# NOTES ON RECENT WORK ON THE ERDŐS–HAJNAL CONJECTURE

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## 1. Introduction, the Alon–Pach–Solymosi theorem, and Rödl's theorem

A cornerstone of Ramsey theory due to Erdős–Szekeres [15] states that every $n$-vertex graph contains a clique or stable set of size at least $\frac{1}{2}\log n$; and random graph examples show that this cannot be substantially improved: indeed, Erdős [14] showed that a typical $n$-vertex graph has no clique or stable set with at least $2\log n$ vertices. The picture, however, seems to change dramatically when one considers graphs with a forbidden induced subgraph. Formally, for a graph $G$, an *induced subgraph* of $G$ is a subgraph obtained from $G$ only by removing vertices. For a graph $H$, say that $G$ is *$H$-free* if $H$ is not isomorphic to any induced subgraph of $G$. Erdős and Hajnal [11], about five decades ago, posed the following problem which remains open:

**Conjecture 1.1** (Erdős–Hajnal)**.** *For every graph $H$, there exists $c > 0$ such that every $H$-free graph $G$ has a clique or stable set of size at least $|G|^c$.*

In other words, this conjecture asserts that in stark contrast to general graphs, graphs with an excluded induced subgraph behave very differently. Conjecture 1.1 lies at the intersection of graph Ramsey theory (diagonal Ramsey numbers under additional restrictions) and structural graph theory (general properties of $H$-free graphs), and exhibits a sharp local-global phenomenon in graph theory.

Say that $H$ has the *Erdős–Hajnal property* if it satisfies Conjecture 1.1. Until now only a few graphs are known to have this property; simple examples include the complete graphs (via Ramsey bounds), the four-vertex path $P_4$ (Exercise 1.1), and the four-cycle $C_4$. The following fundamental result of Alon–Pach–Solymosi [1] gives a way to build graphs with the Erdős–Hajnal property from smaller graphs:

**Theorem 1.2** (Alon–Pach–Solymosi)**.** *If $H_1, H_2$ have the Erdős–Hajnal property, then the graph $H$ obtained from $H_1$ by substituting $H_2$ for $v$ also has the Erdős–Hajnal property.*

Here, the graph $H$ obtained from $H_1$ by *substituting $H_2$ for $v \in V(H_1)$* is the graph obtained from the disjoint union of $H_1 \setminus v$ and $H_2$ by adding all edges between $N_{H_1}(v)$ and $V(H_2)$. In what follows, a *copy* of $H$ in $G$ is an injective map $\varphi\colon V(H) \to V(G)$ such that for all distinct $u, v \in V(H)$, $uv \in E(H)$ if and only if $\varphi(u)\varphi(v) \in E(G)$; then $G$ is $H$-free if and only if there is no copy of $H$ in $G$. Also, for $S \subseteq V(G)$, let $G[S]$ denote the subgraph of $G$ induced on $S$.

**Proof of Theorem 1.2.** Let $f(G) := \max(\alpha(G), \omega(G))$ for every graph $G$, and let $f(S) := f(G[S])$ for every $S \subseteq V(G)$. Then, $H$ has the Erdős–Hajnal property if and only if there is some positive integer $k = k(H)$ such that every $H$-free graph $G$ satisfies $f(G)^k > |G|$, which means that there is a copy of $H$ in $G$ whenever $f(G)^k \le |G|$.

Now, for $i \in \{1, 2\}$, let $k_i$ be such that every $H_i$-free graph $G$ satisfies $f(G)^{k_i} > |G|$. Let $h_1 := |H_1|$; we claim that

$$k = k(H) := k_1 h_1 + k_2$$

verifies that Conjecture 1.1 is upheld for $H$. To this end, let $G$ be a graph with $f(G)^k \le |G| =: n$; we need to show that there is a copy of $H$ in $G$. Put $r := f(G)^{k_1} < f(G)^k \le |G|$, then for every $S \subseteq V(G)$ of size $r$ we see that $f(S)^{k_1} \le f(G)^{k_1} = r = |S|$, so there is a copy of $H_1$ in $G[S]$. By counting the number

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of pairs $(\varphi, S)$ where $\varphi$ is a copy of $H_1$ in $G$ and $S \subseteq V(G)$ such that $\varphi(V(H_1)) \subseteq S$, it follows that the number of copies of $H_1$ in $G$ is at least

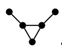
$$\binom{n}{r} \Big/ \binom{n-h_1}{r-h_1} = \frac{n(n-1)\cdots(n-h_1+1)}{r(r-1)\cdots(r-h_1+1)} \geq \left(\frac{n}{r}\right)^{h_1}.$$

Now, observe that the number of copies of $H_1 \setminus v$ in $G$ is at most $n^{h_1-1}$. Hence, by double counting $\#(\varphi, \varphi')$, there is a copy $\varphi$ of $H_1 \setminus v$ in $G$ such that there are at least $(n/r)^{h_1}/n^{h_1-1} = n/r^{h_1}$ copies $\varphi'$ of $H_1$ in $G$ with $\varphi'|_{V(H_1 \setminus v)} = \varphi$. Let $F := G[\varphi(V(H_1) \setminus v)]$, and let $T$ be the set of vertices $u \in V(G) \setminus V(F)$ such that there is a copy $\varphi'$ of $H_1$ in $G$ with $\varphi'|_{V(H_1 \setminus v)} = \varphi$ and $\varphi'(v) = u$; then $|T| \geq n/r^{h_1}$. Thus

$$|T| \geq \frac{n}{r^{h_1}} \geq \frac{f(G)^k}{f(G)^{k_1 h_1}} = f(G)^{k_2} \geq f(T)^{k_2},$$

so there is a copy of $H_2$ in $G[T]$. Therefore there is a copy of $H$ in $G$, as desired. ■

Theorem 1.2 says that graphs with the Erdős–Hajnal property are closed under substitution. Thus, an approach to Conjecture 1.1 is to verify the Erdős–Hajnal property of prime graphs, where a graph $H$ is *prime* if there do not exist graphs $H_1, H_2$ with $|H_1|, |H_2| < |H|$ such that $H$ can be obtained by substituting $H_2$ for any vertex of $H_1$. For example, $P_4$, known to have the Erdős–Hajnal property, is the smallest prime graph with at least two vertices and also the only prime graph on four vertices. What about prime graphs with at least five vertices? Proving the Erdős–Hajnal property of all five-vertex (prime) graphs remained a challenge for a few decades which was only finished very recently:

- the bull , proved by Chudnovsky–Safra 2008 [8] (with optimal exponent 1/4, see Exercise 1.2);
- the five-cycle $C_5$, proved by Chudnovsky–Scott–Seymour–Spirkl 2023 [9]; and
- the five-vertex path $P_5$, proved by Nguyen–Scott–Seymour 2024+ [22].

Regarding prime graphs, there was also a question of Chudnovsky [7] that asks whether there is a prime graph on at least six vertices that satisfies the Erdős–Hajnal property. This was answered recently in the positive [23] via a construction that builds a sequence of infinitely many such graphs.

**Exercise 1.1.** *A* cograph *is a $P_4$-free graph. Show that for every cograph $G$, either $G$ or $\overline{G}$ is disconnected; then deduce that $\alpha(G)\omega(G) \geq |G|$ and $\max(\alpha(G), \omega(G)) \geq \sqrt{|G|}$.*

**Exercise 1.2.** *Show that for every $c > 0$, there is a bull-free graph $G$ with $\max(\alpha(G), \omega(G)) < |G|^{1/4+c}$.*

**Exercise 1.3.** *Show that $H$ has the Erdős–Hajnal property if and only if there are $c, \eta > 0$ such that every $H$-free graph $G$ satisfies $\max(\alpha(G), \omega(G)) \geq \eta|G|^c$.*

Regarding the general properties of $H$-free graphs, perhaps one of the most well-known results in this topic is a theorem of Rödl [27] on the edge distribution of such graphs. To state the result, for $\varepsilon > 0$, say that a graph $G$ is *$\varepsilon$-sparse* if it has maximum degree at most $\varepsilon|G|$, *$\varepsilon$-dense* if its complement $\overline{G}$ is $\varepsilon$-sparse, and *$\varepsilon$-restricted* if it is $\varepsilon$-sparse or $\varepsilon$-dense. Then Rödl's theorem says:

**Theorem 1.3** (Rödl). *Let $\varepsilon \in (0, \frac{1}{2})$ and let $H$ be a graph. Then the following hold:*

- *there is some $\delta = \delta(H, \varepsilon) > 0$ such that every $H$-free graph $G$ has an $\varepsilon$-restricted induced subgraph on at least $\delta|G|$ vertices; and*
- *there is some $\delta = \delta(H, \varepsilon) > 0$ such that for every $H$-free graph $G$, there exists $X \subseteq V(G)$ with $|X| \geq \delta|G|$ and $G[X]$ having at most $\varepsilon|X|^2$ edges.*

**Sketch of proof.** To see how the second statement implies the first statement, we apply the second statement with $\varepsilon' := \varepsilon/4$ and get $\delta' = \delta(H, \varepsilon')$. Put $\delta := \delta'/2$. Let $G$ be an $H$-free graph. We may assume without loss of generality that there is some $X \subseteq V(G)$ such that $|X| \geq \delta'|G|$ and $G[X]$ has at most $\varepsilon'|X|^2 \leq \varepsilon|X|^2/4$ edges. Let $Y$ be the set of vertices in $X$ with degree at least $\varepsilon|X|/2$ in $G[X]$.

Then $|Y| \cdot \varepsilon|X|/2 \leq \varepsilon|X|^2/4$ which implies $|Y| \leq |X|/2$, so $|X \setminus Y| \geq |X|/2 \geq \delta'|G|/2 = \delta|G|$. It follows that $G[X \setminus Y]$ has maximum degree at most $\varepsilon|X|/2 \leq \varepsilon|X \setminus Y|$, as desired.

Here is a sketch to prove the second statement. Given an $H$-free graph $G$, we apply the regularity lemma with parameter $\varepsilon' = \varepsilon'(H, \varepsilon)$ and lower bound $m = m(H, \varepsilon)$ to obtain an $\varepsilon'$-regular partition of $V(G)$ into $k$ subsets where $m \leq k \leq T(H, \varepsilon)$. We then apply Turán's theorem to obtain $R(n, n, n)$ pairwise $\varepsilon'$-regular subsets among them where $n = n(H, \varepsilon)$. Then by Ramsey's theorem, there are $n$ subsets among these such that either every pair of them has density at most $\varepsilon$, or every pair of them has density at least $1 - \varepsilon$, or every pair of them has density between $\varepsilon$ and $1 - \varepsilon$. The first case gives some $X \subseteq V(G)$ with $|X| \geq \delta|G|$ such that $G[X]$ has at most $\varepsilon|X|^2$ edges, the second case gives some $X \subseteq V(G)$ with $|X| \geq \delta|G|$ such that $\overline{G}[X]$ has at most $\varepsilon|X|^2$ edges, and the third case gives a copy of $H$ (cf. the embedding lemma). ■

It is not hard to see that the last step of the above proof sketch actually gives more than just one copy of $H$. By working out the numbers one can recover the following theorem of Nikiforov [26].

**Theorem 1.4** (Nikiforov). *For every $\varepsilon \in (0, \frac{1}{2})$ and every graph $H$, there exists $\delta > 0$ such that every graph $G$ with $\operatorname{ind}_H(G) < (\delta|G|)^{|H|}$ has an $\varepsilon$-restricted induced subgraph on at least $\delta|G|$ vertices.*

How much does $\delta$ depend on $\varepsilon$ in Theorems 1.3 and 1.4? The regularity argument sketched above gives tower-type dependence which perhaps will only be useful when $\varepsilon$ is a fixed constant. Fox and Sudakov [17] conjectured that in Theorem 1.3, $\delta$ can be taken as a power of $\varepsilon$ (whose exponent depends solely on $|H|$); and recently in [23] it was conjectured that the same conclusion holds for Theorem 1.4 as well. More precisely, let us say that $H$ has the *polynomial Rödl* property if there exists $d \geq 1$ such that for every $\varepsilon \in (0, \frac{1}{2})$, every $H$-free graph $G$ contains an $\varepsilon$-restricted induced subgraph on at least $\delta|G|$ vertices; and that $H$ is *viral* if there exists $d \geq 1$ such that for every $\varepsilon \in (0, \frac{1}{2})$, the same conclusion actually holds for every graph $G$ with $\operatorname{ind}_H(G) < (\delta|G|)^{|H|}$. Then it is not hard to check that the viral property implies the polynomial Rödl property, which in turns yields the Erdős–Hajnal property. Here we present an argument by Bucić–Fox–Pham [4] that shows the equivalence of all of these three properties. We start with a primitive version of the Kleitman–Winston graph container method [2, 20].

