OPEN PROBLEMS FOR THE 2024 BARBADOS GRAPH THEORY WORKSHOP

MAINTAINED BY TUNG H. NGUYEN

1. Maria Chudnovsky and Sepehr Hajebi

Given a graph G, let tree-$\alpha(G)$ be the smallest integer $s \geq 1$ for which G admits a tree decomposition where every bag has no stable set of size $s+1$. The following conjecture was posed by Dallard, Krnc, Kwon, Milanič, Munaro, Štorgel, and Wiederrecht:

Problem 1. *Show that for every pair of integers t, d there exists $c(t, d)$ such that every $\{K_{2,d}, P_t\}$-free graph has* tree-α *at most $c(t, d)$.*

A *three-path-configuration* is a graph consisting of three pairwise internally-disjoint paths the union of every two of which is an induced cycle of length at least four. A graph is *3PC-free* if no induced subgraph of it is a three-path-configuration.

Problem 2. *Is there a 3PC-free graph whose* tree-α *is super-logarithmic in the number of vertices?*

It is a recent result of Chudnovsky, Hajebi, Lokshtanov, and Spirkl that every n-vertex 3PC-free graph has tree-α at most $C(\log n)^2$, but the best known lower bound is $\log n$ (due to Sintiari and Trotignon). We would like to know if our upper bound is tight.

A graph class $\mathcal{C}$ is tw-*bounded* if there exists a function $f \colon \mathbb{N} \to \mathbb{N}$ such for every $G \in \mathcal{C}$, we have $\mathrm{tw}(G) \leq f(\omega(G))$ ($\mathrm{tw}(G)$ denotes the treewidth of G except the "-1" will be ignored in the definition).

If $\mathcal{C}$ is a hereditary class of bounded tree-α, then $\mathcal{C}$ is tw-bounded (by Ramsey). Dallad, Milanič, and Štorgel conjectured the converse:

Conjecture 1. *Let $\mathcal{C}$ be a hereditary class that is* tw*-bounded. Then $\mathcal{C}$ has bounded* tree-α*.*

This seems hard to believe. For instance, bounded tree-α has two significant consequences: if $\mathcal{C}$ is a hereditary class of bounded tree-α, then

1. Maximum independent set is polynomial-time solvable in $\mathcal{C}$.
2. $\mathcal{C}$ is *polynomially* tw-*bounded*, that is, there exists a polynomial $f \colon \mathbb{N} \to \mathbb{N}$ such that every graph $G \in \mathcal{C}$ has treewidth at most $f(\omega(G))$ (this is because every graph G satisfies $|V(G)| \leq \omega(G)^{\alpha(G)}$ or something.)

So there are a few weakenings of Conjecture 1, for a hereditary class $\mathcal{C}$:

Problem 3. *Is it true that if $\mathcal{C}$ is* tw*-bounded, then $\mathcal{C}$ is polynomially* tw*-bounded?*

Problem 4. *Is it true that if $\mathcal{C}$ is* tw*-bounded, then* Maximum independent set *is polynomial-time solvable in $\mathcal{C}$?*

Problem 5. *Is it true that if $\mathcal{C}$ is polynomially* tw*-bounded, then $\mathcal{C}$ has bounded* tree-α*?*

Problem 6. *Is it true that if $\mathcal{C}$ is polynomially* tw*-bounded, then* Maximum independent set *is polynomial-time solvable in $\mathcal{C}$?*

Note that Conjecture 1 would yield affirmative answers to Problems 3, 4 and 5, and a "yes" answer to Problem 4 in turn would give a positive answer to Problem 6. In Problems 5 and 6, what if $\mathcal{C}$ is "linearly" tw-bounded? Note that if $\mathrm{tw}(G) \leq \omega(G)$ (in which case we have $\mathrm{tw}(G) = \omega(G)$) for every $G \in \mathcal{C}$, then every graph in $\mathcal{C}$ is chordal and so admits a tree decomposition where each bag is a clique; thus tree-$\alpha(G) = 1$ for all $G \in \mathcal{C}$.

