GROWING BALANCED COVERING SETS

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Abstract. Given a bipartite graph with bipartition (A, B) where B is equipartitioned into k blocks, can the vertices in A be picked one by one so that at every step, the picked vertices cover roughly the same number of vertices in each of these blocks? We show that, if each block has cardinality m, the vertices in B have the same degree, and each vertex in A has at most cm neighbors in every block where $c > 0$ is a small constant, then there is an ordering $v_1, \dots, v_n$ of the vertices in A such that for every $j \in \{1, \dots, n\}$, the numbers of vertices with a neighbor in $\{v_1, \dots, v_j\}$ in every two blocks differ by at most $\sqrt{2(k-1)c} \cdot m$. This is related to a well-known lemma of Steinitz, and partially answers an unpublished question of Scott and Seymour.

1. Introduction

For every integer $n \geq 1$, let $[n] := \{1, \dots, n\}$. Let $\mathbb{N}$ be the set of natural numbers, and let $\mathbb{R}^+$ be the set of nonnegative real numbers. The motivation of this note is an unpublished question of Alex Scott and Paul Seymour [5] on balanced covers of bipartite graphs related to the opening question in the abstract. Here, bipartite graphs have no multiple edges.

Question 1. *Let $k \geq 2$ and $m \geq 1$ be integers, and let $c \in (0, 1/2)$ be a constant independent of k, m. Consider a bipartite graph with bipartition (A, B) where B is partitioned into k blocks $B_1, \dots, B_k$ each of cardinality m, the vertices in B have the same degree $r \geq 1$, and each vertex in A has at most cm neighbors in each B_i. For every $i \in [k]$ and $S \subseteq A$, let $N(S, B_i)$ be the set of vertices in B_i with a neighbor in S. Does there exist $f\colon \mathbb{N} \to \mathbb{R}^+$ such that there is a chain of sets $\emptyset \subsetneq A_1 \subsetneq \dots \subsetneq A_n = A$ where $n = |A|$ satisfying $||N(A_j, B_{i_1})| - |N(A_j, B_{i_2})|| \leq f(k)cm$ for all $i_1, i_2 \in [k]$ and $j \in [n]$?*

To put Question 1 into perspective, suppose that we are given disjoint vertex sets A and $B_1, \dots, B_k$ with $|B_1| = \dots = |B_k| = m$ and each vertex of A only has a small portion of neighbors in each B_j. In some situations (see [4, Section 5] for instance), we hope to find a subset S of A such that $|N(S, B_1)|$ is roughly $m/2$ and $|N(S, B_i)| \leq m/2$ for all $i \in \{2, \dots, k\}$. A moment of thought reveals that this can be achieved if the vertices of A can be picked one by one so that at each step j with A_j the set of picked vertices, $|N(A_j, B_1)|, \dots, |N(A_j, B_k)|$ are roughly the same. Indeed, S can be chosen as A_{j-1} where $j \in [n]$ is the smallest index such that there is some $i \in [k]$ with $|N(A_j, B_i)| > m/2$. As a result, for applications it may be desirable to remove the regularity condition on $B = B_1 \cup \dots \cup B_k$ in Question 1. This condition, unfortunately, is in some sense necessary; if regularity is changed into almost regularity then $f(2)$ might not even exist, as shown by the following proposition.

Proposition 1. *For every c, ε with $0 < c < 1/4$, $0 < \varepsilon < 1$, and $(4c)^{-1}$ an integer, there is some $r_0(c, \varepsilon)$ with the following property. For each integer $r \geq r_0(c, \varepsilon)$, there exists $m_0(r, c, \varepsilon)$ such that for every integer $m \geq m_0(r, c, \varepsilon)$, there is a bipartite graph G with bipartition (A, B) satisfying*

- *$|A| = c^{-1}r$, and B has a partition into two vertex sets B_1, B_2 with $|B_1| = |B_2| = m$,*
- *every vertex in A has at most cm neighbors in each of B_1, B_2,*
- *every vertex in B has degree at least $(1-\varepsilon)r$ and at most r, and*
- *$||N(S, B_1)| - |N(S, B_2)|| \geq \varepsilon m/40$ for every $S \subseteq A$ with $|S| = (4c)^{-1}$.*