**Lemma 1.5.** *Let $0 \leq \ell \leq k$ be integers, let $\varepsilon \in (0, 1)$, and let $G$ be a graph. Let $r \geq (1 - \varepsilon)^{\ell}|G|$, and assume that $G$ has no $\varepsilon$-sparse induced subgraph on at least $r$ vertices. Then $G$ has at most $r^{k-\ell}|G|^{\ell}$ stable sets of size $k$.*

**Proof.** Induction on $\ell + |G|$. If $\ell = 0$ then the lemma is true. Now, assume that $0 < \ell \leq k$ and the lemma is true for $\ell + |G| - 1$; we shall prove it for $\ell + |G|$. Since $G$ is not $\varepsilon$-sparse, there exists $v \in V(G)$ with degree at least $\varepsilon|G|$ in $G$. Let $S$ be the set of nonneighbours of $v$ in $G$; then $|S| \leq (1 - \varepsilon)|G|$ and so $(1 - \varepsilon)^{\ell-1}|S| \leq (1 - \varepsilon)^{\ell}|G| \leq r$. Since $G[S]$ has no $\varepsilon$-sparse induced subgraph on at least $r$ vertices, by induction (with $k, \ell$ replaced by $k - 1, \ell - 1$), $G[S]$ has at most $r^{(k-1)-(\ell-1)}|S|^{\ell-1} \leq r^{k-\ell}|G|^{\ell-1}$ stable sets of size $k - 1$; and so $G$ has at most $r^{k-\ell}|G|^{\ell-1}$ stable sets of size $k$ that contain $v$. Now, by induction, $G \setminus (S \cup \{v\})$ has at most $r^{k-\ell}(|G| - 1)^{\ell}$ stable sets of size $k$. Thus, since $|G|^{\ell-1} + (|G| - 1)^{\ell} \leq |G|^{\ell}$ because $(1 - 1/|G|)^{\ell} \leq 1 - 1/|G|$, the number of stable sets of size $k$ in $G$ is at most $r^{k-\ell}|G|^{\ell-1} + r^{k-\ell}(|G| - 1)^{\ell} \leq r^{k-\ell}|G|^{k-\ell}$. This proves Lemma 1.5. ■

Putting $r := \delta|G|$, $\ell := \lceil \frac{1}{\varepsilon} \log \frac{1}{\delta} \rceil$, and $k := 2\ell$ in Lemma 1.5 gives the following.

**Lemma 1.6.** *Let $\varepsilon, \delta \in (0, \frac{1}{2})$, and let $k := 2\lceil \frac{1}{\varepsilon} \log \frac{1}{\delta} \rceil$. Then every graph $G$ has either an $\varepsilon$-sparse induced subgraph on at least $\delta|G|$ vertices or at most $(\delta^{1/2}|G|)^k$ stable sets of size $k$.*

The following is a product of Lemma 1.6 and an Alon–Pach–Solymosi type argument (see Theorem 1.2).

**Lemma 1.7.** *Let $f\colon [2,\infty)\to[2,\infty)$ be an increasing function. Let $n\ge 2$ be an integer, and let $G$ be a graph such that at least half of the $n$-vertex induced subgraphs of $G$ contain a clique or stable set of size at least $8f(n)\log n$. Then $G$ has a $1/f(n)$-sparse or $1/f(n)$-dense induced subgraph on at least $n^{-4}|G|$ vertices.*

**Proof.** Suppose not. Let $\varepsilon := 1/f(n)$ and $k := 2\lceil 4f(n)\log n\rceil \le \lceil 8f(n)\log n\rceil$; then by Lemma 1.6 (with $\delta = n^{-4}$), there are at most $(n^{-2}|G|)^k$ cliques of size $k$ and at most $(n^{-2}|G|)^k$ stable sets of size $k$ in $G$. Now, for $n=\lceil\frac{1}{\varepsilon}\rceil \le \frac{2}{\varepsilon}$, at least half of the $n$-vertex induced subgraphs of $G$ contain a clique or stable set of size at least $\lceil 8f(n)\log n\rceil \ge k$. Therefore

$$2(n^{-2}|G|)^k\binom{|G|-k}{n-k} \ge \frac{1}{2}\binom{|G|}{n}.$$

Since $\binom{|G|-k}{n-k}/\binom{|G|}{n} \le (n/|G|)^k$, it follows that $2n^{-k}\ge\frac{1}{2}$, a contradiction since $n\ge 2$ and $k\ge 2$. This proves Lemma 1.7. ■

Now, to see how a graph $H$ would be viral if it has the Erdős–Hajnal property, let $c>0$ be such that every $H$-free graph $G$ satisfies $\max(\alpha(G),\omega(G))\ge |G|^c$, let $f(n)$ be any sufficiently small power of $n$ such that $8f(n)\log n\le n^c$ for all large $n$, and let $n$ be a suitable power of $\varepsilon^{-1}$. We remark that the same trick can be applied to the Erdős–Hajnal bound $2^{c\sqrt{\log n}}$ (see Theorem 2.1) to show that in Theorems 1.3 and 1.4 one can take $\delta = 2^{C(\log\frac{1}{\varepsilon})^2}$ for some suitable $C=C(H)<0$ which was previously proved by Fox–Sudakov [17]; and with the $\log\log$ improvement (see Theorem 3.1) this can be strengthened further to $\delta = 2^{C(\log\frac{1}{\varepsilon})^2/\log\log\frac{1}{\varepsilon}}$. We omit the details.

## 2. The Erdős–Hajnal bound and the density Fox–Sudakov theorem

Qualitatively, Conjecture 1.1 implies that $H$-free graphs have much larger cliques or stable sets than a typical graph does. If one does not ask for strict polynomial bounds as the conjecture predicts, then Erdős and Hajnal already verified this qualitative statement when they first posed the problem. In more details, the bound $2^{c\sqrt{\log|G|}}$ was claimed in their 1977 paper [11] where Conjecture 1.1 was first introduced, and was later proved rigorously in a 1989 paper [12].

**Theorem 2.1** (Erdős–Hajnal)**.** *For every graph $H$, there exists $c > 0$ such that every $H$-free graph $G$ has a clique or stable set of size at least $2^{c\sqrt{\log|G|}}$.*

To achieve this, Erdős and Hajnal employed the following "density" result for $H$-free graphs:

**Theorem 2.2** (Erdős–Hajnal)**.** *For every graph $H$, there exists $d \geq 1$ such that for every $x \in (0, \frac{1}{2})$ and every $H$-free graph $G$ with $|G| > x^{-d}$, there are disjoint $A, B \subseteq V(G)$ with $|A|, |B| \geq x^d|G|$ such that $A$ is $x$-sparse or $x$-dense to $B$.*

Let us see how this lemma implies Theorem 2.1. In what follows, for a graph $G$ let $\mu(G) := \max_F|F|$ where the maximum ranges through all induced subgraphs $F$ of $G$ that are cographs. Note that $\mu(G) \geq \max(\alpha(G), \omega(G)) \geq \sqrt{\mu(G)}$ (cf. Exercise 1.1); and so Theorem 2.1 is equivalent to the following.

**Theorem 2.3.** *For every graph $H$, there exists $c > 0$ such that every $H$-free graph $G$ satisfies $\mu(G) \geq 2^{c\sqrt{\log|G|}}$.*

**Proof of Theorem 2.3, assuming Theorem 2.2.** Let $d \geq 1$ be given by Theorem 2.2; we shall prove by induction on $|G|$ that $c := 1/(2d)$ satisfies the theorem. Assume that $|G| \geq 2$, and that the theorem holds for all induced subgraphs of $G$ with fewer than $|G|$ vertices. Note that $\mu(G) \geq 2$ since $|G| \geq 2$.

Let $\mu := \mu(G)$, and let $x := \mu^{-2} \leq 1/4$. If $G \leq x^{-d}$ then $\mu^2 \geq |G|^{1/d}$ so $\mu \geq |G|^{1/(2d)} \geq |G|^c$ and we are done; and so we may assume $|G| > x^{-d}$. By the choice of $d$, there are disjoint $A, B \subseteq V(G)$ with $|A|, |B| \geq x^d|G| > 1$ such that $A$ is $x$-sparse or $x$-dense to $B$. By the symmetry, let us assume that $A$ is $x$-sparse to $B$. Since $|A| < |G|$, the induction hypothesis gives $S \subseteq A$ such that $G[S]$ is a cograph and $|S| \geq 2^{c\sqrt{\log|A|}}$, which is equivalent to $(\log|S|)^2 \geq c^2 \log|A|$. Thus $\log|S| \geq c^2 \log|A|/\log\mu$; that is

$$|S| \geq |A|^{c^2/\log\mu}.$$

Let $B'$ be the set of vertices in $B$ with no neighbour in $S$; then since $|S| \leq \mu = x^{-1/2} \leq \frac{1}{2}x^{-1}$, we have that $|B'| \geq |B| - |S| \cdot x|B| \geq \frac{1}{2}|B| \geq \frac{1}{2}x^d|G|$. Again by induction and in a similar manner as above, there exists $T \subseteq B'$ such that $G[T]$ is a cograph and

$$|T| \geq |B'|^{c^2/\log\mu}.$$

Now, $G[S \cup T]$ is a cograph of $G$ since $G$ has no edge between $S, T$. Hence, because $|A|, |B'| \geq \frac{1}{2}x^d|G| \geq x^{d+1}|G| \geq x^{2d}|G|$, we deduce that

$$\mu(G) \geq |S \cup T| \geq |A|^{c^2/\log\mu} + |B'|^{c^2/\log\mu} \geq 2(x^{2d}|G|)^{c^2/\log\mu} \geq |G|^{c^2/\log\mu}$$

where the last inequality holds since $(x^{2d})^{c^2/\log\mu} = (x^{2d})^{-2c^2/\log x} = 2^{-4c^2d} = 2^{-1/d} \geq 1/2$ by the choice of $c$. Thus $\mu \geq 2^{c\sqrt{\log|G|}}$, completing the induction step and proving Theorem 2.3. ■

It is not hard to see from the above proof that one would only need a $\mu(G)^{-2}$-sparse or $\mu(G)^{-2}$-dense pair of size at least $|G|/\mu(G)^{2d}$ in order to make the induction step work. It is still open whether one can completely get rid of the "noise" between such a pair; that is, every $H$-free graph $G$ with $|G| \geq 2$ contains a pure pair of size at least $|G|/\mu(G)^b$ for some $b \geq 1$ depending on $H$ only. It is worth noting that this is a direct consequence of Conjecture 1.1.

Another well-known weakening of Conjecture 1.1 is the following "one-sided Erdős–Hajnal" theorem of Fox–Sudakov [18] which was first proved by the dependence random choice method.

**Theorem 2.4** (Fox–Sudakov)**.** *For every graph $H$, there exists $c > 0$ such that every $H$-free graph $G$ either has a stable set of size at least $|G|^c$, or a complete bipartite subgraph whose parts each have size at least $|G|^c$.*

In what follows we will present a proof of a common generalization [24] of both Theorem 2.2 and Theorem 2.4:

**Theorem 2.5** (Nguyen–Scott–Seymour)**.** *For every graph $H$, there exists $d \geq 2$ such that for every $x \in (0, \frac{1}{2})$ and every $H$-free graph $G$, either:*

- *$G$ has an $x$-sparse induced subgraph on at least $x^d|G|$ vertices; or*
- *there are disjoint $A, B \subseteq V(G)$ with $|A|, |B| \geq x^d|G|$ such that $A$ is $x$-dense to $B$.*

We shall in fact relax the condition "$H$-free" into "not too many copies of $H$". Formally, let $\operatorname{ind}_H(G)$ denote the number of copies of $H$ in $G$; then our goal in the remainder of this section is the following.