There also might be a connection with χ-boundedness. Note that for every graph G, $\mathrm{tw}(G) \leq$ tree-$\alpha(G)\chi(G)$. What about the converse? Specifically:

Problem 7. *Does there exist a triangle-free graph G with $\chi(G) > 11^{11^{11}}$ such that for every induced subgraph H of G, we have $\mathrm{tw}(H) \leq 2 \cdot$ tree-$\alpha(H)$?*

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2. Linda Cook, Clément Dallard, and Nicolas Trotignon

A *hole* is an induced cycle of length at least four. We say a connected graph G is *coconut-separable* if it has a clique cutset or it has two cliques K_1, K_2 such that $G \setminus (K_1 \cup K_2)$ is chordal. For each integer $\ell \geq 4$, an *ℓ-holed graph* is a graph whose induced cycles of length at least four have length exactly ℓ.

Problem 8. *Is every connected 5-holed graph coconut-separable?*

For each odd $\ell \geq 7$, as a consequence of a previous structure theorem for the class of ℓ-holed graphs, every ℓ-holed graph is coconut-separable. This is false for even ℓ (see Figure 1) so the only open case is $\ell = 5$. Trivially, chordal graphs are coconut-separable.

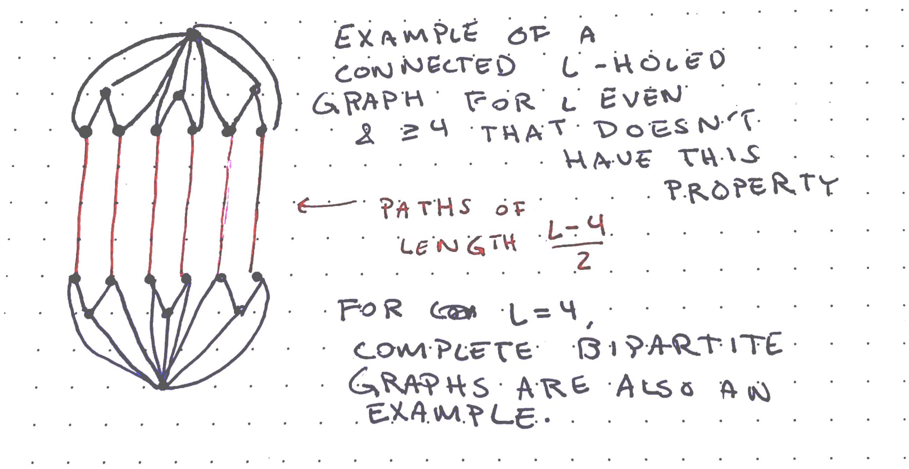

Figure 1. An example of an ℓ-holed graph that is not coconut-separable for each $\ell \geq 4$.

Layered-wheels (introduced by Sintiari and Trotignon) show that not every even-hole-free graph is coconut-separable.

The notion of coconut-separability is connected to the notion of tree-independence number. The *independence number* of a tree decomposition is the maximum of the independence numbers of its bags. The *tree independence number* of a graph G, denoted tree-$\alpha(G)$, is the minimum independence number of a tree decomposition of G. Graph classes with bounded tree-α have nice algorithmic properties, including the polynomial-time solvability of Maximum Weight Independent Set in these classes. For each $\ell \geq 7$, the structure theorem for the class of all ℓ-holed graphs implies that this class has bounded tree-α. On the other hand, 4-holed graphs have unbounded tree-α because of complete bipartite graphs. Whether 5-holed graphs and 6-holed graphs have bounded tree-α is still open. The former is a weakening of Problem 8.

3. Louis Esperet

A graph is said to be *quasi-transitive* if its vertex set has finitely many orbits under the action of its automorphism group (this means that there are only a finite number of essentially different vertices in the graph). Esperet, Giocanti, and Legrand-Duchesne raised the following questions.

Problem 9. *Is it true that any quasi-transitive graph of finite degree has a proper colouring with a finite number of colours such that the resulting coloured graph is still quasi-transitive (where we only consider colour-preserving automorphisms?)*

Problem 10. *Is it true that any quasi-transitive graph of finite degree has an orientation such that the resulting oriented graph is still quasi-transitive (again, where we only consider orientation-perserving automorphisms)?*

Roughly speaking, the main question is whether it is always possible to properly colour a highly symmetric infinite graph, in such a way that the resulting coloured graph is still highly symmetric. Similarly, is it always possible to orient the edges of any highly symmetric infinite graph, in such a way that the resulting oriented graph is still highly symmetric?