Sketch of proof. We make two random bipartite graphs with bipartitions (A, B_1) and (A, B_2) where $|A| = c^{-1}r$ and $|B_1| = |B_2| = m$, such that for $i = 1, 2$, every edge between A and B_i is included independently with probability $c(1 - (2i-1)\varepsilon/4)$. A standard concentration argument shows that there exists $r_0(c, \varepsilon)$ with the property that for every $r \geq r_0(c, \varepsilon)$, there is some $m_0(r, c, \varepsilon)$ such that for each $m \geq m_0(r, c, \varepsilon)$, with positive probability, for every $i = 1, 2$ every vertex in B_i has degree at least $(1 - i\varepsilon/2)r$ and at most $(1 - (i-1)\varepsilon/2)r$, and the number of common neighbors of each subset of A of size at most $(4c)^{-1}$ in B_i is tightly concentrated around its mean. Then an inclusion-exclusion argument finishes the proof. □

Still, we believe that Question 1 is interesting on its own right; our result provides a partial answer to it by asserting that $f(k)$ can be chosen as $\sqrt{2(k-1)}$ if c is replaced by $\sqrt{c}$.

Theorem 2. *Let $k \geq 2$ and $m \geq 1$ be integers, and let $c \in (0, 1/2)$ be a constant independent of k, m. Consider a bipartite graph with bipartition (A, B) where B is partitioned into k blocks $B_1, \ldots, B_k$ each of cardinality m, the vertices in B have the same degree $r \geq 1$, and each vertex in A has at most cm neighbors in each B_i. Then, there is a chain $\emptyset \subsetneq A_1 \subsetneq \ldots \subsetneq A_n = A$ where $n = |A|$ such that for all $i_1, i_2 \in [k]$ and $j \in [n]$, $||N(A_j, B_{i_1})| - |N(A_j, B_{i_2})|| \leq \sqrt{2(k-1)c} \cdot m$.*

In Section 2, we present an equivalent formulation of Question 1 which is Question 2, and an equivalent statement of Theorem 2 which is Theorem 3. In Section 3, we prove Theorem 3. Section 4 is a brief discussion about a variant of Question 2.

2. An equivalent formulation of Question 1

In this section, we introduce an equivalent formulation of Question 1 which is more convenient to work with. For a finite set S and an integer $r \geq 1$, let $S^{(r)}$ be the family of all subsets of cardinality r of S; we identify $S^{(1)}$ with S. For integers n, r with $n \geq r \geq 1$, a *weighted r-uniform hypergraph on* $[n]$ is a function $w\colon [n]^{(r)} \to \mathbb{R}^+$ satisfying $\sum_{R \in [n]^{(r)}} w(R) = 1$. For every $S \subseteq [n]$, let $w^*(S) := \sum_{R \in S^{(r)}} w(R)$. Thus $w^*(S) = 0$ when $|S| < r$. Now, Question 1 can be rephrased as follows.

Question 2. *Let $k \geq 2$ be an integer and $c \in (0, 1/2)$. Let $w_1, \ldots, w_k$ be weighted r-uniform hypergraphs on $[n]$ satisfying $w_i^*([n] \setminus \{j\}) \geq 1 - c$ for all $i \in [k]$ and $j \in [n]$, where n, r are integers with $n \geq r \geq 1$. Does there exist $f\colon \mathbb{N} \to \mathbb{R}^+$ such that there is a chain $\emptyset \subsetneq S_1 \subsetneq \ldots \subsetneq S_n = [n]$ satisfying $|w_{i_1}^*(S_j) - w_{i_2}^*(S_j)| \leq f(k)c$ for all $i_1, i_2 \in [k]$ and $j \in [n]$?*