**Theorem 2.6.** *For every graph $H$, there exists $d \geq 2$ such that for every $x \in (0, \frac{1}{2})$ and every graph $G$ with $\operatorname{ind}_H(G) < x^d|G|^{|H|}$, either:*

- *$G$ has an $x$-sparse induced subgraph on at least $x^d|G|$ vertices; or*
- *there are disjoint $A, B \subseteq V(G)$ with $|A|, |B| \geq x^d|G|$ such that $A$ is $x$-dense to $B$.*

The key idea behind the proof of Theorem 2.6 is to "fill in" the missing edges of $H$ one at a time; more precisely, if $G$ has too few copies of $H$ but enough copies of $H$ with a missing edge added then there has to be a dense pair of polynomial size. This idea is made rigorous by the following lemma.

**Lemma 2.7.** *Let $H$ be a graph with at least one nonedge, and let $e \in E(\overline{H})$. Let $x \in (0, \frac{1}{2})$, and let $G$ be a graph. Then for every $a, b \geq 1$, one of the following holds:*

- *$\operatorname{ind}_{H+e}(G) < x^a|G|^{|H|}$;*
- *$\operatorname{ind}_H(G) \geq x^{2a+b+5}|G|^{|H|}$; and*
- *there are disjoint $A, B \subseteq V(G)$ with $|A|, |B| \geq x^{a+3}|G|$ such that $G$ has at most $x^b|A||B|$ nonedges between $A$ and $B$.*

**Proof.** Assume that the first and the third outcomes do not hold. Let $u, v$ be the endpoints of $e$, and let $H' := H \setminus \{u, v\}$. For every copy $\varphi'$ of $H'$ in $G$, let $S_{\varphi'}$ be the set of copies $\varphi$ of $H$ in $G$ with $\varphi|_{V(H')} = \varphi'$. Let $T$ be the set of copies $\varphi'$ of $H'$ in $G$ with $|S_{\varphi'}| \geq \frac{1}{2}x^a|G|^2$. Since the first outcome of the lemma does not hold, we have that

$$x^a|G|^{|H|} \leq \operatorname{ind}_{H+e}(G) \leq |T| \cdot |G|^2 + |G|^{|H|-2} \cdot \tfrac{1}{2}x^a|G|^2 = |T||G|^2 + \tfrac{1}{2}x^a|G|^{|H|}$$

and so $|T| \geq \frac{1}{2}x^a|G|^{|H|-2}$. We claim the following.

**Claim 2.8.** *For every $\varphi' \in T$, there are at least $x^{a+b+4}|G|^2$ copies $\psi$ of $H$ in $G$ with $\psi|_{V(H')} = \varphi'$.*

*Subproof.* Let $X := \{\varphi(u) : \varphi \in S_{\varphi'}\}$, and let $Y := \{\varphi(v) : \varphi \in S_{\varphi'}\}$.

If $u, v$ are twins[1] in $H$ then $X = Y$. Then, since $G[X] = G[Y]$ has at least $\frac{1}{2}|S_{\varphi'}| \geq \frac{1}{4}x^a|G|^2$ edges, there is a partition $(A, B)$ of $X = Y$ such that $G$ has at least $\frac{1}{4}|S_{\varphi'}| \geq \frac{1}{8}x^a|G|^2$ edges between $A, B$. Thus $|A||B| \geq \frac{1}{8}x^a|G|^2$; and so $|A|, |B| \geq \frac{1}{8}x^a|G| \geq x^{a+3}|G|$.

If $u, v$ are not twins in $H$ then $X \cap Y = \emptyset$. In this case let $A := X$ and $B := Y$; then $G$ has at least $|S_{\varphi'}| \geq \frac{1}{2}x^a|G|$ edges between $A, B$. Thus $|A||B| \geq \frac{1}{2}x^a|G|^2$; and so $|A|, |B| \geq \frac{1}{2}x^a|G| \geq x^{a+1}|G|$.

Hence, since the third outcome of the lemma does not hold, in any case $G$ has at least $x^b|A||B|$ nonedges between $A, B$; and so the number of copies $\psi$ of $H$ in $G$ with $\psi(V(H')) = \varphi'$ is at least

$$\tfrac{1}{2} \cdot x^b|A||B| \geq \tfrac{1}{16}x^{a+b}|G|^2 \geq x^{a+b+4}|G|^2$$

which proves Claim 2.8. □

[1]Two vertices $u, v \in V(H)$ are *twins* in $H$ if they have the same neighbourhood in $V(H) \setminus \{u, v\}$.

Now, Claim 2.8 implies that

$$\text{ind}_H(G) \geq |T| \cdot x^{a+b+4}|G|^2 \geq \tfrac{1}{2}x^{2a+b+4}|G|^{|H|} \geq x^{2a+b+5}|G|^{|H|}.$$

This proves Lemma 2.7. ■

**Lemma 2.9.** *Let $x > y > 0$. Let $G$ be a graph with nonempty disjoint $A, B \subseteq V(G)$ such that $G$ has at most $y|A||B|$ edges between $A$ and $B$. Then there are at least $(1 - \frac{y}{x})|A|$ vertices in $A$ each with at most $x|B|$ neighbours in $B$.*

**Proof.** This is because there are at most $\frac{y}{x}|A|$ vertices in $A$ each with at least $x|B|$ neighbours in $B$. ■

**Lemma 2.10.** *Let $H$ be a graph with at least one edge, and let $e \in E(H)$. Let $x \in (0, \frac{1}{2})$, and let $G$ be a graph. Then for every $a$, one of the following holds:*

- $\text{ind}_{H+e}(G) < x^a|G|^{|H|}$*;*
- $\text{ind}_H(G) \geq x^{2a+7}|G|^{|H|}$*; and*
- *there are disjoint $A, B \subseteq V(G)$ with $|A|, |B| \geq x^{a+4}|G|$ such that $A$ is $x$-dense to $B$ in $G$.*

**Proof.** Assume that the first two outcomes do not hold. Then by Lemma 2.7 with $b = 2$, there are disjoint $A, B \subseteq V(G)$ with $|A|, |B| \geq x^{a+3}|G|$ such that $G$ has at most $x^2|A||B|$ nonedges between $A$ and $B$. Let $A'$ be the set of vertices in $A$ with at most $2x^2|B| \leq x|B|$ nonneighbours in $B$; then $|A'| \geq \frac{1}{2}|A| \geq x^{a+4}|G|$ and $A'$ is $x$-dense to $B$. This proves Lemma 2.10. ■

We are now ready to prove Theorem 2.6.

**Proof of Theorem 2.6.** Let $\{e_1, \ldots, e_k\}$ be the edges of $\overline{H}$. Let $H_0 := K_{|H|}$, and for every $i \in [k]$ let $H_i := H_0 \setminus \{e_1, \ldots, e_i\}$; then $H = H_k$. By Exercise 2.1, there exists $s \geq 1$ such that every graph $G$ with $\text{ind}_{K_{|H|}}(G) < x^s|G|^{|H|}$ has an $x$-sparse induced subgraph with at least $x^s|G|$ vertices. Let $s_0 := s$; and for every $i \in [k]$, let $s_i := 2s_{i-1} + 7$.

We claim that $d := s_k$ satisfies the theorem. To this end, let $G$ be a graph with $\text{ind}_H(G) < x^d|G|^{|H|}$, and suppose that none of the outcomes of the theorem holds. We claim the following.

**Claim 2.11.** $\text{ind}_{H_i}(G) \geq x^{s_i}|G|^{|H|}$ *for every $i \in \{0, 1, \ldots, k\}$.*

*Subproof.* The claim is true for $i = 0$ by the choice of $s$. For $i \geq 1$, assume that the claim is true for $i - 1$; let us prove it for $i$. Since $H_{i-1} = H_i + e_i$, Lemma 2.10 with $a, H, e$ replaced by $s_{i-1}, H_{i-1}, e_i$ respectively implies that one of the following holds:

- $\text{ind}_{H_{i-1}}(G) < x^{s_{i-1}}|G|^{|H|}$;
- $\text{ind}_{H_i}(G) \geq x^{2s_{i-1}+7}|G|^{|H|} = x^{s_i}|G|^{|H|}$; and
- there are disjoint $A, B \subseteq V(G)$ with $|A|, |B| \geq x^{s_{i-1}+3}|G| \geq x^d|G|$ such that $A$ is $x$-dense to $B$.

Since the first bullet cannot hold by the induction hypothesis and the third bullet cannot hold by our supposition, the second bullet holds. This completes the induction step, proving Claim 2.11. □

Now, Claim 2.11 with $i = k$ gives $\text{ind}_H(G) = \text{ind}_{H_k}(G) \geq x^{s_k}|G|^{|H|} = x^d|G|^{|H|}$, contrary to the hypothesis. This proves Lemma 2.10. ■

**Exercise 2.1.** *Without using the equivalence of the Erdős–Hajnal property and the viral property, show that for every $x \in (0, \frac{1}{2})$ and every complete graph $K$, there exists $s \geq 1$ such that every graph $G$ with $\text{ind}_K(G) < x^s|G|^{|K|}$ has an $x$-sparse induced subgraph with at least $x^s|G|$ vertices.*

## 3. A $\log\log$ STEP TOWARDS ERDŐS–HAJNAL

The goal of this section is the following result [5], which improves Theorem 2.1 by a $\log\log$ factor.

**Theorem 3.1** (Bucić–Nguyen–Scott–Seymour)**.** *For every graph $H$, there exists $c > 0$ such that every $H$-free graph $G$ has a clique or stable set of size at least $2^{c\sqrt{\log|G|\log\log|G|}}$.*

In order to approach the bound in Theorem 3.1, Conlon, Fox, and Sudakov [10] proposed the following "polynomial versus linear" conjecture.

**Conjecture 3.2** (Conlon–Fox–Sudakov)**.** *For every graph $H$, there exists $d \ge 2$ such that for every $\varepsilon \in (0, \frac{1}{2})$ and every $H$-free graph $G$ with $|G| \ge 2$, there are disjoint $A, B \subseteq V(G)$ with $|A| \ge x^d|G|$ and $|B| \ge \frac{1}{d}|G|$ such that $B$ is $x$-sparse or $x$-dense to $A$.*

This remains open even when $H$ is a triangle, but is known to be true when $H = C_5$. If Conjecture 3.2 is true, then by taking $x$ to be a suitably small power of $|G|^{-1}$, one could obtain the following which is still open also when $H = C_3$.

**Conjecture 3.3** (Conlon–Fox–Sudakov)**.** *For every graph $H$, there exists $c > 0$ such that every $H$-free graph $G$ with $|G| \ge 2$ contains a pure pair $(A, B)$ with $|A| \ge |G|^c$ and $|B| \ge c|G|$.*

For a graph $G$, a *blockade* in $G$ is a sequence $(B_1, \dots, B_\ell)$ of disjoint (and possibly empty) subsets of $V(G)$; its *length* is $\ell$ and its *width* is $\min_{i\in[\ell]}|B_i|$. For $k, w \ge 0$, this blockade is a $(k, w)$-*blockade* if its length is at least $k$ and its width is at least $w$; and for $x > 0$, it is $x$-*sparse* in $G$ if $B_j$ is $x$-sparse to $B_i$ in $G$ for all $i, j \in [\ell]$ with $i < j$, and $x$-*dense* in $G$ if it is $x$-sparse in $\overline{G}$. While Conjecture 3.2 is open, the following weakening is sufficient to deduce Theorem 3.1 in the same manner that Theorem 2.2 implies Theorem 2.1.

**Theorem 3.4.** *For every graph $H$, there exists $b \ge 2$ such that for every $x \in (0, |H|^{-2})$ and every $H$-free graph $G$, there is an $x$-sparse or $x$-dense $(\frac{1}{2}\log\frac{1}{x}, \lfloor x^b|G|\rfloor)$-blockade in $G$.*

The rest of this section will be devoted to proving Theorem 3.4. The heart of its proof is the following "bipartite" lemma, which extends the idea behind Lemma 2.7 by using a more "local" counting scheme.