Progress.

- (Matthias Hamann and Rögnvaldur Möller) The answer to Problem 9 is no; a counterexample can be found by using infinite simple groups that are finitely generated.
- (Sergey Norin and Piotr Przytycki; Tara Abrishami, Louis Esperet, and Ugo Giocanti) The answers to Problems 9 and 10 are no. Every counterexample to Problem 10 is also a counterexample to Problem 9, because one can get an orientation from every colouring that is invariant under colour-preserving automorphisms. A counterexample to Problem 10 can be found by using the infinite simple Thompson group V, which was shown by Bleak and Quick to have a representation with three generators, two of which have order two.

4. Sepehr Hajebi and Sophie Spirkl

Let $\eta(G)$ be the smallest cardinality of a hitting set of all maximum stable sets of a graph G. Hajebi, Li, and Spirkl introduced the following: A graph class $\mathcal{C}$ is *η-bounded* if there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that $\eta(G) \leq f(\omega(G))$ for every $G \in \mathcal{C}$.

There are several parallels between η-boundedness and χ-boundedness. Notably, an analogue of the Gyárfás–Sumner conjecture was proposed by Hajebi, Li, and Spirkl, that H-free graphs are η-bounded if H is a forest (only forests may have this property). But it seems to call for different methods. For instance, the following is open:

Problem 11. *Let H be a graph and let H^+ be the graph obtained from H by adding an isolated vertex. Assume that H-free graphs are η-bounded. Are H^+-free graphs η-bounded?*

The answer to Problem 11 is "yes" when H is the disjoint union of a star and arbitrarily many isolated vertices.

5. Meike Hatzel

A *directed bramble* in a digraph D is a collection of strongly connected subgraphs such that its elements pairwise intersect or are connected by edges in both directions. The *order* of a directed bramble is the minimum size of a vertex hitting set of all its elements and the bramble number of a digraph is the maximum order of a directed bramble it contains.

Having high bramble number is equivalent to having high directed treewidth. Thus the directed grid theorem by Kawarabayashi and Kreutzer implies that there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every digraph of bramble number at least $f(k)$ contains two directed cycles which are connected by k directed paths in each direction. However this function is highly exponential, so I would like to have a better one.

Problem 12. *Is there a polynomial function $p\colon \mathbb{N} \to \mathbb{N}$ such that every digraph of bramble number at least $p(k)$ contains two directed cycles which are connected by k directed paths in each direction?*

6. Kevin Hendrey

Consider the following variant of the cops and robbers game. There are k cops and one invisible robber, each occupying a vertex of a given graph. On the robber's turn, they may either remain on their current vertex, or move to an adjacent vertex. On the cops' turn, each cop may either remain on their current vertex or teleport to any other vertex of the graph. The position of the cops is always known to the robber, but the position of the robber is not revealed to cops until one of the cops occupies the same position as the robber, at which point the game ends and the cops win. We define the *blind cop-width* of the graph to be the minimum integer k such that k cops can guarantee victory after a finite number of turns.

It follows easily from the definition of pathwidth that this parameter is always at most one more than the pathwidth of a graph. However this bound is sometimes very far from tight. In fact, there are graphs of blind cop-width 3 which contain arbitrarily large complete binary trees as minors, and thus have arbitrarily large pathwidth.

To obtain a lower bound on the blind cop-width of a graph, we introduce the notion of balanced minors. We say that G contains H as a *balanced minor* if there is a minor model of H in G such that all branch sets contain exactly the same number of vertices. In a paper in preparation, Buffiére, Campbell, Hendrey, and Oum prove the following.

Theorem 6.1. *There is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that, given an integer k, every graph G that contains a complete binary tree of depth $f(k)$ as a balanced minor has blind cop-width at least k.*

This leads to the following question:

Problem 13. *Is there a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for every tree T, every graph which does not contain T as a balanced minor has blind cop-width at most $f(|V(T)|)$?*

This seems to be nontrivial even when $T = K_{1,4}$ and the host graph is also a tree.

