left(\log \frac{1}{\varepsilon} \right)^4, F \text{ is } (\varepsilon, t)\text{-dense-minor-free} \right\}.$$

If G is $C_{2.1}t \cdot g(G, \varepsilon, t)$ -chromatic-inseparable and $\chi(G) \geq C_{2.1}t \cdot g_{2.1}(G, \varepsilon, t)$, then G contains an (ε, t) -dense minor.

The (ε, t) -dense minor will be constructed iteratively in the chromatic-inseparable case and recursively in the general case (using the chromatic-inseparable case as a blackbox). These constructions employ similar setups, which involve three vertex-disjoint parts of the graph in consideration:

- the *base*, whose role depends on each case (which we shall explain below);
- the *hub*, which is a small dense subgraph with large connectivity; and
- the *giant*, which is a highly connected subgraph with large chromatic number.

The existence of the hub will be guaranteed by the following result (cf. [4, Theorem 2.3]).

Theorem 2.2. *There is an integer $C_{2.2} > 0$ such that, for every $\varepsilon \in (0, \frac{1}{3})$ and integers $k \geq t \geq 2$, every graph with average degree at least $C_{2.2}k$ contains either an (ε, t) -dense minor or a k -connected subgraph with at most $C_{2.2}t \log^3(1/\varepsilon)$ vertices.*

Informally, we would like to link the base and the giant simultaneously to the hub by a collection \mathcal{P} of vertex-disjoint paths, then group the endpoints of \mathcal{P} in the hub suitably and use high connectivity to connect the vertices within each of these groups. Let U be the set of endpoints of \mathcal{P} in the giant.

In the chromatic-inseparable case, we construct an (ε, t) -dense model after about $(\log \frac{1}{\varepsilon})^{1/2}$ iterations. In each step, the base consists of the roots of the previously constructed model, together with some specified vertices of about $(\log \frac{1}{\varepsilon})^{1/2}$ vertex-disjoint small dense subgraphs with similar features as the hub. We then do:

1. in each of these small dense subgraphs, use high connectivity to generate a small dense model rooted at the endpoints of \mathcal{P} ;
2. “enlarge” each fragment of the existing model by linking it (via \mathcal{P}) to an endpoint in U and several fragments of the small dense models;
3. use the remaining small fragments and vertices in U to form about $\frac{t}{(\log \frac{1}{\varepsilon})^{1/2}}$ new fragments and add them to the above enlarged model to get a new dense model;
4. make U the set of new roots;
5. via chromatic-inseparability, “attach” a highly connected subgraph F that contains the giant and has even larger chromatic number to the new model at U ; then
6. consider F with the new model in the next iteration.

For the general case, the recursion is based on a ternary tree of depth about $\log \log \frac{1}{\varepsilon}$ whose nodes correspond to the graphs in consideration. For each such graph, the base consists of the vertices we want to root our (ε, a) -dense model at (for some a). Let G' be the giant and assume $|U| = a$; we then do:

1. find a subset $V \subseteq V(G') \setminus U$ of size $2a$ which has a partition into a pairs each consisting of two neighbours of a vertex of U ;
2. use Theorem 2.1 as a black box to create three vertex-disjoint subgraphs of high connectivity and chromatic number in $G' \setminus (U \cup V)$;
3. let these subgraphs be L_1, L_2, L_3 and make them the children of our current node in the recursion tree;
4. appropriately link V to $V(L_1) \cup V(L_2) \cup V(L_3)$ by a collection \mathcal{Q} of $2a$ vertex-disjoint paths; then
5. generate an $(\varepsilon, \frac{2}{3}a)$ -dense model within each L_i (by recursion) and combine these models (via \mathcal{Q}) to form an (ε, a) -dense model rooted at U and thus at the designated vertices in the base.

Except we were cheating slightly here; in fact, combining the three $(\varepsilon, \frac{2}{3}a)$ -dense models this way would probably result in an $(\frac{4}{3}\varepsilon, a)$ -dense model, that is, the nonedge density will increase by a factor of $\frac{4}{3}$.

Therefore it is desirable to keep the depth of the recursion tree about $\log \log(1/\varepsilon)$; and after that we should be able to find a small dense model directly within each leaf node of the tree. Starting with $a = t$ at the highest node, we would expect such a small dense model to have about $\frac{t}{(\log \frac{1}{\varepsilon})^{1/2}}$ vertices.

Another technical point that we also cheated in both constructions is that the generated small dense models could intersect \mathcal{P} (in the chromatic-inseparable case) or \mathcal{Q} (in the general case) at many vertices. Thus it is also desirable to show that given high connectivity, it is possible to simultaneously find a dense model and “weave” a given linkage around so that the new linkage only intersects the model at the original endpoints. In particular, in the general case, the base should also contain a couple of pairs of vertices to be linked separately so that the construction can be done smoothly.

All of the above issues, together with Theorem 2.2, will be resolved as long as we have obtained

- density results for (ε, t) -dense minors in general graphs and unbalanced bipartite graphs; and
- tools on rooted dense minors and dense wovenness.

These will be addressed in Sections 3 and 5, respectively; and other technical issues will be handled along the way. Section 4 includes a proof of Theorem 2.2. We shall prove Theorem 2.1 in Section 6, and prove Theorem 1.2 in Section 7.

3. DENSITY RESULTS

In this section, we prove two results providing essentially tight bounds on the density of graphs and the asymmetric density of bipartite graphs for (ε, t) -dense minors.

Theorem 3.1. *There is an integer $C = C_{3.1} > 0$ such that for every $\varepsilon \in (0, \frac{1}{3})$ and every integer $t \geq 2$, every graph with average degree at least $Ct\sqrt{\log(1/\varepsilon)}$ contains an (ε, t) -dense minor.*

Theorem 3.2. *There is an integer $C = C_{3.2} > 0$ such that the following holds. For every $\varepsilon \in (0, \frac{1}{3})$, every integer $t \geq 2$, and every bipartite graph G with bipartition (A, B) , if*

$$e(G) \geq Ct\sqrt{\log(1/\varepsilon)}\sqrt{|A||B|} + t|G|,$$

then G contains an (ε, t) -dense minor.

Theorems 3.1 and 3.2, respectively, are extensions of the Kostochka–Thomason theorem and the asymmetric density theorem developed by Norin and Postle [16, Theorem 3.2]. By random graph considerations mirroring [2], the $t\sqrt{\log(1/\varepsilon)}$ term in these theorems can be seen to be tight up to a constant factor. The $t|G|$ term in Theorem 3.2 is also necessary: indeed, let $|A| = a$ for any $a \geq 1$ and $|B| = \lfloor (1 - \sqrt{2\varepsilon})t \rfloor$; then for every collection of disjoint nonempty connected subsets $\{B_1, \dots, B_t\}$ of G , there are at least $t\sqrt{2\varepsilon}$ sets lying entirely within A and thus are pairwise anticomplete, which implies that the corresponding minor has at least $\binom{t\sqrt{2\varepsilon}}{2} > \varepsilon \binom{t}{2}$ nonedges for all sufficiently large t (as a function of ε). In fact this $t|G|$ term can be improved slightly to $(t - 2)|G|$, but we do not need it for our purposes.

We remark that even though Theorem 3.2 implies Theorem 3.1, the proof of the former makes use of the latter; so it is necessary to prove Theorem 3.1 in the first place. We need the following lemma, which is inspired by the recent idea of Alon, Krivelevich, and Sudakov [1] for the Kostochka–Thomason bound.

Lemma 3.3. *Let $\varepsilon > 0$, let $k \geq 0$ and $n, r \geq 1$ be integers, and let G be a graph with $n/6 \leq |G| \leq n$ and*

$$24^r \left(\frac{\Delta(\overline{G})}{|G| - 1} \right)^{r^2} \leq \varepsilon.$$

Then for $A_1, \dots, A_k \subseteq V(G)$ with $|A_i| \leq \frac{1}{12}\varepsilon^{1/r}n$ for all $i \in [k]$, there exists $S \subseteq V(G)$ with

- $|S| \leq r$;
- $S \subseteq A_i$ for at most εk indices $i \in [k]$; and
- at most $\frac{1}{12}\varepsilon^{1/r}n$ vertices in $V(G) \setminus S$ have no neighbours in S .

Proof. Let S be a random set of r vertices in G chosen uniformly at random with repetitions. For each $i \in [k]$, since $|G| \geq n/6$, the probability that $S \subseteq A_i$ is at most

$$\left(\frac{|A_i|}{|G|} \right)^r \leq \left(\frac{1}{2}\varepsilon^{1/r} \right)^r \leq \frac{1}{2}\varepsilon,$$

and so the expected number of indices $i \in [k]$ with $S \subseteq A_i$ is at most $\frac{1}{2}\varepsilon k$. Thus by Markov’s inequality, with probability more than $1/2$, $S \subseteq A_i$ for at most εk indices $i \in [k]$.

For each $v \in V(G)$, conditioned on $v \notin S$, the probability that v has no neighbours in S is at most

$$\left(\frac{\Delta(\overline{G})}{|G| - 1} \right)^r \leq \frac{1}{24} \varepsilon^{1/r}.$$

Thus, since $|G| \leq n$, the expected number of vertices in $V(G) \setminus S$ with no neighbours in S is at most $\frac{1}{24} \varepsilon^{1/r} n$; hence with probability more than $1/2$, at most $\frac{1}{12} \varepsilon^{1/r} n$ vertices in $V(G) \setminus S$ have no neighbours in S . Consequently there is a choice of S with the desired properties. This proves Lemma 3.3. \blacksquare

We also need the following result of Mader [13] which was mentioned in the Introduction.

Lemma 3.4. *For every integer $d \geq 2$, every graph with average degree at least $d - 1$ contains a minor with at most d vertices and minimum degree at least $d/2$.*

The next result implicitly appears in [1].

Lemma 3.5. *For every integer $d \geq 2$, every graph with average degree at least d contains a minor with at most d vertices, minimum degree at least $d/3$, and connectivity at least $d/6$.*

Proof. Let G be a graph with average degree at least d . Lemma 3.4 gives a minor H of G with $|H| \leq d$ and $\delta(H) \geq d/2$. We shall find a subgraph H_0 of H with $\delta(H_0) \geq d/3$ and $\kappa(H_0) \geq d/6$. Indeed, if $\kappa(H) \geq d/6$ then we can let $H_0 := H$; so we may assume $\kappa(H) < d/6$. Let X be a cutset of H with $|X| < d/6$, and let $A \cup B$ be a partition of $V(H) \setminus X$ with A, B nonempty and anticomplete in H . By symmetry we may assume $|A| \leq |H|/2 \leq d/2$. Since $\delta(H) \geq d/2$ and $|X| < d/6$, each vertex in A has more than $d/3$ neighbours in A . Thus, since $|A| \leq d/2$, every two vertices in A have more than $2d/3 - d/2 = d/6$ common neighbours in A ; so $\kappa(H[A]) > d/6$ and we let $H_0 := H[A]$. This proves Lemma 3.5. \blacksquare

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let $C := 3360$. Let $d := \lceil Ct\sqrt{\log(1/\varepsilon)} \rceil$ and $r := \lceil 20\sqrt{\log(1/\varepsilon)} \rceil$; then $d \geq 84t \cdot 40\sqrt{\log(1/\varepsilon)} \geq 84rt$. Let G be a graph with average degree at least d . By Lemma 3.5, G has a minor H with $|H| \leq d$, $\delta(H) \geq d/3$, and $\kappa(H) \geq d/6$. We shall build an (ε, t) -dense minor in H by the following claim.

Claim 3.6. *There exist disjoint nonempty subsets B_1, \dots, B_t of $V(H)$ such that for each $i \in [t]$, $H[B_i]$ is connected and B_i is anticomplete to at most $\varepsilon(i - 1)$ sets among B_1, \dots, B_{i-1} in H .*

Subproof. Put $\gamma := \frac{1}{12} \varepsilon^{1/r}$. Let k be an integer with $0 \leq k < t$; and assume we have constructed disjoint subsets B_1, \dots, B_k of $V(H)$ such that for each $i \in [k]$,

- $|B_i| \leq 14r$;
- $H[B_i]$ is connected;
- B_i is anticomplete to at most $\varepsilon(i - 1)$ sets among B_1, \dots, B_{i-1} in H ; and
- at most γd vertices in $V(H) \setminus (B_1 \cup \dots \cup B_{i-1})$ have no neighbours in B_i .