**Lemma 3.5.** *Let $H$ be a graph, and let $x \in (0, \frac{1}{2})$. Let $G$ be a graph with nonempty disjoint $A, B \subseteq V(G)$ such that every vertex in $A$ has at least $x|B|$ nonneighbours in $B$. Let $a, b > 0$, and let $c := (a+2)|H| + b - 1$. Then for every $g \in V(H)$, one of the following holds:*

- *there exists $v \in A$ with $\operatorname{ind}_{H\setminus g}(G[B_v]) < x^b|B_v|^{|H|-1}$ where $B_v$ is the nonneighbourhood of $v$ in $B$;*
- *$\operatorname{ind}_H(G) \ge x^c|A||B|^{|H|-1}$; and*
- *there are $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \ge x^c|A|$ and $|B'| \ge x^c|B|$ such that $G$ has at most $x^a|A'||B'|$ edges between $A'$ and $B'$.*

**Proof.** Assume that the first and the third outcomes do not hold; it suffices to prove the second outcome. To this end, let $h := |H|$, let $\{g_1, \dots, g_k\}$ be the neighbours of $g$ in $H$, and for $0 \le i \le k$, let $H_i$ be the graph obtained from $H$ by removing the edges $gg_{i+1}, gg_{i+2}, \dots, gg_k$. Then $H_k = H$ and $H_0$ is the graph obtained from $H \setminus g$ by adding an isolated vertex. For every $i$, a copy $\varphi$ of $H_i$ in $G$ is *split* if $\varphi(g) \in A$ and $\varphi(g') \in B$ for all $g' \in V(H \setminus g)$. Let $d := a + 1$; then $c = (d+1)h + b - 1 \ge kd + b + h - 1$. We shall prove the following claim by induction.

**Claim 3.6.** *For every $0 \le i \le k$, there are at least $x^{id+b+h-1}|A||B|^{h-1}$ split copies of $H_i$ in $G$.*

*Subproof.* Since the first out come of the lemma does not hold, for every $v \in A$ we have

$$\operatorname{ind}_{H\setminus g}(G[B_v]) \ge x^b|B_v|^{h-1} \ge x^b(x|B|)^{h-1} = x^{b+h-1}|B|^{h-1}.$$

Thus there are at least $x^{b+h-1}|A||B|^{h-1}$ split copies of $H_0$ in $G$, proving the base case $i = 0$.

Now, assuming that the claim holds for $i$, we prove it for $i+1$. Put $y := \frac{1}{2}x^{id+b+h-1}$; then by induction there are at least $2y|A||B|^{h-1}$ split copies of $H_i$ in $G$. Let $H_i' := H \setminus \{g, g_{i+1}\}$; and for every copy $\varphi'$ of $H_i'$ in $G[B]$, let $S_{\varphi'}$ be the set of split copies $\varphi$ of $H_i$ in $G$ with $\varphi|_{V(H_i')} = \varphi'$. Let $T$ be the set of copies $\varphi'$ of $H_i'$ in $G[B]$ with $|S_{\varphi'}| \ge y|A||B|$. Since there are at most $|B|^{h-2}$ copies of $H_i'$ in $G[B]$, there are at most $y|A||B|^{h-1}$ split copies $\varphi$ of $H_i$ in $G$ with $\varphi|_{V(H_i')} \notin T$; so there are at least $y|A||B|^{h-1}$ split copies $\varphi$ of $H_i$ in $G$ with $\varphi|_{V(H_i')} \in T$. Hence

$$\sum_{\varphi' \in T} |S_{\varphi'}| \ge y|A||B|^{h-1}.$$

Fix $\varphi' \in T$. Let $A'$ be the set of vertices $v \in A$ for which there exists $\varphi \in S_{\varphi'}$ with $\varphi(g) = v$; and let $B'$ be the set of vertices $u \in B$ for which there exists $\varphi \in S_{\varphi'}$ with $\varphi(g_{i+1}) = u$. Then $|S_{\varphi'}|$ counts the number of nonedges of $G$ between $A'$ and $B'$, and so $|A'||B'| \ge |S_{\varphi'}| \ge y|A||B|$ which gives $|A'| \ge y|A|$ and $|B'| \ge y|B|$. Thus, since (note that $x < \frac{1}{2}$)

$$y = \tfrac{1}{2}x^{id+b+h-1} \ge x^{id+b+h} \ge x^c$$

and the third outcome of the lemma does not hold, $G$ has at least $x^a|A'||B'|$ edges between $A'$ and $B'$. Therefore, by definition, there are at least $x^a|A'||B'| \ge x^a|S_{\varphi'}|$ split copies $\varphi$ of $H_{i+1}$ in $G$ with $\varphi|_{V(H_i')} = \varphi'$. Since this holds for every $\varphi' \in T$, the number of split copies of $H_{i+1}$ in $G$ is thus at least

$$\sum_{\varphi' \in T} x^a|S_{\varphi'}| \ge x^a y|A||B|^{h-1} \ge x^a x^{id+b+h}|A||B|^{h-1} = x^{(i+1)d+b+h-1}|A||B|^{h-1},$$

completing the induction step. This proves Claim 3.6. □

Now, by Claim 3.6 with $i = k$, there are at least $x^{kd+b+h-1}|A||B|^{h-1} \ge x^c|A||B|^{h-1}$ split copies of $H_k = H$ in $G$. This proves Lemma 3.5. ■

Here is a refinement of Lemma 3.5 that strengthens Theorem 2.6 (see Exercise 3.1).

**Lemma 3.7.** *Let $H$ be a graph, and let $x \in (0, \frac{1}{2})$. Let $G$ be a graph with nonempty disjoint $A, B \subseteq V(G)$ such that every vertex in $A$ has at least $x|B|$ nonneighbours in $B$. Let $a, b > 0$, and let $c := (a+3)|H|+b$. Then for every $g \in V(H)$, one of the following holds:*

- *there exists $v \in A$ with $\operatorname{ind}_{H\setminus g}(G[B_v]) < x^b|B_v|^{|H|-1}$ where $B_v$ is the nonneighbourhood of $v$ in $B$;*
- *$\operatorname{ind}_H(G) \ge x^c|A||B|^{|H|-1}$; and*
- *there are $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \ge x^c|A|$ and $|B'| \ge x^c|B|$ such that $A'$ is $x^a$-sparse to $B'$.*

**Proof.** We may assume the first two outcomes do not hold; then Lemma 3.5 gives $A_0 \subseteq A$ and $B \subseteq B$ with $|A_0| \ge x^{c-1}|A|$ and $|B_0| \ge x^{c-1}|B|$ such that $G$ has at most $x^{a+1}|A_0||B'|$ edges between $A_0$ and $B'$. Let $A'$ be the set of vertices in $A_0$ with at most $x^a|B'|$ neighbours in $B'$; then $|A'| \ge (1-x)|A_0| \ge x|A_0| \ge x^c|A|$. Thus the third bullet holds. This proves Lemma 3.7. ■

For graphs $G, H$ with $V(H) = \{g_1, \dots, g_h\}$ and $x > 0$, an *$x$-blowup* of $H$ in $G$ is a sequence $(A_1, \dots, A_h)$ of nonempty disjoint subsets of $V(G)$ such that for all $i, j \in [h]$ with $i < j$,

- if $g_ig_j \in E(H)$, then $A_j$ is $x$-dense to $A_i$; and
- if $g_ig_j \notin E(H)$, then $A_j$ is $x$-sparse to $A_i$.

**Lemma 3.8.** *If $x > 0$ and there is an $x$-blowup $(A_1, \dots, A_h)$ of $H$ in $G$, then there are at least $(1 - \binom{h}{2}x)|A_1| \cdots |A_h|$ copies $\varphi$ of $H$ in $G$ with $\varphi(g_i) \in A_i$ for all $i \in [h]$.*

**Proof.** For each $i \in [h]$, let $v_i$ be a uniformly random element of $A_i$, such that $v_1, \dots, v_h$ are chosen independently. Since $(A_1, \dots, A_h)$ is an $x$-blowup of $H$, for every distinct $i, j \in [h]$, $v_i, v_j$ have the same adjacency type in $G$ as $g_i, g_j$ in $H$ with probability at least $1 - x$. Hence by a union bound,

$\varphi\colon V(H) \to V(G)$ defined by $\varphi(g_i) := v_i$ for all $i \in [h]$ is a copy of $H$ in $G$ with probability at least $1 - \binom{h}{2}x$. This proves Lemma 3.8. ■

**Lemma 3.9.** *Let $H$ be a graph with $V(H) = \{g_1, \ldots, g_h\}$. Then for each $b > 0$, there exists $d > 2$ such that for every graph $G$ and every $x \in (0, h^{-2})$, one of the following holds:*

- *$\operatorname{ind}_{H\setminus g_h}(G[B']) < x^b|B'|^{h-1}$ for some $B' \subseteq V(G)$ with $|B'| \geq x^c|G|$;*
- *$\operatorname{ind}_H(G) \geq (x^c|G|)^h$; and*
- *there are disjoint $A, B \subseteq V(G)$ with $|A| \geq \lfloor x^c|G| \rfloor$ and $|B| \geq \frac{1-hx}{h-1}|G|$ such that $B$ is $x$-sparse or $x$-dense to $A$.*

**Proof.** Let $a_h := 1$; and for $k = h, h-1, \ldots, 1$ in turn, let $c_k := (a_k + 3)b + h$ and $a_{k-1} := a_k + c_k$. Let $d_1 := 2$; and for $k = 2, 3, \ldots, h$ in turn, let $d_k := \max(c_k + d_{k-1}, (k-1)c_k + 2)$. Finally, let $c := d_h + 3$.

Now, let $G$ be a graph, and let $x \in (0, h^{-2})$. If $x^c|G| < 1$ then the third bullet trivially holds; so we may assume $x^c|G| \geq 1$, in particular $|G| > 1/x$. Suppose for the sake of contradiction that all three bullets do not hold. Let $H_k := H[\{g_1, \ldots, g_k\}]$ for all $k \in [h]$; we prove by induction the following claim.

**Claim 3.10.** *For every $k \in [h]$, $G$ contains an $x^{a_k}$-blowup $(A_1, \ldots, A_k)$ of $H_k$ with $x^{d_k}|G| \leq |A_i| \leq x|G|$ for all $i \in [k]$.*

*Subproof.* The case $k = 1$ holds since $\lfloor x|G| \rfloor \geq \frac{1}{2}x|G| \geq x^2|G|$. For $k \leq h$, assume that $G$ contains an $x^{a_{k-1}}$-blowup $(A_1, \ldots, A_{k-1})$ of $H_{k-1}$ with $x^{d_{k-1}}|G| \leq |A_i| \leq x|G|$ for all $i \in [k-1]$. Put $A := A_1 \cup \cdots \cup A_{k-1}$; then $|A| \leq (k-1)x|G|$. For every $i \in [k-1]$, let $C_i$ be the set of vertices in $V(G) \setminus A$ such that

- if $g_ig_k \in E(H)$, then every vertex in $C_i$ has fewer than $x|A_i|$ neighbours in $A_i$; and
- if $g_ig_k \notin E(H)$, then every vertex in $C_i$ has fewer than $x|A_i|$ nonneighbours in $A_i$.

Then $|C_i| < \frac{1-hx}{h-1}|G| \leq \frac{1-kx}{k-1}|G|$ for every $i \in [k-1]$ by the hypothesis. It follows that

$$|G| - |A| - (|C_1| + \cdots + |C_{k-1}|) > |G| - (k-1)x|G| - (1-kx)|G| = x|G|,$$

and so there exists $B \subseteq (V(G) \setminus A) \setminus (C_1 \cup \cdots \cup C_{k-1})$ with $|B| = \lfloor x|G| \rfloor \geq \frac{1}{2}x|G| > x^2|G|$. Note that

- if $g_ig_k \in E(H)$, then every vertex in $B$ has at least $x|A_i|$ neighbours in $A_i$; and
- if $g_ig_k \notin E(H)$, then every vertex in $B$ has at least $x|A_i|$ nonneighbours in $A_i$.