7. Robert Hickingbotham

Let G be a graph. For a set of vertices $S \subseteq V(G)$, and an integer $r > 0$, let $N_G^{\leq r}(S)$ be the set of vertices in $V(G)$ which are at distance at most r from at least one vertex in S. A set of vertices $Z \subseteq V(G)$ is a (k, r)-*centered* set if Z is contained in $N_G^{\leq r}(S)$ for some set S of size at most k. The following two problems are inspired by Georgakopoulos and Papasoglu who initiated the investigation of coarse graph minor theory.

Problem 14. *For every forest F, does there exist $k, r \in \mathbb{N}$ such that every F-induced-minor-free graph G has a path-decomposition where each bag is a (k, r)-centered set?*

It may be true that the above holds with $k = |V(F)| - 1$ and $r = 1$. Analogous questions are also of interest with "forest" replaced by "planar graph" and "path-decomposition" replaced with "tree-decomposition".

Problem 15. *For every graph H, does there exist $r \in \mathbb{N}$ such that every H-induced-minor-free graph G contains an induced r-shallow minor which is H-minor-free?*

Hajebi raised the following conjecture.

Conjecture 2. *For every integer $t \geq 1$, every graph of large enough treewidth has an induced subgraph of treewidth t which is either complete, complete bipartite, or 2-degenerate.*

The *Hadwiger number* of a graph G is the largest integer $s \geq 1$ such that G has a K_s minor. The following is a (very slight) strengthening of Conjecture 2.

Problem 16. *For every integer $t \geq 1$, every graph of large enough Hadwiger number has an induced subgraph which is either K_t, $K_{t,t}$, or 2-degenerate with Hadwiger number t.*

8. Freddie Illingworth

Examples of graphs of large chromatic number seem to fall into roughly four categories: cliques, random (e.g. d-regular random graphs), iterative (e.g. Tutte, Mycielski, Zykov, Burling, ...), and other (e.g. Kneser, Shift, ...). For example, the clique K_d is $(d-1)$-regular and has chromatic number d; and an unusual way of explaining this large chromatic number would be to say that $|K_d|/\alpha(K_d) = d$. Also, the chromatic number of a random d-regular graph G is $(1/2 + o(1))\frac{d}{\log d} = d^{1-o(1)}$ and this again comes from the value of $|G|/\alpha(G)$. However, the graphs in the remaining two categories do not behave in this way: (i) they have large independence number relative to their order; and (ii) their chromatic number, although large, is much smaller than their maximum degree. Is this a general phenomenon?

Problem 17. *For each $c \in (0, 1)$ is there some $\varepsilon > 0$ such that the following holds? If every set of vertices S in a graph G satisfies*

$$\alpha(G[S]) > c \cdot |S|, \tag{1}$$

then $\chi(G) = O(\Delta^{1-\varepsilon})$, where $\Delta = \Delta(G)$.

Note that condition (1) rules out the clique and random examples. The best lower bound on the chromatic number for graphs satisfying (1) is $\Theta(\log \Delta)$ and comes from Kneser and Shift graphs. If $c = 1/2$, then G is bipartite (condition (1) forbids an odd cycle). If $c \geq 1/3$, then G is triangle-free, and so this is a concrete case to start attacking.

It is known that $\chi(G) = o(\Delta)$ for any $c \in (0, 1)$. This is because every graph G satisfying (1) has no clique of size at least c^{-1}; and so a theorem of Molloy gives $\chi(G) \leq 200c^{-1}\Delta\frac{\log\log\Delta}{\log\Delta}$. It is open whether $\chi(G) = O_c(\frac{\Delta}{\log\Delta})$ in general; but another result of Molloy proves this when G is triangle-free and so when $c \geq 1/3$.

Bounded fractional chromatic number implies condition (1) and so the following problem is a nice (weaker) variant of Problem 17.

Problem 18. *For each $C > 0$ is there some $\varepsilon > 0$ such that the following holds? If $\chi^*(G) \leqslant C$, then $\chi(G) = O(\Delta^{1-\varepsilon})$.*

Again Kneser and Shift graphs provide the best lower bound of $\Theta(\log \Delta)$.