Let $F := H \setminus (B_1 \cup \dots \cup B_k)$. Since $|B_1| + \dots + |B_k| \leq 14rk < 14rt \leq d/6$, we have $\delta(F) > \delta(H) - d/6 \geq d/6$ and $\kappa(F) > \kappa(H) - d/6 \geq 0$; thus $d/6 \leq |F| \leq d$, F is connected, and by the choice of r ,

$$24^r \left(\frac{\Delta(\overline{F})}{|F| - 1} \right)^{r^2} = 24^r \left(1 - \frac{\delta(F)}{|F| - 1} \right)^{r^2} \leq 24^r \left(\frac{5}{6} \right)^{r^2} \leq \left(\frac{5}{6} \right)^{r^2/10} \leq \varepsilon.$$

For $i \in [k]$, let A_i be the set of vertices in $V(F)$ with no neighbours in B_i ; then $|A_i| \leq \gamma d$. By Lemma 3.3 applied to F and A_1, \dots, A_k , there exists $S \subseteq V(F)$ such that

- $|S| \leq r$;
- $S \subseteq A_i$ for at most εk indices $i \in [k]$, which means that S is anticomplete to at most εk sets among B_1, \dots, B_k in H ; and
- at most γd vertices in $V(F) \setminus S$ have no neighbours in S .

Since F is connected, every two vertices in S are joined by some shortest path in $V(F)$; and since $\delta(F) \geq d/6$, such a path has length at most 14, because every two vertices of distance at least three on it should have disjoint neighbourhoods in F . So we can add at most $13|S|$ vertices from $V(F) \setminus S$ to S to get a set $B_{k+1} \subseteq V(F)$ with $|B_{k+1}| \leq 14|S| \leq 14r$ such that $F[B_{k+1}] = H[B_{k+1}]$ is connected. Hence,

- $|B_{k+1}| \leq 14r$;
- $H[B_{k+1}]$ is connected;
- B_{k+1} is anticomplete to at most εk sets among B_1, \dots, B_k in H ; and

- at most γd vertices in $V(F) \setminus B_{k+1} = V(H) \setminus (B_1 \cup \dots \cup B_{k+1})$ have no neighbours in B_{k+1} .

Iterating this for $k = 0, 1, \dots, t-1$ in turn proves Claim 3.6. \square

Now, contracting each of B_1, \dots, B_t in H gives a minor of H (and so of G) with t vertices and at least $(1-\varepsilon)(1+2+\dots+(t-1)) = (1-\varepsilon)\binom{t}{2}$ edges. This proves Theorem 3.1. \blacksquare

Theorem 3.1 gives an asymptotically optimal criterion for general graphs containing a dense minor. Here is a related result (cf. [16, Lemma 3.1]) with a similar proof.

Lemma 3.7. *Let $\varepsilon \in (0, 1)$, and let $n, t \geq 2$ be integers with $n \geq 12t$. Let G be a graph with $|G| = n$. Put $r := \lfloor \frac{n}{12t} \rfloor$ and $q := e(\overline{G})/\binom{n}{2}$. If $24^r(12q)^{r^2} \leq \varepsilon$, then G contains an (ε, t) -dense minor.*

Proof. Since $(12q)^{r^2} < \varepsilon < 1$, we have $q < \frac{1}{12}$. Since $e(\overline{G}) \leq \frac{1}{2}qn^2$, there exists $S \subseteq V(G)$ with $|S| = \lfloor \frac{1}{3}n \rfloor$ such that every vertex of S has at most $2qn$ nonneighbours in G ; in particular $\Delta(\overline{G}[S]) \leq 2qn$.

Claim 3.8. *There exist disjoint nonempty subsets S_1, \dots, S_t of S such that for each $i \in [t]$, S_i is anticomplete to at most $\varepsilon(i-1)$ sets among S_1, \dots, S_{i-1} in $G[S]$.*

Subproof. Put $\gamma := \frac{1}{12}\varepsilon^{1/r}$. Let k be an integer with $0 \leq k < t$; and assume we have constructed disjoint nonempty subsets S_1, \dots, S_k of S such that for each $i \in [k]$,

- $|S_i| \leq r$;
- S_i is anticomplete to at most $\varepsilon(i-1)$ sets among S_1, \dots, S_{i-1} ; and
- at most γn vertices in $S \setminus (S_1 \cup \dots \cup S_{i-1})$ have no neighbours in S_i .

Let $F := G[S] \setminus (S_1 \cup \dots \cup S_k)$; then since $|S_1| + \dots + |S_k| \leq rk < rt \leq \frac{1}{12}n$ and $n \geq 12t \geq 24$, we have

$$|F| - 1 \geq |S| - \frac{n}{12} - 1 \geq \frac{n}{3} - 1 - \frac{n}{12} - 1 = \frac{n}{4} - 2 \geq \frac{n}{6}.$$

Hence, since $\Delta(\overline{F}) \leq \Delta(\overline{G}[S]) \leq 2qn$,

$$24^r \left(\frac{\Delta(\overline{F})}{|F| - 1} \right)^{r^2} \leq 24^r (12q)^{r^2} \leq \varepsilon.$$

For each $i \in [k]$, let A_i be the set of vertices in $V(F)$ with no neighbours in S_i ; then $|A_i| \leq \gamma n$. By Lemma 3.3 applied to F with A_1, \dots, A_t , there exists $S_{k+1} \subseteq V(F)$ with

- $|S_{k+1}| \leq r$;
- $S_{k+1} \subseteq A_i$ for at most εk indices $i \in [k]$, which means that S_{k+1} is anticomplete to at most εk sets among S_1, \dots, S_k ; and
- at most γn vertices in $V(F) \setminus S_{k+1} = S \setminus (S_1 \cup \dots \cup S_{k+1})$ have no neighbours in S_{k+1} .

Iterating this for $k = 0, 1, \dots, t-1$ in turn proves Claim 3.8. \square

Now, let S_1, \dots, S_t be obtained from Claim 3.8. Since every vertex in S has at most $2qn$ nonneighbours in G , every two nonadjacent vertices in S have at least $(1-4q)n$ common neighbours in G , and so at least $(1-4q)n - |S| \geq \frac{2}{3}n - |S| \geq |S|$ common neighbours in $V(G) \setminus S$; note that $q < \frac{1}{12}$. Thus we can add vertices from $V(G) \setminus S$ to S_1, \dots, S_t to get disjoint nonempty subsets B_1, \dots, B_t of $V(G)$ such that $S_i \subseteq B_i$ and $G[B_i]$ is connected for all $i \in [t]$. Therefore $\{G[B_1], \dots, G[B_t]\}$ is an H model in G for some graph H with $e(\overline{H}) \leq \varepsilon(1+2+\dots+(t-1)) = \varepsilon\binom{t}{2}$. This proves Lemma 3.7. \blacksquare

We now give a proof of Theorem 3.2, which is adapted from the proof of [16, Theorem 3.2].

Proof of Theorem 3.2. Let $C := 100C_{3.1}$. Let $d := t\sqrt{\log(1/\varepsilon)}$, and let G be a counterexample to Theorem 3.2 with bipartition (A, B) and with $|G| + e(G)$ minimal; then G has no (ε, t) -dense minor while

$$e(G) \geq Cd\sqrt{|A||B|} + t|G|.$$

By symmetry, we may assume that $|A| \geq |B|$. Put $p := \sqrt{|A|/|B|} \geq 1$.

Claim 3.9. *$p \geq 5$, and there exists $u_0 \in A$ with $d_G(u_0) < C_{3.1}d$.*

Subproof. For every $u \in A$, the minimality of G implies that

$$e(G \setminus u) < Cd\sqrt{(|A|-1)|B|} + t(|G|-1),$$

and so

$$d_G(u) = e(G) - e(G \setminus u) > Cd \left(\sqrt{|A|} - \sqrt{|A|-1} \right) \sqrt{|B|} + t > \frac{1}{2}Cdp^{-1} + t.$$

Similarly, we have $d_G(v) > \frac{C}{2}dp + t$ for all $v \in B$. So, since $t \geq 2$, $q \geq 1$, and $C \geq 10C_{3.1}$, we see that $d_G(v) > C_{3.1}d$ for all $v \in B$; thus by Theorem 3.1, since G has no (ε, t) -dense minor, there exists $u_0 \in A$ with $d_G(u_0) < C_{3.1}d$. Thus $C_{3.1}d > \frac{1}{2}Cdp^{-1}$ and so $p > C/(2C_{3.1}) \geq 5$. This proves Claim 3.9. \square

Now, let v_1, v_2 be distinct neighbours of u_0 , and let G' be the graph obtained from G by deleting u_0 and identifying v_1 and v_2 ; then G' is a minor of G . The minimality of G implies that

$$e(G') < Cd\sqrt{(|A|-1)(|B|-1)} + t(|G|-2) < Cd\sqrt{|A|(|B|-1)} + t|G|$$

and so, by Claim 3.9, the number of common neighbours of v_1, v_2 in G is

$$e(G) - e(G') - d_G(u_0) > \frac{1}{2}Cdp - C_{3.1}d \geq \frac{1}{4}Cdp.$$

Hence the number of common neighbours of every two neighbours of u_0 is at least $\lceil \frac{1}{4}Cdp \rceil =: s$.

Now, put $n := \lceil \frac{1}{2}Cdp^{-1} + t \rceil \geq t$, and let S be an arbitrary subset of $N_G(u_0)$ of size n . Let H be a random minor of G obtained as follows: every vertex $u \in A \setminus \{u_0\}$ with a neighbour in S picks one of its neighbours in S , say $\varphi(u)$, independently and uniformly at random; then we contract the edges $\varphi(u)u$. Given distinct $v_1, v_2 \in S$, we see that they are nonadjacent in H if and only if none of their common neighbours picked one of them, which occurs with probability at most

$$\left(1 - \frac{2}{n}\right)^s \leq e^{-2s/n} =: q.$$

Hence, there is a choice of H for which the number of its nonedges is at most $\binom{n}{2}q$. If $q < \varepsilon$ then G would contain an (ε, t) -dense minor by averaging (as $n \geq t$), a contradiction. We may thus assume $q \geq \varepsilon$, and so $\varepsilon^{-1} \geq q^{-1} = e^{2s/n}$ which yields $n \log(1/\varepsilon) \geq 2s$. Since $n \geq \frac{1}{2}Cdp^{-1}$, $s \geq \frac{1}{4}Cdp$, and $C \geq 100$, we have that $n^2 \log(1/\varepsilon) \geq 2ns \geq \frac{1}{4}C^2d^2 \geq 500t^2 \log(1/\varepsilon)$, and so $n \geq 60t$. Put $r := \lfloor \frac{n}{12t} \rfloor$; then $r \geq \max(5, \frac{n}{15t})$. Since $s \geq np \geq 5n$, we have $24q = 24e^{-2s/n} \leq e^{-s/n}$. Also,

$$r^2 \frac{s}{n} \geq \left(\frac{n}{15t}\right)^2 \frac{s}{n} = \frac{ns}{225t^2} \geq \log(1/\varepsilon),$$

and so $24^r(12q)^{r^2} \leq (24q)^{r^2} \leq e^{-r^2s/n} \leq \varepsilon$. By Lemma 3.7, H (and so G) would contain an (ε, t) -dense minor, a contradiction. This proves Theorem 3.2. \blacksquare

4. SMALL DENSE SUBGRAPHS

With Theorems 3.1 and 3.2 in hand, it is not hard to adapt the proof of [4, Theorem 2.3] to prove Theorem 2.2; still, we give a proof for completeness. We need the following lemma from [4, Lemma 4.2].

Lemma 4.1. *Let $r \geq 3$ be an integer, and let $\delta > 0$. Let G be a graph and let $S \subseteq V(G)$ satisfy*

$$(r-2) \cdot e_G(S) > (r-1)\delta|S| + \partial_G(S).$$

Then there exists a nonempty subset S' of S such that $d_{G[S']}(v) \geq \max(\delta, \frac{1}{r}d_G(v))$ for all $v \in S'$.