Let $B_1 := B$. For $i = 2, 3, \ldots, k$ in turn, assume that we are given $B_{i-1} \subseteq B$ with

$$|B_{i-1}| \geq x^{(i-2)c_k}|B| \geq x^{(k-1)c_k+2}|G| \geq x^{d_k}|G| \geq x^{c-1}|G|.$$

In particular, there is no $B' \subseteq B_{i-1}$ with $|B'| \geq x|B_{i-1}|$ and $\operatorname{ind}_{H\setminus g_h}(G[B']) < x^b|B'|^{h-1}$. Since $x|A_i| \geq x^{d_{k-1}+1}|G| \geq x^{c-1}|G|$ and

$$\operatorname{ind}_H(G) < (x^c|G|)^h = x^c(x^{c-1}|G|)^h \leq x^{d_{k-1}}|A_i|^{h-1}|B_{i-1}|,$$

Lemma 3.7 with $(a, b) = (a_k, b)$ gives $A_i' \subseteq A_i$ and $B_i \subseteq B_{i-1}$ with $|A_i'| \geq x^{c_k}|A_i|$ and $|B_i| \geq x^{c_k}|B_{i-1}|$ such that

- if $g_ig_k \in E(H)$, then every vertex in $B_i$ has at most $x^{a_k}|A_i'|$ nonneighbours in $A_i'$; and
- if $g_ig_k \notin E(H)$, then every vertex in $B_i$ has at most $x^{a_k}|A_i'|$ neighbours in $A_i'$.

For every $i \in [k-1]$, we have

$$|A_i'| \geq x^{c_k}|A_i| \geq x^{c_k+d_{k-1}}|G| \geq x^{d_k}|G|.$$

Let $A_k' := B_k$; then since $x|G| \geq |B| > x^2|G|$ and $|B_i| \geq x^{(i-1)c_k}|B_1| = x^{(i-1)c_k}|B|$ for all $i \in [k]$, we have

$$x|G| \geq |B| \geq |A_k'| = |B_k| \geq x^{(k-1)c_k}|B| > x^{(k-1)c_k+2}|G| \geq x^{d_k}|G|.$$

For all $i, j \in [k-1]$ with $i < j$, since $x^{a_{k-1}}|A_i| \leq x^{a_{k-1}-c_k}|A_i| = x^{a_k}|A_i'|$ and $(A_1, \ldots, A_{k-1})$ is an $x^{a_{k-1}}$-blowup of $H_{k-1}$,

- if $g_ig_j \in E(H)$, then every vertex in $A'_j$ has at most $x^{a_{k-1}}|A_i| \le x^{a_k}|A'_i|$ nonneighbours in $A'_i$; and
- if $g_ig_j \notin E(H)$, then every vertex in $A'_j$ has at most $x^{a_{k-1}}|A_i| \le x^{a_k}|A'_i|$ neighbours in $A'_i$.

Therefore $(A'_1, \dots, A'_{k-1}, A'_k)$ is an $x^{a_k}$-blowup of $H_k$ in $G$ with $x^{d_k}|G| \le |A'_i| \le x|G|$ for all $i \in [k]$, completing the induction step. This proves Claim 3.10. □

Now, Claim 3.10 with $k = h$ yields an $x$-blowup $(A_1, \dots, A_h)$ of $H$ in $G$ with $|A_i| \ge x^{d_k+2}|G|$ for all $i \in [h]$. Thus, since $1 - \binom{h}{2}x \ge \frac{1}{2} > x$ for all $i \in [h-1]$ (recall that $x \in (0, h^{-2})$), Lemma 3.8 implies that

$$\operatorname{ind}_H(G) \ge x \cdot (x^{d_h+2}|G|)^h \ge (x^{d_h+3}|G|)^h = (x^c|G|)^h,$$

a contradiction. This proves Lemma 3.9. ■

For $x > 0$, an *$x$-threshold blockade* in a graph $G$ is a blockade $(B_1, \dots, B_\ell)$ in $G$ such that for every $i \in [\ell]$, $B_{i+1} \cup \dots \cup B_\ell$ is $x$-sparse or $x$-dense to $B_i$. (This is related to the notion of *threshold graphs*, where a graph $F$ is *threshold* if every induced subgraph $J$ of $F$ contains $v \in V(J)$ that is complete or anticomplete to $V(J) \setminus \{v\}$.) We now iterate Lemma 3.9 to obtained the following "symmetric" result.

**Lemma 3.11.** *For every graph $H$ with $h = |H| \ge 1$, there exists $d \ge 2$ such that for every graph $G$ and every $x \in (0, h^{-2})$, if $\operatorname{ind}_H(G) < (x^d|G|)^h$ then $G$ contains an $x$-threshold $(\log \frac{1}{x}, \lfloor x^d|G| \rfloor)$-blockade.*

**Proof.** We proceed by induction on $h$. For $h = 1$ we can choose $d = 1$; so let us assume $h \ge 2$. Let $g \in V(H)$, and let $d' > 0$ be chosen for $H \setminus g$ by induction. Let $c > 0$ be chosen from Lemma 3.9 with $b = d'(h-1)$, let $s := \log h$, and let $d := d' + c + s$. We claim that $d$ satisfies the theorem.

To this end, let $x \in (0, h^{-2})$, and let $G$ be a graph with $\operatorname{ind}_H(G) < (x^d|G|)^h$. If $x^d|G| < 1$ then we are done since $G$ contains an $x$-threshold blockade of any given length and width at least zero. Thus we may assume $x^d|G| \ge 1$. Let $\ell \ge 0$ be maximal such that $G$ contains an $x$-threshold blockade $(B_1, \dots, B_{\ell-1}, B_\ell)$ with $|B_i| \ge \lfloor x^d|G| \rfloor$ for all $i \in [\ell-1]$ and $|B_\ell| \ge h^{1-\ell}|G|$. If $\ell - 1 \ge \log(1/x)$ then we are done; so we may assume $\ell - 1 < \log(1/x)$. Then

$$|B_\ell| \ge h^{1-\ell}|G| > h^{-\log(1/x)}|G| = x^s|G|.$$

By Lemma 3.9, one of the following holds:

- $\operatorname{ind}_{H\setminus g}(G[B']) < x^b|B'|^{h-1} = (x^{d'}|B'|)^{h-1}$ for some $B' \subseteq B_\ell$ with $|B'| \ge x^c|B_\ell| \ge x^{c+s}|G|$;
- $\operatorname{ind}_H(G) \ge (x^c|G|)^h \ge (x^d|G|)^h$; and
- there are disjoint $A, B \subseteq B_\ell$ with $|A| \ge \lfloor x^c|B_\ell| \rfloor$ and $|B| \ge \frac{1-hx}{h-1}|B_\ell|$ such that either every vertex in $B$ has at most $x|A|$ neighbours in $A$ or every vertex in $B$ has at most $x|A|$ nonneighbours in $A$.

The second bullet cannot happen since $\operatorname{ind}_H(G) < (x^d|G|)^h$; and the third bullet cannot happen by the maximality of $\ell$ and

$$|A| \ge \lfloor x^c|B_\ell| \rfloor \ge \lfloor x^{c+s}|G| \rfloor \ge \lfloor x^d|G| \rfloor,$$
$$|B| \ge \frac{1-hx}{h-1}|B_\ell| \ge h^{-\ell}|G|,$$

where the last inequality holds since $\frac{1-hx}{h-1} > \frac{1}{h}$ (note that $x \in (0, h^{-2})$). Thus the first bullet holds, and so the choice of $d'$ gives an $x$-threshold blockade in $G[B']$ of length at least $\log(1/x)$ and width at least $\lfloor x^{d'}|B'| \rfloor \ge \lfloor x^{d'+c+s}|G| \rfloor = \lfloor x^d|G| \rfloor$. This completes the induction step, proving Lemma 3.11. ■

It is now not hard to derive Theorem 3.4 from Lemma 3.11 by noting that every $n$-vertex threshold graph has a clique or stable set of size at least $\frac{1}{2}n$.

**Exercise 3.1.** *Deduce Theorem 2.6 from Lemma 3.7.*

## 4. Graphs of bounded VC-dimension and ultra-strong regularity

For a family $\mathcal{F}$ of subsets of a ground set $V$, a subset $S \subseteq V$ is *shattered* by $\mathcal{F}$ if for every $A \subseteq S$ there exists $F \in \mathcal{F}$ with $A = F \cap S$. The *VC-dimension* of $\mathcal{F}$ (with respect to $V$) is the largest size of a subset of $V$ shattered by $\mathcal{F}$. For a graph $G$, its *VC-dimension* is the VC-dimension of the family $\{N_G(v) : v \in V(G)\}$ with respect to $V(G)$. We start with the following simple characterization of graphs of bounded VC-dimension; and in what follows, a *bigraph* is a bipartite graph $H$ with a fixed bipartition.

**Lemma 4.1.** *For every bigraph $H$, there exists $d \geq 1$ such that every graph of VC-dimension at least $d$ contains a bi-induced copy of $H$. Conversely, for every $d \geq 1$, there is a bigraph $H$ such that every graph with a bi-induced copy of $H$ has VC-dimension at least $d$.*

**Proof.** The second statement is trivial. To prove the first statement, it suffices to show that for every bigraph $H$ with bipartition $(X, Y)$ where $|X| = |Y| =: k$, there is a bi-induced copy of $H$ in every graph $G$ of VC-dimension at least $k + \log(2k)$. To see this, let $S \subseteq V(G)$ with $|S| \geq k + \log(2k)$ such that $S$ is shattered by $\{N_G(v) : v \in V(G)\}$. Let $A \cup B$ be a partition of $S$ with $|A| = k$ and $|B| \geq \log(2k)$; then there are distinct subsets $B_1, \ldots, B_{2k}$ of $B$. Let $\psi$ be a bijection from $X$ to $A$. For every $u \in Y$ and every $i \in [2k]$, there exists $v_i^u \in V(G)$ such that $N_G(v_i^u) \cap S = \psi(N_H(u)) \cup B_i$; then $v_1^u, \ldots, v_{2k}^u$ are distinct and so at least $k$ of them are not in $A$, say $v_1^u, \ldots, v_k^u \notin A$. Now it is not hard to greedily construct an injective map $\phi\colon Y \to V(G) \setminus A$ such that $\phi(u) \in \{v_1^u, \ldots, v_k^u\}$ for all $u \in Y$. Then $\psi$ and $\phi$ together give a bi-induced copy of $H$ in $G$, as desired. This proves Lemma 4.1. ■

For two bigraphs $F$ with bipartition $(A, B)$ and $H$ with bipartition $(X, Y)$, say that $(F, A, B)$ *bi-contains* $(H, X, Y)$ if there is a bi-induced copy $\varphi$ of $H$ in $F$ with $\varphi(X) \subseteq A, \varphi(Y) \subseteq B$. To refine Lemma 4.1 into a "proper" forbidden induced subgraphs characterization of graphs of bounded VC-dimension, we need the following two results, the first of which was proved by Erdős–Hajnal–Pach [13] and the second of which results from a simple random bipartite graph argument.

**Theorem 4.2** (Erdős–Hajnal–Pach)**.** *For every bigraph $H$ with bipartition $(X, Y)$, there exists $a \geq 1$ such that for every bigraph $F$ with bipartition $(A, B)$ where $|A|, |B| \geq 2^a$ and $(F, A, B)$ does not bi-contain $(H, X, Y)$, there is a pure pair $(C, D)$ in $F$ with $C \subseteq A$, $D \subseteq B$, $|C|^a \geq |A|$, and $|D|^a \geq |B|$.*

**Lemma 4.3.** *For every integer $n \geq 2$, there is a bigraph $F$ with bipartition $(A, B)$ such that $|A| = |B| = n$ and there is no pure pair $(C, D)$ in $F$ with $C \subseteq A$, $D \subseteq B$, and $|C|, |D| \geq 2 \log n$.*

**Exercise 4.1.** *Prove Theorem 4.2 and Lemma 4.3.*

A *split* graph is a graph whose vertex set is the disjoint union of a clique and a stable set. Via an application of the Erdős–Hajnal theorem 2.1, a graph missing a bi-induced copy of a bigraph if and only if it avoids a bipartite graph, the complement of a bipartite graph, and a split graph as induced subgraphs. The following proof can also be found in [3].