9. Nina Kamčev

For a fixed graph H, we propose studying the existence of (n,H)*-local colourings* defined as follows. A collection $(f_v)_{v\in[n]}$ of n edge-colourings of K_n is called (n,H)*-local* if any (not necessarily induced) copy of H in K_n contains a vertex u such that f_u assigns different colours to all the edges in this H-copy. We define $g(n,H)$ as the smallest k for which there is an (n,H)-local collection of colourings using k colours. Clearly $g(n,H) \le \binom{n}{2}$, but Alon and Ben-Eliezer proved that $g(n,H) = O(n^{1-2/|V(H)|})$ for all H using the Local Lemma.

The function $g(n,H)$ was introduced by Alon and Ben-Eliezer, motivated by applications in Theoretical Computer Science (more specifically, computation of Boolean functions). The parameter $g(n,H)$ does look somewhat unnatural, but recently Janzer and Janzer published a very interesting paper applying a number of existing tools from extremal combinatorics and Ramsey theory.

Problem 19. *Estimate $g(n,K_r)$ and/or $g(n,C_r)$ up to a constant factor.*

For even r, Janzer and Janzer have shown (using a strengthening of the Bondy–Simonovits bound on the Turán number of C_r) that

$$g(n,K_r) \ge g(n,C_r) \ge \Omega(n^{1-\frac{4}{r+2}}),$$

so there is a gap in the exponent. For odd r, the gap is even larger, and it is intriguing that e.g. we do not know a priori that $g(n,K_{r+1}) \ge g(n,K_r)$.

10. Tung Nguyen

A classical argument shows that every n-vertex graph with no induced C_4 has a clique or stable set of size at least $n^{1/3}$. Can this be significantly improved? More precisely:

Problem 20. *Does there exist $\varepsilon > 0$ such that every n-vertex graph with no induced C_4 has a clique or stable set of size at least $n^{1/3+\varepsilon}$?*

This problem seems well-known and is connected to the problem of bounding the Ramsey numbers $R(C_4,K_t)$, as follows. Let H be a graph with $R(C_4,K_t)-1$ vertices, no C_4 subgraph, and no stable set of size s; in particular $\omega(H) \le 3$ and H has no induced C_4. Let G be the graph obtained from H by blowing up each vertex into a copy of $K_{\lfloor t/3\rfloor}$; then $\alpha(G),\omega(G) \le t$ and G has no induced C_4 while $|G| = \lfloor t/3\rfloor(R(C_4,K_t)-1)$. Thus, an affirmative answer to Problem 20 would confirm a conjecture of Erdős that $R(C_4,K_t) = o(t^{2-c})$ for some universal $c > 0$. On the other hand, if there is no $\varepsilon > 0$ satisfying Problem 20, this would be a (weak) evidence that $R(C_4,K_t) = t^{2-o(1)}$.

11. Tung Nguyen, Alex Scott, and Paul Seymour

For a tournament T, its *chromatic number* $\chi(T)$ is the smallest $k \ge 0$ such that there is partition of $V(T)$ into k subsets each inducing a transitive subtournament of T; and its *domination number* $\mathrm{dom}(T)$ is the smallest $\ell \ge 1$ such that for some vertex subset X of ℓ vertices, every vertex of T either is in X or has an inneighbour in X. Harutyunan, Le, Thomassé, and Wu proved that every tournament with huge domination number contains a bounded size subtournament with large chromatic number, which implies that every tournament with huge chromatic number contains a vertex whose outneighbourhood has large chromatic number. These results motivate the following three problems, the last two of which appeared in a survey of Nguyen, Scott, and Seymour.

Problem 21. *If T is a tournament, is there necessarily a vertex whose outneighbours include a cyclic triangle and whose inneighbours include a cyclic triangle? The answer is no if $\chi(T) \le 3$, yes if $\chi(T) \ge 5$, but what about when $\chi(T) = 4$?*

Problem 22. *If T is a tournament, is there necessarily a vertex v such that its outneighbourhood induces a subtournament with chromatic number at least $\frac{1}{2}\chi(T)$? Note that this is false if "at least" is replaced by "more than" (the 11-vertex Paley tournament is a counterexample).*

Problem 23. *If $\mathrm{dom}(T) \ge 10^{10^{10}}$, is it necessary that the tournament T' obtained by reversing all edges of T has a subtournament T'' with $\mathrm{dom}(T'') \ge 3$? More generally, for every $k \ge 3$, does there exist $\ell \ge 3$ such that T' has a subtournament T'' with $\mathrm{dom}(T'') \ge k$ whenever $\mathrm{dom}(T) \ge \ell$?*