We also need a classical result of Mader [13] (see also [5, Theorem 1.4.3]).

Lemma 4.2. *For integer $k \geq 1$, every graph with average degree at least $4k$ has a k -connected subgraph.*

We are now ready to prove Theorem 2.2, which we restate here for the convenience of the readers.

Theorem 4.3. *There is an integer $C > 0$ such that, for every $\varepsilon \in (0, \frac{1}{3})$ and integers $k \geq t \geq 2$, every graph with average degree at least Ck contains either an (ε, t) -dense minor or a k -connected subgraph with at most $C^2t \log^3(1/\varepsilon)$ vertices.*

Proof. Let $C := 400 \max(C_{3.1}, C_{3.2})$. Put $\Gamma := \log(1/\varepsilon)$; then by Theorem 3.1, we may assume $k \leq \Gamma^{1/2}t$. Let G be a graph with average degree $d \geq Ck$; we may assume $Ck \leq d \leq 2Ck$. Let $m \geq 0$ be maximal such that there exist vertex-disjoint subgraphs F_1, \dots, F_m of G satisfying

- F_i is connected and $|F_i| = \lceil 10\Gamma^2(t/k)^2 \rceil$ for all $i \in [m]$; and
- the graph G' obtained from G by contracting each F_i into a vertex x_i satisfies

$$e(G) - e(G') \leq \frac{d}{20}(|G| - |G'|).$$

Such a collection F_1, \dots, F_m always exists because the empty collection satisfies the two bullets. Let

$$\begin{aligned} X &:= \{x_i : i \in [m]\}, \\ Y &:= \{v \in V(G') \setminus X : |N_{G'}(v) \cap X| \geq \frac{1}{30}d\}, \\ Z &:= \{v \in V(G') \setminus (X \cup Y) : d_{G'}(v) \geq 20d\Gamma\}. \end{aligned}$$

Note that G' is (ε, t) -dense-minor-free since G is, and that $|X| = m \leq \frac{1}{10}\Gamma^{-2}(k/t)^2|G| \leq \frac{1}{10}\Gamma^{-1}|G|$. In what follows, for disjoint $S, T \subseteq V(G')$, let $e_{G'}(S, T)$ be the number of edges of G' between S and T .

Claim 4.4. $|Y| \leq \Gamma \cdot (t/k)^2|X|$, and so $|Y| \leq \frac{1}{10}\Gamma^{-1}|G|$.

Subproof. Since $k \leq \Gamma^{-1/2}t$, we may assume $|Y| \geq |X|$. By Lemma 3.7 and since G' is (ε, t) -dense-minor-free,

$$e_{G'}(X, Y) \leq C_{3.2}\Gamma^{1/2}t\sqrt{|X||Y|} + t(|X| + |Y|) \leq C_{3.2}\Gamma^{1/2}t\sqrt{|X||Y|} + 2t|Y|.$$

Now, since $|N_{G'}(v) \cap X| \geq d/30$ for all $v \in Y$, since $C \geq 60C_{3.2}$, and since $d \geq C_{3.2}k \geq 2t$, we see that

$$e_{G'}(X, Y) \geq \frac{d}{30}|Y| \geq 2C_{3.2}k|Y| \geq C_{3.2}k|Y| + 2t|Y|.$$

It follows that $k|Y| \leq \Gamma^{1/2}t\sqrt{|X||Y|}$ and so $|Y| \leq \Gamma \cdot (t/k)^2|X|$. This proves Claim 4.4. \square

Now, let $T := X \cup Y \cup Z$ and $S := V(G') \setminus T$.

Claim 4.5. $e(G'[S]) > \frac{2d}{5}|S| + e_{G'}(S, T)$.

Subproof. Since $d_{G'}(v) \geq 20d\Gamma$ for all $v \in Z$, we have $|Z| \leq 2 \cdot e(G)/(20d\Gamma) = \frac{1}{10}\Gamma^{-1}|G|$. Thus, Claim 4.4 implies that $|T| = |X| + |Y| + |Z| \leq \frac{3}{10}\Gamma^{-1}|G|$. Since $|S| \leq |G|$, by Theorem 3.2, we have

$$e_{G'}(S, T) \leq C_{3.2}\Gamma^{1/2}t\sqrt{|S||T|} + t(|S| + |T|) \leq C_{3.2}\Gamma^{1/2}t\sqrt{|G| \cdot \frac{3}{10}\Gamma^{-1}|G|} + 2t|G| \leq C_{3.2}t|G|.$$

Since $G'[T]$ is (ε, t) -dense-minor-free, we also have

$$e(G'[T]) \leq C_{3.1}\Gamma^{1/2}t|T| \leq C_{3.1}\Gamma^{1/2}t \cdot \Gamma^{-1}|G| \leq C_{3.1}t|G|.$$

Hence, because $e(G') > e(G) - \frac{1}{20}d|G| = \frac{9}{20}d|G|$, $k \geq t$, and $C_{3.1}k, C_{3.2}k \leq \frac{1}{80}d$, it follows that

$$\begin{aligned} e(G'[S]) - e_{G'}(S, T) &= e(G') - 2 \cdot e_{G'}(S, T) - e_{G'}(T) \\ &> \frac{9}{20}d|G| - 2C_{3.2}k|G| - C_{3.1}k|G| \geq \frac{9}{20}d|G| - \frac{1}{40}d|G| - \frac{1}{80}d|G| > \frac{2}{5}d|G|. \quad \square \end{aligned}$$

By Claim 4.5 and Lemma 4.1 applied to $G'[S]$ with $r = 3$ and $\delta = \frac{1}{5}d$, there exists $S' \subseteq S$ such that $d_{G'}[S'](v) \geq \max(\frac{1}{5}d, \frac{1}{3}d_{G'}(v))$ for all $v \in S'$. Let F be a nonempty subgraph of $G'[S']$ such that

- F is connected and $|F| \leq \lceil 10\Gamma^2(t/k)^2 \rceil$;
- the graph G'' obtained from G' by contracting F into a vertex x satisfies

$$e(G') - e(G'') \leq \frac{d}{20}(|F| - 1);$$

- subject to the above two bullets, $|F|$ is maximal.

Such an F always exists since every one-vertex subgraph of $G'[S']$ satisfies the first two bullets. Let $R := N_{G''}(x) \setminus X$, and let $R' := N_{G''}(x) \cap S' \subseteq R \setminus Y$.

Claim 4.6. $|R'| \geq \frac{1}{6}|R|$.

Subproof. Because $d_{G'}[S'](v) \geq \frac{1}{5}d$ for all $v \in V(F) \subseteq S'$, $\sum_{v \in V(F)} d_{G'}[S'](v) \geq \frac{1}{5}d|F| \geq 4(e(G') - e(G''))$. It follows that

$$|R'| \geq \sum_{v \in V(F)} d_{G'}[S'](v) - 2(e(G') - e(G'')) \geq \frac{1}{2} \sum_{v \in V(F)} d_{G'}[S'](v).$$

Now, since $d_{G'}[S'](v) \geq \frac{1}{3}d_{G'}(v)$ for all $v \in V(F) \subseteq S'$ and since $R \subseteq \bigcup_{v \in V(F)} N_{G'}(v)$, we obtain

$$|R'| \geq \frac{1}{2} \sum_{v \in V(F)} d_{G'}[S'](v) \geq \frac{1}{6} \sum_{v \in V(F)} d_{G'}(v) \geq \frac{1}{6}|R|. \quad \square$$

Now, let $v \in R'$. The maximality of m yields $|F| < \lceil 10\Gamma^2(t/k)^2 \rceil$. So, by the maximality of $|F|$, the graph obtained from G' by contracting $G'[V(F) \cup \{v\}]$ has fewer than $e(G') - \frac{1}{20}d|F| \leq e(G'') - \frac{1}{20}d$ edges, which yields $|N_{G''}(x) \cap N_{G''}(v)| > \frac{1}{20}d - 1$. Since $v \in R' \subseteq R \setminus Y$, $|N_{G''}(v) \cap X| \leq \frac{1}{30}d$; hence

$$d_{G[R]}(v) \geq |N_{G''}(v) \cap N_{G''}(x)| - |N_{G''}(v) \cap X| > \frac{d}{20} - 1 - \frac{d}{30} = \frac{d}{60} - 1 \geq \frac{d}{120}.$$

Therefore, by Claim 4.6,

$$2 \cdot e(G[R]) \geq \sum_{v \in R'} d_{G[R]}(v) \geq \frac{1}{120}d|R'| \geq \frac{1}{720}d|R|$$

so $G[R]$ has average degree at least $\frac{1}{720}d \geq 4k$. By Lemma 4.2, $G[R]$ contains a k -connected subgraph H . Finally, since $d_{G'}(v) \leq 20\Gamma d$ for all $v \in V(F) \subseteq S$, we have

$$|R| \leq \sum_{v \in V(F)} d_{G'}(v) \leq 20\Gamma d|F| \leq 400C\Gamma^3 k(t/k)^2 \leq C^2\Gamma^3 t$$

where the third inequality holds as $d \leq 2Ck$ and $|F| \leq 10\Gamma^2(t/k)^2$, and the last inequality holds as $k \geq t$. Hence $|H| \leq |R| \leq C^2\Gamma^3 t$. This proves Theorem 4.3. \blacksquare

The following handy consequence of Theorem 2.2 (cf. [4, Corollary 6.1]) will be needed later on.

Corollary 4.7. *Let $k \geq t \geq 2$ be integers. Let $\varepsilon > 0$, and let $\Gamma := \log(1/\varepsilon)$. For every graph G , put*

$$g(G, \varepsilon, t) = g_{4.7}(G, \varepsilon, t) := 1 + \max_{F \subseteq G} \left\{ \frac{\chi(F)}{t} : |F| \leq C_{2.2}^2 \Gamma^4 t, F \text{ is } (\varepsilon, t)\text{-dense-minor-free} \right\}.$$

If $\chi(G) \geq 3C_{2.2}k \cdot g(G, \varepsilon, t)$, then G contains at least Γ vertex-disjoint k -connected subgraphs each with at most $C_{2.2}^2 \Gamma^3 t$ vertices.

Proof. Let $m \geq 0$ be maximal such that there is a collection F_1, \dots, F_m of vertex-disjoint k -connected subgraphs of G each with at most $C_{2.2}^2 \Gamma^3 t$ vertices. We may assume $m < \Gamma$. Let F be the subgraph of G induced by $\bigcup_{i \in [m]} V(F_i)$; then $|F| \leq C_{2.2}^2 \Gamma^3 t \cdot m \leq C_{2.2}^2 \Gamma^4 t$ and so $\chi(F) \leq t \cdot g(G, \varepsilon, t)$. Let $F' := G \setminus F$; then by Theorem 2.2 and the maximality of m , every subgraph of F' has average degree at most $C_{2.2}k$, and so F' has degeneracy at most $C_{2.2}k$ which yields $\chi(F') \leq C_{2.2}k + 1$. It follows that

$$\chi(G) \leq \chi(F) + \chi(F') \leq t \cdot g(G, \varepsilon, t) + C_{2.2}k + 1 < 3C_{2.2}k \cdot g(G, \varepsilon, t),$$

a contradiction. This proves Corollary 4.7. \blacksquare

5. CONNECTIVITY TOOLS

5.1. Basic tools. In this subsection, we recall several definitions and known results on connectivity. For a graph G and subsets $S, T \subseteq V(G)$, a *path between S and T* is a path with one endpoint in S , the other endpoint in T , and no interval vertex in $S \cup T$; thus an edge between S and T is a path of length one between S and T . A *separation* of G is a pair (A, B) of subsets of $V(G)$ with $A \cup B = V(G)$ and $A \setminus B, B \setminus A$ anticomplete in G ; and the *order* of (A, B) is $|A \cap B|$. Thus $A \cap B$ is a cutset of G if and only if $A \setminus B, B \setminus A \neq \emptyset$. We start with Menger's theorem [14] and its ‘‘redundancy’’ version developed by Delcourt and Postle [4, Lemma 5.3].