**Lemma 4.4.** *For every two bipartite graphs $H_1, H_2$ and every split graph $J$, there is a bigraph $F$ such that there is no bi-induced copy of $F$ in any $\{H_1, \overline{H_2}, J\}$-free graph.*

**Proof.** Let $c > 0$ be given by Theorem 2.1 with $H = H_1$. Let $(X_1, Y_1)$ be a bipartition of $H_1$, $(X_2, Y_2)$ be a bipartition of $H_2$, and $U \cup V$ be a partition of $V(J)$ such that $U$ is a clique in $J$ and $V$ is stable in $J$. Let $H_2'$ be the bigraph obtained from $\overline{H_2}$ by making $X_2, Y_2$ stable, and let $J'$ be the bigraph obtained from $J$ by making $U$ stable. By Theorem 4.2, there exists $a \geq 1$ such that for every bigraph $F$ with bipartition $(A, B)$ for which $|A|, |B| \geq 2^a$ and $(F, A, B)$ does not bi-contain at least one of $(H_1, X_1, Y_1)$, $(H_2', X_2, Y_2)$, $(J', U, V)$, and $(J', V, U)$, there is a pure pair $(C, D)$ in $F$ with $C \subseteq A$, $D \subseteq B$, $|C|^a \geq |A|$, and $|D|^a \geq |B|$.

Now, let $n \ge 1$ be such that $2^{c\sqrt{\log n}} \ge (2\log n)^a$, and let $F$ be a bigraph with bipartition $(A, B)$ and $|A| = |B| = n$ given by Lemma 4.3; we claim that $F$ satisfies the lemma. To this end, let $G$ be a $\{H_1, \overline{H_2}, J\}$-free graph; and suppose that there is a bi-induced copy of $F$ in $G$. Thus we may assume $V(F) \subseteq V(G)$. By the choice of $c$, since each of $G[A], G[B]$ is $H_1$-free, there are $P \subseteq A, Q \subseteq B$ with $|P|, |Q| \ge 2^{c\sqrt{\log n}} \ge (2\log n)^a \ge 2^a$ such that each of $P, Q$ is a clique or stable set in $G$. Let $F' := F[P \cup Q]$; then since $G$ is $\{H_1, \overline{H_2}, J\}$-free, $(F', P, Q)$ does not bi-contain at least one of $(H_1, X_1, Y_1)$, $(H_2', X_2, Y_2)$, $(J', U, V)$, and $(J', V, U)$. Hence, by the choice of $a$, there is a pure pair $(C, D)$ in $F'$ (and so in $F$ and $G$) with $C \subseteq P$, $D \subseteq Q$, $|C|^a \ge |P|$, and $|D|^a \ge |Q|$; but then $|C|^a, |D|^a \ge 2^{c\sqrt{\log n}} \ge (2\log n)^a$ which yields $|C|, |D| \ge 2\log n$, contrary the choice of $F$. This proves Lemma 4.4. ■

By combing Lemmas 4.1 and 4.4, we obtain the following forbidden induced subgraphs characterization of graphs of bounded VC-dimension.

**Lemma 4.5.** *For every bigraphs $H_1, H_2$ and every split graph $J$, there exists $d \ge 1$ such that every $\{H_1, \overline{H_2}, J\}$-free graph has VC-dimension at most $d$. Conversely, for every $d \ge 1$, there are bigraphs $H_1, H_2$ and a split graph $J$ such that every graph of VC-dimension at most $d$ is $\{H_1, \overline{H_2}, J\}$-free.*

For $\varepsilon > 0$ and a graph $G$, an *ultra-strong $\varepsilon$-partition* of $G$ is an equipartition $(V_1, \dots, V_L)$ of $G$ such that all but at most an $\varepsilon$ fraction of the pairs $(V_i, V_j)$ are $\varepsilon$-pure. The rest of this section deals with the following partitioning result of Lovász–Szegedy [21] for graphs of bounded VC-dimension, which significantly strengthens the regularity lemma for these graphs.

**Theorem 4.6** (Lovász–Szegedy)**.** *For every $d \ge 1$, there exists $K \ge 2$ such that for every $\varepsilon \in (0, \frac{1}{2})$, every graph $G$ of VC-dimension at most $d$ admits an ultra-strong $\varepsilon$-regular partition into $L$ parts for some integer $L \in [\varepsilon^{-1}, \varepsilon^{-K}]$.*

We shall present a proof of this result by Fox–Pach–Suk [16]. First, we require the well-known Sauer–Shelah–Perles lemma [28, 29].

**Theorem 4.7** (Sauer–Shelah–Perles)**.** *Let $\mathcal{F}$ be a family of subsets of VC-dimension $d \ge 1$ of a ground set $V$. Then for every $S \subseteq V$,*

$$|\{F \cap S : F \in \mathcal{F}\}| \le \sum_{j=0}^{d} \binom{|S|}{j} \le 2|S|^d.$$

Next we require a simple application of Theorem 4.7, which is a weaker version (with an additional polylog factor) of Haussler's packing lemma [19].

**Lemma 4.8.** *For every $d \ge 1$, there exists $C_d \ge 1$ for which the following holds for every $q \ge 1$. Let $\mathcal{F}$ be a family of subsets of VC-dimension at most $d$ of a ground set $V$, such that $|A \Delta B| \ge |V|/q$ for all distinct $A, B \in \mathcal{F}$. Then $|\mathcal{F}| \le C_d (q \ln q)^d \le C_d q^{2d}$.*

**Proof.** Let $k := \lceil 2q \ln|\mathcal{F}| \rceil \le 4q \ln|\mathcal{F}|$. Let $v_1, \dots, v_k$ be uniformly random elements of $V$ chosen independently with repetition. For every two distinct $A, B \in \mathcal{F}$, the probability that $A \Delta B$ contains none of $v_1, \dots, v_k$ is at most

$$(1 - 1/q)^k \le e^{-k/q} \le |\mathcal{F}|^{-2}.$$

Thus, with positive probability, $A \Delta B$ contains at least one of $v_1, \dots, v_k$ for all distinct $A, B \in \mathcal{F}$. Hence there exists $S \subseteq V$ with $|S| \le k$ such that $(A \Delta B) \cap S \ne \emptyset$ for all distinct $A, B \in \mathcal{F}$; and so

$$|\mathcal{F}| = |\{F \cap S : F \in \mathcal{F}\}| \le 2k^d \le 2^{2d+1}(q \ln|\mathcal{F}|)^d$$

where the second equation is due to Theorem 4.7. Thus there exists $C_d \ge 1$ such that

$$|\mathcal{F}| \le C_d (q \ln q)^d \le C_d q^{2d}$$

as was to be shown. This proves Lemma 4.8. ■

Intuitively, a bigraph with not too few edges and not too few nonedges should have one side with a decent number of vertices of not too small degree and antidegree. This intuition is made rigorous by the following lemma.

**Lemma 4.9.** *Let $G$ be a bigraph with bipartition $(A, B)$. Let $\varepsilon \in (0, 1)$, and assume that $G$ has at least $\varepsilon|A||B|$ and at most $(1-\varepsilon)|A||B|$ edges between $A, B$. Let $a$ denote the number of pairs $(e, e')$ such that $e \in E(G)$, $e' \notin E(G)$, and $e \cap e' \subseteq A$, and let $b$ denote the number of pairs $(e, e')$ such that $e \in E(G)$, $e' \notin E(G)$, and $e \cap e' \subseteq B$. Then $a|A|+b|B| \geq \varepsilon(1-\varepsilon)|A|^2|B|^2$ and so $a+b \geq \varepsilon(1-\varepsilon)|A||B|\min(|A|, |B|)$.*

**Proof.** Let $a_1, a_2$ be random vertices of $A$ chosen independently and uniformly with repetition, and let $b_1, b_2$ be random vertices of $B$ chosen independently and uniformly with repetition. Then

$$\mathsf{P}[a_1b_1 \in E(G), a_2b_2 \notin E(G)] = (|A||B|)^{-1}|E(G)| \cdot (|A||B|)^{-1}|E(\overline{G})| \geq \varepsilon(1-\varepsilon).$$

Observe that if $a_1b_1 \in E(G)$ and $a_2b_2 \notin E(G)$ then either $a_1b_1 \in E(G)$ and $a_1b_2 \notin E(G)$, or $a_1b_2 \in E(G)$ and $a_2b_2 \notin E(G)$. Therefore, since $a = |A||B|^2 \cdot \mathsf{P}[a_1b_1 \in E(G), a_1b_2 \notin E(G)]$ and $b = |A|^2|B| \cdot \mathsf{P}[a_1b_2 \in E(G), a_2b_2 \notin E(G)]$, the proof of Lemma 4.9 is complete. ■

We are now ready to prove Theorem 4.6.

**Proof of Theorem 4.6 by Fox–Pach–Suk.** Let $C_d \geq 1$ be given by Lemma 4.8; we claim that $K := \log C_d + 18d + 5$ satisfies the theorem. To see this, let $G$ be a graph with VC-dimension at most $d$. We may assume $|G| > \varepsilon^{-K}$, for otherwise letting $L := \max(|G|, \lceil \varepsilon^{-1} \rceil)$ would work. In what follows, let $N(v) := N_G(v)$ for all $v \in V(G)$. Let $q := 2^7\varepsilon^{-2}$, and let $k \geq 1$ be maximal such that there exists $S = \{v_1, \dots, v_k\} \subseteq V(G)$ with $|N(v_i)\Delta N(v_j)| \geq |G|/q$ for all $i, j \in [k]$. Then $k = |S| \leq C_d q^{2d} = C_d 2^{14d}\varepsilon^{-4d}$ by the choice of $C_d$. By the maximality of $k$, there is a partition $(A_1, \dots, A_k)$ of $V(G)$ with $|N(v)\Delta N(v_i)| \leq |G|/q$ for all $i \in [k]$ and $v \in A_i$; then $|N(u)\Delta N(v)| \leq 2|G|/q$ for all $u, v \in A_i$. Let $L := \lceil 8k\varepsilon^{-1} \rceil$; then $\varepsilon^{-1} \leq L \leq 16k\varepsilon^{-1} \leq C_d 2^{14d+4}\varepsilon^{-4d-1} \leq \varepsilon^{-K}$ since $\varepsilon \in (0, \frac{1}{2})$ and by the choice of $K$. Let $(u_1, \dots, u_{|G|})$ be an ordering of $V(G)$ such that for all $1 \leq i < j \leq |G|$, if there exists $\ell \in [k]$ with $u_i, u_j \in A_\ell$ then $u_p \in A_\ell$ for all $p \in \{i, i+1, \dots, j\}$. Let $(V_1, \dots, V_L)$ be an equipartition of $V(G)$ such that for all $1 \leq i < j \leq |G|$, if there exists $\ell \in [L]$ with $u_i, u_j \in A_\ell$ then $u_p \in V_\ell$ for all $p \in \{i, i+1, \dots, j\}$. Let $I$ be the set of indices $\ell \in L$ for which $V_\ell \subseteq A_i$ for some $i \in [k]$, and let $J := [L] \setminus I$; then $|J| \leq k$. Thus there are at most $kL$ pairs $(V_i, V_j)$ that are not $\varepsilon$-pure and satisfy $j \in J$.