12. Sergey Norin

Georgakopoulos and Papasoglu proposed systematic investigation of coarse graph minor theory. The coarse notions corresponding to graph minor and subgraph are the following. Let H and G be graphs. A *K-fat H-minor in G* is a collection of connected subgraphs $\{B_v : v \in V(H)\}$ and paths $\{P_e : e \in e(H)\}$ in G such that

- $V(B_v) \cap V(P_e) \neq \emptyset$ whenever v is an end of e; and
- for all pairs of distinct $X, Y \in \{B_v : v \in V(H)\} \cup \{P_e : e \in e(H)\}$ not covered by the above condition, we have $d_G(X, Y) \geq K$, where $d_G(\cdot, \cdot)$ denotes the usual graph distance in G.

(Then a graph G has a 1-fat H-minor if and only if G has a (usual) minor isomorphic to H.)
An (M, A)-*quasi-isometric embedding* of a graph H into a graph G is a map $\phi\colon V(H) \to V(G)$ such that

$$M^{-1}d_H(x, y) - A \leq d_G(\phi(x), \phi(y)) \leq Md_H(x, y) + A.$$

Georgakopoulos and Papasoglu conjectured the following coarse version of the Kuratowski–Wagner theorem.

Conjecture 3. *For every $K > 0$ there exist $M, A > 0$ such that if H is a graph with no K-fat K_5-minor and no K-fat $K_{3,3}$-minor then there exists an (M, A)-quasi-isometric embedding of H into a planar graph.*

They have also conjectured a coarse version of Menger's theorem, which was proven to be false by Nguyen, Scott, and Seymour.

Problem 24. *Is the family of Nguyen–Scott–Seymour counterexamples to the coarse Menger conjecture also a counterexample to Conjecture 3?*

Progress.

- (James Davies) The answer to Problem 24 is no; these graphs contain increasingly fat K_5-minors.

13. Sang-il Oum

Dabrowski, Dross, Jeong, Kanté, Kwon, Oum, and Paulusma posed the following conjecture on algorithms for pivot-minors.

Problem 25. *Prove that for each fixed graph H, there is a polynomial-time algorithm to decide whether the input graph has a pivot-minor isomorphic to H.*

Smallest unknown cases are: $H = K_4$ and H is the disjoint union of K_1 and K_3.

14. Vaidy Sivaraman

A graph is *squco* (*square-complementary*) if its square is isomorphic to its complement. This was first investigated by Seymour Schuster in 1980. Recently, Darda, Milanič, and Pizaña proved certain structural features about these graphs. In particular, they proved that such graphs cannot have girth 6. It is not hard to show that nontrivial squco graphs must contain an induced path of length 4.

Problem 26. *Does every nontrivial squco graph contain an induced path of length 5?*

Problem 27. *Is the chromatic number of squco graphs bounded?*

15. Jane Tan

A *c-strong colouring* (for $c \geq 2$ fixed) of a hypergraph $H = (V, E)$ is an assignment of colours to V such that every edge $e \in E$ has at least $\min(c, |e|)$ distinct colours. When $c = 2$ this coincides with weak colourings, whilst strong colourings can be thought of as being ∞-strong. The *c-strong chromatic number* of H, denoted $\chi(H, c)$, is then the minimum number of colours required to c-strongly colour H. Blais, Weinstein, and Yoshida studied the c-strong chromatic number of t-intersecting hypergraphs. Specifically, they asked: letting $\chi(t, c)$ denote the minimum number of colours that suffice to c-strong colour any t-intersecting hypergraph, for which $t \geq 0$ and $c \geq 2$ is $\chi(t, c)$ finite? They proved that $\chi(t, c)$ is finite whenever $t \geq c$ and $(t, c) = (1, 2)$, and unbounded whenever $t \leq c - 2$. Colucci and Gyárfás have also verified that $\chi(2, 3)$ is finite (in fact, 5 colours suffice and this is tight). This leaves the remaining $t = c - 1$ cases (with $t \geq 3$) open. The proofs of the known cases are not very long, so maybe the following is approachable:

Problem 28. *Determine whether $\chi(c - 1, c)$ is finite for every $c \geq 4$, or at least for $c = 4$.*

If we can do this, the next goal is to determine the values $\chi(t, c)$ precisely. For $c \geq 2$, Blais, Weinstein, and Yoshida have a general lower bound $\chi(c-1, c) \geq 2c-1$, which they believe could be tight although the evidence is just that $\chi(1, 2) = 3$ and $\chi(2, 3) = 5$.