Theorem 5.1. *Let G be a graph, and let $S, T \subseteq V(G)$. Then for every integer $k \geq 0$, either*

- *there are k vertex-disjoint paths of G between S and T ; or*
- *there is a separation (A, B) of G of order less than k with $S \subseteq A$ and $T \subseteq B$.*

Lemma 5.2. *Let G be a graph, and let S_1, S_2, T be disjoint subsets of $V(G)$ such that for each $i \in \{1, 2\}$, there are $2|S_i|$ paths between S_i and T in $G \setminus T_{3-i}$ which pairwise share no vertex in $V(G) \setminus S_i$, such that each vertex in S_i is the endpoint of exactly two of them. Then there are $|S_1| + |S_2|$ vertex-disjoint paths between $S_1 \cup S_2$ and T in G .*

For an integer $k \geq 0$, a graph G , and subsets $\{s_1, \dots, s_k\}$ and $\{t_1, \dots, t_k\}$ of $V(G)$ such that $s_i \neq t_j$ for all distinct $i, j \in [k]$, a *linkage in G linking $\{(s_i, t_i) : i \in [k]\}$* is a collection $\mathcal{P} = \{P_1, \dots, P_k\}$ of vertex-disjoint paths in G such that P_i has endpoints s_i and t_i for all $i \in [k]$; and G is *k -linked* if for every such choice of $\{s_1, \dots, s_k\}$ and $\{t_1, \dots, t_k\}$, there is a linkage in G linking $\{(s_i, t_i) : i \in [k]\}$. Bollobás and Thomason [3] were the first to prove that linear connectivity is sufficient to force high linkedness; and the current record on this problem is due to Thomas and Wollan [24] who proved that $10k$ -connectivity yields k -linkedness.

Theorem 5.3. *For every integer $k \geq 1$, every $10k$ -connected graph is k -linked.*

For integers $k \geq \ell \geq 1$, G is (k, ℓ) -knot if for every integer m with $\ell \leq m \leq k$ and every $S \subseteq V(G)$ with $|S| = k$ and every partition of S into nonempty subsets S_1, \dots, S_m , there are vertex-disjoint connected subgraphs G_1, \dots, G_m of G such that $S_i \subseteq V(G_i)$ for all $i \in [m]$. Thus G is k -linked if it is $(2k, k)$ -knot. Note that connectivity linear in $k - \ell$ is insufficient to force the (k, ℓ) -knot property: indeed, given integers $C, a \geq 1$, let $k \geq (C + 1)a + 1$, and let G be K_k with all edges in a clique of size $a + 1$ removed; then $\kappa(G) \geq k - a - 1 \geq Ca$ but G is not $(k, k - a)$ -knot. Bollobás and Thomason [3] observed that their method for proving linkedness given linear connectivity can be adapted to show the existence of some constant $C > 0$ such that every Ck -connected graph is (k, ℓ) -knot. Here we give a quick proof, using Theorem 5.3, that $11k$ -connectivity is enough.

Theorem 5.4. *For all integers $k \geq \ell \geq 1$, every $11k$ -connected graph is (k, ℓ) -knot.*

Proof. Let G be a graph with $\kappa(G) \geq 11k$. Let m be an integer with $\ell \leq m \leq k$, let $S \subseteq V(G)$ with $|S| = k$ and let $S_1 \cup \dots \cup S_m$ be a partition of S into nonempty subsets. For every $i \in [m]$, let $k_i := |S_i|$ and $S_i = \{u_i^j : j \in [k_i]\}$; and we may assume $k_1 \geq \dots \geq k_m \geq 1$. Let $n \geq 0$ be maximal such that $k_n \geq 2$. Then it suffices to construct vertex-disjoint connected subgraphs G_1, \dots, G_n of $G \setminus (\bigcup_{i=n+1}^m S_i)$ with $S_i \subseteq V(G_i)$ for all $i \in [n]$. Since $\delta(G) \geq \kappa(G) \geq 11k \geq 3|S|$, there exists a set of $2n$ vertices $\bigcup_{i=1}^n \{v_{i1}^j, v_{i2}^j : j \in [k_i]\} \subseteq V(G) \setminus S$ such that for every $i \in [n]$, u_i^j is adjacent to v_{i1}^j, v_{i2}^j for all $j \in [k_i]$. Since $\kappa(G \setminus S) \geq 11k - k = 10k$, $G \setminus S$ is k -linked by Theorem 5.3; thus there is a linkage in $G \setminus S$ linking $\bigcup_{i=1}^n \{(v_{i2}^j, v_{i1}^{j+1}) : j \in [k_i - 1]\}$. This linkage gives rise to the desired vertex-disjoint connected subgraphs G_1, \dots, G_n of G , proving Theorem 5.4. ■

We remark that the above proof actually gives a stronger conclusion, of independent interest: given any linear ordering of the vertices in S , there is a path P of G such that $S \subseteq V(P)$ and the vertices of S lie on P in the given order.

We also need the following result [15] on highly connected subgraphs of large chromatic number, which is an improvement on the constant factors of a result of Girão and Narayanan [7]. (In fact, it is proved in [15] that the constant 4 can be replaced by $3 + \frac{1}{16}$, but we choose 4 since this has a simple proof.)

Theorem 5.5. *For every integer $k \geq 1$, every graph G with $\chi(G) \geq 4k$ contains a k -connected subgraph with chromatic number at least $\chi(G) - 2k$.*

Last but not least, we require the following result [4, Lemma 6.2], which will be helpful to construct models with fragments of reasonably small chromatic number.

Lemma 5.6. *Let G be a connected graph. Then for every $S \subseteq V(G)$, there is $S' \subseteq V(G)$ with $S \subseteq S'$ and $|S'| \leq 3|S|$ and an connected induced subgraph H of G containing S' such that $\chi(H \setminus S') \leq 2$.*

5.2. Rooted dense minors and dense wovenness. We now develop tools on rooted minors which will be necessary for our (ε, t) -dense minors construction. Given graphs G, H and $S \subseteq V(G)$ with $|S| \leq |H|$, an H model attached to S in G is an H model $\mathcal{M} = \{M_1, \dots, M_{|H|}\}$ in G with $|S \cap V(M_i)| = 1$ for all $i \in \{1, 2, \dots, |S|\}$; thus \mathcal{M} is an H model rooted at S in G if $|S| = |H|$. A separation (A, B) of G is an S -separation if $S \subseteq A$; and (A, B) avoids a subset $D \subseteq V(G)$ if $D \subseteq B \setminus A$. The following lemma (with a simple proof, cf. [3, Lemma 2]) is an extension of an argument by Kawarabayashi [10], which in turn is based on earlier work of Robertson and Seymour [21, Theorem 5.3].

Lemma 5.7. *Let $m, t \geq 1$ and $n \geq 0$ be integers with $m \geq n + 2t$. Let G be a graph and $S \subseteq V(G)$ with $|S| = t$. Let $D_1, \dots, D_m \neq \emptyset$ be disjoint subsets of $V(G)$ and $I := \{i \in [m] : D_i \cap S = \emptyset\}$, such that*

- for every $i \in I$, $G[D_i]$ is connected;
- for every $j \in [m] \setminus I$, each component of $G[D_j]$ intersects S ;
- for every $j \in [m]$, D_j is anticomplete in G to at most n sets in $\mathcal{D} := \{D_i : i \in I\}$; and
- there is no S -separation of G of order less than t avoiding more than n sets in \mathcal{D} .

Then G has an H model attached to S for some graph H with $|H| = m - t$ and $\Delta(\overline{H}) \leq n$.

Proof. Suppose not. Let G be a counterexample with $|G| + e(G)$ minimal; let $D := \bigcup_{j \in [m]} D_j$. If $E(G[S]) = \emptyset$, then $G \setminus E(G[S])$ would be a smaller counterexample, a contradiction; so $E(G[S]) \neq \emptyset$.

Claim 5.8. *Every edge of G is an edge between two distinct members of D_1, \dots, D_m .*

Subproof. Suppose there is $e \in E(G)$ not satisfying that property. For every $j \in [m]$, let D'_j be obtained from D_j after the contraction of e ; then D'_1, \dots, D'_m are nonempty disjoint connected subsets of G/e . By the minimality of G , G/e has an S -separation of order less than t avoiding more than n sets in $\{D'_j : j \in [m]\}$. Thus G has an S -separation (A, B) of order t avoiding more than n sets in \mathcal{D} such that $e \subseteq A \cap B =: S'$. Since $E(G[S]) = \emptyset$, $S \neq S'$ which yields $B \subsetneq V(G)$. If there are fewer than t vertex-disjoint paths from S to S' within $G[A]$, then Theorem 5.1 would yield an S -separation (A_1, A_2) of $G[A]$ of order less than t ; so $(A_1, A_2 \cup B)$ would be an S -separation of G of order less than t avoiding more than n sets in \mathcal{D} , a contradiction. Thus there are t vertex-disjoint paths from S to S' within $G[A]$. If there is an S' -separation (B_1, B_2) of $G[B]$ of order less than t avoiding more than n sets in \mathcal{D} , then $(A \cup B_1, B_2)$ would be an S -separation of G of order less than t avoiding more than n sets in \mathcal{D} , a contradiction. Now, note that for each $j \in [m]$, $D_j \cap B \neq \emptyset$ since D_j is anticomplete to at most n sets in \mathcal{D} ; in particular either $G[D_j \cap B]$ is connected or its components intersect S' . Thus, the minimality of G applied to $G[B]$ with S' and $\{D_j \cap B : j \in [m]\}$ yields an H model attached to S' in $G[B]$ for some H with $|H| = m - t$ and $\Delta(\overline{H}) \leq n$. This model together with the above t vertex-disjoint paths from S to S' yields an H model attached to S in G , a contradiction. This proves Claim 5.8. \square

Now, by the minimality of G , it has no isolated vertex outside of S . So Claim 5.8 yields $V(G) \setminus S \subseteq D$; and for every $j \in [m]$, if $|D_j| \geq 2$ then $D_j \subseteq S$. Let $T := V(G) \setminus S = \bigcup_{i \in I} D_i$; then $|T| \geq m - t \geq t + n$. If there are no t disjoint edges between $D \cap S$ and T , then since the vertices in $S \setminus D$ are isolated, Theorem 5.1 would give an S -separation (A, B) of order less than t in G with $T \subseteq B$ which avoids at least $|T \setminus (A \cap B)| > t + n - t = n$ sets in \mathcal{D} , a contradiction. Thus there are t disjoint edges between $D \cap S$ and T whose contraction would yield a graph H with $\Delta(\overline{H}) \leq n$, a contradiction. This proves Lemma 5.7. \blacksquare

The following result is a straightforward consequence of Lemma 5.7.

Corollary 5.9. *Let $m, t \geq 1$ and $n \geq 0$ be integers with $m \geq n + 2t$. Let G be a t -connected graph, and let $S \subseteq V(G)$ with $|S| = t$. If G contains a minor J with $|J| = m$ and $\Delta(\overline{J}) \leq n$, then it contains an H model attached to S for some graph H with $|H| = m - t$ and $\Delta(\overline{H}) \leq n$.*

For $\varepsilon > 0$ and integers $a \geq 1, b \geq 0$, a graph G is (ε, a, b) -woven if for every subsets $R = \{r_1, \dots, r_a\}$, $S = \{s_1, \dots, s_b\}$, and $T = \{t_1, \dots, t_b\}$ of $V(G)$ such that $s_i \neq t_j$ for all distinct $i, j \in [b]$, there is an (ε, a) -dense graph H such that G has an H model \mathcal{M} rooted at R and a linkage \mathcal{P} linking $\{(s_i, t_i) : i \in [b]\}$ such that $V(\mathcal{M}) \cap V(\mathcal{P}) = R \cap (S \cup T)$. Two remarks:

- the existence of H depends entirely on R, S , and T in this definition; and
- this definition can also be made for a much stronger notion of rooted minors which requires each fragment of the H model to contain a prespecified vertex in R (see [26]), but we do not need it here.