Proof of the equivalence of Questions 1 and 2. First, assume that Question 2 has a positive answer with some $f\colon \mathbb{N} \to \mathbb{R}^+$. To see that f answers Question 1 in the positive, we identify A with $[n]$, and let $w_i(R) := 1 - |N(A \setminus R, B_i)|/m$ for all $i \in [k]$ and $R \in A^{(r)}$; then $w_i^*(S) = 1 - |N(A \setminus S, B_i)|/m$ for all $i \in [k]$ and $S \subseteq A = [n]$, in particular $w_i^*([n] \setminus \{j\}) = 1 - |N(j, B_i)|/m \geq 1 - c$ for all $j \in [n]$. Let $S_n := [n]$ and $S_j := [n] \setminus A_{n-j}$ for every $j \in [n-1]$, then $\emptyset \subsetneq S_1 \subsetneq \ldots \subsetneq S_n = [n]$, hence

$$||N(A_j, B_{i_1})| - |N(A_j, B_{i_2})|| = m|w_{i_1}^*(S_{n-j}) - w_{i_2}^*(S_{n-j})| \leq f(k)cm \quad \text{for all } i_1, i_2 \in [k] \text{ and } j \in [n-1].$$

Moreover, $|N(A_n, B_i)| = |N(A, B_i)| = |B_i| = m$ for all $i \in [k]$ as $r \geq 1$. Therefore f answers Question 1 in the positive.

Now, assume that Question 1 has a positive answer with some $f\colon \mathbb{N} \to \mathbb{R}^+$. To see that f answers Question 2 in the positive, observe that it suffices to consider when each w_i assumes rational values, in which case there exists m such that $m \cdot w_i(R)$ is an integer for all $i \in [k]$ and $R \in [n]^{(r)}$. We then let $A := [n]$, and let each B_i have cardinality m and have precisely $m \cdot w_i(R)$ vertices each having neighborhood R for every $R \in [n]^{(r)}$. Since $\sum_{R \in [n]^{(r)}} w_i(R) = 1$, every vertex in B_i has degree r and $|N(A \setminus S, B_i)|/m = 1 - w_i^*(S)$ for all $i \in [k]$ and $S \subseteq [n] = A$, thus $|N(j, B_i)|/m = 1 - w_i^*([n] \setminus \{j\}) \leq c$ for all $j \in [n]$. Let $A_n := A$ and $A_j := A \setminus S_{n-j}$ for every $j \in [n-1]$, then $\emptyset \subsetneq A_1 \subsetneq \ldots \subsetneq A_n = A$, so

$$|w_{i_1}^*(S_j) - w_{i_2}^*(S_j)| = ||N(A_{n-j}, B_{i_1})| - |N(A_{n-j}, N_{i_2})||/m \leq f(k)c \quad \text{for all } i_1, i_2 \in [k] \text{ and } j \in [n-1].$$

Moreover, $w_i^*(S_n) = w_i^*([n]) = 1$ for all $i \in [k]$. Therefore f answers Question 2 in the positive. □

The above proof also shows that Theorem 2 is equivalent to a result that partially answers Question 2; we shall prove this result in Section 3.

Theorem 3. *Let $k \geq 2$ be an integer and $c \in (0, 1/2)$. Let $w_1, \ldots, w_k$ be weighted r-uniform hypergraphs on $[n]$ for some $n \geq r \geq 1$. If $w_i^*([n] \setminus \{j\}) \geq 1 - c$ for every $i \in [k]$ and $j \in [n]$, then there is a chain of sets $\emptyset \subsetneq S_1 \subsetneq \ldots \subsetneq S_n = [n]$ such that $|w_{i_1}^*(S_j) - w_{i_2}^*(S_j)| \leq \sqrt{2(k-1)c}$ for all $i_1, i_2 \in [k]$ and $j \in [n]$.*

We would like to make three remarks. First, in Questions 1 and 2, $f(2)$ can be chosen to be 1, but it is still open whether $f(3)$ exists. It would also be helpful to know whether f exists when $r = 2$.

Second, if one restricts Question 2 to the case $r = 1$ only, then one can choose $f(k) = 2(k-1)$; this follows from a well-known lemma of Steinitz (see [1] for its history and related results), which we state here. In what follows, $\|.\|_\infty$ denotes the ∞-norm.