Problem 29. *Does $\chi(c, c) = 2(c-1)$ for every $c > 2$?*

16. Stephan Thomassé

An (infinite) graph G is *rigid* if G is not an induced subgraph of $G \setminus v$, for all vertices v of G. For instance, the two-way infinite path is rigid, but the one-way is not. A (countably infinite) graph G is *rich* if, with positive probability, a random induced subgraph S of G contains an induced copy of G. For instance every countable cograph is rich (not obvious). Given a class $\mathcal{C}$ of finite graphs, a *limit* is a countably infinite graph with all finite induced subgraphs in $\mathcal{C}$.

I propose the following "rich versus rigid dichotomy" conjecture:

Problem 30. *For every class $\mathcal{C}$, either there exists a rigid limit, or all limits are rich.*

Some remarks:

(1) I spent more time finding a fancy name to the question than actually trying to solve it. This is maybe nonsensical, but otherwise we can speak of "rich" or "rigid" classes of graphs.
(2) The probability makes sense and corresponds to keeping every vertex with probability 1/2. Indeed, the collection of subsets of vertices of G containing a copy of G is measurable.
(3) If you do not like probabilities, a Maker/Breaker definition of "rich" is possible: each player alternatively select a finite subset of vertices to keep/delete. A rich G would be: if Maker starts, she can keep a subset of vertices inducing a copy of G. The fact that one of the players has a winning strategy follows from Baire property. The two definitions of "rich" (Baire/Measure) do not coincide in general, but could be the same in "non-rigid classes".
(4) The fact that cographs are rich follows (I sort of remember) from the fact that they are well-quasi-ordered under induced subgraphs. It could even be true that a class is rich iff it is well-quasi-ordered for every finite vertex label, but here we need to define "class" as the set of all finite induced subgraphs of a countable graph. Note that this is reminiscent of the existence of a "strict minor" in every countable graph, as conjectured by Seymour.
(5) This question is very much influenced by the work of Maurice Pouzet, who recently passed away (on December 31). He spent a considerable part of his research on understanding well-quasi-ordered classes, and brought many researchers to the subject.

17. Nicolas Trotignon

Problem 31. *Is there a polynomial time algorithm that decides whether an input graph contains $K_{3,3}$ as an induced minor?*

This question may be harder than it looks. This is because in a recent joint work with Dallard, Dumas, Hilaire, Milanič, and Perez, we found a polynomial algorithm to detect $K_{2,3}$ as an induced minor, and this was already nontrivial, at least to us.

18. David Wood

Distel et al. asked the following question:

Problem 32. *Is there a constant c such that every n-vertex planar graph G is contained in $H \boxtimes K_m$ where H has pathwidth at most c and $m \in O(\sqrt{n})$?*

Reasons to think the answer is "yes":

- It is true with "pathwidth" replaced by "treewidth".
- It would imply that n-vertex planar graphs have pathwidth $O(\sqrt{n})$, which is true.
- It is true for graphs of bounded treewidth (without the planar assumption), where H is a star and thus $c = 1$.

The above question can also be asked for K_t-minor-free graphs (either allowing c to depend on t or as an absolute constant). Also note that Dvořák and Wood showed that the answer is "no" if "pathwidth" is replaced by "treedepth", where grid graphs are the negative examples.

Dujmović–Joret–Micek–Morin–Wood worked on this last week, and made significant progress. So any resulting paper would include all these authors.