The following useful lemma, an analogue of [4, Lemma 5.15], allows us to generate a dense minor and “weave” a given linkage around at the same time, as mentioned in Section 2.

Lemma 5.10. *Let $\varepsilon > 0$, and let $a \geq 1, b \geq 0$ be integers. Let G be a graph with an (ε, a, b) -woven subgraph F . Let $S = \{s_1, \dots, s_b\}$ and $T = \{t_1, \dots, t_b\}$ be subsets of $V(G)$ such that $s_i \neq t_j$ for all distinct $i, j \in [b]$. Let \mathcal{P}' be a linkage in G linking $\{(s_i, t_i) : i \in [b]\}$, and let $\{r_1, \dots, r_a\} \subseteq V(F)$. Then there is an (ε, a) -dense graph H such that there is an H model \mathcal{M} rooted at $\{r_1, \dots, r_a\}$ in F and a linkage \mathcal{P} in G linking $\{(s_i, t_i) : i \in [b]\}$ such that $V(\mathcal{M}) \cap V(\mathcal{P}) \subseteq R \cap (S \cup T)$ and $V(\mathcal{P}) \subseteq V(F) \cup V(\mathcal{P}')$.*

Proof. For each $i \in [a]$, let P'_i be the path in \mathcal{P}' linking s_i, t_i . Let $I := \{i \in [b] : V(P'_i) \cap V(F) \neq \emptyset\}$. For every $i \in I$, let s'_i, t'_i , respectively, be the vertices nearest to s_i, t_i on P'_i . Since F is (ε, a, b) -woven, there exists an (ε, a) -dense graph H and a linkage $\mathcal{Q} = \{Q_i : i \in I\}$ in F linking $\{(s'_i, t'_i) : i \in I\}$ such that there is an H model \mathcal{M} rooted at $\{r_1, \dots, r_a\}$ in F which is vertex-disjoint from \mathcal{Q} . For every $i \in I$, let $P_i := s_i P'_i s'_i Q_i t'_i P'_i t_i$; then \mathcal{M} together with the linkage $\mathcal{P} = \{P_i : i \in I\} \cup \{P'_i : i \in [b] \setminus I\}$ satisfy the conclusion of Lemma 5.10. \blacksquare

We now give two applications of Corollary 5.9 which will be needed later on. The first one says that linear connectivity together with a very dense minor yield dense wovenness.

Lemma 5.11. *For every $\varepsilon > 0$ and integer $a \geq 2$, every $8a$ -connected graph G containing a $(\frac{1}{256}\varepsilon, 32a)$ -dense minor is $(\varepsilon, a, 3a)$ -woven.*

Proof. Since G has a $(\frac{1}{256}\varepsilon, 32a)$ -dense minor, it has a minor J with $16a \leq |J| \leq 32a$ and $\Delta(\overline{J}) \leq \frac{1}{2}\varepsilon a$. Let $b := 3a$; let $R = \{r_1, \dots, r_a\}$, $S = \{s_1, \dots, s_b\}$, and $T = \{t_1, \dots, t_b\}$ be subsets of $V(G)$ such that $s_i = t_j$ for some $i, j \in [b]$ only if $i = j$. We may assume $S \cap T = \emptyset$. Since G is $8a$ -connected and so

has minimum degree at least $8a$, there is $R' = \{r'_1, \dots, r'_a\} \subseteq V(G) \setminus (R \cup S \cup T)$ with $r_i r'_i \in E(G)$ for all $i \in [a]$. Now, let $G' := G \setminus R$; then G' contains a minor J' which is an induced subgraph of J with $15a \leq |J'| \leq 32a$ and $\Delta(\overline{J'}) \leq \Delta(\overline{J}) \leq \frac{1}{2}\varepsilon a$. Let $m := |J'|$, $n := \Delta(\overline{J'})$, and $t := a + 2b = 7a$; then $m - n \geq (15 - \varepsilon)a \geq 14a = 2t$. So by Corollary 5.9, G' has an F model $\{G_1, \dots, G_{m-t}\}$ for some F with

- $V(F) = [m - t]$ and $\Delta(\overline{F}) \leq \Delta(\overline{J'}) \leq \frac{1}{2}\varepsilon a$;
- $r'_i \in V(G_i)$ for all $i \in [a]$; and
- $s_i \in V(G_{a+i})$ and $t_i \in V(G_{a+2i})$ for all $i \in [b]$.

Observe that every two vertices of F have at least $m - t - 1 - \varepsilon a$ common neighbours, and so at least $m - 2t - 1 - \varepsilon a = m - (10 + \varepsilon)a - 1 \geq 3a = b$ common neighbours in $V(F) \setminus [t]$. It follows that there are distinct $j_1, \dots, j_b \in V(F) \setminus [t]$ such that for every $i \in [b]$, j_i is adjacent to both $a + i$ and $a + 2i$.

Now, let H be the subgraph of F induced by $[a]$; then $|H| = a$ and $\Delta(\overline{H}) \leq \frac{1}{2}\varepsilon a \leq \varepsilon(a - 1)$, in particular H is (ε, a) -dense. For every $i \in [b]$, let P_i be some path between s_i and t_i in the subgraph of G' induced by $V(G_{a+i}) \cup V(G_{j_i}) \cup V(G_{a+2i})$; then $\{G_1, \dots, G_a\}$ is an H model in G' rooted at R' and is vertex-disjoint from the linkage $\{P_1, \dots, P_b\}$ linking $\{(s_i, t_i) : i \in [b]\}$ in G' . Combining $\{G_1, \dots, G_a\}$ with the edges $\{r_i r'_i : i \in [a]\}$ gives an H model in G rooted at R and vertex-disjoint from $\{P_1, \dots, P_b\}$. This proves Lemma 5.11. \blacksquare

Note the same argument shows that for every integer $k \geq 1$, there is $K > 0$ such that for every $\varepsilon > 0$ and integer $a \geq 2$, every Ka -connected graph containing a $(K^{-1}\varepsilon, Ka)$ -dense minor is (ε, a, ka) -woven.

We also need the following result of Bollobás and Thomason [3, Lemma 1] (cf. Lemmas 3.4 and 3.5).

Lemma 5.12. *Let $d \geq 3$ be an integer. Then every graph with average degree at least d contains a minor J with $|J| \leq \frac{1}{2}d$ and $\Delta(\overline{J}) \leq \frac{1}{2}|J| - \frac{3}{40}d$.*

Now we come the second application of Corollary 5.9, which implies that high connectivity yields dense woveness.

Lemma 5.13. *There is an integer $C = C_{5.13} > 10$ such that the following holds. Let $a \geq 1$ and $b \geq 0$ be integers, and let $\varepsilon \in (0, \frac{1}{3})$. Then every $(2a + 2b)$ -connected graph G with average degree at least $d = \lceil C \max(a\sqrt{\log(1/\varepsilon)}, b) \rceil$ is (ε, a, b) -woven.*

Proof. The constant C is chosen implicitly to satisfy the inequalities throughout the proof.

Let $R = \{r_1, \dots, r_a\}$, $S = \{s_1, \dots, s_b\}$, and $T = \{t_1, \dots, t_b\}$ be subsets of $V(G)$ such that $s_i = t_j$ for some $i, j \in [b]$ only if $i = j$. We may assume $S \cap T = \emptyset$. Since G has minimum degree at least $2a + 2b$, there is $R' = \{r'_1, \dots, r'_a\} \subseteq V(G) \setminus (R \cup S \cup T)$ with $r_i r'_i \in E(G)$ for all $i \in [a]$. Let $G' := G \setminus R$; then G' is $(a + 2b)$ -connected and has average degree at least $d' := d - a$. By Lemma 5.12, G' has a minor J with $|J| \leq \frac{1}{2}d'$ and $\Delta(\overline{J}) \leq \frac{1}{2}|J| - \frac{3}{40}d'$. Let $m := |J|$, $n := \Delta(\overline{J})$, and $t := a + 2b$; then $m - n \geq \frac{1}{2}m + \frac{3}{40}d' \geq 2t$. So by Corollary 5.9, G' has an F model $\{G_1, \dots, G_{m-t}\}$ for some graph F with

- $V(F) = [m - t]$ and $\Delta(\overline{F}) \leq \Delta(\overline{J}) \leq \frac{1}{2}m - \frac{3}{40}d'$;
- $r'_i \in V(G_i)$ for all $i \in [a]$; and
- $s_i \in V(G_{a+i})$ and $t_i \in V(G_{a+2i})$ for all $i \in [b]$.

Note that every two vertices of F have at least $m - t - 1 - 2(\frac{1}{2}m - \frac{3}{40}d') = \frac{3}{40}d' - t - 1$ common neighbours, and so at least $\frac{3}{40}d' - 2t - 1 = \frac{3}{40}(d - a) - 2t - 1 \geq \frac{1}{20}d$ common neighbours in $V(F) \setminus [t]$.

Let I be a random subset of $V(F) \setminus [t]$ where each element is included independently with probability $1/3$. Then by Markov's inequality, $|I| \leq \frac{1}{2}(|F| - t)$ with probability at least $1/3$. By Hoeffding's inequality, two given vertices of F have at most $\frac{1}{80}d$ common neighbours in I with probability at most $e^{-d/3000}$. So by a union bound, with probability at least $1 - d^2 e^{-d/3000} > 2/3$, every two vertices of F have at least $\frac{1}{80}d$ common neighbours in I . Hence there is a choice of I such that $|I| \leq \frac{1}{2}(|F| - t)$ and every two vertices of F have at least $\frac{1}{80}d$ common neighbours in I .

Let $L := F \setminus (I \cup [t])$; then $|L| \geq |F| - \frac{1}{2}(|F| + t) = \frac{1}{2}(|F| - t) = \frac{1}{2}m - t$, and so

$$\delta(L) \geq |L| - 1 - \Delta(\overline{F}) \geq \frac{1}{2}m - t - 1 - \left(\frac{1}{2}m - \frac{3}{40}d'\right) = \frac{3}{40}(d - a) - t - 1 \geq \frac{1}{20}d \geq C_{3.1}a\sqrt{\log(1/\varepsilon)}.$$

Thus by Theorem 3.1, L contains an H model $\{L_1, \dots, L_a\}$ for some (ε, a) -dense graph H .

Now, since every two vertices of F (thus L) have at least $\frac{1}{80}d \geq a + b$ common neighbours in I , there exist distinct $j_1, \dots, j_a, k_1, \dots, k_b \in I$ such that in F ,

- for every $i \in [a]$, j_i is adjacent to i and has a neighbour in $V(L_i)$ for all $i \in [a]$; and
- for every $i \in [b]$, k_i is adjacent to $a + i$ and $a + 2i$.

For every $i \in [a]$, let M'_i be the subgraph of G' induced by $V(G_i) \cup V(G_{j_i}) \cup \bigcup_{j \in V(L_i)} V(G_j)$; and for every $i \in [b]$, let P_i be some path between s_i and t_i in the subgraph of G' induced by $V(G_{a+i}) \cup V(G_{k_i}) \cup V(G_{a+2i})$. Then $\{M'_1, \dots, M'_a\}$ is an H model rooted at R' in G' and is vertex-disjoint from the linkage $\{P_1, \dots, P_b\}$ linking $\{(s_i, t_i) : i \in [b]\}$. Let $M_i := M'_i \cup r_i r'_i$ for every $i \in [a]$; then $\{M_1, \dots, M_a\}$ is an H model rooted at R in G which is vertex-disjoint from $\{P_1, \dots, P_b\}$. This proves Lemma 5.13. \blacksquare

6. CHROMATIC-INSEPARABILITY

This section presents the proof of Theorem 2.1. We need more definitions. For graphs G, H and an H model $\mathcal{M} = \{M_1, \dots, M_k\}$ in G where $k = |H| \geq 1$, a subset S of $V(G)$ is a *core* of \mathcal{M} if for all distinct $i, j \in [k]$, $S \cap V(M_i)$ and $S \cap V(M_j)$ are not anticomplete in G if $ij \in E(H)$. For $U \subseteq V(\mathcal{M})$ such that \mathcal{M} is rooted at U , a subgraph F of G is *tangent to \mathcal{M} at U* if $V(F) \cap V(M_i) = U \cap V(M_i)$ for all $i \in [k]$. The following lemma formalizes the iterative construction of an (ε, t) -dense minor.