Theorem 4. *Let $k \geq 1$ be an integer. Then for every finite subset V of $\mathbb{R}^k$ with $\|\mathbf{v}\|_\infty \leq 1$ for all $\mathbf{v} \in V$ and $\sum_{\mathbf{v}\in V} \mathbf{v} = \mathbf{0}$, there is an ordering $\mathbf{v}_1, \dots, \mathbf{v}_n$ of the vectors in V with $\|\mathbf{v}_1 + \dots + \mathbf{v}_j\|_\infty \leq k$ for all $j \in [n]$.*

To see how $f(k)$ can be chosen as $2(k-1)$ when $r = 1$ in Question 2, for each $j \in [n]$ let $\mathbf{v}_j$ be the vector in $\mathbb{R}^{k-1}$ whose i-th component is $w_i(j) - w_k(j)$ for all $i \in [k-1]$, then $\|\mathbf{v}_j\|_\infty \leq c$. We apply Theorem 4 to $V = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, noting that $w_i^*(S) = \sum_{j\in S} w_i(j)$ and $|w_{i_1}^*(S) - w_{i_2}^*(S)| \leq |w_{i_1}^*(S) - w_k^*(S)| + |w_{i_2}^*(S) - w_k^*(S)|$ for all $i, i_1, i_2 \in [k]$ and $S \subseteq [n]$. Theorem 3 does not imply Theorem 4 as far as we know.

Third, every function f answering Question 2 in the positive, if exists, satisfies $f(k) = \Omega(\sqrt{k})$. To see this, for integer $k \geq 1$ let $g(k)$ be the smallest constant such that for each finite $V \subseteq \mathbb{R}^k$ with $\|\mathbf{v}\|_\infty \leq 1$ for all $\mathbf{v} \in V$ and $\sum_{\mathbf{v}\in V} \mathbf{v} = \mathbf{0}$, there is an ordering $\mathbf{v}_1, \dots, \mathbf{v}_n$ of the vectors in V with $\|\mathbf{v}_1 + \dots + \mathbf{v}_j\|_\infty \leq g(k)$ for all $j \in [n]$; then $g(k) \leq k$ by Theorem 4.

Proposition 5. *$f(k) \geq g(k-1)$ for all $k \geq 2$.*

Proof. Fix $\theta \in (0,1)$, then by the definition of $g(k-1)$, there exist an integer $n \geq 1$ and $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^{k-1}$ with $\|\mathbf{v}_j\|_\infty \leq \theta c$ for all $j \in [n]$ and $\sum_{j=1}^n \mathbf{v}_j = \mathbf{0}$, such that for every chain $\emptyset \subsetneq S_1 \subsetneq \dots \subsetneq S_n = [n]$, there is some $j_0 \in [n]$ with $\|\sum_{j\in S_{j_0}} \mathbf{v}_j\|_\infty > (\theta g(k-1))(\theta c) = \theta^2 g(k-1)c$. By adding zero vectors if necessary, we may assume $n \geq ((1-\theta)c)^{-1}$. Define $w_1, \dots, w_k \colon [n] \to \mathbb{R}^+$ by the rule that for every $j \in [n]$, $w_k(j) := 1/n$ and $w_i(j) := w_k(j) + v_{ji}$ where v_{ji} is the i-th coordinate of $\mathbf{v}_j$ for all $i \in [k-1]$. Then $w_1, \dots, w_k$ are weighted 1-uniform hypergraphs on $[n]$; and as $n \geq ((1-\theta)c)^{-1}$, $w_i(j) = w_k(j) + v_{ji} \leq (1-\theta)c + \theta c = c$ for every $i \in [k-1]$ and $j \in [n]$. But $\mathbf{v}_j = (w_1(j) - w_k(j), \dots, w_{k-1}(j) - w_k(j))$ for all $j \in [n]$, so for every chain $\emptyset \subsetneq S_1 \subsetneq \dots \subsetneq S_n = [n]$, there exists $j_0 \in [n]$ with $\max_{i\in[k-1]} |w_i^*(S_{j_0}) - w_k^*(S_{j_0})| > \theta^2 g(k-1)c$.

Thus, $f(k) > \theta^2 g(k-1)$ for all $\theta \in (0,1)$ hence $f(k) \geq g(k-1)$. This completes the proof. □

By the construction in [1, Section 3], $g(k) \geq \sqrt{k}/2$ whenever k is the order of some Hadamard matrix. For $k \geq 1$ in general, by Sylvester's construction which yields a Hadamard matrix of order equal to an arbitrary power of two, in particular of order $2^{\lfloor \log_2 k \rfloor}$, it is not hard to see that $g(k) \geq 2^{\lfloor \log_2 k \rfloor/2}/2 \geq \sqrt{2k}/4$. Hence $f(k) \geq g(k-1) \geq \sqrt{2(k-1)}/4$ for all $k \geq 2$ so $f(k) = \Omega(\sqrt{k})$. We note that showing the existence of $f(k)$ for $k \geq 2$ is equivalent to proving that $f(k)$ is bounded from above by a function of $g(k)$; and there is also a long-standing conjecture (according to [1]) that $g(k) = O(\sqrt{k})$.