Reed and Seymour asked the following question: For any graph G, is there a partition $\mathcal{P}$ of $V(G)$ such that the quotient of $\mathcal{P}$ is chordal, and for each part $X \in \mathcal{P}$, the induced subgraph $G[X]$ is connected and bipartite? A positive answer to this question would imply that K_{t+1}-minor-free graphs are $2t$-colourable, which would be a major breakthrough on Hadwiger's conjecture. Scott, Seymour, and Wood showed that the answer is "no", and remains "no" with "chordal" replaced by "perfect" (and under various other weakenings). However, their construction relies heavily on clique-separators, which one can assume do not exist when colouring in a hereditary family. The following question arises:

Problem 33. *For any graph G with no clique separator, is there a partition $\mathcal{P}$ of $V(G)$ such that the quotient of $\mathcal{P}$ is perfect, and for each part $X \in \mathcal{P}$, the induced subgraph $G[X]$ is connected and bipartite?*

A positive answer to this question would still imply that K_{t+1}-minor-free graphs are $2t$-colourable. It would still be interesting with "bipartite" replaced by "c-colourable" for some constant c.

The *growth* of a graph G is the function $f_G\colon \mathbb{N} \to \mathbb{N}$ where $f_G(r)$ is the maximum number of vertices in a subgraph of G with radius at most r. We say G has *degree-d polynomial growth* if $f_G(r) \leq cr^d$ for all r (where c, d are constants). For example, the d-dimensional grid (with crosses) $P_1 \boxtimes \cdots \boxtimes P_d$ has growth $(2r+1)^d$. (Here each P_i is the 2-way infinite path.) Conversely, Krathgammer and Lee showed that every graph with degree d-polynomial growth is a subgraph of the infinite $O(d \log d)$-dimensional grid, and $O(d \log d)$ is best possible. Campbell et al. conjectured that every graph with degree-d polynomial growth is a subgraph of the product of d trees, each with linear growth, and a complete graph of bounded-size. More precisely:

Problem 34. *There are functions g and h such that every graph with growth cr^d (for all r) is a subgraph of $T_1 \boxtimes \cdots \boxtimes T_d \boxtimes K_{g(c,d)}$, where each T_i is a tree with growth at most $h(c,d)r$.*

This would be a rough characterisation, in the sense that $T_1 \boxtimes \cdots \boxtimes T_d \boxtimes K_{g(c,d)}$ also has degree-d polynomial growth. This conjecture is open even in the linear growth case ($d = 1$), although Campbell et al. did prove that every such graph is a subgraph of $T \boxtimes K_{882c^3}$ where T is a tree, but they say nothing about the growth of T.

Comments.

- (Rose McCarty) Problem 34 aims to give a precise characterisation - in d - of graphs whose balls of radius r have size $\mathcal{O}(r^d)$. (Here, $r \to \infty$ and d is a fixed positive integer.) Owen Huang and I have been working on this problem and suspect that it is actually false. Our intuition is that the problem statement is missing something like a semidirect product.

  For Cayley graphs of finitely generated groups, we can show that Problem 34 is true for $d = 1, 2, 3$, and that each of the trees T_i is just the two-way infinite path P_∞ (see Owen's talk at JMM). Our proof is just an application of the fact that, for $d = 1, 2, 3$, any group of growth $\mathcal{O}(r^d)$ is virtually $\mathbb{Z}^d$; this is stated somewhere in the book by Meier. For $d = 4$, however, the discrete Heisenberg group $H_3(\mathbb{Z})$ has growth of order $\mathcal{O}(r^4)$ but is *not* virtually $\mathbb{Z}^4$. I would be surprised if this group does not yield a counterexample.

  Problem 35. *Show that any Cayley graph of the discrete Heisenberg group $H_3(\mathbb{Z})$ is a counterexample to Problem 34.*

  Owen and I have some ideas about how to use the vertex-transitivity of Cayley graphs. And we know that it does not matter which Cayley graph you take. So we really just want to prove the following.

  Problem 36. *Let G be the infinite graph depicted in Fig. 2, which is a Cayley graph of the discrete Heisenberg group $H_3(\mathbb{Z})$. Show that there is no integer N so that G is a subgraph of $P_\infty \boxtimes P_\infty \boxtimes P_\infty \boxtimes P_\infty \boxtimes K_N$.*

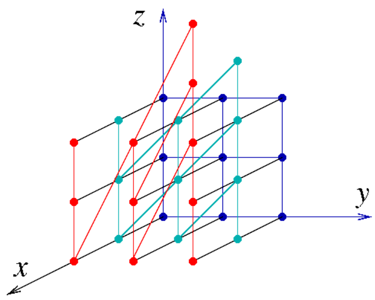

FIGURE 2. A Cayley graph of $H_3(\mathbb{Z})$.