Lemma 6.1. *There is an integer $C = C_{6.1} > 0$ such that the following holds. Let $\varepsilon \in (0, \frac{1}{2})$ and $\Gamma := \log \frac{1}{\varepsilon}$. Let $t \geq \Gamma^{1/2}$ be an integer, let G be a graph, and let*

$$g(G, \varepsilon, t) = g_{6.1}(G, \varepsilon, t) = 1 + \max_{F \subseteq G} \left\{ \frac{\chi(F)}{t} : |F| \leq C^2 \Gamma^4 t, F \text{ is } (\varepsilon, t)\text{-dense-minor-free} \right\}.$$

If G is $(C^2 t \cdot g(G, \varepsilon, t))$ -chromatic-inseparable and $\chi(G) \geq 2C^2 t \cdot g(G, \varepsilon, t)$, then for every integer q with $0 \leq q \leq \lceil \Gamma^{1/2} \rceil$, G contains an (ε, t) -dense minor or contains both of the following

- an H model \mathcal{A} rooted at U with a core $S \subseteq V(G)$ where $U \subseteq S$, $|S| \leq q^2 C_{2.2}^2 \Gamma^3 t$, and H is an (ε, s) -dense graph with $s = q \lceil \Gamma^{-1/2} t \rceil$; and
- a Ct -connected subgraph F tangent to \mathcal{A} at U where $\chi(F) \geq \chi(G) - \frac{1}{2} C^2 t \cdot g(G, \varepsilon, t)$.

Proof. The constant C is chosen implicitly to satisfy the inequalities throughout the proof.

Induction on q . For $q = 0$, the subgraph F exists by Theorem 5.5. So we may assume $q \geq 1$ and the lemma holds for $q - 1$. Let $r := \lceil \Gamma^{-1/2} t \rceil \geq 1$ and $s' := s - r = (q - 1)r$; then $s' \leq \Gamma^{1/2} (\Gamma^{-1/2} t + 1) \leq 2t$ since $q - 1 \leq \Gamma^{1/2} \leq t$. We may assume G is (ε, t) -dense-minor-free; then by induction, G contains

- an H' model $\mathcal{A}' = \{A'_1, \dots, A'_{s'}\}$ rooted at U' with a core $S' \subseteq V(G)$ where $U' \subseteq S'$, $|S'| \leq (q - 1)^2 C_{2.2}^2 \Gamma^3 t$, and H is an (ε, s') -dense graph; and
- a Ct -connected subgraph F' tangent to \mathcal{A}' at U' where $\chi(F') \geq \chi(G) - \frac{1}{2} C^2 t \cdot g(G, \varepsilon, t)$.

For every $i \in [s']$, let u_i be the unique vertex in $V(F') \cap A'_i$; then $U' = \{u_1, \dots, u_{s'}\}$. Let $k := Ct$; then

$$\begin{aligned} \chi(F' \setminus U') &\geq \chi(F') - s' \geq \chi(G) - \frac{1}{2} C^2 t \cdot g(G, \varepsilon, t) - 2t \\ &\geq C^2 t \cdot g(G, \varepsilon, t) \geq C^2 t \cdot g_{4.7}(G, \varepsilon, t) \geq 3C_{2.2}^2 k. \end{aligned}$$

By Corollary 4.7, $F' \setminus U'$ has q vertex-disjoint k -connected subgraphs F_0, F_1, \dots, F_{q-1} with $|F_i| \leq C_{2.2}^2 \Gamma^3 t$ for all $i \in \{0, 1, \dots, q-1\}$. For every $i \in [q-1]$, let $U_i \subseteq V(F_i)$ with $|U_i| = 2r$, and let $T := U' \cup \bigcup_{i=1}^{q-1} U_i$; then $|T| = s' + 2r(q-1) = 3s'$. Since $\kappa(F') \geq Ct \geq 6s'$, by Theorem 5.1 and by duplicating each vertex of T , we get a set \mathcal{P}_1 of $6s'$ paths in F' between T and $V(F_0)$ which pairwise share no vertex in $V(F') \setminus T$, such that each vertex of T is the endpoint of exactly two paths in \mathcal{P}_1 . Let $D := \bigcup_{i=0}^{q-1} V(F_i)$; then

$$|D| \leq |F_0| + |F_1| + \dots + |F_{q-1}| \leq q \cdot C_{2.2}^2 \Gamma^3 t \leq C^2 \Gamma^4 t \quad (1)$$

so $\chi(F[D]) = \chi(G[D]) \leq t \cdot g(G, \varepsilon, t)$ by the definition of $g(G, \varepsilon, t)$. It follows that

$$\chi(F' \setminus D) \geq \chi(F') - t \cdot g(G, \varepsilon, t) \geq C^2 t \cdot g(G, \varepsilon, t) \geq 4k,$$

and so by Theorem 5.5, $F' \setminus D$ contains a k -connected subgraph G_1 with

$$\chi(G_1) \geq \chi(F' \setminus D) - 2k \geq \chi(F') - t \cdot g(G, \varepsilon, t) - 2Ct \geq \chi(G) - C^2 t \cdot g(G, \varepsilon, t). \quad (2)$$

Since $\kappa(F' \setminus T) \geq \kappa(F') - |T| \geq Ct - 3s' \geq 2s$, Theorem 5.1 yields a set \mathcal{P}_2 of $2s$ vertex-disjoint paths between $V(F_0)$ and $V(G_1)$ in $F' \setminus T$. By Lemma 5.2 applied to $\mathcal{P}_1, \mathcal{P}_2$ and by identifying in pairs the endpoints in $V(G_1)$ of the paths of \mathcal{P}_2 , we get a set \mathcal{P}' of $3s' + s$ vertex-disjoint paths between $T \cup V(G_1)$ and $V(F_0)$ such that each vertex in T is the endpoint of exactly one path in \mathcal{P}' . For each $i \in [q-1]$, since $2r \log^{1/2}(2/\varepsilon) \leq 4r \Gamma^{1/2} \leq 8t$ and $3s' + s \leq 10t$, we have that

$$\kappa(F_i) \geq k = Ct \geq C_{5.13} \cdot 10t \geq C_{5.13} \cdot \max(2r \log^{1/2}(2/\varepsilon), 3s' + s),$$

so by Lemma 5.13, F_i is $(\frac{1}{2}\varepsilon, 2r, 3s' + s)$ -woven. Thus, by applying Lemma 5.10 iteratively to F_1, \dots, F_{q-1} , we get a set \mathcal{P} of $3s' + s$ vertex-disjoint paths between $T \cup V(G_1)$ and $V(F_0)$ such that

- each vertex in T is the endpoint of exactly one path in \mathcal{P} ; and
- for every $i \in [q-1]$, F_i has an H_i model \mathcal{M}_i rooted at U_i for some $(\frac{1}{2}\varepsilon, 2r)$ -dense graph H_i such that $V(\mathcal{M}_i) \cap V(\mathcal{P}) = U_i$.

For every $u \in T \cup V(G_1)$, let P_u be the path of \mathcal{P} having u as an endpoint. For every $i \in [q-1]$, for each $u \in U_i$, let M_u be the member of \mathcal{M}_i containing u ; and also, let $X_i \cup Y_i$ be a partition of U_i with $|X_i| = |Y_i| = r$, and let $Y_i = \{y_j^i : j \in [r]\}$. Let $\bigcup_{i=1}^{q-1} X_i = \{x_1, \dots, x_{s'}\}$. Let $U = \{v_1, \dots, v_s\}$ be the vertices in $V(G_1)$ that are the endpoints of some paths in \mathcal{P} . For every $i \in [s']$, let V_i be the set of three vertices in $V(F_0)$ linked to $\{u_i, x_i, v_i\}$ via \mathcal{P} ; and for every $j \in [r]$, let W_j be the set of q vertices in $V(F_0)$ linked to $\{y_j^i : i \in [q-1]\} \cup \{v_{s'+j}\}$ via \mathcal{P} . Since $\kappa(F_0) \geq k = Ct \geq 11(3s' + s)$, Theorem 5.4 implies that F_0 is $(3s' + s, s)$ -knot, and so F_0 contains vertex-disjoint connected subgraphs J_1, \dots, J_s such that $V_i \subseteq V(J_i)$ for all $i \in [s']$ and $W_j \subseteq V(J_{s'+j})$ for all $j \in [r]$. Now, let

- $B_i := (A'_i \cup P_{u_i}) \cup (M_{x_i} \cup P_{x_i}) \cup J_i \cup P_{v_i}$ for every $i \in [s']$; and
- $B_{s'+j} := J_{s'+j} \cup P_{v_{s'+j}} \cup \bigcup_{i \in [q-1]} (M_{y_j^i} \cup P_{y_j^i})$ for all $j \in [r]$.

Then $\mathcal{B} := \{B_1, \dots, B_s\}$ is an H model with $S := S' \cup D \cup U$ as a core, for some H with $|H| = s$. By (1),

$$|S| = |S'| + |D| + |U| \leq (q-1)^2 C_{2.2}^2 \Gamma^3 t + q \cdot C_{2.2}^2 \Gamma^3 t + s \leq q^2 C_{2.2}^2 \Gamma^3 t. \quad (3)$$

Claim 6.2. H is (ε, s) -dense.

Subproof. Since $\{A'_1, \dots, A'_{s'}\}$ is an H' model in G and H' is (ε, s') -dense, there are at most $\varepsilon \binom{s'}{2}$ anti-complete pairs among $\{B_i : i \in [s']\}$. Moreover, the number of anticomplete pairs of the form $\{V(B_i), V(B_{s'+j})\}$ for $i \in [s'], j \in [r]$ plus the number of anticomplete pairs among $\{V(B_{s'+j}) : j \in [r]\}$ is at most $\sum_{i=1}^{q-1} e(\overline{H_i}) \leq \frac{1}{2}(q-1)\varepsilon \binom{2r}{2}$, since each of H_1, \dots, H_{q-1} is $(\frac{1}{2}\varepsilon, 2r)$ -dense and every nonedge of each H_i contributes to at most one such anticomplete pair. Thus, in order to show that H is (ε, s) -dense, it suffices to prove $\binom{s'}{2} + \frac{1}{2}(q-1)\binom{2r}{2} \leq \binom{s}{2}$, which holds since (recall that $s = s' + r$)

$$\binom{s}{2} - \binom{s'}{2} = \frac{1}{2}(s-s')(s+s'-1) \geq rs' = (q-1)r^2 \geq \frac{1}{2}(q-1)\binom{2r}{2}. \quad \square$$

Note that while G_1 is k -connected and tangent to \mathcal{B} at U , it only satisfies $\chi(G_1) \geq \chi(G) - C^2 t \cdot g(G, \varepsilon, t)$ by (2). For every $i \in [s]$, by Lemma 5.6, there is an induced connected subgraph A_i of $G[V(B_i)]$ and $Z_i \subseteq V(A_i)$ with $S \cap V(B_i) \subseteq Z_i$, $|Z_i| \leq 3|S \cap V(B_i)|$, and $\chi(A_i \setminus Z_i) \leq 2$. Then $\mathcal{A} = \{A_1, \dots, A_s\}$ is an H model in G with S as a core such that G_1 is tangent to \mathcal{A} at $U \subseteq S \cap V(\mathcal{A})$. Let $Z := \bigcup_{i=1}^s Z_i$; then

$$|Z| \leq 3|S| \leq 3q^2 C_{2.2}^2 \Gamma^3 t \leq C^2 \Gamma^4 t,$$

where the second inequality holds by (3), and so $\chi(G[Z]) \leq t \cdot g(G, \varepsilon, t)$. Let $A := G[\bigcup_{i=1}^s V(A_i)]$; then

$$\chi(A) \leq 2s + \chi(G[Z]) \leq 2s + t \cdot g(G, \varepsilon, t) \leq 7t \cdot g(G, \varepsilon, t).$$

It follows that

$$\chi(G \setminus A) \geq \chi(G) - \chi(A) \geq C^2 t \cdot g(G, \varepsilon, t) - 7t \cdot g(G, \varepsilon, t) \geq 4Ct,$$

and so by Theorem 5.5, $G \setminus A$ has a k -connected subgraph G_2 with

$$\chi(G_2) \geq \chi(G \setminus A) - 2k \geq \chi(G) - 7t \cdot g(G, \varepsilon, t) - 2Ct \geq \chi(G) - \frac{1}{2}C^2 t \cdot g(G, \varepsilon, t).$$

Thus, if $|V(G_1) \cap V(G_2)| \leq k$, then

$$\chi(G_2 \setminus G_1) \geq \chi(G) - 7t \cdot g(G, \varepsilon, t) - 2Ct - k \geq \chi(G) - C^2 t \cdot g(G, \varepsilon, t),$$

and so G_1 and $G_2 \setminus G_1$ would be two vertex-disjoint subgraphs of G each with chromatic number at least $\chi(G) - C^2 t \cdot g(G, \varepsilon, t)$, contradicting that G is $(C^2 t \cdot g(G, \varepsilon, t))$ -chromatic-inseparable. Hence G_1, G_2 have more than k common vertices, which implies that $F := G_1 \cup G_2$ is a k -connected subgraph tangent to \mathcal{A} at U with $\chi(F) \geq \chi(G_2) \geq \chi(G) - \frac{1}{2}C^2 t \cdot g(G, \varepsilon, t)$. This proves Lemma 6.1. \blacksquare

We are now ready to prove Theorem 2.1, which we restate here (in a slightly different form).