3. Proof of Theorem 3

In this section, we prove Theorem 3. For every $i \in [k]$, $S \subseteq [n]$, and $j \in S$, let

$$\delta_i(j, S) := w_i^*(S) - w_i^*(S \setminus \{j\}) = \sum_{R\in S^{(r)}, j\in R} w_i(R).$$

Then $0 \leq \delta_i(j, S) \leq \delta_i(j, [n]) \leq c$; and moreover, for all $i \in [k]$ and $S \subseteq [n]$,

$$\sum_{j\in S} \delta_i(j, S) = r \cdot w_i^*(S). \tag{1}$$

For $S \subseteq [n]$, let the *unbalance* of S be the quantity $\max_{i_1,i_2\in[k]} |w_{i_1}^*(S) - w_{i_2}^*(S)|$. To prove theorem 3, we shall build the desired chain $\emptyset \subsetneq S_1 \subsetneq \dots \subsetneq S_n = [n]$ in reverse so that S_j has unbalance at most $\sqrt{2(k-1)c}$ for all $j \in [n]$. So, given a nonempty subset S of $[n]$ whose unbalance is reasonably small, it might be helpful to see how we can remove some $j \in S$ while maintaining reasonably small unbalance. When $k = 2$, by (1) there is some $j \in S$ for which $\delta_1(j, S) - \delta_2(j, S)$ has the same sign as $w_1^*(S) - w_2^*(S)$ has, and thus $S \setminus \{j\}$ has unbalance at most c if S has unbalance at most c. (This explains why one can choose $f(2) = 1$ in Question 1 and Question 2.) When $k \geq 3$, note that

$$(w_{i_1}^*(S) - w_{i_2}^*(S))^2 \leq 2[(w_{i_1}^*(S) - w_k^*(S))^2 + (w_{i_2}^*(S) - w_k^*(S))^2] \leq 2\sum_{i=1}^{k-1} (w_i^*(S) - w_k^*(S))^2, \tag{2}$$

which leads us to pay attention to how the quantity

$$\|\boldsymbol{\varphi}(S)\|^2 = \sum_{i=1}^{k-1} (w_i^*(S) - w_k^*(S))^2$$

changes when we delete an element from S. Here, $\boldsymbol{\varphi}(S)$ is the vector in $\mathbb{R}^{k-1}$ whose i-th component is $w_i^*(S) - w_k^*(S)$, and $\|.\|$ is the Euclidean norm in $\mathbb{R}^{k-1}$. The following lemma is motivated by this idea, showing that if $\|\boldsymbol{\varphi}(S)\|$ is reasonably small and j is a uniformly random element of S, then $\|\boldsymbol{\varphi}(S \setminus \{j\})\|$ is reasonably small in expectation.

Lemma 6. *If S is a nonempty subset of $[n]$, then*

$$\frac{1}{|S|}\sum_{j\in S}\|\boldsymbol{\varphi}(S\setminus\{j\})\|^2 \le \|\boldsymbol{\varphi}(S)\|^2 - \frac{2r}{|S|}(\|\boldsymbol{\varphi}(S)\|^2 - (k-1)c).$$

Proof. For every $j \in S$, let

$$\mathbf{x}_j := \boldsymbol{\varphi}(S) - \boldsymbol{\varphi}(S\setminus\{j\}) = (\delta_1(j,S) - \delta_k(j,S), \ldots, \delta_{k-1}(j,S) - \delta_k(j,S)),$$

then (1) yields

$$\sum_{j\in S}\mathbf{x}_j = r(w_1^*(S) - w_k^*(S), \ldots, w_{k-1}^*(S) - w_k^*(S)) = r\boldsymbol{\varphi}(S).$$