Now, let $Q$ denote the set of pairs $\{i, j\} \in \binom{I}{2}$ such that $(V_i, V_j)$ are not $\varepsilon$-pure. Let $R := \{(e, e') : e \in E(G), e \in E(\overline{G}), |e \cap e'| = 1\}$. By Lemma 4.9, for each $\{i, j\} \in Q$, since $|V_i|, |V_j| \geq \lfloor |G|/L \rfloor \geq \lceil |G|/L \rceil - 1 \geq \frac{1}{2}\lceil |G|/L \rceil$, there are at least

$$\varepsilon(1-\varepsilon)\lfloor |G|/L \rfloor^3 \geq 2^{-4}\varepsilon\lceil |G|/L \rceil^3$$

pairs $(e, e') \in R$ with $e, e' \in V_i \times V_j$. On the other hand, by the definition of $I$, for every $i \in I$ there are at most

$$(2|G|/q)\binom{\lceil |G|/L \rceil}{2} \leq (|G|/q)\lceil |G|/L \rceil^2$$

pairs $(e, e') \in R$ with $|e \cap V_i| = |e' \cap V_i| = 1$ and $e \cap e' \not\subseteq V_i$. Hence there are at most $L(|G|/q)\lceil |G|/L \rceil^2$ pairs $(e, e') \in R$ such that there exists $i \in I$ with $|e \cap V_i| = |e' \cap V_i| = 1$ and $e \cap e' \not\subseteq V_i$. It follows that

$$|Q| \cdot 2^{-4}\varepsilon\lceil |G|/L \rceil^3 \leq L(|G|/q)\lceil |G|/L \rceil^2$$

and so $|Q| \leq 2^4\varepsilon^{-1}L^2/q$. Therefore, since $q = 2^7\varepsilon^{-2}$ and $L = \lceil 8k\varepsilon^{-1} \rceil$, the number of pairs $\{i, j\} \in \binom{[L]}{2}$ such that $(V_i, V_j)$ are not $\varepsilon$-pure is at most

$$2^4\varepsilon^{-1}L^2/q + kL \leq (2^6\varepsilon^{-1}/q + 4k/L)\binom{L}{2} = \varepsilon\binom{L}{2}.$$

This proves Theorem 4.6. ■

**Exercise 4.2.** *Show that Theorem 4.6 is no longer true without the bounded VC-dimension assumption. More precisely, show that if $\mathcal{C}$ is a hereditary class of graphs such that for every $d \geq 1$ there is a graph in $\mathcal{C}$ of VC-dimension at least $d$, then for every $K \geq 2$ there exist $G \in \mathcal{C}$ and $\varepsilon \in (0, \frac{1}{2})$ so that $G$ admits no ultra-strong $\varepsilon$-regular partition into $L$ parts for any integer $L \in [\varepsilon^{-1}, \varepsilon^{-K}]$.*

**Theorem 4.10.** *For every bigraph $H$, there exists $K \geq 2$ such that for every $\varepsilon \in (0, \frac{1}{2})$ and every graph $G$ with no bi-induced copy of $H$, there exists $L \in [\varepsilon^{-1}, \varepsilon^{-K}]$ for which there is an equipartition $V(G) = V_1 \cup \cdots \cup V_L$ such that all but at most an $\varepsilon$ fraction of the pairs $(V_i, V_j)$ are $\varepsilon$-pure.*

We use Theorem 4.10 to obtain a blockade that is (strongly) $\varepsilon$-pure and large (more specifically, we want the length and width to have polynomial dependence on $\varepsilon$). This will be useful in the proof of the Erdős–Hajnal conjecture for graphs with bounded VC-dimension (Theorem 5.1).

**Theorem 4.11.** *For every bigraph $H$, there exists $b \geq 1$ such that for every $\varepsilon \in (0, \frac{1}{2})$ and every graph $G$ with $|G| \geq \varepsilon^{-b}$ and no bi-induced copy of $H$, there is an $(\varepsilon^{-1}, \varepsilon^{b}|G|)$-blockade $(B_1, \ldots, B_\ell)$ in $G$ such that*

- *$|B_1| = \cdots = |B_\ell| \leq \varepsilon^2|G|$; and*
- *for all distinct $i, j \in [\ell]$, $B_i, B_j$ are $\varepsilon$-sparse to each other in $G$ or $\overline{G}$.*

**Proof.** Let $K \geq 2$ be given by Theorem 4.10; we claim that $b := 5K$ satisfies the theorem. To this end, by Theorem 4.10 with $\varepsilon^4$ in place of $\varepsilon$, if $G$ is a graph with no bi-induced copy of $H$, then there is an equipartition $V(G) = V_1 \cup \cdots \cup V_L$ with $L \in [\varepsilon^{-4}, \varepsilon^{-4K}]$ such that all but at most an $\varepsilon^4$ fraction of the pairs $(V_i, V_j)$ are weakly $\varepsilon^4$-pure. By Turán's theorem, there exists $J \subseteq [L]$ with $|J| \geq \frac{1}{2}\varepsilon^{-4}$ such that $(V_i, V_j)$ is weakly $\varepsilon^4$-pure for all distinct $i, j \in J$. Then there exists $I \subseteq J$ with $|I| \geq \frac{1}{2}|J| \geq \frac{1}{4}\varepsilon^{-4} \geq \varepsilon^{-1}$ such that $|V_i| = |V_j|$ for all distinct $i, j \in I$. Let $\ell := \lceil \varepsilon^{-1} \rceil$; it follows that there exists $I \subseteq J$ with $I = [\ell]$ such that $|V_i| = |V_j|$ for all distinct $i, j \in I$. For every $i \in I$, let $B_i$ be the set of vertices $v$ in $V_i$ such that for every $j \in I \setminus \{i\}$,

- $v$ has at most $\frac{1}{2}\varepsilon|V_j|$ neighbours in $V_j$ if $(V_i, V_j)$ is weakly $\varepsilon^4$-sparse in $G$; and
- $v$ has at most $\frac{1}{2}\varepsilon|V_j|$ nonneighbours in $V_j$ if $(V_i, V_j)$ is weakly $\varepsilon^4$-sparse in $\overline{G}$.

Then
$$|B_i| \geq |V_i| - (\ell - 1) \cdot 2\varepsilon^3|V_i| \geq |V_i| - \varepsilon^{-1} \cdot 2\varepsilon^3|V_i| = (1 - 2\varepsilon^2)|V_i| \geq |V_i|/2$$
and by removing vertices if necessary we may assume that $|B_i| = \lceil |V_i|/2 \rceil = \lceil m/2 \rceil$ where $m = |V_i|$. It follows that for all distinct $i, j \in I$, $B_i, B_j$ are $\varepsilon$-sparse to each other in $G$ or $\overline{G}$. Also, since
$$m \geq \lfloor |G|/L \rfloor \geq |G|/(2L) \geq \varepsilon^{4K+1}|G|$$
(as $|G| \geq \varepsilon^{-b} = \varepsilon^{-5K}$), it follows that for each $i \in I$,
$$|B_i| \geq m/2 \geq \varepsilon^{4K+2}|G| \geq \varepsilon^{5K}|G| = \varepsilon^{b}|G|$$
and
$$|B_i| \leq m \leq \lceil |G|/L \rceil \leq 2|G|/L \leq 2\varepsilon^4|G| \leq \varepsilon^2|G|.$$
This proves Theorem 4.11. ■

## 5. Iterative sparsification and Erdős–Hajnal for graphs of bounded VC-dimension

The goal of this section is to prove the following theorem [25], which was conjectured independently by Fox–Pach–Suk [16] and Chernikov–Starchenko–Thomas [6].

**Theorem 5.1** (Nguyen–Scott–Seymour 2024+)**.** *For every $d \geq 1$, there exists $c > 0$ such that every graph $G$ of VC-dimension at most $d$ has a clique or stable set of size at least $|G|^c$.*

Via Lemma 4.1, Theorem 5.1 is equivalent to the following result.

**Theorem 5.2.** *For every bigraph $H$, there exists $c > 0$ such that every graph $G$ with no bi-induced copy of $H$ has a clique or stable set of size at least $|G|^c$.*

The proof of Theorem 5.2 uses the framework of *iterative sparsification*, which was introduced in [23, 24] and later also employed in [22]. Recall that a graph is *$\varepsilon$-restricted* if either the graph or its complement is $\varepsilon$-sparse. The goal is to find an $\varepsilon$-restricted induced subgraph of size at least $\operatorname{poly}(\varepsilon)|G|$. Rather than doing this in one step, we will instead attempt to move through a sequence of induced subgraphs that are successively more restricted: given a $y$-restricted induced subgraph $F$, we search for an induced subgraph that is $\operatorname{poly}(y)$-restricted and is at most a $\operatorname{poly}(y)$ factor smaller. Provided we can start the process, and it does not get stuck on the way, the following lemma shows that the process gives the required subset.

**Lemma 5.3.** *Let $c \in (0, 1)$, $a \geq 2$, and $t \geq 1$. Suppose that $x \in (0, c)$, and $G$ is a graph satisfying:*

- *there is a $c$-restricted induced subgraph of $G$ with at least $c^t|G|$ vertices; and*
- *for every $y \in [x, c]$ and every $y$-restricted induced subgraph $F$ of $G$ with $|F| \geq y^{2t}|G|$, there is a $y^a$-restricted induced subgraph of $F$ with at least $y^{at}|F|$ vertices.*

*Then $G$ contains an $x$-restricted induced subgraph with at least $x^{2at}|G|$ vertices.*

**Proof.** By the first condition of the lemma, there exists $y \in [x^a, c]$ minimal such that $G$ has a $y$-restricted induced subgraph $F$ with $|F| \geq y^{2t}|G|$. If $y \geq x$, then by the second condition of the lemma and since $a \geq 2$, $F$ has a $y^a$-restricted induced subgraph with at least $y^{at}|F| \geq y^{at+2t}|G| \geq y^{2at}|G|$ vertices; but this contradicts the minimality of $y$ since $x^a \leq y^a < y$. Thus $x^a \leq y < x$; and so $F$ is $x$-restricted, and $|F| \geq y^{2t}|G| \geq x^{2at}|G|$. This proves Lemma 5.3. ∎

We will find a subgraph satisfying the first bullet by using Rödl's theorem 1.3, with a suitable $t = t(c)$. However, finding a subgraph that satisfies the second bullet is more challenging, and we need to allow for an alternative "good" outcome. We will show in Lemma 5.8 that if we get stuck then we can instead find a large complete or anticomplete blockade (note that this is much stronger than being pure; and the blockades given by Theorem 4.11 are only $\varepsilon$-pure). Let us show that, if we can find sufficiently large complete or anticomplete blockades, then we can obtain the Erdős–Hajnal result (see [30] for an early version of this idea).

**Lemma 5.4.** *Let $\mathcal{C}$ be a hereditary class of graphs. Suppose that there exists $d \geq 2$ such that for every $x \in (0, 2^{-d})$ and every $G \in \mathcal{C}$, either:*

- *$G$ has an $x$-restricted induced subgraph with at least $x^d|G|$ vertices; or*
- *there is a complete or anticomplete $(k, |G|/k^d)$-blockade in $G$, for some $k \in [2, 1/x]$.*

*Then there exists $a \geq 2$ such that every $n$-vertex graph in $\mathcal{C}$ has a clique or stable set of size at least $n^{1/a}$.*

**Proof.** A *cograph* is a graph with no induced four-vertex path; and it is well-known that every $k$-vertex cograph has a clique or stable set of size at least $k^{1/2}$. Thus, it suffices to prove by induction that every $G \in \mathcal{C}$ contains an induced cograph of size at least $|G|^{1/(2d^2)}$. We may assume $|G| > 2^{2d^2}$. Let $x := |G|^{-1/(2d)} \in (0, 2^{-d})$. By the hypothesis, either:

- there exists $S \subseteq V(G)$ with $|S| \geq x^d|G|$ such that $G[S]$ is $x$-threshold; or

- there is a complete or anticomplete $(k, |G|/k^d)$-blockade in $G$, for some $k \in [2, 1/x]$.

If the first bullet holds, then since $|S| \geq x^d|G| \geq x^{-1}$, Turán's theorem gives a clique or stable set in $G[S]$ of size at least $(2x)^{-1} > x^{-1/2} = |G|^{1/(4d)}$. If the second bullet holds, then by induction and since $k \geq 2$, $G$ contains an induced cograph of size at least

$$k(|G|/k^d)^{1/(2d^2)} = k^{1-1/(2d)}|G|^{1/(2d^2)} > |G|^{1/(2d^2)}.$$

This proves Lemma 5.4. ∎

The key step in making the iterative sparsification strategy work is therefore to show that if the second bullet of Lemma 5.3 does not hold then we can find a sufficiently large complete or anticomplete blockade. We will show this in Lemma 5.7: given a $y$-restricted graph $F$ with no bi-induced $H$, we will prove that we can either pass to the desired poly$(y)$-restricted subgraph or find a complete or anticomplete blockade whose length and width depend polynomially on $y$. We will argue by induction on $|H|$, and grow the blockade one block at a time. The first step (Lemma 5.5) is to find a complete or anticomplete pair $(A, B)$, where $A$ has size poly$(y)|F|$ and $B$ contains all but a small fraction of the rest of $F$; we then (Lemma 5.7) repeat the argument inside $B$, continuing until we obtain a blockade that is long enough. Restricted graphs can either be dense or sparse: we will assume for the moment that our restricted graph is sparse, and handle the dense case later by taking complements.