Theorem 6.3. *There is an integer $C > 0$ such that the following holds. Let $\varepsilon \in (0, \frac{1}{2})$ and $\Gamma := \log \frac{1}{\varepsilon}$. Let $t \geq \Gamma^{1/2}$ be an integer, let G be a graph, and let*

$$g(G, \varepsilon, t) := 1 + \max_{F \subseteq G} \left\{ \frac{\chi(F)}{t} : |F| \leq C\Gamma^4 t, F \text{ is } (\varepsilon, t)\text{-dense-minor-free} \right\}.$$

If G is $(Ct \cdot g(G, \varepsilon, t))$ -chromatic-inseparable and $\chi(G) \geq 2Ct \cdot g(G, \varepsilon, t)$, then G has a (ε, t) -dense-minor.

Proof. Let $C := C_{6.1}^2$. Theorem 6.3 follows from Lemma 6.1 with $q = \lceil \Gamma^{1/2} \rceil$, as $s = q \lceil \Gamma^{-1/2} t \rceil \geq t$. \blacksquare

7. FINISHING THE PROOF

This section deals with the general case. The recursive construction of an (ε, t) -dense minor will be made rigorous by the following lemma.

Lemma 7.1. *There is an integer $C = C_{7.1} > 0$ such that the following holds. Let $\varepsilon \in (0, \frac{1}{2})$, and let $\Gamma := \log \frac{1}{\varepsilon}$. Let $t \geq \Gamma^2$ be an integer, let G be a graph, and let*

$$f(G, \varepsilon, t) = f_{7.1}(G, \varepsilon, t) := 1 + \max_{F \subseteq G} \left\{ \frac{\chi(F)}{a} : \Gamma^{-1/2}t \leq a \leq t, |F| \leq C\Gamma^4 a, F \text{ is } (\varepsilon, a)\text{-dense-minor-free} \right\}.$$

If t is a power of 3, s is an integer with $0 \leq s \leq \log_3 t$, and G is Ca -connected with $a = (\frac{2}{3})^{st}$ and

$$\chi(G) \geq C(t + 11a \cdot f(G, \varepsilon, t)),$$

then for $\xi = \log_3(\frac{4}{3}) = 0.7095\dots$,

- *if $s \geq \frac{1}{2} \log_3 \Gamma$, then G is $(\varepsilon, a, 3a)$ -woven; and*
- *if $s \leq \frac{1}{2} \log_3 \Gamma$, then G is $(\frac{3}{4})^s \Gamma^\xi \varepsilon, a, 3a)$ -woven.*

In particular, G is always $(\Gamma\varepsilon, a, 3a)$ -woven.

Proof. The constant C is chosen implicitly to satisfy the inequalities throughout the proof.

Since $t \geq \Gamma^2$, $\log_3 t \geq 2 \log_3 \Gamma > \frac{1}{2} \log_{3/2} \Gamma$. We proceed by backward induction on s . If $\frac{1}{2} \log_3 \Gamma \leq s \leq \log_3 t$ then $a \leq \Gamma^{-1/2}t$. Since $\chi(G) \geq Ct + 1$, by Lemma 4.2, G contains a subgraph with connectivity at least $\frac{1}{4}Ct \geq C_{5.13}t \geq C_{5.13}\Gamma^{1/2}a$, thus is $(\varepsilon, a, 3a)$ -woven by Lemma 5.13.

We may thus assume that $s < \frac{1}{2} \log_3 \Gamma$ and that the lemma is true for $s + 1$; note that $(\frac{3}{4})^{s+1} \Gamma^\xi > \frac{3}{4} \Gamma^{\xi/2} > 1$. Put $\varepsilon' := \Gamma^\xi \varepsilon$. Let $b := 3a$, and let $R = \{r_1, \dots, r_a\}$, $S = \{s_1, \dots, s_b\}$, $T = \{t_1, \dots, t_b\}$ be subsets of $V(G)$ such that $s_i = t_j$ for some $i, j \in [b]$ only if $i = j$; we may assume R, S, T are pairwise disjoint. Let $Z := R \cup S \cup T$, and let $G_1 := G \setminus Z$; then $\kappa(G_1) \geq (11C - 7)a \geq 8a$, in particular G_1 has average degree at least $(11C - 7)a \geq C_{2.2}a$. If G_1 has a $(\frac{1}{256}\varepsilon, 32a)$ -dense minor, then by Lemma 5.11, it is $(\varepsilon, a, 3a)$ -woven and we are done. Thus we may assume G_1 is $(\frac{1}{256}\varepsilon, 32a)$ -dense-minor-free; and so by Theorem 2.2 with $k = 40a$, G_1 has a subgraph F_0 with $\kappa(F_0) \geq 40a$ and (note that $\varepsilon < \frac{1}{256}$)

$$|F_0| \leq C_{2.2}^2 \cdot 32a \cdot \log^3(256/\varepsilon) \leq 256C_{2.2}^2 a \cdot \log^3(1/\varepsilon) = 256C_{2.2}^2 \Gamma^3 a \leq C\Gamma^4 a.$$

Since $a = (\frac{2}{3})^{st} \geq \Gamma^{-1/2}t$, $\chi(F_0) \leq a \cdot f(G, \varepsilon, t)$ by the definition of $f(G, \varepsilon, t)$. Since $\kappa(G) \geq Ca \geq 14a$, Theorem 5.1 yields a set \mathcal{P}_1 of $14a$ paths in G between Z and $V(F_0)$ which pairwise share no vertex in $V(G) \setminus Z$, such that each vertex of Z is the endpoint of exactly two paths in \mathcal{P}_1 . We may assume the paths in \mathcal{P}_1 are induced paths in G ; and so $\chi(G[V(\mathcal{P}_1)]) \leq 28a$. Let $F := G_1 \setminus (V(F_0) \cup V(\mathcal{P}_1))$; then

$$\begin{aligned} \chi(F) &\geq \chi(G_1) - \chi(G[V(F_0) \cup V(\mathcal{P}_1)]) \geq \chi(G) - 35a - a \cdot f(G, \varepsilon, t) \\ &\geq Ct + (11C - 36)a \cdot f(G, \varepsilon, t) \geq 4Ca. \end{aligned}$$

By Theorem 5.5, F contains a Ca -connected subgraph G' with

$$\chi(G') \geq \chi(F) - 2Ca \geq Ct + (9C - 36)a \cdot f(G, \varepsilon, t). \quad (4)$$

Claim 7.2. *G' is $(\frac{3}{4})^s \varepsilon', a, 0)$ -woven.*

Subproof. Let $R' = \{r'_1, \dots, r'_a\} \subseteq V(G')$; we need to show there is an H model in G' rooted at R' for some $((\frac{3}{4})^s \varepsilon', a)$ -dense graph H . Let $G_0 := G' \setminus R'$. Since $\kappa(G') \geq Ca$, there is $U = \{u_1, \dots, u_{2a}\} \subseteq V(G_0)$ with r'_i adjacent to u_i and u_{a+i} for all $i \in [a]$. Let $G'_0 := G_0 \setminus U$. Put $m := C_{6.3} \cdot 32a \cdot g_{6.3}(G, \frac{1}{256}\varepsilon, 32a)$. Since

$$C_{6.3} \cdot 32a \cdot \log^4(256/\varepsilon) \leq 512C_{6.3} a \cdot \log^4(1/\varepsilon) \leq C\Gamma^4 a,$$

we have $f(G, \varepsilon, t) \geq g_{2.1}(G, \frac{1}{256}\varepsilon, 32a)$; so by (4),

$$\chi(G'_0) \geq \chi(G') - |R' \cup U| = \chi(G') - 3a \geq Ct + (9C - 39)a \cdot f(G, \varepsilon, t) \geq 3m.$$

Since G'_0 is $(\frac{1}{256}\varepsilon, 32a)$ -dense-minor-free, and since $32a \geq 32\Gamma^{-1/2}t \geq 32\Gamma^{1/2} \geq \log^{1/2}(256/\varepsilon)$, Theorem 6.3 implies that G'_0 is m -chromatic-separable. Thus, G'_0 has vertex-disjoint subgraphs F_1, F'_1 with $\chi(F_1), \chi(F'_1) \geq \chi(G'_0) - m \geq 2m$. By Theorem 6.3 again, G'_1 has vertex-disjoint subgraphs F_2, F_3 with $\chi(F_2), \chi(F_3) \geq \chi(F'_1) - m \geq \chi(G'_0) - 2m$. Put $a' := (\frac{2}{3})^{s+1} \varepsilon' = \frac{2}{3}a$; and observe that for every $j \in \{1, 2, 3\}$,

$$\chi(F_j) \geq \chi(G'_0) - 2m \geq Ct + (9C - 39 - 64C_{6.3})a \cdot f(G, \varepsilon, t) \geq 4Ca',$$

so by Theorem 5.5, F_j has a Ca' -connected subgraph L_j with (note that $\frac{3}{2}(9C - 29 - 64C_{6.3}) - 2C \geq 11C$)

$$\chi(L_j) \geq \chi(F_j) - 2Ca' \geq Ct + (9C - 29 - 64C_{6.3})a \cdot f(G, \varepsilon, t) - 2Ca' \geq C(t + 11a' \cdot f(G, \varepsilon, t)).$$

Hence, by induction, L_j is $((\frac{3}{4})^{s+1}\varepsilon', a', 3a')$ -woven; let $V_j = \{v_{(j-1)a'+1}, v_{(j-1)a'+2}, \dots, v_{ja'}\} \subseteq V(L_j)$. Since $\kappa(G_0) \geq (C-1)a \geq 20a$, G_0 is $2a$ -linked by Theorem 5.3; so there is a linkage \mathcal{P}' in G_0 linking $\{(u_i, v_i) : i \in [2a]\}$. By Lemma 5.10 applied iteratively to each of J_1, J_2, J_3 (note that $3a' = 2a$), there is a linkage $\mathcal{P} = \{P_1, \dots, P_{2a}\}$ linking $\{(u_i, v_i) : i \in [2a]\}$ in G_0 and $((\frac{3}{4})^{s+1}\varepsilon', a')$ -dense graphs H_1, H_2, H_3 such that for each $j \in \{1, 2, 3\}$, there is an H_j model \mathcal{M}_j rooted at V_j in L_j such that $V(\mathcal{M}_j) \cap V(\mathcal{P}) = V_j$.