Write $\langle .,. \rangle$ for the standard inner product in $\mathbb{R}^{k-1}$. It follows that

$$\sum_{j\in S}\|\boldsymbol{\varphi}(S\setminus\{j\})\|^2 = |S|\cdot\|\boldsymbol{\varphi}(S)\|^2 - 2\sum_{j\in S}\langle\boldsymbol{\varphi}(S),\mathbf{x}_j\rangle + \sum_{j\in S}\|\mathbf{x}_j\|^2 = |S|\cdot\|\boldsymbol{\varphi}(S)\|^2 - 2r\|\boldsymbol{\varphi}(S)\|^2 + \sum_{j\in S}\|\mathbf{x}_j\|^2.$$

To conclude the proof of the lemma, it suffices to show that $\sum_{j\in S}\|\mathbf{x}_j\|^2 \le 2r(k-1)c$. To this end, let $x_{ij} := \delta_i(j,S) - \delta_k(j,S)$ for every $i \in [k-1]$ and $j \in S$. Observe that, for each $i \in [k-1]$,

$$\sum_{j\in S}x_{ij}^2 \le \sum_{j\in S}(\delta_i(j,S)^2 + \delta_k(j,S)^2) \le c\sum_{j\in S}(\delta_i(j,S) + \delta_k(j,S)) \overset{(1)}{=} rc(w_i(S) + w_k(S)) \le 2rc.$$

Therefore $\sum_{j\in S}\|\mathbf{x}_j\|^2 = \sum_{j\in S}\sum_{i=1}^{k-1}x_{ij}^2 = \sum_{i=1}^{k-1}\sum_{j\in S}x_{ij}^2 \le 2r(k-1)c$, as claimed. ☐

We are now ready to finish the proof of Theorem 3.

Proof of Theorem 3. If $n \le 2r$, we order the elements of $[n]$ arbitrarily, obtaining the chain $\emptyset \subsetneq S_1 \subsetneq \ldots \subsetneq S_n = [n]$. For every $j \in [n]$, $w_i^*(S_j) \le 2c$ for each $i \in [k]$ by (1), so S_j has unbalance at most $2c \le \sqrt{2(k-1)c}$ where we assumed $k \ge 3$. If $n > 2r$, we first construct $S_{2r} \subsetneq \ldots \subsetneq S_{n-1} \subsetneq S_n = [n]$ by backward induction where $|S_j| = j$ and $\|\boldsymbol{\varphi}(S_j)\|^2 \le (k-1)c$ for all j with $2r < j \le n$; by (2), it follows that S_j has unbalance at most $\sqrt{2(k-1)c}$ for such j. Initially $\|\boldsymbol{\varphi}(S_n)\| = 0$ as $S_n = [n]$. For $2r < j \le n$, assume that we have constructed $S_j, \ldots, S_{n-1}, S_n$. By Lemma 6 with $S = S_j$, there exists $j_0 \in S_j$ with $\|\boldsymbol{\varphi}(S_j \setminus \{j_0\})\|^2 \le (k-1)c$, and we let $S_{j-1} := S_j \setminus \{j_0\}$. This finishes the construction of $S_{2r}, \ldots, S_{n-1}, S_n$. We then order the elements of S_{2r} arbitrarily, obtaining the chain $\emptyset \subsetneq S_1 \subsetneq \ldots \subsetneq S_{2r-1} \subsetneq S_{2r}$, and arguing similarly as in the case $n \le 2r$. This completes the construction and the proof of Theorem 3. ☐

4. Additional remarks

We would like to discuss a variant of Question 2 that corresponds to a result of Bárány and Grinberg [1, Theorem 4.1]. For integer $n \ge 1$, let $\mathcal{F}_n$ be the family of all subsets of $[n]$. A *weighted hypergraph on* $[n]$ is a function $w\colon \mathcal{F}_n \to \mathbb{R}^+$ with $w(\emptyset) = 0$ and $\sum_{X\subseteq[n]} w(X) = 1$; a subset X of $[n]$ is called an *edge* of w if $w(X) > 0$. Thus, if the edges of w have the same cardinality r for some $r \in [n]$, then w can be viewed as a weighted r-uniform hypergraph on $[n]$. For every $S \subseteq [n]$, let $w^*(S) := \sum_{X\subseteq S} w(X)$.