**Lemma 5.5.** *Let $H$ be a bigraph, and let $v \in V(H)$. Let $b \geq 1$ be given by Theorem 4.11. Assume there exists $a \geq 2$ such that every $n$-vertex graph with no bi-induced copy of $H \setminus v$ contains a clique or stable set of size at least $n^{1/a}$. Let $y \in (0, 1/|H|)$, and let $F$ be a $y$-sparse graph with no bi-induced copy of $H$. Then either:*

- *$F$ has a $y^{2a}$-restricted induced subgraph with at least $y^{3ba^2}|F|$ vertices; or*
- *there is an anticomplete pair $(A, B)$ in $F$ with $|A| \geq y^{3ba^2}|F|$ and $|B| \geq (1-3y)|F|$.*

**Proof.** We have a sparse graph, and want to find an anticomplete pair $(A, B)$. We will do this by first using ultraregularity to find a large, nearly-pure blockade and then looking at how the rest of the graph attaches to it. So let $\varepsilon := y^{3a^2}$, and suppose that the first outcome does not hold; then $|F| > y^{-3ba^2} = \varepsilon^{-b}$. By Theorem 4.11, $F$ has a $(\varepsilon^{-1}, \varepsilon^b|F|)$-blockade $(B_1, \dots, B_\ell)$ with $\ell = \lceil \varepsilon^{-1} \rceil$, such that:

- $|B_1| = \cdots = |B_\ell| \leq 2\varepsilon^2|F|$; and
- for all distinct $i, j \in [\ell]$, $B_i, B_j$ are $\varepsilon$-sparse to each other in $F$ or $\overline{F}$.

Let $D := V(F) \setminus (B_1 \cup \cdots \cup B_\ell)$ and $m := |B_1|$. For $i \in [\ell]$, a vertex $v \in D$ is *mixed* on $B_i$ if it has a neighbour and a nonneighbour in $B_i$.

**Claim 5.6.** *Every vertex in $D$ is mixed on at most $y\ell$ of the blocks $B_1, \dots, B_\ell$.*

*Subproof.* Suppose there is a vertex $w \in D$ mixed on at least $y\ell$ of the blocks $B_1, \dots, B_\ell$, say $B_1, \dots, B_r$ where $r \geq y\ell \geq y\varepsilon^{-1} = y^{1-3a^2}$. Let $J$ be the graph with vertex set $[r]$ where for all distinct $i, j \in [r]$, $ij \notin E(J)$ if and only if $B_i, B_j$ are $\varepsilon$-sparse to each other in $F$.

We claim that there is no bi-induced copy of $H \setminus v$ in $J$. Suppose that there is; and we may assume $V(H \setminus v) \subseteq V(J)$. We assume that $v \in V_1(H)$ without loss of generality. For each $u \in V_2(H)$, let $w_u$ be a neighbour of $w$ in $B_u$ if $uv \in E(H)$ and a nonneighbour of $w$ in $B_u$ if $uv \notin E(H)$. For each $z \in V_1(H)\setminus\{v\}$ and $u \in V_2(H)$, since $uz \notin E(H)$ if and only if $uz \notin E(J)$ if and only if $B_u, B_z$ are $\varepsilon$-sparse to each other in $F$, $w_u$ is adjacent in $F$ to at most $\varepsilon|B_z|$ vertices in $B_z$ if $uz \notin E(H)$ and nonadjacent in $G$ to at most $\varepsilon|B_z|$ vertices in $B_z$ if $uz \in E(H)$. Thus, for each $z \in V_1(H) \setminus \{v\}$, there are at least (note that $\varepsilon \leq y < 1/|H|$)

$$|B_z| - |V_2(H)|\varepsilon|B_z| \geq |B_z| - |H|y|B_z| > 0$$

vertices $z' \in B_z$ such that for every $u \in Y$, $w_u z' \in E(F)$ if and only if $uz \in E(H)$; let $w_z$ be such a vertex. It follows that $\{w\} \cup \{w_z : z \in V_1(H) \setminus \{v\}\} \cup \{w_u : u \in Y\}$ forms a bi-induced copy of $H$ in $F$, a contradiction.

Thus, there is no bi-induced copy of $H \setminus v$ in $J$. By the choice of $a$, $J$ thus contains a clique or stable set $I$ with $|I| \ge |J|^{1/a} \ge (y^{1-3a^2})^{1/a} = y^{-3a+1/a}$. Let $S := \bigcup_{i\in I} B_i$; then $|S| = |I|m$ since $|B_i| = m$ for all $i \in I$. If $I$ is a stable set in $J$, then $F[S]$ has maximum degree at most

$$m + |I|\varepsilon m = (1/|I| + \varepsilon)|I|m \le (y^{3a-1/a} + y^{3a})|S| \le 2y^{3a-1}|S| \le y^{2a}|S|;$$

and similarly, if $I$ is a clique in $J$, then $\overline{F}[S]$ has maximum degree at most $y^{2a}|S|$. Thus $F[S]$ is $y^{2a}$-restricted which is the first outcome of the lemma, a contradiction. This proves the claim. □

Now, by the claim, there exists $i \in [\ell]$ such that there are at most $y|D|$ vertices in $D$ that are mixed on $B_i$. Since $F$ is $y$-sparse, there are at most $y|F|$ vertices in $D$ that are complete to $B_i$. Thus, since

$$|B_1| + \cdots + |B_\ell| \le \ell\varepsilon^2|F| \le 2\varepsilon|F| = 2y^{3a^2}|F| \le y|F|,$$

there are at least

$$|F| - y|D| - y|F| - (|B_1| + \cdots + |B_\ell|) \ge (1 - 3y)|F|$$

vertices in $F$ with no neighbour in $B_i$. Because $|B_i| \ge \varepsilon^b|F| = y^{3ba^2}|F|$, the second outcome of the lemma holds. This proves Lemma 5.5. ■

We now apply Lemma 5.5 repeatedly to move from an anticomplete pair to an anticomplete blockade.

**Lemma 5.7.** *Let $H$ be a bigraph, and let $v \in V(H)$. Let $b \ge 1$ be given by Theorem 4.11. Assume there exists $a \ge 2$ such that every $n$-vertex graph with no bi-induced copy of $H \setminus v$ contains a clique or stable set of size at least $n^{1/a}$. Let $0 < y \le 2^{-12|H|}$, and let $F$ be a $y$-sparse graph with no bi-induced copy of $H$. Then either:*

- *$F$ has a $y^a$-restricted induced subgraph with at least $y^{2ba^2}|F|$ vertices; or*
- *there is an anticomplete $(y^{-1/2}, y^{2ba^2}|F|)$-blockade in $F$.*

**Proof.** Suppose that the second outcome does not hold. Let $n \ge 0$ be maximal such that there is a blockade $(B_0, B_1, \ldots, B_n)$ in $F$ with $|B_n| \ge (1 - 3y^{1/2})^n|F|$ and $|B_{i-1}| \ge y^{2ba^2}|F|$ for all $i \in [n]$. Since the second outcome does not hold, $n < y^{-1/2}$; and so, since $y \le 2^{-12}$,

$$|B_n| \ge (1 - 3y^{1/2})^n|F| \ge 4^{-3y^{1/2}n}|F| \ge 4^{-3}|F| \ge y^{1/2}|F|.$$

Hence $F[B_n]$ has maximum degree at most $y|F| \le y^{1/2}|B_n|$; and so Lemma 5.5 (with $y^{1/2}$ in place of $y$, note that $y^{1/2} \le 2^{-6|H|} < 1/|H|$) implies that either:

- there exists $S \subseteq B_n$ with $|S| \ge y^{3ba^2/2}|B_n|$ such that $F[S]$ is $y^a$-restricted; or
- there is an anticomplete pair $(A, B)$ in $F[B_n]$ with $|A| \ge y^{3ba^2/2}|B_n|$ and $|B| \ge (1 - 3y^{1/2})|B_n|$.

If the second bullet holds, then $(B_0, B_1, \ldots, B_{n-1}, A, B)$ would be a blockade contradicting the maximality of $n$ since $|A| \ge y^{3ba^2/2}|B_n| \ge y^{3ba^2/2+1/2}|F| \ge y^{2ba^2}|F|$. Thus the first bullet holds; and so $S \subseteq V(F)$ is $y^a$-restricted in $F$ and satisfies $|S| \ge y^{3ba^2/2}|B_n| \ge y^{3ba^2/2+1/2}|G| \ge y^{2ba^2}|F|$. Hence the first outcome of the lemma holds. This proves Lemma 5.7. ■

For a bigraph $H$, its *bicomplement* is the bigraph $\overline{\overline{H}}$ with the same bipartition and edge set $\{uv : u \in V_1(H), v \in V_2(H), uv \notin E(H)\}$. We can now prove that the conditions of Lemma 5.4 are satisfied.

**Lemma 5.8.** *For every bigraph $H$, there exists $d \ge 2$ such that for every $x \in (0, 2^{-d})$ and every graph $G$ with no bi-induced copy of $H$, either:*

- *$G$ has an $x$-restricted induced subgraph with at least $x^d|G|$ vertices; or*
- *there is a complete or anticomplete $(k, |G|/k^d)$-blockade in $G$, for some $k \in [2, 1/x]$.*

**Proof.** We argue by induction on $|H|$. We may assume that $|H| \geq 2$. Choose $v \in V(H)$. By Lemma 5.4 and the induction hypothesis applied to $H \setminus v$, there exists $a \geq 4$ such that every $n$-vertex graph with no bi-induced copy of $H \setminus v$ contains a clique or stable set of size at least $n^{1/a}$. By taking complements, it follows that every $n$-vertex graph with no bi-induced copy of $\overline{\overline{H}} \setminus v$ contains a clique or stable set of size at least $n^{1/a}$. Let $c := 2^{-12|H|}$, and let $b \geq 1$ be given by Theorem 4.11. By Rödl's theorem 1.3, we can choose some $t \geq ba^2$ such that every graph $G$ with no bi-induced copy of $H$ contains a $c$-restricted induced subgraph with at least $c^t|G|$ vertices. We claim that $d := 2\max(at, |H|) \geq 8t$ satisfies the theorem.

To show this, let $x \in (0, 2^{-d}) \subseteq (0, c)$, and suppose that $G$ has no bi-induced copy of $H$. Suppose that the second outcome of the lemma does not hold; that is, there is no $k \in [2, 1/x]$ such that there is a complete or anticomplete $(k, |G|/k^d)$-blockade in $G$.

**Claim 5.9.** *For every $y \in [x, c]$ and every $y$-restricted induced subgraph $F$ of $G$ with $|F| \geq y^{2t}|G|$, there is a $y^a$-restricted induced subgraph of $F$ with at least $y^{at}|F|$ vertices.*

*Subproof.* We claim that either:

- $F$ has a $y^a$-restricted induced subgraph with at least $y^{2ba^2}|F| \geq y^{2t}|F| \geq y^{at}|F|$ vertices; or
- there is a complete or anticomplete $(y^{-1/2}, y^{2ba^2}|F|)$-blockade in $F$.

If $F$ is $y$-sparse, then one of the bullets holds by Lemma 5.7. If not, then $\overline{F}$ is $y$-sparse and $\overline{\overline{H}}$-free, and contains no bi-induced copy of $\overline{\overline{H}} \setminus v$, and so one of the bullets holds by Lemma 5.7 applied to $\overline{F}$.

If the second bullet holds, then since $y^{2ba^2}|F| \geq y^{4t}|G| \geq y^{d/2}|G|$ by the choice of $d$, there would be a complete or anticomplete $(y^{-1/2}, y^{d/2}|G|)$ blockade in $G$, which contradicts that the second outcome of the lemma does not hold (note that $y^{1/2} \leq c^{1/2} \leq \frac{1}{2}$). Thus the first bullet holds, proving the claim. □

Lemma 5.3 and the claim imply that $G$ has an $x$-restricted induced subgraph with at least $x^{2at}|G| \geq x^d|G|$ vertices, which is the first outcome of the theorem. This proves Lemma 5.8. ■

**Proof of Theorem 5.2.** Combine Lemmas 5.4 and 5.8. ■

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