Now, for every $i \in [2a]$, let $j \in \{1, 2, 3\}$ be such that $v_i \in V_j$, and let M_i be the member of \mathcal{M}_j containing v_i . For every $i \in [a]$, let

$$D_i := M_i \cup P_i \cup r'_i u_i \cup r'_i u_{a+i} \cup P_{a+i} \cup M_{a+i},$$

then it is not hard to see that $\{D_1, \dots, D_a\}$ is an H model rooted at R' in G' for some graph H with

$$e(\overline{H}) \leq e(\overline{H_1}) + e(\overline{H_2}) + e(\overline{H_3}) \leq 3 \left(\frac{3}{4}\right)^{s+1} \varepsilon' \binom{a'}{2} \leq \left(\frac{3}{4}\right)^s \varepsilon' \binom{a}{2},$$

where the last inequality holds because $\frac{9}{4} \binom{a'}{2} = \frac{1}{8}(3a')(3a' - 3) < \frac{1}{8}(2a)(2a - 2) = \binom{a}{2}$. Therefore H is $((\frac{3}{4})^s \varepsilon', a)$ -dense, proving Claim 7.2. \square

The rest of the proof is now straightforward. Recall that \mathcal{P}_1 is a set of $14a$ paths between Z and $V(F_0)$ in G which pairwise share no vertex in $G_1 = G \setminus Z$, such that each vertex of Z is the endpoint of exactly two paths in \mathcal{P}_1 . Since $\kappa(G_1) \geq \kappa(G) - |Z| \geq (C-7)a \geq 2a$, Theorem 5.1 yields a set \mathcal{P}_2 of $2a$ vertex-disjoint paths between $V(G')$ and $V(F_0)$ in G_1 . By Lemma 5.2 applied to $\mathcal{P}_1, \mathcal{P}_2$ and by identifying in pairs the endpoints in $V(G')$ of the paths of \mathcal{P}_2 , we obtain a set \mathcal{Q} of $|Z| + a = 8a$ vertex-disjoint paths between $Z \cup V(G')$ and $V(F_0)$ such that each vertex of Z is an endpoint of some path of \mathcal{Q} . For each $i \in [a]$, let x_i be the vertex in $V(F_0)$ linked to r_i by some path $Q_i \in \mathcal{Q}$; and for each $i \in [b]$ (recall that $b = 3a$), let y_i, z_i , respectively, be the vertices in $V(F_0)$ linked to s_i, t_i by some paths $P_i, P'_i \in \mathcal{Q}$. Let $\{r'_1, \dots, r'_a\}$ be the set of endpoints in $V(G')$ of the paths of \mathcal{Q} ; and for every $i \in [a]$, let w_i be the vertex in $V(F_0)$ linked to r'_i by some path $P'_i \in \mathcal{Q}$. Since $\kappa(F_0) \geq 40a$, F_0 is $4a$ -linked by Theorem 5.3; so there is a linkage $\mathcal{N} = \{N_1, \dots, N_{4a}\}$ linking $\{(x_i, w_i) : i \in [a]\} \cup \{(y_i, z_i) : i \in [b]\}$ in F_0 . By Claim 7.2, G' has an H model $\{M_1, \dots, M_a\}$ rooted at $\{r'_1, \dots, r'_a\}$ for some $((\frac{3}{4})^s \varepsilon', a)$ -dense graph H . Now, let

- $A_i := (Q_i \cup N_i) \cup (Q'_i \cup M_i)$ for every $i \in [a]$; and
- $B_i := P_i \cup N_{a+i} \cup P'_i$ for every $i \in [b]$.

Then $\mathcal{A} = \{A_1, \dots, A_a\}$ is an H model rooted at R in G and $\mathcal{B} = \{B_1, \dots, B_b\}$ is a linkage in G linking $\{(s_i, t_i) : i \in [b]\}$ such that $V(\mathcal{A}) \cap V(\mathcal{B}) = \emptyset$. This proves Lemma 7.1. \blacksquare

We are now ready to prove Theorem 1.2, which we restate here (in a slightly different form) for the convenience of the readers.

Theorem 7.3. *There is an integer $C > 0$ such that the following holds. Let $\varepsilon \in (0, \frac{1}{256})$, and let $\Gamma := \log(1/\varepsilon)$. Let $t \geq 4\Gamma^2$ be an integer, let G be a graph, and let*

$$f(G, \varepsilon, t) := 1 + \max_{F \subseteq G} \left\{ \frac{\chi(F)}{a} : a \geq (2\Gamma)^{-1/2}t, |F| \leq C\Gamma^4 a, F \text{ is } (\varepsilon, a)\text{-dense-minor-free} \right\}.$$

If $\chi(G) \geq Ct \cdot f(G, \varepsilon, t)$, then G has an (ε, t) -dense minor.

Proof. Let $C := 80C_{7.1}$. Let $\varepsilon' \in (0, \frac{1}{256})$ satisfy $\varepsilon = \varepsilon' \log(1/\varepsilon')$; then $\Gamma = \log(1/\varepsilon') - \log \log(1/\varepsilon')$, and so $\Gamma \leq \log(1/\varepsilon') \leq 2\Gamma$. Thus $t \geq 4\Gamma^2 \geq \log^2(1/\varepsilon')$. Let $t' := 3^{\lceil \log_3 t \rceil}$; then $t \leq t' \leq 3t$, and so

$$f(G, \varepsilon, t) \geq f_{7.1}(G, \varepsilon', t) \geq f_{7.1}(G, \varepsilon', t') \geq 1.$$

Hence $\chi(G) \geq Ct \cdot f(G, \varepsilon, t) \geq 4C_{7.1}t'$; so by Theorem 5.5, G has a $C_{7.1}t'$ -connected subgraph G' with

$$\chi(G') \geq \chi(G) - 2C_{7.1}t' \geq Ct \cdot f(G, \varepsilon, t) - 2C_{7.1}t' \geq 24C_{7.1}t' \cdot f(G, \varepsilon', t').$$

By Lemma 7.1 with $s = 0$ and by the choice of ε' , G' is $(\varepsilon, t', 3t')$ -woven, and so contains a (ε, t') -dense minor. By averaging, G' contains a (ε, t) -dense minor, and so does G . This proves Theorem 7.3. \blacksquare

8. ADDITIONAL REMARKS

An *odd minor* of a graph G is a graph obtained from a subgraph of G by a sequence of edge cut contractions; and so every odd minor of G is a minor of G . Odd Hadwiger’s conjecture, a strengthening of the original one suggested by Gerards and Seymour (see [9, Section 6.5]), says that every graph with chromatic number at least t has an odd K_t minor, and is still open. There is also the odd variant of linear Hadwiger’s conjecture which is open as well, which asserts that for some universal constant $C > 0$, every graph with chromatic number at least Ct contains an odd K_t minor. Linear Hadwiger’s conjecture and its odd variant are recently shown to be equivalent by Steiner [23], who actually proved the following stronger result via an elegant argument.

Theorem 8.1. *Let H be a graph, and let $m > 0$ be such that every graph with no H minor has chromatic number at most m . Then every graph with no odd H minor has chromatic number at most $2m$.*

By Theorem 8.1, Theorem 1.1 is equivalent to its odd version, as follows.

Theorem 8.2. *There is an integer $C > 0$ such that for every $\varepsilon \in (0, \frac{1}{2})$ and every integer $t \geq 2$, every graph with chromatic number at least $Ct \log \log \frac{1}{\varepsilon}$ contains an odd (ε, t) -dense minor.*

Acknowledgements. This work was partially done at the workshop “Seymour is $70 + 2\varepsilon$ ” at LIP, ENS de Lyon in June, 2022. The author would like to thank the participants for creating a hospitable working environment, and to thank Paul Seymour, in particular, for helpful remarks and encouragement. He would also like to thank Luke Postle for useful discussions.

REFERENCES

[1] N. Alon, M. Krivelevich, and B. Sudakov. Complete minors and average degree: a short proof. *J. Graph Theory*, 103(3):599–602, 2023. 4, 5

[2] B. Bollobás, P. A. Catlin, and P. Erdős. Hadwiger’s conjecture is true for almost every graph. *European J. Combin.*, 1(3):195–199, 1980. 4

[3] B. Bollobás and A. Thomason. Highly linked graphs. *Combinatorica*, 16(3):313–320, 1996. 9, 10, 12

[4] M. Delcourt and L. Postle. Reducing linear Hadwiger’s conjecture to coloring small graphs. [arXiv:2108.01633](https://arxiv.org/abs/2108.01633), 2023. 1, 2, 3, 7, 9, 10, 11

[5] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Berlin, fifth edition, 2017. 7

[6] P. Duchet and H. Meyniel. On Hadwiger’s number and the stability number. In *Graph theory (Cambridge, 1981)*, volume 13 of *North-Holland Mathematics Studies, 62; Annals of Discrete Mathematics*, pages 71–73. North-Holland, Amsterdam-New York, 1982. 2

[7] A. Girão and B. Narayanan. Subgraphs of large connectivity and chromatic number. *Bull. Lond. Math. Soc.*, 54(3):868–875, 2022. 10

[8] H. Hadwiger. Über eine Klassifikation der Streckenkomplexe. *Vierteljahrsschr. Naturforsch. Ges. Zürich*, 88:133–142, 1943. 1

[9] T. R. Jensen and B. Toft. *Graph coloring problems*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., New York, 1995. A Wiley-Interscience Publication. 17

[10] K.-i. Kawarabayashi. On the connectivity of minimum and minimal counterexamples to Hadwiger’s conjecture. *J. Combin. Theory Ser. B*, 97(1):144–150, 2007. 10

[11] A. V. Kostochka. The minimum Hadwiger number for graphs with a given mean degree of vertices. *Metody Diskret. Analiz.*, (38):37–58, 1982. 1

[12] A. V. Kostochka. Lower bound of the Hadwiger number of graphs by their average degree. *Combinatorica*, 4(4):307–316, 1984. 1

[13] W. Mader. Homomorphiesätze für Graphen. *Math. Ann.*, 178:154–168, 1968. 1, 5, 7

[14] K. Menger. Zur allgemeinen kurventheorie. *Fund. Math.*, 10:96–115, 1927. 9

[15] T. H. Nguyen. Highly connected subgraphs with large chromatic number. *SIAM J. Discrete Math.*, 38(1):243–260, 2024. 10

[16] S. Norin and L. Postle. Connectivity and choosability of graphs with no K_t minor. *J. Combin. Theory Ser. B*, 158(part 1):283–300, 2023. 4, 6

[17] S. Norin, L. Postle, and Z.-X. Song. Breaking the degeneracy barrier for coloring graphs with no K_t minor. *Adv. Math.*, 422:Paper No. 109020, 23, 2023. 1

[18] S. Norin and P. Seymour. Dense minors of graphs with independence number two. [arXiv:2206.00186](https://arxiv.org/abs/2206.00186), 2022. 1

[19] M. D. Plummer, M. Stiebitz, and B. Toft. On a special case of Hadwiger’s conjecture. *Discuss. Math. Graph Theory*, 23(2):333–363, 2003. 1

[20] B. Reed and P. Seymour. Fractional colouring and Hadwiger’s conjecture. *J. Combin. Theory Ser. B*, 74(2):147–152, 1998. 1

[21] N. Robertson and P. D. Seymour. Graph minors. XIII. The disjoint paths problem. *J. Combin. Theory Ser. B*, 63(1):65–110, 1995. 10

[22] P. Seymour. Hadwiger’s conjecture. In *Open Problems in Mathematics*, pages 417–437. Springer, 2016. 1, 2

[23] R. Steiner. Asymptotic equivalence of Hadwiger’s conjecture and its odd minor-variant. *J. Combin. Theory Ser. B*, 155:45–51, 2022. 17

- [24] R. Thomas and P. Wollan. An improved linear edge bound for graph linkages. *European J. Combin.*, 26(3-4):309–324, 2005. [9](#)
- [25] A. Thomason. An extremal function for contractions of graphs. *Math. Proc. Cambridge Philos. Soc.*, 95(2):261–265, 1984. [1](#)
- [26] P. Wollan. Extremal functions for rooted minors. *J. Graph Theory*, 58(2):159–178, 2008. [11](#)
- [27] D. R. Woodall. Subcontraction-equivalence and Hadwiger’s conjecture. *J. Graph Theory*, 11(2):197–204, 1987. [2](#)

PRINCETON UNIVERSITY, PRINCETON, NJ 08544, USA

Email address: tunghn@math.princeton.edu