Question 3. *Let $k \ge 1$ be an integer, and let $c > 0$. Let $w_1, \ldots, w_k$ be weighted hypergraphs on $[n]$ with $w_i^*([n]\setminus\{j\}) \ge 1 - c$ for all $i \in [k]$ and $j \in [n]$, for some integer $n \ge 1$. Does there exist $f\colon \mathbb{N} \to \mathbb{R}^+$ such that there is a partition $[n] = S \cup T$ with $|w_i^*(S) - w_i^*(T)| \le f(k)c$ for all $i \in [k]$?*

Dömötör Pálvölgyi [2] observed that $f(k) = 2k$ answers Question 3 in the positive. With his permission, we present his argument here.

Proposition 7. *Let $k \ge 1$ be an integer, and let $c > 0$. Let $w_1, \ldots, w_k$ be weighted hypergraphs on $[n]$ with $w_i^*([n]\setminus\{j\}) \ge 1 - c$ for all $i \in [k]$ and $j \in [n]$, for some $n \ge 1$. Then there is a partition $[n] = S \cup T$ with $|w_i^*(S) - w_i^*(T)| \le 2kc$ for all $i \in [k]$.*

Proof. Let $\mathcal{U} := \{(A,B) : A, B \subseteq [n], A\cap B = \emptyset, A\cup B \neq \emptyset\}$. The proof makes use of the octahedral Tucker lemma [3, 7], which we state here.

Lemma 8. *For an integer $n \geq 1$, if there exists a function $\lambda\colon \mathcal{U} \to \{-n+1, -n+2, \dots, -1, 1, 2, \dots, n-1\}$ such that $\lambda(S,T) = -\lambda(T,S)$ for all $(S,T) \in \mathcal{U}$, then there exist $(S_1,T_1),(S_2,T_2) \in \mathcal{U}$ with $S_1 \subseteq S_2$ and $T_1 \subseteq T_2$ such that $\lambda(S_1,T_1) = -\lambda(S_2,T_2)$.*

Now, we may assume that $n \geq 2k$. For every $(S,T) \in \mathcal{U}$, define $\lambda(S,T)$ by the rule that

- if $|S| + |T| < n - k$, then let $\lambda(S,T)$ be either $k + |S| + |T|$ or $-(k + |S| + |T|)$ (sign chosen arbitrarily so that $\lambda(S,T) = -\lambda(T,S)$ in this case);
- if $|S| + |T| \geq n - k$ and $w_i^*(S) - w_i^*([n] \setminus S) > 0$ for some $i \in [k]$, then let $\lambda(S,T)$ be such an i; and
- if $|S| + |T| \geq n - k$ and $w_i^*(T) - w_i^*([n] \setminus T) > 0$ for some $i \in [k]$, then let $-\lambda(S,T)$ be such an i.

Now, if every $\lambda(S,T)$ were defined, then it would not be difficult to verify that $\lambda(S,T) = -\lambda(T,S)$ for all $(S,T) \in \mathcal{U}$ and that there would not exist a complementary containment pair relative to λ, a contradiction by Lemma 8. So there exists $(S,T) \in \mathcal{U}$ with $|S| + |T| \geq n - k$ such that $w_i^*(S) - w_i^*([n] \setminus S) \leq 0$ and $w_i^*(T) - w_i^*([n] \setminus T) \leq 0$ for all $i \in [k]$. Put $Z := [n] \setminus (S \cup T)$, then $|Z| \leq k$. For every $i \in [k]$, the condition $w_i^*([n] \setminus \{j\}) \geq 1 - c$ for all $j \in [n]$ yields $w_i^*(S \cup Z) \leq w_i^*(S) + c|Z|$ and $w_i^*(T \cup Z) \leq w_i^*(T) + c|Z|$ which together imply $w_i^*(S \cup Z) - w_i^*(T) \leq w_i^*(S) + c|Z| - w_i^*([n] \setminus S) + c|Z| \leq 2kc$. Similarly, for all $i \in [k]$, $w_i^*(T \cup Z) - w_i^*(S) \leq 2kc$ thus $w_i^*(T) - w_i^*(S \cup Z) \leq 2kc$. Hence $[n] = (S \cup Z) \cup T$ is a desired partition. □

It is unknown whether $f(k)$ can be chosen to be of order $o(k)$ in Question 3, but if one requires $w_1, \dots, w_k$ to have edges of cardinality at most two, then one can choose $f(k) = 6\sqrt{k}$; we present a proof of this result.

Proposition 9. *Let $k \geq 1$ be an integer, and let $c > 0$. Let $w_1, \dots, w_k$ be weighted hypergraphs on $[n]$ with $w_i^*([n] \setminus \{j\}) \geq 1 - c$ for all $i \in [k]$ and $j \in [n]$, for some $n \geq 1$. If the edges of w_i have cardinality at most two for each $i \in [k]$, then there is a partition $[n] = S \cup T$ with $|w_i^*(S) - w_i^*(T)| \leq 6\sqrt{k} \cdot c$ for all $i \in [k]$.*

Proof. We follow the idea illustrated in [1, Section 4]. For every $j \in [n]$, let $\mathbf{x}_j := (x_{1j}, \dots, x_{kj}) \in \mathbb{R}^k$ where

$$x_{ij} = w_i(j) + \frac{1}{2} \sum_{j' \in [n] \setminus \{j\}} w_i(\{j, j'\}) \quad \text{for every } i \in [k],$$

then $0 \leq x_{ij} \leq w_i^*([n]) - w_i^*([n] \setminus \{j\}) \leq c$ for all $i \in [k]$; thus $\|\mathbf{x}_j\|_\infty \leq c$. Define the convex polytope

$$\mathcal{P} := \{(a_1, \dots, a_n) \in \mathbb{R}^n : -1 \leq a_1, \dots, a_n \leq 1 \text{ and } a_1\mathbf{x}_1 + \cdots + a_n\mathbf{x}_n = \mathbf{0}\}.$$

Observe that $\mathcal{P} \neq \emptyset$ since $\mathbf{0} \in \mathcal{P}$; thus let $(a_1, \dots, a_n) \neq \mathbf{0}$ be an extreme point of $\mathcal{P}$. The system of linear equations defining $\mathcal{P}$ has k equations, so $|J| \leq k$ where $J = \{j \in [n] : -1 < a_j < 1\}$. We may assume $J = [p]$ for some $p \in [n]$. Put $\mathbf{y} := \sum_{j=p+1}^n a_j\mathbf{x}_j = -\sum_{j=1}^p a_j\mathbf{x}_j$; then by Spencer's six standard deviations theorem [6], there exists $\mathbf{z} = \sum_{j=1}^p b_j\mathbf{x}_j$ with $b_1, \dots, b_p \in \{-1, 1\}$ such that $\|\mathbf{z} - \mathbf{y}\|_\infty \leq 6\sqrt{p} \cdot c \leq 6\sqrt{k} \cdot c$. Let $\xi_j := b_j$ for each $j \in [p]$ and let $\xi_j := a_j$ for each $j \in [n] \setminus [p]$, then $\sum_{j=1}^n \xi_j\mathbf{x}_j = \mathbf{z} - \mathbf{y}$ has ∞-norm at most $6\sqrt{k} \cdot c$. Put $S := \{j \in [n] : \xi_j = 1\}$ and $T := \{j \in [n] : \xi_j = -1\}$; then for every $i \in [k]$,

$$\sum_{j \in S} x_{ij} = \sum_{j \in S} \left(w_i(j) + \frac{1}{2} \sum_{j' \in [n] \setminus \{j\}} w_i(\{j, j'\}) \right) = w_i^*(S) + \frac{1}{2} \sum_{j \in S, j' \in T} w_i(\{j, j'\}),$$

and similarly,

$$\sum_{j \in T} x_{ij} = w_i^*(T) + \frac{1}{2} \sum_{j \in T, j' \in S} w_i(\{j, j'\}).$$

It follows that

$$|w_i^*(S) - w_i^*(T)| = \left| \sum_{j \in S} x_{ij} - \sum_{j \in T} x_{ij} \right| = |\xi_1 x_{i1} + \cdots + \xi_n x_{in}| \leq 6\sqrt{k} \cdot c$$

for all $i \in [k]$. This completes the proof. □

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